Sudoku and Matrices

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Outline

Introduction

Doing Sudoku is

1. Intellectually interesting,
2. Mathematically interesting,
3. Fun.

Conventions

- $\mathcal{S}$: set of all the (solved) Sudokus. We have $|\mathcal{S}| \approx 6.771 \cdot 10^{21}$, according to [3],
- $S_{i,j}$: element at the ith row and the jth column of the general S Sudoku square matrix. ($1 \leq i \leq n$, $1 \leq j \leq n$, n is the number of rows (or columns).) S is thus like

$$S = \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,n} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,1} & S_{n,2} & \cdots & S_{n,n} \end{pmatrix},$$

- $\sum_{j=1}^{n} S_{i,j} = \frac{n(n+1)}{2}$, $1 \leq i \leq n$ (i being free in this interval): the sum of all the elements on one line i of S,
- $\mathbb{Z}^*$: the set of the integers without 0; that is, $\mathbb{Z} \setminus \{0\}$,
- $\widetilde{C}$: the transpose of the matrix C,
- C^*: the adjoint of the matrix C, *i.e.* $\widetilde{\overline{C}}$,
- $\mathrm{Tr}(C) = \sum_{i=1}^{n} C_{i,i}$, $\mathrm{antiTr}(C) = \sum_{i=1}^{n} C_{i,n-i+1}$,
- $M^{i,j}$: the algebraic minor of the element at the ith row and the jth column of a given matrix,
- $\mathrm{cofact}(C_{i,j}) = (-1)^{i+j} \cdot M^{i,j}$,
- The dimension of a Sudoku matrix is expressed as a product $n \times n$, where n denotes its dimension, and n^2 the number of elements it is filled with,
- When taking matrices with a $n > 9$ (*i.e.* 16×16 or 25×25), we assume there exists a sufficiently complete alphabet $\mathcal{A}$ which has n distinct symbols. For example, when speaking about 16×16 Sudokus, "A" is used for "10," "B" is used for "11," ..., until "G" is used for "16."

Determinant

Theorem (Determinant of S sometimes equals 0)

The determinant of the S matrix, $\det S$, formed by the elements of a complete Sudoku, can equal 0. □

Proof (First part).

If $\det(S) = 0$, the rows of S are linearly dependent. We prove first that there exists at least a situation where $\det(S) \neq 0$, S still being a valid Sudoku matrix. It is shown above for at least two rows (it is the same for columns, just transpose). There are $n!$ possible orders of the elements of a row of S. From these possibilities, Let's take

$$\det(B) = \begin{vmatrix} 9 & 4 & 7 & 2 & 5 & 8 & 1 & 3 & 6 \\ 1 & 2 & 3 & 4 & 6 & 7 & 9 & 8 & 5 \\ 6 & 5 & 8 & 1 & 9 & 3 & 7 & 2 & 4 \\ 8 & 9 & 5 & 6 & 4 & 2 & 3 & 7 & 1 \\ 7 & 6 & 4 & 9 & 3 & 1 & 2 & 5 & 8 \\ 3 & 1 & 2 & 8 & 7 & 5 & 4 & 6 & 9 \\ 4 & 8 & 9 & 7 & 2 & 6 & 5 & 1 & 3 \\ 2 & 3 & 6 & 5 & 1 & 9 & 8 & 4 & 7 \\ 5 & 7 & 1 & 3 & 8 & 4 & 6 & 9 & 2 \end{vmatrix} \neq 0.$$

□

Determinant (2)

Proof (Second part).

It is shown in [2] that

$$\begin{vmatrix} a & b & c & d \\ c & d & a & b \\ d & c & b & a \\ b & a & d & c \end{vmatrix} = \begin{vmatrix} a & b & d & c \\ c & d & b & a \\ b & a & c & d \\ d & c & a & b \end{vmatrix} = a^4 + b^4 + c^4 + d^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 - 2a^2d^2 - 2b^2d^2 - 2c^2d^2 + 8abcd,$$

as there is an even number of permutations of rows (and columns) between the first and the second matrix. In the standard case, *i.e.* if (a, b, c, d) are mapped bijectively to $(1, 2, 3, 4)$, we have

$$a^4 + b^4 + c^4 + d^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 - 2a^2d^2 - 2b^2d^2 - 2c^2d^2 + 8abcd = 0.$$

Note that

$$\begin{vmatrix} 6 & 7 & 3 & 5 & 1 & 4 & 9 & 8 & 2 \\ 9 & 2 & 1 & 3 & 6 & 8 & 7 & 5 & 4 \\ 5 & 8 & 4 & 7 & 9 & 2 & 1 & 3 & 6 \\ 8 & 9 & 6 & 2 & 3 & 7 & 4 & 1 & 5 \\ 2 & 1 & 5 & 9 & 4 & 6 & 3 & 7 & 8 \\ 3 & 4 & 7 & 1 & 8 & 5 & 2 & 6 & 9 \\ 4 & 5 & 2 & 8 & 7 & 1 & 6 & 9 & 3 \\ 1 & 6 & 9 & 4 & 5 & 3 & 8 & 2 & 7 \\ 7 & 3 & 8 & 6 & 2 & 9 & 5 & 4 & 1 \end{vmatrix} = 0.$$

This result will be used later. □

Erroneous Sudoku

Theorem (Finding if a Sudoku matrix is erroneous or not)

There is no way to be sure about the correctness of a Sudoku matrix by only computing its determinant. □

Proof (First part).

An erroneous Sudoku can have a determinant of zero, but it is not always the case. Furthermore, having a determinant of zero cannot even be a sufficient condition to be an erroneous Sudoku matrix, as there are correct Sudoku matrices which have a determinant of 0.
Let's take two erroneous Sudokus:

1. The first will verify,

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 6 & 5 & 4 & 3 & 7 & 9 & 8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \neq 0,$$

   assuming the $\cdots$ can be filled correctly to make no other mistake. At least a given Sudoku can thus have a non-zero determinant,

2. The second is "more" erroneous:

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

□

Proof (Second part).

We have also seen before that

$$\begin{vmatrix} a & b & c & d \\ c & d & a & b \\ d & c & b & a \\ b & a & d & c \end{vmatrix} = \begin{vmatrix} a & b & d & c \\ c & d & b & a \\ b & a & c & d \\ d & c & a & b \end{vmatrix} = a^4 + b^4 + c^4 + d^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 - 2a^2d^2 - 2b^2d^2 - 2c^2d^2 + 8abcd,$$

which equals 0 iff (a, b, c, d) are mapped bijectively to $(1, 2, 3, 4)$, leading however to a correct Sudoku. □

Eigenvalues

Theorem (Perron)

If all of the entries of a matrix are positive, then the matrix has a dominant eigenvalue that is real and has multiplicity 1. □

Theorem (Max eigenvalue of a square and positive matrix)

The dominant eigenvalue of any square and positive matrix where each row and column have the same sum, will equal that sum. □

Eigenvalues (2)

Corollary (Eigenvalues of S)

A Sudoku matrix S with a dimension of n always verifies

$$\max_{\substack{1 \leq i \leq n, \\ \lambda_i \in \mathbb{R}}} \lambda_i = \sum_{j=1}^{n} S_{i,j} = \frac{n(n+1)}{2}$$

for exactly one λ_i. This is thus the dominant eigenvalue of the matrix S. This eigenvalue has multiplicity 1.

Proof.

Trivial using Theorem 5.1 and Theorem 5.2. □ □

Eigenvalues (3)

Corollary

A Sudoku square matrix S, of dimension n, and of determinant $\det(S)$, *always verifies*

$$\det(S) \mod \left(\max_{\substack{1 \le i \le n, \\ \lambda_i \in \mathbb{R}}} \lambda_i \right) = 0,$$

which is equivalent to ask

$$\det(S) \mod \left(\frac{n(n+1)}{2} \right) = 0.$$

□

Proof.

The determinant of a (square) matrix is the product of its eigenvalues. □

Eigenvalues (4)

Example (Divisibility of the determinant of S if it represents a 4×4 Sudoku)

Whatever the 4×4 Sudoku, its determinant is always divisible by $\frac{4(4+1)}{2} = 10$. Evidently, it *can* be divisible by 45, for example, but it might not always be the case. □

Transpose

Theorem (Transpose of a Sudoku matrix)

The transpose of a Sudoku matrix is still a correct, but different, Sudoku matrix (of the same dimension).

Proof.

According to the rules of Sudoku, at least one number cannot be repeated at least two times on a row, a column, or in a $n^{\frac{1}{2}} \times n^{\frac{1}{2}}$ square, where n is the dimension of the Sudoku. The matrix which is the result of the transposition is *still a valid Sudoku matrix*, as

1. $\widetilde{B_{i,j}} = B_{i,j}$ if $i = j$ (*i.e.* (i, j) is a diagonal couple),
2. If we let $j_{\mathrm{old}} := j$, $j := i$ and $i := j_{\mathrm{old}}$, everywhere in B, we obtain $\widetilde{B}$. By reversing rows and columns, rules of Sudoku are still respected, as the rules of Sudoku are applicable on both,
3. For subsquares, rules are still respected, because of the last point.

We now have to prove that *it is always different*. We have to prove that *there is no* Sudoku matrix in $\mathcal{S}$ which verifies $S_{i,j} = \widetilde{S_{i,j}}$ with $1 \leq i \leq n$, and $1 \leq j \leq n$. Let's reason by contradiction. All the S matrices verify

$$S_{i,j} = \widetilde{S_{i,j}}$$

for $i = j$. Let $1 \leq i \leq n$, $1 \leq j \leq n$. Is the same equality possible with these conventions? We shall now try to construct such a matrix. Let's reason first with a 4×4 matrix.

Transpose (2)

Proof (Continued).

We define

$$B := \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} \\ B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} \\ B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} \\ B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4} \end{pmatrix} \quad \text{and} \quad \widetilde{B} := C := \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} \\ C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} \\ C_{4,1} & C_{4,2} & C_{4,3} & C_{4,4} \end{pmatrix}.$$

It is clear that $B_{i,j} = \widetilde{B_{i,j}}$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$ iff $B_{i,j} = C_{i,j}$ for all i and j. The C matrix has thus to be exactly B. It is the case iff $B_{i,j} = B_{j,i}$, with $1 \leq i \leq n$ and $1 \leq j \leq n$, thus iff B is symmetric. Let's try to construct a symmetric B. Such a B would be symmetric:

$$\begin{pmatrix} B_{1,1} & B_{2,1} & B_{3,1} & B_{4,1} \\ B_{2,1} & B_{2,2} & B_{3,2} & B_{4,2} \\ B_{3,1} & B_{3,2} & B_{3,3} & B_{4,3} \\ B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4} \end{pmatrix},$$

but it does not respect the rules of Sudoku, as there is at least one subsquare of dimension $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} \equiv 2 \times 2$ where two elements are the same (here, there is only one: $B_{4,3}$). Here is the generalization of this fact. If we take

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,n} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n,1} & B_{n,2} & \cdots & B_{n,n} \end{pmatrix},$$

□

Transpose (3)

Proof (Continued).

B should verify

$$B = \begin{pmatrix} B_{1,1} & B_{2,1} & \cdots & B_{n,1} \\ B_{2,1} & B_{2,2} & \cdots & B_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n,1} & B_{n,2} & \cdots & B_{n,n} \end{pmatrix}$$

to be symmetric. It is symmetric, but does not respect the rules of Sudoku anmyore, as, at the place of the $\vdots$ and the $\cdots$, zooming at the end would lead to the submatrix

$$\begin{pmatrix} \ddots & \cdots & \cdots & S_{n,2} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \ddots & S_{n,n-1} \\ S_{n,2} & \cdots & S_{n,n-1} & S_{n,n} \end{pmatrix}.$$

□

Determinant Of the Transpose

Theorem (Determinant of the transpose of a square matrix)

For every square matrix C, $\det(C) = \det(\widetilde{C})$. □

Corollary (Determinant of the transpose of a Sudoku matrix)

If S is the matrix of a given Sudoku, $\det(S) = \det(\widetilde{S})$. □

Determinant Of the Transpose (2)

Corollary

If S is the matrix of a given Sudoku, $\widetilde{S}$ its transpose, and n their dimension (which is the same),

$$\det(S) \mod \left(\frac{n(n+1)}{2}\right) = \det(\widetilde{S}) \mod \left(\frac{n(n+1)}{2}\right) = 0.$$

□

Proof.

Trivial, as $\det(S) = \det(\widetilde{S})$. □

Trace Of a Transpose

Theorem (Trace of a transpose)

Whatever the matrix E, $\mathrm{Tr}(E) = \mathrm{Tr}(\widetilde{E})$. □

Corollary (Trace of the transpose of a Sudoku matrix)

If S is the matrix of a given Sudoku, $\mathrm{Tr}(S) = \mathrm{Tr}(\widetilde{S})$. □

Corollary (Every Sudoku matrix is not Hermitian)

There is no Hermitian Sudoku matrix. □

Proof.

Direct using Theorem 6.1, as $\overline{S_{i,j}} = S_{i,j}$ for $1 \leq i \leq n$, $1 \leq j \leq n$, as $S_{i,j} \in \mathbb{R}$. □

Antisymmetricity

Corollary (Every Sudoku matrix is not antisymmetric)

S is not antisymmetric, for clear reasons: $S_{i,j} \neq -S_{j,i}$. □

Proof.

The $S_{i,j}$ are in $\mathbb{Z}^*$. If we take $-S_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$, all the $S_{i,j}$ will lie in $\mathbb{Z}^*_-$, or no Sudoku matrix can have negative elements. □

Non-Normality

Theorem (Every Sudoku matrix is not normal)

There is no normal Sudoku matrix. □

Proof.

Let's try to build such a matrix. If we have

$$S = \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n,1} & s_{n,2} & \cdots & s_{n,n} \end{pmatrix} \quad \text{and} \quad S' := \widetilde{S} = \begin{pmatrix} s_{1,1} & s_{2,1} & \cdots & s_{n,1} \\ s_{1,2} & s_{2,2} & \cdots & s_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,n} & s_{2,n} & \cdots & s_{n,n} \end{pmatrix},$$

we have $S\widetilde{S} = \widetilde{S}S$ iff

$$S = \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n,1} & s_{n,2} & \cdots & s_{n,n} \end{pmatrix} \begin{pmatrix} s_{1,1} & s_{2,1} & \cdots & s_{n,1} \\ s_{1,2} & s_{2,2} & \cdots & s_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,n} & s_{2,n} & \cdots & s_{n,n} \end{pmatrix} \text{ equals}$$

$$\begin{pmatrix} s_{1,1} & s_{2,1} & \cdots & s_{n,1} \\ s_{1,2} & s_{2,2} & \cdots & s_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,n} & s_{2,n} & \cdots & s_{n,n} \end{pmatrix} \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n,1} & s_{n,2} & \cdots & s_{n,n} \end{pmatrix},$$

□

Non-Normality (2)

Proof (Continued).

Thus asking an equality between

$$\begin{pmatrix} S_{1,1}^2 + S_{1,2}^2 + \cdots + S_{1,n}^2 & S_{1,1}S_{2,1} + S_{1,2}S_{2,2} + \cdots + S_{1,n}S_{2,n} & \cdots & S_{1,1}S_{n,1} + S_{1,2}S_{n,2} + \cdots + S_{1,n}S_{n,n} \\ S_{2,1}S_{1,1} + S_{2,2}S_{1,2} + \cdots + S_{2,n}S_{1,n} & S_{2,1}^2 + S_{2,2}^2 + \cdots + S_{2,n}^2 & \cdots & S_{2,1}S_{n,1} + S_{2,2}S_{n,2} + \cdots + S_{2,n}S_{n,n} \\ \cdots & \cdots & \cdots & \cdots \\ S_{n,1}S_{1,1} + S_{n,2}S_{1,2} + \cdots + S_{n,n}S_{1,n} & S_{n,1}S_{2,1} + S_{n,2}S_{2,2} + \cdots + S_{n,n}S_{2,n} & \cdots & S_{n,1}^2 + S_{n,2}^2 + \cdots + S_{n,n}^2 \end{pmatrix}$$

and

$$\begin{pmatrix} S_{1,1}^2 + S_{2,1}^2 + \cdots + S_{n,1}^2 & S_{1,1}S_{1,2} + S_{2,1}S_{2,2} + \cdots + S_{n,1}S_{n,2} & \cdots & S_{1,1}S_{1,n} + S_{2,1}S_{2,n} + \cdots + S_{n,1}S_{n,n} \\ S_{1,2}S_{1,1} + S_{2,2}S_{2,1} + \cdots + S_{n,2}S_{n,1} & {S_{1,2}}^2 + S_{2,2}^2 + \cdots + {S_{n,2}}^2 & \cdots & S_{1,2}S_{1,n} + S_{2,2}S_{2,n} + \cdots + S_{n,2}S_{n,n} \\ \cdots & \cdots & \cdots & \cdots \\ S_{1,n}S_{1,1} + S_{2,n}S_{2,1} + \cdots + S_{n,n}S_{n,1} & S_{1,n}S_{1,2} + S_{2,n}S_{2,2} + \cdots + S_{n,n}S_{n,2} & \cdots & S_{n,1}^2 + {S_{2,n}}^2 + \cdots + S_{n,n}^2 \end{pmatrix},$$

leading to such a system:

$$\begin{cases} S_{1,n}^2 &= S_{n,1}^2 \\ S_{1,2} &= S_{2,1} \\ S_{n,1} &= S_{1,n} \\ S_{n,2} &= S_{2,n} \\ & \vdots \end{cases}.$$

The condition $S_{1,2} = S_{2,1}$ already goes against Sudoku's rules. Anyway, it is clear that S has to be symmetric to respect these equalities. However, it was proven at Theorem 6.1 that S cannot be symmetric. □

Non-Orthogonality

Theorem (Non-orthogonality of S if $\det(S) = 0$)

If $\det(S) = 0$, S *cannot be orthogonal.* □

Proof.

We have

$$\det(\widetilde{S} \cdot S) = \det(\widetilde{S}) \det(S),$$

which equals 0 if $\det(S) = 0$, and $\det(I_n) = 1$. □

See Conjecture 1 of [5] for the $\det(S) \neq 0$ case.

Order

Theorem (Every invertible matrix has a finite order)

Every invertible matrix has a finite order. □

Corollary

If $\det(S) \neq 0$, *S has a finite order.* □

Example

The determinant

$$\det(B) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{vmatrix}$$

equalling 0, there is no smallest $k > 0$ such that $B^k = I_n$. □

However, there is no special remark to do about this and Sudokus: there seems to be no link between the order of a Sudoku matrix S and other concepts related to Sudoku's matrices.

Remark: Possible Configurations

Given a matrix S, [at least] two rows in S will never be formed by the same elements in the same order than another row. If this happens, S is not a valid Sudoku anymore, as it leads to a part of S (the example is for the rows of S, but it is as trivial for the columns of S) like

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $1, 2, \cdots, n$ can be in a different order. This would also be shown by the determinant of a submatrix of

$$\begin{pmatrix} k & k+1 & k+2 & \cdots & n \\ k & k+1 & k+2 & \cdots & n \end{pmatrix},$$

which always equals 0 (thus leading to a rank ρ less than 2),
Beginning by the end, *i.e.* using the following configuration:

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & n \\ n & n-1 & n-2 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

can only lead to linearly independent rows.
The rank ρ of a matrix A being defined as the biggest dimension of the square submatrices extracted from A, we know that

$$\rho\left(\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & n \\ n & n-1 & n-2 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}\right) = 2,$$

under some given conditions which will be determined now.

Remark: Possible Configurations (2)

For $k \in \{1, \cdots, n-1\}$, we have

$$\begin{vmatrix} k & k+1 \\ n & n-k \end{vmatrix} = k(n-k) - n(k+1) = kn - k^2 - kn - n = -k^2 - n;$$

That is, $-k^2 - n \leq 2$ iff $-n \leq 2 + k^2$, leading to $n \geq -2 - k^2$. As $-2 - k^2 = -3$ iff $k = 1$ and

$$-2 - k^2 \overset{k=(n-1)}{=} -2 - (n-1)^2 = -n^2 + 2n + 3,$$

we must ask

$$\begin{cases} n \geq -n^2 + 2n + 3 \\ n \geq -3 \end{cases},$$

which is equivalent to

$$\begin{cases} n & \in \left] -\infty, \frac{1-\sqrt{5}}{2} \right] \cup \left[\frac{1+\sqrt{5}}{2}, +\infty \right[\\ n & \geq -3 \end{cases},$$

thus giving

$$\begin{cases} n & \in \left] -\infty, \frac{1-\sqrt{5}}{2} \right] \cup \left[\frac{1+\sqrt{5}}{2}, +\infty \right[\\ n & \geq -3 \end{cases},$$

and

$$n \in \left[\frac{1+\sqrt{5}}{2}, +\infty \right[,$$

which is always the case, as

$$\frac{1+\sqrt{5}}{2} \simeq 1.6180339887498948482045868343656,$$

and the dimension of a Sudoku is always the square of a number. Thus, the first interesting Sudoku's dimension would be $4 = 2^2$ (thus greater than the golden ratio). □

Condition Number (1)

Theorem (No well-conditioning for S)

The system $Sx = b$ can be else than well-conditioned (assuming $b \neq \mathbf{0}_n$). □

Condition Number (2)

Example (Extreme example)

Consider the matrix

$$P := \begin{pmatrix} 6 & 7 & 3 & 5 & 1 & 4 & 9 & 8 & 2 \\ 9 & 2 & 1 & 3 & 6 & 8 & 7 & 5 & 4 \\ 5 & 8 & 4 & 7 & 9 & 2 & 1 & 3 & 6 \\ 8 & 9 & 6 & 2 & 3 & 7 & 4 & 1 & 5 \\ 2 & 1 & 5 & 9 & 4 & 6 & 3 & 7 & 8 \\ 3 & 4 & 7 & 1 & 8 & 5 & 2 & 6 & 9 \\ 4 & 5 & 2 & 8 & 7 & 1 & 6 & 9 & 3 \\ 1 & 6 & 9 & 4 & 5 & 3 & 8 & 2 & 7 \\ 7 & 3 & 8 & 6 & 2 & 9 & 5 & 4 & 1 \end{pmatrix}.$$

It verifies

$$\kappa(P)_\infty \approx (2.052475887800532) \cdot 10^{17} \gg$$
$$\det(P) = 0.$$

As $\kappa(P) \gg$, trying to solve

$$\begin{pmatrix} 6 & 7 & 3 & 5 & 1 & 4 & 9 & 8 & 2 \\ 9 & 2 & 1 & 3 & 6 & 8 & 7 & 5 & 4 \\ 5 & 8 & 4 & 7 & 9 & 2 & 1 & 3 & 6 \\ 8 & 9 & 6 & 2 & 3 & 7 & 4 & 1 & 5 \\ 2 & 1 & 5 & 9 & 4 & 6 & 3 & 7 & 8 \\ 3 & 4 & 7 & 1 & 8 & 5 & 2 & 6 & 9 \\ 4 & 5 & 2 & 8 & 7 & 1 & 6 & 9 & 3 \\ 1 & 6 & 9 & 4 & 5 & 3 & 8 & 2 & 7 \\ 7 & 3 & 8 & 6 & 2 & 9 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \\ \eta \\ \theta \\ \iota \end{pmatrix} = \begin{pmatrix} \kappa \\ \lambda \\ \mu \\ \nu \\ \xi \\ \chi \\ \rho \\ \sigma \\ \tau \end{pmatrix}$$

is error-prone.

Condition Number (3)

Example (Extreme example (Continued).)

Let's take

$$\begin{aligned}
\kappa &= 1 \\
\lambda &= 2 \\
\mu &= 3 \\
\nu &= 4 \\
\xi &= 5 \\
\chi &= 6 \\
\rho &= 7 \\
\sigma &= 8 \\
\tau &= 9.
\end{aligned}$$

Using numerical analysis, we can now solve

$$\begin{pmatrix}
6 & 7 & 3 & 5 & 1 & 4 & 9 & 8 & 2 \\
9 & 2 & 1 & 3 & 6 & 8 & 7 & 5 & 4 \\
5 & 8 & 4 & 7 & 9 & 2 & 1 & 3 & 6 \\
8 & 9 & 6 & 2 & 3 & 7 & 4 & 1 & 5 \\
2 & 1 & 5 & 9 & 4 & 6 & 3 & 7 & 8 \\
3 & 4 & 7 & 1 & 8 & 5 & 2 & 6 & 9 \\
4 & 5 & 2 & 8 & 7 & 1 & 6 & 9 & 3 \\
1 & 6 & 9 & 4 & 5 & 3 & 8 & 2 & 7 \\
7 & 3 & 8 & 6 & 2 & 9 & 5 & 4 & 1
\end{pmatrix}
\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \\ \eta \\ \theta \\ \iota \end{pmatrix}
=
\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}.$$

Condition Number (4)

Example (Extreme example (Continued).)

The result is roughly

$$10^{15} \cdot \begin{Bmatrix} -6.259323629462618 \\ 3.426654756202998 \\ -2.859392164516785 \\ -0.213786330057330 \\ 2.235038905144813 \\ 6.094125101691045 \\ 0.146978101914415 \\ 0.070452313314347 \\ -2.640747054230881 \end{Bmatrix}.$$

Let's now compute $A := A + (0.1)_9$ and $b := b + (0.1)_9$. We now want to solve

$$\begin{pmatrix} 6.1 & 7.1 & 3.1 & 5.1 & 1.1 & 4.1 & 9.1 & 8.1 & 2.1 \\ 9.1 & 2.1 & 1.1 & 3.1 & 6.1 & 8.1 & 7.1 & 5.1 & 4.1 \\ 5.1 & 8.1 & 4.1 & 7.1 & 9.1 & 2.1 & 1.1 & 3.1 & 6.1 \\ 8.1 & 9.1 & 6.1 & 2.1 & 3.1 & 7.1 & 4.1 & 1.1 & 5.1 \\ 2.1 & 1.1 & 5.1 & 9.1 & 4.1 & 6.1 & 3.1 & 7.1 & 8.1 \\ 3.1 & 4.1 & 7.1 & 1.1 & 8.1 & 5.1 & 2.1 & 6.1 & 9.1 \\ 4.1 & 5.1 & 2.1 & 8.1 & 7.1 & 1.1 & 6.1 & 9.1 & 3.1 \\ 1.1 & 6.1 & 9.1 & 4.1 & 5.1 & 3.1 & 8.1 & 2.1 & 7.1 \\ 7.1 & 3.1 & 8.1 & 6.1 & 2.1 & 9.1 & 5.1 & 4.1 & 1.1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \\ \eta \\ \theta \\ \iota \end{pmatrix} = \begin{pmatrix} 1.1 \\ 2.1 \\ 3.1 \\ 4.1 \\ 5.1 \\ 6.1 \\ 7.1 \\ 8.1 \\ 9.1 \end{pmatrix}.$$

Condition Number (5)

Example (Extreme example (Continued).)

The result is now

$$10^{17} \cdot \begin{pmatrix} -1.201790136856824 \\ 0.657917713190976 \\ -0.549003295587223 \\ -0.041046975371007 \\ 0.429127469787804 \\ 1.170072019524681 \\ 0.028219795567568 \\ 0.013526844156355 \\ -0.507023434412329 \end{pmatrix}.$$

□

Trace

Theorem (The trace of a Sudoku matrix is not constant)

The trace of a Sudoku matrix is not constant. □

Proof.

Consider

$$P = \begin{pmatrix} 6 & 7 & 3 & 5 & 1 & 4 & 9 & 8 & 2 \\ 9 & 2 & 1 & 3 & 6 & 8 & 7 & 5 & 4 \\ 5 & 8 & 4 & 7 & 9 & 2 & 1 & 3 & 6 \\ 8 & 9 & 6 & 2 & 3 & 7 & 4 & 1 & 5 \\ 2 & 1 & 5 & 9 & 4 & 6 & 3 & 7 & 8 \\ 3 & 4 & 7 & 1 & 8 & 5 & 2 & 6 & 9 \\ 4 & 5 & 2 & 8 & 7 & 1 & 6 & 9 & 3 \\ 1 & 6 & 9 & 4 & 5 & 3 & 8 & 2 & 7 \\ 7 & 3 & 8 & 6 & 2 & 9 & 5 & 4 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 9 & 4 & 7 & 2 & 5 & 8 & 1 & 3 & 6 \\ 1 & 2 & 3 & 4 & 6 & 7 & 9 & 8 & 5 \\ 6 & 5 & 8 & 1 & 9 & 3 & 7 & 2 & 4 \\ 8 & 9 & 5 & 6 & 4 & 2 & 3 & 7 & 1 \\ 7 & 6 & 4 & 9 & 3 & 1 & 2 & 5 & 8 \\ 3 & 1 & 2 & 8 & 7 & 5 & 4 & 6 & 9 \\ 4 & 8 & 9 & 7 & 2 & 6 & 5 & 1 & 3 \\ 2 & 3 & 6 & 5 & 1 & 9 & 8 & 4 & 7 \\ 5 & 7 & 1 & 3 & 8 & 4 & 6 & 9 & 2 \end{pmatrix},$$

whose traces are 32 and 44, respectively. □

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