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Some characterizations about Generalized Hölder-Zygmund Spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$

Preprint/Working Paper
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February 2016

Abstract

Generalized Hölder-Zygmund spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ were recently introduced in [42] and are based on a generalization of Besov spaces (see e.g. [21]). Under some conditions, generalized Hölder-Zygmund and Besov spaces are equal ([56]). It has been proved that most properties of classical Hölder-Zygmund spaces are held for spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$, which constitute a particular case of spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ with $N_j = 2^j$ ([42, 43]). The goal of the present document is to prove that most of these properties are kept for $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ spaces.

Contents

Some basic notations

- $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ denotes the set of natural numbers.
- $\mathbb{N}^* = \{1, 2, 3, ...\}$.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R}^d = \{(x_1, ..., x_d) : x_i \in \mathbb{R} \quad \forall i \in \{1, ..., d\}\}$ denotes the euclidean space of dimension $d \in \mathbb{N}^*$.
- $\mathbb{N}_0^d = \{(\alpha_1, ..., \alpha_d) : \alpha_i \in \mathbb{N}_0 \quad \forall i \in \{1, ..., d\}\}$ denotes the set of natural numbers of dimension $d \in \mathbb{N}^*$ (also called the set of *multi-indices*).
- $B(x, R)$ denotes the open ball with center $x \in \mathbb{R}^d$ and radius $R > 0$.
- $B(x, \leq R)$ denotes the closed ball with center $x \in \mathbb{R}^d$ and radius $R > 0$.
- $\lfloor x \rfloor = \sup\{m \in \mathbb{Z} : m \leq x\}$ denotes the floor of $x \in \mathbb{R}$.
- $\lceil x \rceil = \inf\{m \in \mathbb{Z} : m \geq x\}$ denotes the ceiling of $x \in \mathbb{R}$.
- $(x)_+ = \max\{x, 0\}$ denotes the positive value of a real number $x \in \mathbb{R}$.
- $\alpha!$ denotes the value $\alpha! = \alpha_1!...\alpha_d!$ if $\alpha \in \mathbb{N}_0^d$ is a multi-index.
- $|\alpha|$ denotes the value $\alpha_1 + ... + \alpha_d$ if $\alpha \in \mathbb{N}_0^d$ is a multi-index.
- $\binom{m}{j} = \frac{m!}{(m-j)!j!}$ where $m, j \in \mathbb{N}_0$ and $m \geq j$.
- $C(A)$ denotes the space of continuous functions defined on $A \subseteq \mathbb{R}^d$.
- $C^p(\Omega)$ ($p \in \mathbb{N}_0 \cup \{\infty\}$) denotes the space of functions which are p-times continuously differentiable on Ω (where Ω is an open set of $\mathbb{R}^d$).
- $D(\Omega)$ denotes the subspace of $C^\infty(\Omega)$ made of compactly supported functions on $\Omega \subseteq \mathbb{R}^d$.
- $L^p(A)$ denotes the space of measurable functions on A satisfying $\|f\|_{L^p(A)} = \left(\int_A |f(x)|^p \, dx\right)^{1/p} < \infty$ (where $p \in]0, \infty[$ and A is a measurable set of $\mathbb{R}^d$).
- $L^\infty(A)$ denotes the space of measurable functions on A satisfying $\|f\|_{L^\infty(A)} = \sup_{ppA} |f| < \infty$ (where A is a measurable set of $\mathbb{R}^d$).
- $\|f\|_E = \sup_{x \in E} |f(x)|$ where f is function defined on $E \subseteq \mathbb{R}^d$.

- $L^p = L^p(\mathbb{R}^d)$ if $p \in]0, \infty]$.
- $l^p = l^p(\mathbb{N}_0)$ denotes the space of sequences $(a_n)_{n\in\mathbb{N}_0}$ such that $\|(a_n)_{n\in\mathbb{N}_0}\|_{l^p} = (\sum_{n=0}^{\infty} |a_n|^p)^{1/p} < \infty$ ($p \in [0, \infty[$).
- $l^\infty = l^\infty(\mathbb{N}_0)$ denotes the space of sequences $(a_n)_{n\in\mathbb{N}_0}$ such that $\|(a_n)_{n\in\mathbb{N}_0}\|_{l^\infty} = \sup_{n\in\mathbb{N}_0} |a_n| < \infty$.
- $D'(\mathbb{R}^d)$ denotes the space of distributions on $\mathbb{R}^d$.
- $\mathcal{S}(\mathbb{R}^d) = \mathcal{S}$ denotes the Schwartz space, composed of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$.
- $\mathcal{S}'(\mathbb{R}^d) = \mathcal{S}'$ denotes the topological dual of the space $\mathcal{S}$, i.e. the space of all tempered distributions on $\mathbb{R}^d$.
- $\mathcal{F}f$ denotes the Fourier transform of the distribution $f \in \mathcal{S}'(\mathbb{R}^d)$. If the function f belongs to $L^1(\mathbb{R}^d)$, this expression is equal to $\mathcal{F}f(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\, dx$.
- $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of the distribution $f \in \mathcal{S}'(\mathbb{R}^d)$. If the function f belongs to $L^1(\mathbb{R}^d)$, this expression is equal to $\mathcal{F}^{-1}f(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{+ix\xi} f(x)\, dx$.

Chapter 1

Definition of generalized Hölder-Zygmund spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$

The core of generalized Besov and Hölder spaces relies on the notion of admissible sequence. We briefly recall the concept.

Definition 1. A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ of real positive numbers is called an *admissible sequence* if there exists two positive constants d_0 and d_1 such that

$$d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j, \qquad j \in \mathbb{N}_0\,. \tag{1.1}$$

In the following, we will only consider admissible sequences which are not identically zero. This implies in particular that no element can be equal to 0.

For an admissible sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$, let

$$\underline{\sigma}_j := \inf_{k\geq 0} \frac{\sigma_{j+k}}{\sigma_k} \qquad \text{and} \qquad \overline{\sigma}_j := \sup_{k\geq 0} \frac{\sigma_{j+k}}{\sigma_k}, \qquad j \in \mathbb{N}_0\,.$$

The *lower and upper Boyd indices* are respectively defined by

$$\underline{s}(\sigma) := \lim_{j\to+\infty} \frac{\log_2(\underline{\sigma}_j)}{j} \quad \text{and} \quad \overline{s}(\sigma) := \lim_{j\to+\infty} \frac{\log_2(\overline{\sigma}_j)}{j}.$$

It is known that two previous limits exist and are finite (see for example [41, 42]).

The Boyd index $\overline{s}(\sigma)$ of an admissible sequence σ describes the asymptotic behaviour of $\overline{\sigma}_j$; similarly, the index $\underline{s}(\sigma)$ describes the asymptotic behaviour of $\underline{\sigma}_j$. We notice that for $\varepsilon > 0$, there exist two positive constants $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$ such that

$$c_1 2^{(\underline{s}(\sigma)-\varepsilon)j} \leq \underline{\sigma}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\sigma}_j \leq c_2 2^{(\overline{s}(\sigma)+\varepsilon)j}, \qquad j,k \in \mathbb{N}_0\,. \tag{1.2}$$

Conversely, $\underline{s}(\sigma)$ and $\overline{s}(\sigma)$ are respectively the biggest and the lowest real numbers satisfying inequalities (1.2) for every $\varepsilon > 0$.

Definition 2. An admissible sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ is *strong of order* $N \in \mathbb{N}^*$ if it satisfies

$$\sum_{j=0}^{J} 2^{Nj}\sigma_j \leq C2^{NJ}\sigma_J, \tag{1.3}$$

$$\sum_{j=J}^{+\infty} 2^{(N-1)j}\sigma_j \leq C2^{(N-1)J}\sigma_J \tag{1.4}$$

for all $J \in \mathbb{N}_0$.

We have the following results.

Lemma 3. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be two admissible sequences such that $\underline{s}(N) > 0$. If $M \in \mathbb{N}_0$ is such that $M > \overline{s}(\sigma^{-1})\underline{s}(N)^{-1}$, then there exists a constant $C > 0$ such that*

$$\sum_{j=0}^{J} N_j^M \sigma_j \leq C N_J^M \sigma_J \quad \forall J \in \mathbb{N}_0\,.$$

Lemma 4. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be two admissible sequences such that $\overline{s}(N) > 0$. If $L \in \mathbb{N}_0$ is such that $L < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1}$, then there exists a constant $C > 0$ such that*

$$\sum_{j=J}^{+\infty} N_j^L \sigma_j \leq C N_J^L \sigma_J \quad \forall J \in \mathbb{N}_0\,.$$

Let us recall the definition of generalized Hölder-Zygmund spaces. This definition relies on the principles exposed in [41].

Definition 5. Let $\alpha > 0$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be two admissible sequences. The *generalized Hölder-Zygmund space* $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ is defined by

$$\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d) = \{f \in L^\infty(\mathbb{R}^d) : \sup_{j\in\mathbb{N}_0} \sigma_j^{-1} \sup_{|h|\leq N_j^{-1}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty} < \infty\}.$$

If $N_j = 2^j$ ($j \in \mathbb{N}_0$), then we note $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ instead of $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ to simplify notations.

Proposition 6. *Let $\alpha > 0$, σ and N be two admissible sequences. The space $(\Lambda^{\alpha}_{\sigma,N}, \|.\|_{\Lambda^{\alpha}_{\sigma,N}})$ is a Banach space.*

The next result links the generalized Hölder-Zygmund spaces and the generalized Besov spaces studied in [21].

Proposition 7. *1. Let σ and N be two admissible sequences such that $\underline{N}_1 > 1$ and $\underline{s}(\sigma^{-1})\overline{s}(N)^{-1} > 0$. We have*

$$B^{\sigma^{-1},N}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d) = \Lambda^{M-1}_{\sigma,N}(\mathbb{R}^d)$$

for all $M \in \mathbb{N}_0$ such that $M > \overline{s}(\sigma^{-1})\underline{s}(N)^{-1}$.

2. Let σ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$. We have

$$B^{\sigma^{-1}}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = \Lambda^{\sigma,M-1}(\mathbb{R}^d)$$

for all $M \in \mathbb{N}_0$ such that $M > \overline{s}(\sigma^{-1})$.

Chapter 2

Characterizations and basic properties of generalized Hölder-Zygmund spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$

The purpose of this section is to generalize the results obtained in [41, 42, 43] to the more general spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$. Basically, we show that main properties of spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ can be transposed to spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$. This indicates that those spaces provide a generalization of Hölder spaces at a higher level that keeps them interesting. Moreover, these properties hold for spaces $B^{\sigma,N}_{\infty,\infty}$ ([56]).

2.1 Some preliminary results

In this section, we present two basic results that are useful in the sequel. Let $\rho \in C^{\infty}(\mathbb{R}^d)$ a compactly supported function whose support is included in the closed ball $B(0,<1)$ and which satisfies the following conditions:

1. $0 \leq \rho \leq 1$;
2. $\int_{\mathbb{R}^d} \rho(x)dx = 1$;
3. ρ is a radial function, i.e. $|x| = |y| \Rightarrow \rho(x) = \rho(y)$.

Let us denote $\rho_\delta(x) := \delta^{-d}\rho(x/\delta)$ $\forall \delta > 0$. The same proof as lemma 4.1 in [42] leads to the following result.

Proposition 8. *Let $m \in \mathbb{N}^*$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence, $N = (N_j)_{j\in\mathbb{N}^*}$ be a sequence of positive numbers and $f \in L^1_{loc}(\mathbb{R})$ be a function such that $\sup_{|h|\leq N_j^{-1}} \|\Delta_h^m f\|_{L^\infty} \leq C\sigma_j$ for all $j \in \mathbb{N}_0$. There exists $\Phi \in D(\mathbb{R})$ such that*

$$\sup_{\delta \leq N_j^{-1}} \|f \star \Phi_\delta - f\|_{L^\infty} \leq C\sigma_j, \;\; \forall j \in \mathbb{N}_0 .$$

Lemma 9. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and $N = (N_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers such that there exists $d_0 > 0$ satisfying*

$$d_0 N_j \leq N_{j+1} \quad \forall j \in \mathbb{N}_0 \,.$$

If $f \in L^1_{loc}(\mathbb{R})$ is a function satisfying

$$\|f \star \rho_{N_j^{-1}} - f\|_{L^\infty(\mathbb{R})} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0$$

then, for all $k \in \mathbb{N}_0$, we have

$$\|D^k \left(f \star \rho_{N_j^{-1}} - f \star \rho_{N_{j-1}^{-1}}\right) \|_{L^\infty(\mathbb{R})} \leq C N_j^k \sigma_j \qquad \forall j \in \mathbb{N}^* \,.$$

Proof. Let us note that

$$\begin{aligned} f \star \rho_{N_j^{-1}} - f \star \rho_{N_{j-1}^{-1}} =& \rho_{N_j^{-1}} \star (f \star \rho_{N_j^{-1}} - f \star \rho_{N_{j-1}^{-1}}) \\ &+ \rho_{N_j^{-1}} \star (f - f \star \rho_{N_j^{-1}}) \\ &- \rho_{N_{j-1}^{-1}} \star (f - f \star \rho_{N_j^{-1}}). \end{aligned}$$

Using Hausdorff-Young inequalities (see appendix), one gets

$$\begin{aligned} \|D^k(\rho_{N_j^{-1}} \star (f \star \rho_{N_j^{-1}} - f \star \rho_{N_{j-1}^{-1}}))\|_{L^\infty} &\leq ||D^k \rho_{N_j^{-1}}||_{L^1} ||f \star \rho_{N_j^{-1}} - f \star \rho_{N_{j-1}^{-1}}||_{L^\infty} \\ &\leq C N_j^k \left(||f \star \rho_{N_j^{-1}} - f||_{L^\infty} + ||f - f \star \rho_{N_{j-1}^{-1}}||_{L^\infty}\right). \end{aligned}$$

Then, we have

$$\|D^k(\rho_{N_j^{-1}} \star (f \star \rho_{N_j^{-1}} - f \star \rho_{N_{j-1}^{-1}}))\|_{L^\infty} \leq C N_j^k (\sigma_j + \sigma_{j-1}) \leq C N_j^k \sigma_j$$

for all $j \in \mathbb{N}_0$. The two other terms in the decomposition of $f \star \rho_{N_j^{-1}} - f \star \rho_{N_{j-1}^{-1}}$ can be handled in the same way. □

2.2 Generalized Hölder spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ and $C^k(\mathbb{R}^d)$ spaces

It is known that spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ can be linked with $C^k(\mathbb{R}^d)$ spaces ([42]). The goal of this section is to show a similar result for spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$.

Proposition 10. *Let $m, k \in \mathbb{N}^*$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and $N = (N_j)_{j\in\mathbb{N}_0}$ be an non-decreasing sequence of positive numbers such that*

$$\sum_{j=1}^{+\infty} N_j^k \sigma_j < \infty.$$

If the function $f \in L^\infty(\mathbb{R}^d)$ satisfies

$$\sup_{|h|\leq N_j^{-1}} \|\Delta_h^m f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0$$

then f is k-times continuously differentiable (in the sense that f coincides almost everywhere on $\mathbb{R}^d$ with a k-times continuously differentiable function).

Proof. Let Φ be the function given by proposition 8 and let us set

$$f_1 := f \star \Phi_{N_1^{-1}}, \quad f_j := f \star (\Phi_{N_j^{-1}} - \Phi_{N_{j-1}^{-1}}) \quad \forall j \in \mathbb{N}_0, \quad j > 1.$$

We have $||f_j||_{L^\infty} \leq C\sigma_j$ for all $j \in \mathbb{N}^*$, where the constant C does not depend on j. So, the series $\sum_{j=1}^{+\infty} f_j$ converges uniformly on $\mathbb{R}^d$ to a function which coincides almost everywhere with f. Moreover,

$$\|D^\alpha f_j\|_{L^\infty} \leq CN_j^k \sigma_j \quad \forall j \in \mathbb{N}^*, |\alpha| \leq k.$$

One can conclude, since the series $\sum_{j=1}^{\infty} D^\alpha f_j$ converges uniformly. □

We have the following result without any assumption on the sequence N.

Proposition 11. *Let $\alpha > 0$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and $N = (N_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers such that*

$$\sum_{j=1}^{+\infty} \sigma_j < \infty.$$

We have

$$\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$$

(in the sense that each element of $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ coincides almost everywhere with a continuous function on $\mathbb{R}^d$).

Under the assumptions of theorem 4.1 in [56], the result can be rewritten in the following way.

Corollary 12. *Let $k \in \mathbb{N}_0$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}^*}$ be two admissible sequences such that $\underline{N}_1 > 1$, $\underline{s}(\sigma^{-1}) > 0$ and*

$$\sum_{j=1}^{+\infty} N_j^k \sigma_j < \infty.$$

We have

$$B^{\sigma^{-1},N}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\bar{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d) \subseteq C^k(\mathbb{R}^d).$$

Remark 13. Let $m \in \mathbb{N}_0$ and N_j be a non-decreasing sequence such that $N_j \to +\infty$. If σ is an admissible sequence satisfying

$$\sum_{j=1}^{+\infty} N_j^{m+1}\sigma_j < \infty,$$

then the space $\Lambda^{m}_{\sigma,N}(\mathbb{R}^d)$ is composed of constant functions.

Indeed, if the function f belongs to this space, we know that f belongs to $C^{m+1}(\mathbb{R}^d)$ by proposition 10. Moreover, we have

$$\frac{|\Delta^{m+1}_{N_j^{-1}e_i} f(x)|}{N_j^{-(m+1)}} \to |D^{m+1}_{x_i} f(x)|$$

and

$$\frac{|\Delta^{m+1}_{N_j^{-1}e_i} f(x)|}{N_j^{-(m+1)}} \leq C N_j^{m+1}\sigma_j \to 0 \quad \text{if } j \to +\infty.$$

So, we have $D^{m+1}_{x_i} f = 0$ for all $i \in \{1,\dots,d\}$. For all $j \in \{1,\dots,d\}$, the function f can be written as

$$f(x_1,...,x_j,...,x_d) = \sum_{i=0}^{m} a^{(i)}_{x_1,\dots,x_{j-1},x_{j+1},\dots,x_d} x_j^i.$$

Since $f \in L^\infty$, we thus get

$$f(x_1,...,x_j,...,x_d) = a^{(0)}_{x_1,\dots,x_{j-1},x_{j+1},\dots,x_d},$$

so that $D_{x_j} f = 0$. This implies that f is a constant function.

2.3 A characterization of the spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ in terms of convolution

It is shown in [42] that spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ can be characterized through approximations by the convolution product of their own elements with a smooth function. The quality of the approximation is directly linked with the sequence σ. The goal of this section is to generalize this result to the spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$.

Proposition 14. *Let $m \in \mathbb{N}^*$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and $N = (N_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers such that*

$$\sum_{j=1}^{J} N_j^m \sigma_j \leq C N_J^m \sigma_J \quad \forall J \in \mathbb{N}_0 \tag{2.1}$$

and

$$\sum_{j=J}^{+\infty} \sigma_j \leq C\sigma_J \quad \forall J \in \mathbb{N}_0 \,. \tag{2.2}$$

If $f \in L^\infty(\mathbb{R}^d)$ is a function for which there exists $\Phi \in D(\mathbb{R}^d)$ satisfying

$$\|f \star \Phi_{N_j^{-1}} - f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0,$$

then we have

$$\sup_{|h|\leq N_j^{-1}} \|\Delta_h^m f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0 \,.$$

Proof. We keep the same notations as in proof of proposition 10. We know that $\Delta_h^m f = \sum_{j=1}^{+\infty} \Delta_h^m f_j$ with uniform convergence on $\mathbb{R}^d$. For all $J \in \mathbb{N}^*$, we have

$$\begin{aligned}
\|\Delta_h^m f\|_{L^\infty} &\leq \sum_{j=1}^{J} \|\Delta_h^m f_j\|_{L^\infty} + \sum_{j=J+1}^{+\infty} \|\Delta_h^m f_j\|_{L^\infty} \\
&\leq C \sum_{j=1}^{J} |h|^m \sup_{|\alpha|=m} \|D^\alpha f_j\|_{L^\infty} + \sum_{j=J+1}^{+\infty} 2^m \|f_j\|_{L^\infty} \\
&\leq C|h|^m \sum_{j=1}^{J} N_j^m \sigma_j + C \sum_{j=J+1}^{+\infty} \sigma_j,
\end{aligned}$$

so

$$\|\Delta_h^m f\|_{L^\infty} \leq C(1 + |h|^m N_J^m)\sigma_J.$$

One can conclude, since we have

$$\sup_{|h|<N_J^{-1}} \|\Delta_h^m f\|_{L^\infty} \leq C\sigma_J.$$

□

Corollary 15. (D.K., S. Nicolay) Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be two admissible sequences such that $\underline{N}_1 > 1$ and $\underline{s}(\sigma^{-1}) > 0$. We have

$$B^{\sigma^{-1},N}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d) = \left\{ f \in L^\infty(\mathbb{R}^d) : \exists \Phi \in D(\mathbb{R}^d) \quad \text{such that} \quad \sup_{j\in\mathbb{N}_0} \left(\sigma_j^{-1} \sup_{\delta \leq N_j^{-1}} \|f \star \Phi_\delta - f\|_{L^\infty} \right) < \infty \right\}.$$

Proof. This is a consequence of proposition 14. Inequality (2.2) is satisfied because $\underline{s}(\sigma^{-1}) > 0$ and inequality (2.1) is satisfied for any natural numbers m satisfying $m > \overline{s}(\sigma^{-1})\underline{s}(N)^{-1}$.
□

2.4 A polynomial characterization of the spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$

It is shown in [42] that spaces $\Lambda^{\sigma,\alpha}(\mathbb{R}^d)$ can be characterized in terms of polynomial approximations, where the quality of the approximation is linked with the sequence σ. We prove in this section that this result can be extended to the spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$.

Notation 16. Let $\mathbb{P}_m$ denote the set of polynomials of degree less or equal to $m \in \mathbb{N}_0$.

Theorem 17. *Let $m \in \mathbb{N}^*$, $f \in L^\infty(\mathbb{R}^d)$ be a continuous function on $\mathbb{R}^d$, $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and $N = (N_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers such that there exists $k_0 \in \mathbb{N}_0$ satisfying*

$$2N_j \leq N_k \quad \text{for all } j \text{ and } k \quad \text{such that } j + k_0 \leq k.$$

The following assertions are equivalent:

1. *there exists a constant $C > 0$ such that $\sup_{|h|\leq N_j^{-1}} \|\Delta_h^m f\|_{L^\infty} \leq C\sigma_j \ \forall j \in \mathbb{N}^*$;*
2. *there exist a constant $C > 0$ and a natural number J such that*

$$\inf_{P\in\mathbb{P}_{m-1}} \|f - P\|_{L^\infty(B(x,N_j^{-1}))} \leq C\sigma_j \quad \forall x \in \mathbb{R}^d, j \geq J.$$

Proof. The proof that $1 \Rightarrow 2$ is immediate from Whitney theorem ([11]). Let us prove that $2 \Rightarrow 1$. For all $x \in \mathbb{R}^d$ and $j \geq J$, there exists a polynomial $P \in \mathbb{P}_{m-1}$ such that

$$\sup_{y\in B(x,N_j^{-1})} |f(y) - P(y)| \leq C\sigma_j.$$

For any polynomial $P \in \mathbb{P}_{m-1}$, we have

$$\begin{aligned} |\Delta_h^m f(x)| &= |\Delta_h^m (f - P)(x)| \\ &\leq 2^m \sup_{y\in\{x,\ldots,x+mh\}} |f(y) - P(y)|. \end{aligned}$$

By assumption, there exists a natural number k_1 such that

$$N_k^{-1} \leq \frac{N_j^{-1}}{m} \quad \forall j + k_1 \leq k.$$

For all $|h| \leq N_{j+k_1}^{-1}$, we find

$$\begin{aligned} |\Delta_h^m f(x)| &\leq 2^m C\sigma_j \\ &\leq 2^m C d_0^{-k_1} \sigma_{j+k_1} \end{aligned}$$

which is sufficient to conclude. □

Remark 18. The assumption on N in theorem 17 is satisfied by all strongly increasing sequences.

Corollary 19. (D.K., S. Nicolay) Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be two admissible sequences such that $\underline{N}_1 > 1$ and $\underline{s}(\sigma^{-1}) > 0$. If $M \in \mathbb{N}_0$ is such that $M > \overline{s}(\sigma^{-1})\underline{s}(N)^{-1}$, then

$$B^{\sigma^{-1},N}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d) = \left\{ f \in L^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left(\sup_{j\in\mathbb{N}_0} \left(\sigma_j^{-1} \inf_{P\in\mathbb{P}_{M-1}} \|f-P\|_{L^\infty(B(x,N_j^{-1}))} \right) \right) < \infty \right\}.$$

Proof. This is a consequence of theorem 17, where the continuity of the elements of $B^{\sigma^{-1},N}_{\infty,\infty}(\mathbb{R}^d)$ results from corollary 12. □

2.5 A characterization of the spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ in terms of derivatives

It is shown in [42] that spaces $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ can be characterized in terms of derivatives of their elements. The goal of this section is to generalize this result to the spaces $\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$.

Proposition 20. *Let σ, N be two admissible sequences such that $\overline{s}(N) > 0$ and L, M be two natural numbers such that $L < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1} \leq \overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < M$. A function $f \in \Lambda^{M-1}_{\sigma,N}(\mathbb{R}^d)$ satisfies the following properties:*

1. *$f \in C^L(\mathbb{R}^d)$;*
2. *$D^\nu f \in L^\infty(\mathbb{R}^d)$ $\forall |\nu| \leq L$;*
3. *$\sup_{|h|\leq N_j^{-1}} \|\Delta_h^{M-|\nu|} D^\nu f\|_{L^\infty} \leq C N_j^{|\nu|}\sigma_j \quad \forall j \in \mathbb{N}_0, |\nu| \leq L$.*

Conversely, if a function $f \in C^L(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ satisfies

$$\sup_{|h|\leq N_j^{-1}} \|\Delta_h^{M-|\nu|} D^\nu f\|_{L^\infty} \leq C N_j^{|\nu|}\sigma_j \qquad \forall j \in \mathbb{N}_0, |\nu| = L,$$

then $f \in \Lambda^{M-1}_{\sigma,N}(\mathbb{R}^d)$.

Proof. Let $f \in \Lambda^{M-1}_{\sigma,N}(\mathbb{R}^d)$. Using the same notations as in proposition 10, we have

$$\sum_{j=1}^{+\infty} D^\nu f_j = D^\nu f \text{ (uniformly)} \quad \forall |\nu| \leq L.$$

Using lemmata 9 and 4, we get

$$\sum_{j=1}^{+\infty} \|D^\nu f_j\|_{L^\infty} \leq C \sum_{j=1}^{+\infty} N_j^{|\nu|}\sigma_j < +\infty,$$

which proves the two first assertions. Let $\nu \in \mathbb{N}_0^d$ be a multi-index such that $|\nu| \leq L$, $h \in \mathbb{R}^d$ and $J \in \mathbb{N}_0$ such that $|h| \leq N_J^{-1}$. By the mean value theorem and by lemmata 3 and 4, we have

$$\begin{aligned}
\|\Delta_h^{M-|\nu|} D^\nu f\|_{L^\infty} &\leq \sum_{j=1}^{J} \|\Delta_h^{M-|\nu|} D^\nu f_j\|_{L^\infty} + \sum_{j=J+1}^{+\infty} \|\Delta_h^{M-|\nu|} D^\nu f_j\|_{L^\infty} \\
&\leq \sum_{j=1}^{J} |h|^{M-|\nu|} \sup_{|\alpha|=M-|\nu|} \|D^{\alpha+\nu} f_j\|_{L^\infty} + C \sum_{j=J+1}^{+\infty} |h|^{L-|\nu|} \sup_{|\alpha|=L-|\nu|} \|D^{\alpha+\nu} f_j\|_{L^\infty} \\
&\leq C \sum_{j=1}^{J} |h|^{M-|\nu|} N_j^M \sigma_j + C \sum_{j=J+1}^{+\infty} |h|^{L-|\nu|} N_j^L \sigma_j \\
&\leq C N_J^{|\nu|} \sigma_J.
\end{aligned}$$

Let us prove the converse result. Let $|h| \leq N_j^{-1}$. By the mean value theorem, we have

$$\begin{aligned}
\|\Delta_h^M f\|_{L^\infty} &\leq C|h|^L \sup_{|\nu|=L} \|\Delta_h^{M-L} D^\nu f\|_{L^\infty} \\
&\leq C N_j^{-L} N_j^L \sigma_j = C\sigma_j.
\end{aligned}$$

□

So, the value $\underline{s}(\sigma^{-1})\overline{s}(N)^{-1}$ characterizes the level of differentiability of $f \in \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$, while the value $\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}$ determines the order of the finite difference for $D^\nu f$ ($|\nu| < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1}$).

Proposition 21. *Let $L \in \mathbb{N}_0$, σ, N be two admissible sequences and $f \in C^L(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. If there exists a natural number $M > L$ such that*

$$\sup_{|h|\leq N_j^{-1}} \|\Delta_h^{M-L} D^\nu f\|_{L^\infty} \leq C\sigma_j N_j^L \quad \forall |\nu| = L,$$

then $f \in \Lambda^{M-1}_{\sigma,N}(\mathbb{R}^d)$.

Let us now give a characterization of spaces $\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$ in terms of derivatives.

Corollary 22. (D.K., S. Nicolay) Let σ, N be two admissible sequences and let L, M be two natural numbers such that $L < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1} \leq \overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < M$ and $\underline{N}_1 > 1$. We have

$$\begin{aligned}
B^{\sigma^{-1},N}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d) = \{f \in L^\infty(\mathbb{R}^d) \cap C^L(\mathbb{R}^d) : \\
\sup_{|h|\leq N_j^{-1}} \|\Delta_h^{M-L} D^\nu f\|_{L^\infty} \leq C\sigma_j N_j^L \quad \forall j \in \mathbb{N}_0, |\nu| = L\}. \quad (2.3)
\end{aligned}$$

2.6 A characterization of the spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ in terms of Taylor decomposition

Under strong assumptions on admissible sequences, spaces $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ can be characterized in terms of Taylor decomposition of their elements ([42]). The goal of this section is to generalize this result to the spaces $\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$.

Theorem 23. (D.K., S. Nicolay) Let $L \in \mathbb{N}_0$, σ and N be two admissible sequences such that $\overline{s}(N) > 0$ and

$$L < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1} \leq \overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < L+1.$$

If $f \in \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$, then for all $x \in \mathbb{R}^d$ we have

$$f(x+h) = \sum_{|\nu|\leq L} D^\nu f(x)\frac{h^\nu}{|\nu|!} + R_L(x,h)\frac{|h|^L}{L!}, \quad \forall h \in \mathbb{R}^d$$

where $|R_L(x,h)| \leq C\sigma_j N_j^L$, $\forall |h| \leq N_j^{-1}$.

Conversely, if $f \in L^\infty(\mathbb{R}^d) \cap C^L(\mathbb{R}^d)$ satisfies

$$f(x+h) = \sum_{|\nu|\leq L} D^\nu f(x)\frac{h^\nu}{|\nu|!} + R_L(x,h)\frac{|h|^L}{L!} \qquad \forall x,h \in \mathbb{R}^d \tag{2.4}$$

with $\sup_{x,|h|\leq N_j^{-1}} |R_L(x,h)| \leq C\sigma_j N_j^L$ $\forall j \in \mathbb{N}_0$, then $f \in \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$.

Proof. Let $f \in \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$. As $f \in C^L(\mathbb{R}^d)$, we get, using lemma **??**, that

$$f(x+h) = \sum_{|\nu|\leq L} D^\nu f(x)\frac{h^\nu}{|\nu|!} + R_L(x,h)\frac{|h|^L}{L!},$$

where $|R_L(x,h)| \leq C \sup_{\substack{x,|l|\leq|h| \\ |\nu|=L}} \|\Delta^1_l D^\nu f\|_{L^\infty}$. Proposition 20 leads to the conclusion.

The converse result is a consequence of theorem 17. □

The following result is immediate:

Corollary 24. *Let L be a natural number, σ and N be two admissible sequences such that $\overline{s}(N) > 0$ and*

$$L < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1} \leq \overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < L+1.$$

We have

$$\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d) = \{f \in L^\infty(\mathbb{R}^d)\cap C^L(\mathbb{R}^d) : \ f \text{ can be written as (2.4) with } \sup_{x,|h|\leq N_j^{-1}} |R_L(x,h)| \leq C\sigma_j N_j^L \quad \forall j \in \mathbb{N}_0\}.$$

Moreover, if $\underline{N}_1 > 1$, then this space is equal to $B^{\sigma^{-1},N}_{\infty,\infty}(\mathbb{R}^d)$.

2.7 A characterization of the spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ in terms of Littlewood-Paley decomposition

It is shown in [41, 43] that spaces $B^{\sigma}_{\infty,\infty}(\mathbb{R}^d)$ and $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ can be characterized in terms of Littlewood-Paley decomposition. The goal of this section is to generalize these results to the spaces $B^{\sigma,N}_{\infty,\infty}(\mathbb{R}^d)$ and $\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$. For that purpose, we need to slightly modify the definition of Littlewood-Paley decomposition in order to take into account the presence of the sequence N.

Let $J \in \mathbb{N}^*$ and N be a sequence of bounded growth that is also strongly increasing (we define k_0 as the natural number associated with the strongly increasing sequence N). Let us construct a sequence of functions $(\varphi_j^{N,J})_{j\in\mathbb{N}_0}$ which belong to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ in the following way ([21]). Let $\rho \in D(\mathbb{R})$ be a function such that

$$\rho(t) = 1 \quad \forall |t| \leq 1 \quad , \quad \operatorname{supp}\rho \subset \{t \in \mathbb{R} : |t| \leq 2\},$$

and such that ρ is non-increasing for $t \geq 0$. Let us set

$$\varphi_j^{N,J}(\xi) = \rho(N_j^{-1}|\xi|) \qquad j = 0, 1, \cdots, Jk_0 - 1$$

and

$$\varphi_j^{N,J}(\xi) = \rho(N_j^{-1}|\xi|) - \rho(N_{j-Jk_0}^{-1}|\xi|) \quad \forall j \geq Jk_0.$$

We easily check that this sequence of functions belongs to the set $\Phi^{N,J}$ (defined in [21, 41]) where we can suppose that[1] $c_\varphi = 1$ (we can divide the function ρ by $c_\varphi = k_0 J$ for this purpose). We set

$$\Delta_j^{N,J}(f) = \mathcal{F}^{-1}(\varphi_j^{N,J}\mathcal{F}f)$$

for all $j \in \mathbb{N}_0$. These functions belong to the space $C^\infty(\mathbb{R}^d)$. One gets

$$Id = \Delta_0^{N,J} + \Delta_1^{N,J} + \dots$$

(with convergence in $\mathcal{S}'(\mathbb{R}^d)$).

We have the following result.

Lemma 25. *If $f \in L^p(\mathbb{R}^d)$ where $p \in [1, +\infty]$, then the functions $\Delta_j^{N,J}(f)$ belong to $L^p(\mathbb{R}^d)$ and we have*

$$\Delta_j^{N,J}(f) = (\mathcal{F}^{-1}\varphi_j^{N,J}) \star f$$

for all $j \in \mathbb{N}_0$.

Let $\rho' : \xi \in \mathbb{R}^d \mapsto \rho(|\xi|)$. Let us remark that

$$\mathcal{F}^{-1}\varphi_j^{N,J}(\xi) = N_j^d \mathcal{F}^{-1}\rho'(N_j\xi) \qquad j = 0, \cdots, Jk_0 - 1,$$

and

$$\mathcal{F}^{-1}\varphi_j^{N,J}(\xi) = N_j^d \mathcal{F}^{-1}\rho'(N_j\xi) - N_{j-Jk_0}^d \mathcal{F}^{-1}\rho'(N_{j-Jk_0}\xi) \qquad \forall j \geq Jk_0.$$

[1] This assumption is not necessary but it simplifies notations.

Remark 26. Let $C_1 := \inf\{1, d_1^{-Jk_0}\}$, $C_2 := \sup\{2, 2d_0^{-Jk_0}\}$ and $j \geq Jk_0$. Let us remark that

$$|\xi| \leq C_1 N_j \Rightarrow \varphi_j^{N,J}(\xi) = 0.$$

Similarly,

$$|\xi| \geq C_2 N_j \Rightarrow \varphi_j^{N,J}(\xi) = 0.$$

So, the support of the function $\varphi_j^{N,J}$ is included in the annulus $B(0, \leq C_2 N_j) \backslash B(0, \leq C_1 N_j)$ for all $j \geq Jk_0$.

Proposition 27. *Let* $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ *and* $N = (N_j)_{j\in\mathbb{N}_0}$ *be two admissible sequences such that* $\underline{N}_1 > 1$ *and* $\underline{s}(\sigma^{-1}) > 0$*. If* $f \in L^{\infty}(\mathbb{R}^d)$ *satisfies*

$$\|\Delta_j^{N,J} f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0$$

then $f \in \Lambda_{\sigma,N}^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}(\mathbb{R}^d)$*.*

Proof. Let $M \in \mathbb{N}_0$ such that $\overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < M$. As $\underline{s}(\sigma^{-1}) > 0$, we have

$$f = \sum_{j\in\mathbb{N}_0} \Delta_j^{N,J} f \quad \text{uniformly on } \mathbb{R}^d.$$

Let $x_0 \in \mathbb{R}^d$, $J' \in \mathbb{N}_0$ and set

$$P_j(x - x_0) = \sum_{|\alpha|\leq M-1} \frac{(x-x_0)^\alpha}{|\alpha|!} D^\alpha \Delta_j^{N,J} f(x_0) \quad \forall j \in \mathbb{N}_0 \tag{2.5}$$

and

$$P_{x_0,J'}(x - x_0) = \sum_{j=0}^{J'} P_j(x - x_0). \tag{2.6}$$

The degree of the last polynomial is less or equal to $M - 1$. Let $x \in \mathbb{R}^d$ be such that $|x - x_0| \leq N_{J'}^{-1}$. We have

$$|f(x) - P_{x_0,J'}(x - x_0)| \leq \left| \sum_{j=0}^{J'} \left(\Delta_j^{N,J} f(x) - \sum_{|\alpha|\leq M-1} \frac{(x-x_0)^\alpha}{|\alpha|!} D^\alpha \Delta_j^{N,J} f(x_0) \right) \right| + \left| \sum_{j=J'+1}^{+\infty} \Delta_j^{N,J} f(x) \right|.$$

Since $\underline{s}(\sigma^{-1}) > 0$, the second term is bounded by $C\sigma_{J'}$. Based on the Taylor formula, the first term is bounded by

$$\begin{aligned}
&\sum_{j=0}^{J'} |x - x_0|^M \sup_{|\alpha|=M} \|D^\alpha \Delta_j^{N,J} f\|_{L^\infty} \\
&\leq C N_{J'}^{-M} \sum_{j=0}^{J'} N_{j+Jk_0}^M \sigma_j \quad \text{(by S. Bernstein's inequalities)} \\
&\leq C\sigma_{J'}
\end{aligned}$$

thanks to lemma 3, where the constant C is independent of x and J'. Theorem 17 leads to the conclusion. □

Remark 28. The sequence of polynomials given by

$$P_{x_0,J'}(x-x_0)=\sum_{j=0}^{J'}P_j(x-x_0) \quad (J'\in\mathbb{N}_0) \tag{2.7}$$

where

$$P_j(x-x_0)=\sum_{|\alpha|\leq M-1}\frac{(x-x_0)^\alpha}{|\alpha|!}D^\alpha\Delta_j^{N,J}f(x_0) \tag{2.8}$$

can be used in the approximation given by corollary 19. Moreover, if $M\in\mathbb{N}^*$ is such that

$$M-1<\underline{s}(\sigma^{-1})\overline{s}(N)^{-1}\leq\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}<M,$$

then the polynomial P_{x_0} given by

$$P_{x_0}(x-x_0)=\sum_{j=0}^{+\infty}P_j(x-x_0) \tag{2.9}$$

where

$$P_j(x-x_0)=\sum_{|\alpha|\leq M-1}\frac{(x-x_0)^\alpha}{|\alpha|!}D^\alpha\Delta_j^{N,J}f(x_0) \quad (j\in\mathbb{N}_0) \tag{2.10}$$

satisfies corollary 19 for every scale $j\in\mathbb{N}_0$.

Remark 29. The previous proof gives interesting information about the spaces $\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$. Indeed, equations (2.8) and (2.7) give some polynomials that can be used in the approximation given by corollary 19. Moreover, if $M\in\mathbb{N}^*$ is such that

$$M-1<\underline{s}(\sigma^{-1})\overline{s}(N)^{-1}\leq\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}<M,$$

then the polynomial P_{x_0} given by

$$P_{x_0}(x-x_0)=\sum_{j=0}^{+\infty}P_j(x-x_0) \tag{2.11}$$

where

$$P_j(x-x_0)=\sum_{|\alpha|\leq M-1}\frac{(x-x_0)^\alpha}{|\alpha|!}D^\alpha\Delta_j^{N,J}f(x_0) \quad (j\in\mathbb{N}_0) \tag{2.12}$$

satisfies corollary 19 for every scale $j\in\mathbb{N}_0$.

Proposition 30. *Let $L \in \mathbb{N}_0$ and $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$, $N = (N_j)_{j\in\mathbb{N}_0}$ be two admissible sequences such that $\underline{N}_1 > 1$ and*

$$L < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1} \leq \overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < L+2.$$

If $f \in \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$ then

$$\|\Delta_j^{N,J} f\|_{L^\infty} \leq C\sigma_j \quad \forall j \in \mathbb{N}_0\,.$$

Proof. Using proposition 20, we have $f \in C^L(\mathbb{R}^d)$,

$$D^\alpha f \in L^\infty(\mathbb{R}^d) \quad \text{and} \quad \sup_{|h|\leq N_j^{-1}} \|\Delta_h^2 D^\alpha f\|_{L^\infty} \leq C\sigma_j N_j^L \quad \forall |\alpha| = L.$$

One gets

$$\begin{aligned}\Delta_j^{N,J} D_{y_k} f(x) &= \mathcal{F}^{-1}(iy_k \varphi_j^{N,J}\mathcal{F}f)(x)\\ &= D_{y_k}\Delta_j^{N,J} f(x)\end{aligned}$$

for all $k \in \{1,...,d\}$. By induction, we find

$$\Delta_j^{N,J} D^\alpha f(x) = D^\alpha \Delta_j^{N,J} f(x) \quad \forall |\alpha| \leq L.$$

Let $j \geq Jk_0$. We have

$$\begin{aligned}\Delta_j^{N,J} D^\alpha f(x) &= \int_{\mathbb{R}^d} D^\alpha f(x-y)\mathcal{F}^{-1}\varphi_j^{N,J}(y)dy\\ &= \frac{1}{2}\int_{\mathbb{R}^d}(D^\alpha f(x+y) - 2D^\alpha f(x) + D^\alpha f(x-y))\mathcal{F}^{-1}\varphi_j^{N,J}(y)dy\end{aligned}$$

because $F^{-1}\varphi_j^{N,J}$ is even with a vanishing integral (its Fourier transform is equal to zero at the origin and the function $\varphi_j^{N,J}$ is even). We know that the support of $\varphi_j^{N,J}$ is included in the set $B(0, C_2N_j)\backslash B(0, C_1N_j)$ for some constants C_1, C_2 such that $0 < C_1 < C_2$ (remark 26). One of S. Bernstein's inequalities states that

$$\|\Delta_j^{N,J} f\|_{L^\infty} \leq CN_j^{-L} \sup_{|\alpha|=L}\|D^\alpha \Delta_j^{N,J} f\|_{L^\infty}.$$

We get

$$\begin{aligned}\|\Delta_j^{N,J} f\|_{L^\infty} &\leq CN_j^{-L}\int_{\mathbb{R}^d}\sup_{|\alpha|=L}\|\Delta_y^2 D^\alpha f\|_{L^\infty}|\mathcal{F}^{-1}\varphi_j^{N,J}(y)|dy\\ &\leq CN_j^{-L}\int_{\mathbb{R}^d}\sup_{|\alpha|=L}\|\Delta_y^2 D^\alpha f\|_{L^\infty}|N_j^d\mathcal{F}^{-1}\rho'(N_jy) - N_{j-Jk_0}^d\mathcal{F}^{-1}\rho'(N_{j-Jk_0}y)|dy\\ &\leq CN_j^{-L}\int_{\mathbb{R}^d}\sup_{|\alpha|=L}\|\Delta_y^2 D^\alpha f\|_{L^\infty}|N_j^d\mathcal{F}^{-1}\rho'(N_jy)|dy\\ &\qquad + CN_j^{-L}\int_{\mathbb{R}^d}\sup_{|\alpha|=L}\|\Delta_y^2 D^\alpha f\|_{L^\infty}|N_{j-Jk_0}^d\mathcal{F}^{-1}\rho'(N_{j-Jk_0}y)|dy.\end{aligned}$$

Let us consider the first term of the last sum. One can proceed similarly for the second one. We have

$$N_j^{-L}\int_{\mathbb{R}^d}\sup_{|\alpha|=L}\|\Delta_y^2 D^\alpha f\|_{L^\infty}|N_j^d\mathcal{F}^{-1}\rho'(N_jy)|dy$$
$$= N_j^{-L}\int_{\mathbb{R}^d}\sup_{|\alpha|=L}\|\Delta^2_{yN_j^{-1}}D^\alpha f\|_{L^\infty}|\mathcal{F}^{-1}\rho'(y)|dy.$$

We get

$$\int_{|y|\leq 1}\sup_{|\alpha|=L}\|\Delta^2_{yN_j^{-1}}D^\alpha f\|_{L^\infty}|\mathcal{F}^{-1}\rho'(y)|dy \leq C\sigma_j N_j^L$$

and

$$\begin{aligned}
&\int_{N_m\leq|y|\leq N_{m+1}}\sup_{|\alpha|=L}\|\Delta^2_{yN_j^{-1}}D^\alpha f\|_{L^\infty}|\mathcal{F}^{-1}\rho'(y)|dy\\
&\leq \int_{N_m\leq|y|\leq N_{m+1}}\sup_{\substack{|\alpha|=L\\|h|\leq N_j^{-1}}}\|\Delta^2_{\lceil N_{m+1}\rceil h}D^\alpha f\|_{L^\infty}|\mathcal{F}^{-1}\rho'(y)|dy\\
&\leq C\lceil N_{m+1}\rceil^2\int_{N_m\leq|y|\leq N_{m+1}}\sup_{\substack{|\alpha|=L\\|h|\leq N_j^{-1}}}\|\Delta^2_h D^\alpha f\|_{L^\infty}|\mathcal{F}^{-1}\rho'(y)|dy\\
&\leq CN^2_{m+1}\sigma_j N_j^L\int_{N_m\leq|y|\leq N_{m+1}}\frac{C_M}{(1+|y|)^M}dy \quad (\text{as seen that } \underline{N}_1>1)\\
&\leq C_M\sigma_j N_j^L N_m^{-M+d+2}
\end{aligned}$$

for $M\in\mathbb{N}_0$ arbitrarily large. Let us take e.g. $M=d+3$. One can conclude, since the series $\sum_{j=0}^{+\infty}N_j^{-1}$ converges because $\underline{N}_1>1$; indeed, we have

$$\underline{N}_1 N_j \leq N_{j+1} \qquad \forall j\in\mathbb{N}_0$$

and

$$N_j^{-1}\leq \underline{N}_1^{-j}N_0^{-1} \qquad \forall j\in\mathbb{N}_0,$$

so that

$$\sum_{j=0}^{+\infty}N_j^{-1}\leq N_0^{-1}\sum_{j=0}^{+\infty}\underline{N}_1^{-j}.$$

□

2.8 Wavelet coefficients and spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$

It is known that spaces $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ can be completely characterized by the decreasing properties of their wavelet coefficients ([43]). In this section, we show that wavelet coefficients of elements of $\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$ keep their decreasing properties, under strong assumptions on σ and N. We consider Lemarié-Meyer wavelets or Daubechies wavelets.

Wavelet coefficients considered in this section are modified versions of the classical concept. This modification takes into account the presence of the sequence $(N_j)_j$ which replaces the usual dyadic sequence $(2^j)_j$: we set

$$\psi^i_{N_j,k}(x) = \psi^i(N_j x - k) \qquad \text{and} \qquad c^i_{N_j,k} = N_j^d \int_{\mathbb{R}^d} f(x)\psi^i_{N_j,k}(x)dx,$$

for all $i \in \{1, ..., 2^d - 1\}$, $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$.

Proposition 31. *Let σ, N be two admissible sequences such that $\underline{N}_1 > 1$ and $M \in \mathbb{N}_0$ such that*

$$M < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1} \leq \overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < M + 1.$$

If $f \in \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$ then there exists a constant $C > 0$ such that

$$\begin{cases} |C_k| \leq C & \forall k \in \mathbb{Z}^d \\ |c^i_{N_j,k}| \leq C\sigma_j & \forall j \in \mathbb{N}_0, \forall i \in \{1, \ldots, 2^d - 1\}, \forall k \in \mathbb{Z}^d. \end{cases}$$

Proof. Suppose that $f \in \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$. We get

$$|C_k| = \left| \int_{\mathbb{R}^d} f(x)\phi(x-k)dx \right| \leq C\|f\|_{L^\infty}.$$

We also have

$$\begin{aligned} |c^i_{Nj,k}| &= N_j^d \left| \int_{\mathbb{R}^d} f\left(kN_j^{-1} + (x - kN_j^{-1})\right) \psi(N_j x - k)dx \right| \\ &= N_j^d | \int_{\mathbb{R}^d} R_M\left(kN_j^{-1}, x - kN_j^{-1}\right) \frac{|x - kN_j^{-1}|^M}{M!} |\psi(N_j x - k)dx| \\ &= N_j^d | \int_{\mathbb{R}^d} R_M\left(kN_j^{-1}, y\right) \frac{|y|^M}{M!} |\psi(N_j y)dy| \\ &\leq C \int_{\mathbb{R}^d} \sup_{\substack{|h| \leq |y|N_j^{-1} \\ |\alpha| = M}} \|\Delta^1_h D^\alpha f\|_{L^\infty} |y|^M N_j^{-M} |\psi(y)|dy, \end{aligned}$$

using the same notations as in theorem 23. A similar proof to proposition 30 allows to conclude. □

2.9 Generalized Hölder spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$ as a generalized (real) interpolation of Sobolev spaces

It is shown in [43] that spaces $\Lambda^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ can be expressed as a generalized interpolation of classical Sobolev spaces. The goal of this section is to generalize this result to spaces $\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$. For a reminder about interpolation theory, the reader can refer to [43] and references therein.

First of all, we need a preliminary result.

Proposition 32. *Let L, $M \in \mathbb{N}_0$ and σ, N be two admissible sequences such that $\underline{N}_1 > 1$ and*

$$L < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1} \leq \overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < M.$$

If A_1 is continuously embedded in A_0, one has

$$[A_0, A_1]^{\star}_{\theta,\psi,J} = [A_0, A_1]^{\star}_{\theta,\psi,K}$$

where θ is the admissible sequence defined by

$$\theta_j = \begin{cases} N_{-j}^{-L}\sigma_{-j}^{-1} & \forall j \in -\mathbb{N}_0, \\ (\theta_{-j})^{-1} = N_j^L \sigma_j & \forall j \in \mathbb{N}^*, \end{cases}$$

and ψ is the admissible sequence defined by

$$\psi_j = \begin{cases} N_{-j}^{-(M-L)} & \forall j \in -\mathbb{N}_0, \\ (\psi_{-j})^{-1} = N_j^{M-L} & \forall j \in \mathbb{N}^*. \end{cases}$$

Proof. Let $f \in [A_0, A_1]^{\star}_{\theta,\psi,J}$. This function can be written as $f = \sum_{j\in\mathbb{Z}} f_j$ where the series converges in A_0 and the functions f_j satisfy

$$\|f_j\|_{A_0} + \psi_j \|f_j\|_{A_1} \leq C\theta_j^{-1} \qquad \forall j \in \mathbb{Z}.$$

Let us set $b_j = \sum_{l=-\infty}^{j-1} f_l$ and $c_j = \sum_{l=j}^{+\infty} f_l$ for all $j \in \mathbb{Z}$. We have $b_j \in A_0$ and $c_j \in A_1$. Let us now prove the following inequality:

$$\theta_j(\|b_j\|_{A_0} + \psi_j\|c_j\|_{A_1}) \leq C \qquad \forall j \in \mathbb{Z}.$$

1. If $j < 0$, then

$$\begin{aligned} \|b_j\|_{A_0} &\leq \sum_{l=-\infty}^{j-1} \|f_l\|_{A_0} \\ &\leq C \sum_{l=-j+1}^{+\infty} \theta_{-l}^{-1} = C \sum_{l=-j+1}^{+\infty} N_l^L \sigma_l \\ &\leq C N_{-j}^L \sigma_{-j} = C\theta_j^{-1} \end{aligned}$$

   and

$$\begin{aligned} \|c_j\|_{A_1} &\leq \sum_{l=j}^{+\infty} \|f_l\|_{A_1} \\ &\leq C \sum_{l=j}^{+\infty} \theta_l^{-1}\psi_l^{-1} \\ &\leq C \sum_{l=1}^{-j} \psi_{-l}^{-1}\theta_{-l}^{-1} + C \sum_{l=0}^{+\infty} \psi_l^{-1}\theta_l^{-1} \\ &\leq C N_{-j}^M \sigma_{-j} + C \leq C\psi_j^{-1}\theta_j^{-1}. \end{aligned}$$

2. If $j \geq 0$, then

$$\begin{aligned}\|b_j\|_{A_0} &\leq \sum_{l=-\infty}^{0} \|f_l\|_{A_0} + \sum_{l=1}^{j-1} \|f_l\|_{A_1} \\ &\leq C + C\sum_{l=1}^{j-1} N_l^{-M}\sigma_l^{-1} \\ &\leq C \leq C\theta_j^{-1}\end{aligned}$$

and

$$\begin{aligned}\|c_j\|_{A_1} &\leq \sum_{l=j}^{+\infty} \|f_l\|_{A_1} \leq C\sum_{l=j}^{+\infty} N_l^{-M}\sigma_l^{-1} \\ &\leq CN_j^{-M}\sigma_j^{-1},\end{aligned}$$

where we used the relation $\overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < M$ in the last inequality.

Let $f \in [A_0, A_1]^{\star}_{\theta,\psi,K}$. For all $j \in \mathbb{Z}$, there exist $b_j \in A_0$ and $c_j \in A_1$ such that $f = b_j + c_j$ and

$$\|b_j\|_{A_0} + \psi_j\|c_j\|_{A_1} \leq C\theta_j^{-1}.$$

Let us write $b_0 = \sum_{j=-\infty}^{-1}(b_{j+1} - b_j)$, with convergence in A_0. Similarly, we have $c_0 = \sum_{j=0}^{+\infty}(c_j - c_{j+1})$, with convergence in A_1. Let us set

$$f_j = \begin{cases} b_{j+1} - b_j & \text{if } j \in -\mathbb{N}^*, \\ c_j - c_{j+1} & \text{if } j \in \mathbb{N}_0. \end{cases}$$

As $b_{j+1} - b_j = c_j - c_{j+1}$ for all $j \in \mathbb{Z}$, we have $f = \sum_{j\in\mathbb{Z}} f_j$ in A_0, where $f_j \in A_1$ for all $j \in \mathbb{Z}$. Moreover, we have

$$\|f_j\|_{A_0} = \|b_{j+1} - b_j\|_{A_0} \leq C\theta_j^{-1}$$

and

$$\|f_j\|_{A_1} = \|c_{j+1} - c_j\|_{A_1} \leq C\psi_j^{-1}\theta_j^{-1},$$

which lead to the conclusion. □

The following result rewrites spaces $\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$ as a generalized interpolation of Sobolev spaces:

Theorem 33. (D.K., S. Nicolay) Let L, $M \in \mathbb{N}_0$ and σ, N be two admissible sequences such that $\underline{N}_1 > 1$ and

$$L < \underline{s}(\sigma^{-1})\overline{s}(N)^{-1} \leq \overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < M.$$

We have

$$B^{\sigma^{-1},N}_{\infty,\infty}(\mathbb{R}^d) = \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d) = [W^{\infty}_L, W^{\infty}_M]^{\star}_{\theta,\psi,J} = [W^{\infty}_L, W^{\infty}_M]^{\star}_{\theta,\psi,K}$$

where θ is the admissible sequence defined by

$$\theta_j = \begin{cases} N_{-j}^{-L}\sigma_{-j}^{-1} & \forall j \in -\mathbb{N}_0, \\ (\theta_{-j})^{-1} = N_j^L \sigma_j & \forall j \in \mathbb{N}^*, \end{cases}$$

and ψ is the admissible sequence defined by

$$\psi_j = \begin{cases} N_{-j}^{-(M-L)} & \forall j \in -\mathbb{N}_0, \\ (\psi_{-j})^{-1} = N_j^{M-L} & \forall j \in \mathbb{N}^*. \end{cases}$$

Proof.

Let us prove that $\Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d) = [W^{\infty}_L, W^{\infty}_M]^{\star}_{\theta,\psi,J}$. Let $f \in \Lambda^{\overline{s}(\sigma^{-1})\underline{s}(N)^{-1}}_{\sigma,N}(\mathbb{R}^d)$ and set

$$u_j = \begin{cases} 0 & \text{if } j \in \mathbb{Z},\ j > 1, \\ S_0(f) & \text{if } j = 1, \\ \Delta_{-j}(f) & \text{if } j \in \mathbb{Z},\ j < 1. \end{cases}$$

By Bernstein inequalities, the series $\sum_{j\in\mathbb{Z}} u_j$ converges in W^L_∞, and $u_j \in W^M_\infty$. Moreover, we have $\theta_j J(\psi_j, u_j) \leq C$ using theorem 4.1 of [56].

Let $f \in [W^{\infty}_L, W^{\infty}_M]^{\star}_{\theta,\psi,J}$. Let us check that the assumptions of proposition 20 are satisfied. Let $(f_j)_{j\in\mathbb{Z}}$ be a sequence of functions of $W^{\infty}_M(\mathbb{R}^d)$ such that $\sum_{j\in\mathbb{Z}} f_j = f$ in $W^{\infty}_N(\mathbb{R}^d)$ and such that

$$\theta_j J(\psi_j, f_j) \in l^{\infty}(\mathbb{Z}).$$

As we modify the functions f_j on some negligible set, we can suppose that they belong to $C^{M-1}(\mathbb{R}^d)$. Let $|\alpha| \leq L$. We have

$$\sum_{l=0}^{+\infty} \|D^\alpha f_l\|_{L^\infty} \leq C \sum_{l=0}^{+\infty} \psi_l^{-1}\theta_l^{-1} = C \sum_{l=0}^{+\infty} N_l^{-M}\sigma_l^{-1}$$

which is bounded because of $\overline{s}(\sigma^{-1})\underline{s}(N)^{-1} < M$. Moreover, we have

$$\sum_{l=-\infty}^{-1} \|D^\alpha f_l\|_{L^\infty} \leq C \sum_{l=-\infty}^{-1} \theta_l^{-1} = C \sum_{l=1}^{+\infty} N_l^L \sigma_l.$$

Putting these inequalities together, we find that $f \in C^L(\mathbb{R}^d)$ and $D^\alpha f \in L^\infty$ $\forall |\alpha| \leq L$. Let $h \in \mathbb{R}^d$ be such that $|h| \leq N_j^{-1}$ and $|\alpha| = L$. We have

$$\Delta_h^{M-L} D^\alpha f = \sum_{l\in\mathbb{Z}} \Delta_h^{M-L} D^\alpha f_l \qquad \text{(uniformly)}.$$

Using successive applications of the mean value theorem and by Morrey inequality (see e.g. [43]), we have

$$\begin{aligned}\sum_{l=0}^{+\infty}\|\Delta_h^{M-L}D^{\alpha}f_l\|_{L^\infty} &\le C|h|^{M-L}\sum_{l=0}^{+\infty}\|f_l\|_{W^{\infty}_{M}}\\ &\le CN_j^{-(M-L)}\sum_{l=0}^{+\infty}\psi_l^{-1}\theta_l^{-1}\\ &\le CN_j^{-(M-L)}\le CN_j^{L}\sigma_j\end{aligned}$$

and

$$\begin{aligned}\sum_{l=-\infty}^{-1}\|\Delta_h^{M-L}D^{\alpha}f_l\|_{L^\infty} &= \sum_{l=-j}^{-1}\|\Delta_h^{M-L}D^{\alpha}f_l\|_{L^\infty}+\sum_{l=-\infty}^{-j-1}\|\Delta_h^{M-L}D^{\alpha}f_l\|_{L^\infty}\\ &\le C|h|^{M-L}\sum_{l=-j}^{-1}\|f_l\|_{W^{\infty}_{M}}+C\sum_{l=-\infty}^{-j-1}\|f_l\|_{W^{\infty}_{L}}\\ &\le CN_j^{-(M-L)}\sum_{l=-j}^{-1}\psi_l^{-1}\theta_l^{-1}+C\sum_{l=-\infty}^{-j-1}\theta_l^{-1}\\ &\le CN_j^{-(M-L)}\sum_{l=1}^{j}N_l^{L}\sigma_l+C\sum_{l=j+1}^{+\infty}N_l^{L}\sigma_l\\ &\le CN_j^{L}\sigma_j\end{aligned}$$

which implies $\sup_{|h|\le N_j^{-1}}\|\Delta_h^{M-L}D^{\alpha}f\|_{L^\infty}\le CN_j^{L}\sigma_j$.

□

2.10 A weak result of Lions-Peetre type for spaces $\Lambda^{\alpha}_{\sigma,N}(\mathbb{R}^d)$

Proposition 34. *Let $m\in\mathbb{N}^*\backslash\{1\}$, $\alpha>0$ satisfying $1\le\alpha\le m$, $\sigma=(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence, $N=(N_j)_{j\in\mathbb{N}_0}$ be a non-decreasing sequence of positive numbers and $f\in L^\infty(\mathbb{R})$ such that*

$$\sup_{|h|\le N_j^{-1}}\|\Delta_h^m f\|_{L^\infty}\le C\sigma_j\quad\forall j\in\mathbb{N}_0\,.$$

If

$$\sum_{j=1}^{+\infty}N_j^{m-\alpha}\sigma_j<+\infty,$$

then, for all $n \in \mathbb{N}^*$, there exist two functions $F_1^n \in \Lambda^{m-(\alpha-1)}(\mathbb{R})$, $F_2^n \in \Lambda^{m-\alpha}(\mathbb{R})$ such that $f = F_1^n + F_2^n$ and

$$\|\Delta_h^m F_2^n\|_{L^\infty} \leq C_1 |h|^{m-\alpha} \sum_{j=n+1}^{+\infty} N_j^{m-\alpha} \sigma_j \quad \forall |h| < 1$$

and

$$\|\Delta_h^m F_1^n\|_{L^\infty} \leq C_2 |h|^{m-\alpha+1} \sum_{j=1}^{n} N_j^{m-\alpha+1} \sigma_j \quad \forall |h| < 1,$$

where C_1 and C_2 are two constants independent of n.

Proof. Let Φ be the function defined in proposition 8 and

$$f_1 := f \star \Phi_{N_1^{-1}}, \quad f_j := f \star (\Phi_{N_j^{-1}} - \Phi_{N_{j-1}^{-1}}) \quad (j > 1).$$

We have $||f_j||_{L^\infty} \leq C\sigma_j$ for all $j \in \mathbb{N}^*$, where the constant C does not depend on j. We thus get

$$\sum_{j=1}^{k} ||f_j||_{L^\infty} \leq C \sum_{j=1}^{k} \sigma_j,$$

for all $k \in \mathbb{N}^*$, which implies $f = \sum_{j=1}^{+\infty} f_j$ with uniform convergence. By the mean value theorem and lemma 9, one has

$$|\Delta_h^m f_j(x)| \leq C|h|^m \|D^m f_j\|_{L^\infty} \leq C|h|^m N_j^m \sigma_j$$

and

$$|\Delta_h^m f_j(x)| \leq 2^m \|f_j\|_{L^\infty} \leq 2^m C\sigma_j,$$

for all $j \in \mathbb{N}^*$. Therefore,

$$\begin{aligned} |\Delta_h^m f_j(x)| &= |\Delta_h^m f_j(x)|^{1-\alpha/m} |\Delta_h^m f_j(x)|^{\alpha/m} \\ &\leq C|h|^{m-\alpha} N_j^{m-\alpha} \sigma_j \end{aligned}$$

for all $0 \leq \alpha \leq m$.

For $n \in \mathbb{N}^*$ let us set $F_1^n := \sum_{j=1}^{n} f_j$ and $F_2^n := \sum_{j=n+1}^{+\infty} f_j$. We have

$$\|\Delta_h^m F_2^n\|_{L^\infty} \leq C|h|^{m-\alpha} \sum_{j=n+1}^{+\infty} N_j^{m-\alpha} \sigma_j$$

and

$$\|\Delta_h^m F_1^n\|_{L^\infty} \leq C|h|^{m-\alpha+1} \sum_{j=1}^{n} N_j^{m-\alpha+1} \sigma_j$$

which ends the proof. □

Remark 35. This result can be adapted to $\mathbb{R}^d$.

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