A Sudoku Matrix Study: Determinant, Golden Ratio, Eigenvalues, Transpose, Non-Hermitianity, Non-Normality, Orthogonality Discussion, Order, Condition Number

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Abstract. The Sudoku game has gained much interest for a dozen years. It is now put in various magazines and mathematical research is more and more interested in it. This document aims at providing some newer information about the mathematical properties of the Sudoku, but not according to graphs' theory. We do not speak about the "minimum number of clues," but about Sudokus' matrix interpretation: general properties of its determinant, in relation to eigenvalues; transpose; non-Hermitian character; neither symmetric nor antisymmetric character; non-normal character; non-orthogonal character when the matrix has a determinant of 0, conjecture when the determinant is not equal to 0; order; condition number.

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1 Introduction

The Sudoku game has gained much interest for a dozen years. It is now put in various magazines and mathematical research is more and more interested in it. This document aims at providing some newer information about the mathematical properties of the Sudoku, but not according to graphs' theory. We do not speak about the "minimum number of clues," but about Sudokus' matrix interpretation: general properties of its determinant, in relation to eigenvalues; transpose; non-Hermitian character; neither symmetric nor antisymmetric character; non-normal character; non-orthogonal character when the matrix has a determinant of 0, conjecture when the determinant is not equal to 0; order; condition number.

2 Conventions

Through this paper, we use the following *conventions*:

- $\mathcal{S}$ denotes the set of all the (solved) Sudokus. We have $|\mathcal{S}| \approx 6.771 \cdot 10^{21}$, according to [3],
- $S_{i,j}$ denotes the element at the ith row and the jth column of the general S Sudoku square matrix. We shall use $1 \leq i \leq n$, $1 \leq j \leq n$, where n denotes the number of rows (or columns, as it is always a square matrix) of the Sudoku (*we always consider* $n \geq 4$). S has thus the following form:

$$S = \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,n} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,1} & S_{n,2} & \cdots & S_{n,n} \end{pmatrix},$$

- $\sum_{j=1}^{n} S_{i,j} = \frac{n(n+1)}{2}$, $1 \leq i \leq n$ (i being free in this interval) denotes the sum of all the elements on one line i of S. Evidently,
$$\sum_{j=1}^{n} S_{i,j} = \frac{n(n+1)}{2} = \sum_{i=1}^{n} S_{i,j} = \frac{n(n+1)}{2},$$
because of the fundamental properties of the Sudoku,
- $\mathbb{Z}^*$ denotes the set of the integers without 0; that is, $\mathbb{Z} \setminus \{0\}$,
- $\widetilde{C}$ is the transpose of the matrix C,
- C^* is the adjoint of the matrix C, *i.e.* $\overline{\widetilde{C}}$,
- $\mathrm{Tr}(C) = \sum_{i=1}^{n} C_{i,i}$, $\mathrm{antiTr}(C) = \sum_{i=1}^{n} C_{i,n-i+1}$,
- $M^{i,j}$ is the algebraic minor of the element at the ith row and the jth column of a given matrix,
- $\mathrm{cofact}(C_{i,j}) = (-1)^{i+j} \cdot M^{i,j}$,
- The dimension of a Sudoku matrix is expressed as a product $n \times n$, where n denotes its dimension, and n^2 the number of elements it is filled with,
- When taking matrices with a $n > 9$ (*i.e.* 16×16 or 25×25), we assume there exists a sufficiently complete alphabet $\mathcal{A}$ which has n distinct symbols. For example, when speaking about 16×16 Sudokus, "A" is used for "10," "B" is used for "11," ..., until "G" is used for "16."

Evidently, we assume the reader is familiar with the notions of Sudoku, Latin Square, and related concepts.

3 Determinants

3.1 General Case

As S is a square matrix, one can wonder about the value of $\det(S)$, for a given n. Based on many experiments, $\det S$ has the following properties:

- It is in $\mathbb{Z}^*$ or in $\mathbb{Z}$,
- The assertion
$$(\det(S) \pmod 2 = 0) \vee (\det(S) \pmod 2 \neq 0) \tag{1}$$
is always true, but $\det S$ is sometimes even, and sometimes odd.

3.2 Why it Sometimes Does Not Equal Zero

Theorem 1 (Determinant of S sometimes equals 0). *The determinant of the S matrix, $\det S$, formed by the elements of a complete Sudoku, can equal 0.* □

Proof. Consider the two following cases:

- Let's take
$$\det(B) = \begin{vmatrix} 9&4&7&2&5&8&1&3&6 \\ 1&2&3&4&6&7&9&8&5 \\ 6&5&8&1&9&3&7&2&4 \\ 8&9&5&6&4&2&3&7&1 \\ 7&6&4&9&3&1&2&5&8 \\ 3&1&2&8&7&5&4&6&9 \\ 4&8&9&7&2&6&5&1&3 \\ 2&3&6&5&1&9&8&4&7 \\ 5&7&1&3&8&4&6&9&2 \end{vmatrix} \neq 0,$$

– It is shown in [2] that

$$\begin{vmatrix} a & b & c & d \\ c & d & a & b \\ d & c & b & a \\ b & a & d & c \end{vmatrix} = \begin{vmatrix} a & b & d & c \\ c & d & b & a \\ b & a & c & d \\ d & c & a & b \end{vmatrix} = a^4 + b^4 + c^4 + d^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 - 2a^2d^2 - 2b^2d^2 - 2c^2d^2 + 8abcd,$$

which is logical, as there is an even number of permutations of rows (and columns) between the first and the second matrix. In the standard case, *i.e.* if (a, b, c, d) are mapped bijectively to $(1, 2, 3, 4)$, we have

$$a^4 + b^4 + c^4 + d^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 - 2a^2d^2 - 2b^2d^2 - 2c^2d^2 + 8abcd = 0.$$

Note that

$$\begin{vmatrix} 6 & 7 & 3 & 5 & 1 & 4 & 9 & 8 & 2 \\ 9 & 2 & 1 & 3 & 6 & 8 & 7 & 5 & 4 \\ 5 & 8 & 4 & 7 & 9 & 2 & 1 & 3 & 6 \\ 8 & 9 & 6 & 2 & 3 & 7 & 4 & 1 & 5 \\ 2 & 1 & 5 & 9 & 4 & 6 & 3 & 7 & 8 \\ 3 & 4 & 7 & 1 & 8 & 5 & 2 & 6 & 9 \\ 4 & 5 & 2 & 8 & 7 & 1 & 6 & 9 & 3 \\ 1 & 6 & 9 & 4 & 5 & 3 & 8 & 2 & 7 \\ 7 & 3 & 8 & 6 & 2 & 9 & 5 & 4 & 1 \end{vmatrix} = 0.$$

(This result will be used later.)

□

Remark 1 (Possible configurations). Given a matrix S, [at least] two rows in S will never be formed by the same elements in the same order than another row. If this happens, S is not a valid Sudoku anymore, as it leads to a part of S (the example is for the rows of S, but it is as trivial for the columns of S) like

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $1, 2, \cdots, n$ can be in a different order. This would also be shown by the determinant of a submatrix of

$$\begin{pmatrix} k & k+1 & k+2 & \cdots & n \\ k & k+1 & k+2 & \cdots & n \end{pmatrix},$$

which always equals 0 (thus leading to a rank ρ less than 2),

Beginning by the end, *i.e.* using the following configuration:

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots \cdots \\ 1 & 2 & 3 & \cdots & n \\ n & n-1 & n-2 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \cdots \end{pmatrix}$$

can only lead to linearly independent rows. The rank ρ of a matrix A being defined as the biggest dimension of the square submatrices of $\neq 0$ det extracted from A, we know that

$$\rho\left(\begin{pmatrix} \cdots & \cdots & \cdots & \cdots \cdots \\ 1 & 2 & 3 & \cdots & n \\ n & n-1 & n-2 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \cdots \end{pmatrix}\right) = 2,$$

under some given conditions which will be determined now. For $k \in \{1, \cdots, n-1\}$, we have

$$\begin{vmatrix} k & k+1 \\ n & n-k \end{vmatrix} = k(n-k) - n(k+1) = kn - k^2 - kn - n = -k^2 - n;$$

That is, $-k^2 - n \leq 2$ iff $-n \leq 2 + k^2$, leading to $n \geq -2 - k^2$. As $-2 - k^2 = -3$ iff $k = 1$ and

$$-2 - k^2 \overset{k=(n-1)}{=} -2 - (n-1)^2 = -n^2 + 2n + 3,$$

we must ask

$$\begin{cases} n \geq -n^2 + 2n + 3 \\ n \geq -3 \end{cases},$$

which is equivalent to

$$\begin{cases} n & \in \left] -\infty, \frac{1-\sqrt{5}}{2} \right] \bigcup \left[\frac{1+\sqrt{5}}{2}, +\infty \right[\\ n & \geq -3 \end{cases},$$

thus giving

$$\begin{cases} n & \in \left] -\infty, \frac{1-\sqrt{5}}{2} \right] \bigcup \left[\frac{1+\sqrt{5}}{2}, +\infty \right[\\ n & \geq -3 \end{cases},$$

and

$$n \in \left[\frac{1+\sqrt{5}}{2}, +\infty \right[,$$

which is always the case, as

$$\frac{1+\sqrt{5}}{2} \simeq 1.618,$$

and the dimension of a Sudoku is always the square of a number. Thus, the first interesting Sudoku's dimension would be $4 = 2^2$ (thus greater than the golden ratio). $\square$

4 Erroneous Sudokus

Theorem 2 (Finding if a Sudoku matrix is erroneous or not). *There is no way to be sure about the correctness of a Sudoku matrix by only computing its determinant.* $\square$

Proof. An erroneous Sudoku can have a determinant of zero, but it is not always the case. Furthermore, having a determinant of zero cannot even be a sufficient condition to be an erroneous Sudoku matrix, as there are correct Sudoku matrices which have a determinant of 0.

Let's take two erroneous Sudokus:

1. The first will verify,

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \color{red}{1} & \color{red}{2} & 6 & 5 & 4 & 3 & \color{red}{7} & 9 & 8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \neq 0,$$

   assuming the $\cdots$ can be filled correctly to make no other mistake. At least a given Sudoku can thus have a non-zero determinant,

2. The second is "more" erroneous:

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} & \color{red}{6} & \color{red}{7} & \color{red}{8} & \color{red}{9} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0,$$

which is very clear, as

$$\rho\left(\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \color{red}{1\,2\,3\,4\,5\,6\,7\,8\,9} \end{pmatrix}\right) = 1.$$

We have also seen before that

$$\begin{vmatrix} a & b & c & d \\ c & d & a & b \\ d & c & b & a \\ b & a & d & c \end{vmatrix} = \begin{vmatrix} a & b & d & c \\ c & d & b & a \\ b & a & c & d \\ d & c & a & b \end{vmatrix} = a^4 + b^4 + c^4 + d^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 - 2a^2d^2 - 2b^2d^2 - 2c^2d^2 + 8abcd,$$

which equals 0 iff (a, b, c, d) are mapped bijectively to $(1, 2, 3, 4)$, leading however to a correct Sudoku. □

5 Sudoku Eigenvalues

5.1 General Case

Let's take the eigenvalues of S; that is, let's compute $\det(S - \lambda I)$. Depending on the dimension of S, denoted by n, $\det(S - \lambda I)$ will be a polynom of a different order: this order will always be of n. That is,

$$\det(S - \lambda I) = \alpha\lambda^n + \beta\lambda^{n-1} + \cdots + \zeta\lambda^0 = \sum_{i=0}^{n} \alpha_i \lambda^i \tag{2}$$

for a general S matrix of dimension n.

5.2 Example: Dimension Nine

As a 9×9 Sudoku is also a magic square, $\det(S - \lambda I) = 0$ for exactly a $\lambda_i = 45$ (found in [3]). The spectrum of S (restricted to a real spectrum), composed of the λ_i also verifies

$$\max_{\substack{1 \leq i \leq n, \\ \lambda_i \in \mathbb{R}}} \lambda_i = 45.$$

Equation 2 can be rewritten (for a 9×9 Sudoku) as

$$\det(S - \lambda I) = (\lambda - 45) \prod_{i=1}^{n-1} (\lambda - \lambda_i). \tag{3}$$

the n roots being counted with their multiplicities.

Why 45? One can notice easily two facts:

- $9 \times 5 = 45$,
- $1 + 2 + 3 + \cdots + 9 = \frac{9(9+1)}{2} = \frac{90}{2} = 45$.

Thus, the dominant eigenvalue in the spectrum of S, with a size of 9, equals the sum of all the elements of a given line (the first fact cannot be extended to other dimensions).

It is said in [3] that the determinant of a Sudoku 9×9 is always divisible by 45, and it is even the case if at least one eigenvalue is in $\mathbb{C}$.

5.3 Return to General Case

Theorem 3 (Perron). *If all of the entries of a matrix are positive, then the matrix has a dominant eigenvalue that is real and has multiplicity* 1. $\square$

The proof of this theorem is not given here.

Theorem 4 (Max eigenvalue of a square and positive matrix). *The dominant eigenvalue of any square and positive matrix where each row and column have the same sum, will equal that sum.* $\square$

Proof. Easily stated by using Theorem 3. $\square$

Corollary 1 (Eigenvalues of S). *A Sudoku matrix S with a dimension of n always verifies*

$$\max_{\substack{1 \leq i \leq n, \\ \lambda_i \in \mathbb{R}}} \lambda_i = \sum_{j=1}^{n} S_{i,j} = \frac{n(n+1)}{2}$$

for exactly one λ_i. This is thus the dominant eigenvalue of the matrix S. This eigenvalue has multiplicity 1. $\square$

Proof. Trivial using Theorem 3 and Theorem 4. $\square$

Corollary 2. *A Sudoku square matrix S, of dimension n, and of determinant* $\det(S)$, *always verifies*

$$\det(S) \mod \left(\max_{\substack{1 \leq i \leq n, \\ \lambda_i \in \mathbb{R}}} \lambda_i \right) = 0,$$

which is equivalent to ask

$$\det(S) \mod \left(\frac{n(n+1)}{2} \right) = 0.$$

$\square$

Proof. The determinant of a (square) matrix is the product of its eigenvalues.

Example 1 (Divisibility of the determinant of S if it represents a 4×4 Sudoku). Whatever the 4×4 Sudoku, its determinant is always divisible by $\frac{4(4+1)}{2} = 10$. Evidently, it *can* be divisible by 45, for example, but it might not always be the case. $\square$

6 Transpose

One could ask if

$$\widetilde{S} = \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,n} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,1} & S_{n,2} & \cdots & S_{n,n} \end{pmatrix} = \begin{pmatrix} S_{1,1} & S_{2,1} & \cdots & S_{n,1} \\ S_{1,2} & S_{2,2} & \cdots & S_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{1,n} & S_{2,n} & \cdots & S_{n,n} \end{pmatrix}$$

is still a valid Sudoku matrix (*i.e.* it fulfills the rules of Sudoku).

An example would be given by

$$B = \widetilde{\begin{pmatrix} 6\,2\,9\,3\,8\,5\,1\,4\,7 \\ 5\,1\,3\,7\,6\,4\,8\,2\,9 \\ 8\,4\,7\,1\,2\,9\,6\,5\,3 \\ 4\,9\,6\,2\,3\,1\,7\,8\,5 \\ 2\,7\,5\,9\,4\,8\,3\,6\,1 \\ 1\,3\,8\,6\,5\,7\,4\,9\,2 \\ 3\,8\,1\,4\,9\,2\,5\,7\,6 \\ 9\,6\,4\,5\,7\,3\,2\,1\,8 \\ 7\,5\,2\,8\,1\,6\,9\,3\,4 \end{pmatrix}} = \begin{pmatrix} 6\,5\,8\,4\,2\,1\,3\,9\,7 \\ 2\,1\,4\,9\,7\,3\,8\,6\,5 \\ 9\,3\,7\,6\,5\,8\,1\,4\,2 \\ 3\,7\,1\,2\,9\,6\,4\,5\,8 \\ 8\,6\,2\,3\,4\,5\,9\,7\,1 \\ 5\,4\,9\,1\,8\,7\,2\,3\,6 \\ 1\,8\,6\,7\,3\,4\,5\,2\,9 \\ 4\,2\,5\,8\,6\,9\,7\,1\,3 \\ 7\,9\,3\,5\,1\,2\,6\,8\,4 \end{pmatrix}$$

It is noticeable that $\widetilde{B}$ is still a valid Sudoku matrix, but $\widetilde{B} \neq B$.

Theorem 5 (Transpose of a Sudoku matrix). *The transpose of a Sudoku matrix is still a correct, but different, Sudoku matrix (of the same dimension).* □

Proof. According to the rules of Sudoku, at least one number cannot be repeated at least two times on a row, a column, or in a $n^{\frac{1}{2}} \times n^{\frac{1}{2}}$ square, where n is the dimension of the Sudoku. The matrix which is the result of the transposition is *still a valid Sudoku matrix*, as

1. $\widetilde{B_{i,j}} = B_{i,j}$ if $i = j$ (*i.e.* (i, j) is a diagonal couple),
2. If we let $j_{\text{old}} := j$, $j := i$ and $i := j_{\text{old}}$, everywhere in B, we obtain $\widetilde{B}$. By reversing rows and columns, rules of Sudoku are still respected, as the rules of Sudoku are applicable on both,
3. For subsquares, rules are still respected, because of the last point.

We now have to prove that *it is always different*. We have to prove that *there is no* Sudoku matrix in $\mathcal{S}$ which verifies $S_{i,j} = \widetilde{S_{i,j}}$ for all $1 \leq i \leq n$, and $1 \leq j \leq n$. Let's reason by contradiction. All the S matrices verify

$$S_{i,j} = \widetilde{S_{i,j}}$$

for $i = j$. Let $1 \leq i \leq n$, $1 \leq j \leq n$. Is the same equality possible with these conventions? We shall now try to construct such a matrix. Let's reason first with a 4×4 matrix. We define

$$B := \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} \\ B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} \\ B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} \\ B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4} \end{pmatrix} \quad \text{and} \quad \widetilde{B} := C := \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} \\ C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} \\ C_{4,1} & C_{4,2} & C_{4,3} & C_{4,4} \end{pmatrix}.$$

It is clear that $B_{i,j} = \widetilde{B_{i,j}}$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$ iff $B_{i,j} = C_{i,j}$ for all i and j. The C matrix has thus to be exactly B. It is the case iff $B_{i,j} = B_{j,i}$, with $1 \leq i \leq n$ and $1 \leq j \leq n$, thus iff B is symmetric. Let's try to construct a symmetric B. Such a B would be symmetric:

$$\begin{pmatrix} B_{1,1} & B_{2,1} & B_{3,1} & B_{4,1} \\ B_{2,1} & B_{2,2} & B_{3,2} & B_{4,2} \\ B_{3,1} & B_{3,2} & B_{3,3} & B_{4,3} \\ B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4} \end{pmatrix},$$

but it does not respect the rules of Sudoku, as there is at least one subsquare of dimension $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} \equiv 2 \times 2$ where two elements are the same (here, there is only one: $B_{4,3}$). Here is the generalization of this fact. If we take

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,n} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n,1} & B_{n,2} & \cdots & B_{n,n} \end{pmatrix},$$

B should verify

$$B = \begin{pmatrix} B_{1,1} & B_{2,1} & \cdots & B_{n,1} \\ B_{2,1} & B_{2,2} & \cdots & B_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n,1} & B_{n,2} & \cdots & B_{n,n} \end{pmatrix}$$

to be symmetric. It is symmetric, but does not respect the rules of Sudoku anmyore, as, at the place of the $\vdots$ and the $\cdots$, zooming at the end would lead to the submatrix

$$\begin{pmatrix} \ddots & \cdots & \cdots & S_{n,2} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \ddots & S_{n,n-1} \\ S_{n,2} & \cdots & S_{n,n-1} & S_{n,n} \end{pmatrix}.$$

□

6.1 Determinant

Theorem 6 (Determinant of the transpose of a square matrix). *For every square matrix C, $\det(C) = \det(\widetilde{C})$.* □

This theorem is not proved here, even if it is evident.

Corollary 3 (Determinant of the transpose of a Sudoku matrix). *If S is the matrix of a given Sudoku, $\det(S) = \det(\widetilde{S})$.* □

Corollary 4. *If S is the matrix of a given Sudoku, $\widetilde{S}$ its transpose, and n their dimension (which is the same),*

$$\det(S) \mod \left(\frac{n(n+1)}{2}\right) = \det(\widetilde{S}) \mod \left(\frac{n(n+1)}{2}\right) = 0.$$

□

Proof. Trivial, as $\det(S) = \det(\widetilde{S})$. □

6.2 Trace

Theorem 7 (Trace of a transpose). *Whatever the matrix E, $\mathrm{Tr}(E) = \mathrm{Tr}(\widetilde{E})$.* □

This theorem is not proved here, even if it is evident.

Corollary 5 (Trace of the transpose of a Sudoku matrix). *If S is the matrix of a given Sudoku, $\mathrm{Tr}(S) = \mathrm{Tr}(\widetilde{S})$.* □

7 Non-Hermitianity

A matrix C is *Hermitian* if and only if

$$C = C^*.$$

As, for any Sudoku matrix S, $S = \overline{S}$, as $\bar{\cdot}$ is the complex conjugate of $\cdot$, and the $S_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq n$, asking if any Sudoku matrix is Hermitian or not is equivalent to ask if, for S, $S = \widetilde{S}$. However, this is not true, as we have

$$S \neq \widetilde{S},$$

thanks to Theorem 5.

Corollary 6 (Every Sudoku matrix is not Hermitian). *There is no Hermitian Sudoku matrix.* □

Proof. Direct using Theorem 5, as $\overline{S_{i,j}} = S_{i,j}$ for $1 \leq i \leq n$, $1 \leq j \leq n$, as $S_{i,j} \in \mathbb{R}$. □

Corollary 7 (Every Sudoku matrix is not antisymmetric). *S is not antisymmetric, for clear reasons: $S_{i,j} \neq -S_{j,i}$.* □

Proof. The $S_{i,j}$ are in $\mathbb{Z}^*$. If we take $-S_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$, all the $S_{i,j}$ will lie in $\mathbb{Z}^*_-$, or no Sudoku matrix can have negative elements. □

8 Non-Normality

A matrix C is *normal* if and only if

$$CC^* = C^*C.$$

It is equivalent to ask, for now clear reasons,

$$C\widetilde{C} = \widetilde{C}C,$$

if C is a Sudoku matrix.

Theorem 8 (Every Sudoku matrix is not normal). *There is no normal Sudoku matrix.* □

Proof. Let's try to build such a matrix. If we have

$$S = \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,n} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,1} & S_{n,2} & \cdots & S_{n,n} \end{pmatrix} \qquad \text{and} \qquad S' := \widetilde{S} = \begin{pmatrix} S_{1,1} & S_{2,1} & \cdots & S_{n,1} \\ S_{1,2} & S_{2,2} & \cdots & S_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{1,n} & S_{2,n} & \cdots & S_{n,n} \end{pmatrix},$$

we have $S\widetilde{S} = \widetilde{S}S$ iff

$$S = \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,n} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,1} & S_{n,2} & \cdots & S_{n,n} \end{pmatrix} \begin{pmatrix} S_{1,1} & S_{2,1} & \cdots & S_{n,1} \\ S_{1,2} & S_{2,2} & \cdots & S_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{1,n} & S_{2,n} & \cdots & S_{n,n} \end{pmatrix} = \begin{pmatrix} S_{1,1} & S_{2,1} & \cdots & S_{n,1} \\ S_{1,2} & S_{2,2} & \cdots & S_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{1,n} & S_{2,n} & \cdots & S_{n,n} \end{pmatrix} \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,n} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,1} & S_{n,2} & \cdots & S_{n,n} \end{pmatrix},$$

thus asking an equality between

$$\begin{pmatrix} S_{1,1}^2 + S_{1,2}^2 + \cdots + S_{1,n}^2 & S_{1,1}S_{2,1} + S_{1,2}S_{2,2} + \cdots + S_{1,n}S_{2,n} & \cdots & S_{1,1}S_{n,1} + S_{1,2}S_{n,2} + \cdots + S_{1,n}S_{n,n} \\ S_{2,1}S_{1,1} + S_{2,2}S_{1,2} + \cdots + S_{2,n}S_{1,n} & S_{2,1}^2 + S_{2,2}^2 + \cdots + S_{2,n}^2 & \cdots & S_{2,1}S_{n,1} + S_{2,2}S_{n,2} + \cdots + S_{2,n}S_{n,n} \\ \cdots & \cdots & \cdots & \cdots \\ S_{n,1}S_{1,1} + S_{n,2}S_{1,2} + \cdots + S_{n,n}S_{1,n} & S_{n,1}S_{2,1} + S_{n,2}S_{2,2} + \cdots + S_{n,n}S_{2,n} & \cdots & S_{n,1}^2 + S_{n,2}^2 + \cdots + S_{n,n}^2 \end{pmatrix}$$

and

$$\begin{pmatrix} S_{1,1}^2 + S_{2,1}^2 + \cdots + S_{n,1}^2 & S_{1,1}S_{1,2} + S_{2,1}S_{2,2} + \cdots + S_{n,1}S_{n,2} & \cdots & S_{1,1}S_{1,n} + S_{2,1}S_{2,n} + \cdots + S_{n,1}S_{n,n} \\ S_{1,2}S_{1,1} + S_{2,2}S_{2,1} + \cdots + S_{n,2}S_{n,1} & {S_{1,2}}^2 + S_{2,2}^2 + \cdots + {S_{n,2}}^2 & \cdots & S_{1,2}S_{1,n} + S_{2,2}S_{2,n} + \cdots + S_{n,2}S_{n,n} \\ \cdots & \cdots & \cdots & \cdots \\ S_{1,n}S_{1,1} + S_{2,n}S_{2,1} + \cdots + S_{n,n}S_{n,1} & S_{1,n}S_{1,2} + S_{2,n}S_{2,2} + \cdots + S_{n,n}S_{n,2} & \cdots & S_{n,1}^2 + {S_{2,n}}^2 + \cdots + S_{n,n}^2 \end{pmatrix},$$

leading to such a system:

$$\begin{cases} S_{1,n}^2 & = S_{n,1}^2 \\ S_{1,2} & = S_{2,1} \\ S_{n,1} & = S_{1,n} \\ S_{n,2} & = S_{2,n} \\ & \vdots \end{cases}.$$

The condition $S_{1,2} = S_{2,1}$ already goes against Sudoku's rules. Anyway, it is clear that S has to be symmetric to respect these equalities. However, it was proven at Theorem 5 that S cannot be symmetric. □

9 Orthogonality Discussion

A matrix C is *orthogonal* if and only if

$$\widetilde{C}C = I_n.$$

Theorem 9 (Non-orthogonality of S if $\det(S) = 0$). *If $\det(S) = 0$, S cannot be orthogonal.* □

Proof. We have

$$\det(\widetilde{S} \cdot S) = \det(\widetilde{S}) \det(S),$$

which equals 0 if $\det(S) = 0$, and $\det(I_n) = 1$. □

If $\det(S) \neq 0$, $\widetilde{S}$ has to coincide with S^{-1} to make S orthogonal. Using Theorem 5, $\widetilde{S} \neq S$. But one might ask the question "Can $\widetilde{S}$ equal S^{-1}?" The answer to this question is not known yet.

Conjecture 1 (Non-orthogonality of S if $\det(S) \neq 0$). If $\det(S) \neq 0$, S cannot be orthogonal. □

***Suggestion* 1** *The only way to make S orthogonal is to have $\widetilde{S} = S^{-1}$. Or this equality should be impossible: as*

$$S^{-1} = \widetilde{\mathrm{cofact}(S)} \cdot (\det(S))^{-1},$$

it would be equivalent to ask

$$\widetilde{S} = \widetilde{\mathrm{cofact}(S)} \cdot (\det(S))^{-1},$$

which should never be true for a S Sudoku matrix. □

10 Order

Theorem 10 (Every invertible matrix has a finite order). *Every invertible matrix has a finite order.* □

This theorem is proved in many books, and its proof does not enter in the scope of this paper.

Corollary 8. *Iff $\det(S) \neq 0$, S has a finite order.* □

Example 2. The determinant

$$\det(B) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{vmatrix}$$

equalling 0, there is no smallest $k > 0$ such that $B^k = I_n$. □

However, there is no special remark to do about this and Sudokus: there seems to be no link between the order of a Sudoku matrix S and other concepts related to Sudoku's matrices.

11 Condition Number

Theorem 11 (No well-conditioning for S can arise). *The system $Sx = b$ can be else than well-conditioned (assuming $b \neq 0_n$).* □

Proof. Consider Example 3. □

Example 3 (Extreme example). Consider the matrix

$$P := \begin{pmatrix} 6\,7\,3\,5\,1\,4\,9\,8\,2 \\ 9\,2\,1\,3\,6\,8\,7\,5\,4 \\ 5\,8\,4\,7\,9\,2\,1\,3\,6 \\ 8\,9\,6\,2\,3\,7\,4\,1\,5 \\ 2\,1\,5\,9\,4\,6\,3\,7\,8 \\ 3\,4\,7\,1\,8\,5\,2\,6\,9 \\ 4\,5\,2\,8\,7\,1\,6\,9\,3 \\ 1\,6\,9\,4\,5\,3\,8\,2\,7 \\ 7\,3\,8\,6\,2\,9\,5\,4\,1 \end{pmatrix}.$$

It verifies

$$\begin{aligned} &\kappa(P)_\infty \approx (2.052475887800532) \cdot 10^{17} \gg \\ &\det(P) = 0. \end{aligned}$$

As $\kappa(P) \gg$, trying to solve

$$\begin{pmatrix} 6\,7\,3\,5\,1\,4\,9\,8\,2 \\ 9\,2\,1\,3\,6\,8\,7\,5\,4 \\ 5\,8\,4\,7\,9\,2\,1\,3\,6 \\ 8\,9\,6\,2\,3\,7\,4\,1\,5 \\ 2\,1\,5\,9\,4\,6\,3\,7\,8 \\ 3\,4\,7\,1\,8\,5\,2\,6\,9 \\ 4\,5\,2\,8\,7\,1\,6\,9\,3 \\ 1\,6\,9\,4\,5\,3\,8\,2\,7 \\ 7\,3\,8\,6\,2\,9\,5\,4\,1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \zeta \\ \eta \\ \theta \\ \iota \end{pmatrix} = \begin{pmatrix} \kappa \\ \lambda \\ \mu \\ \nu \\ \xi \\ \chi \\ \rho \\ \sigma \\ \tau \end{pmatrix}$$

is *really* error-prone. Let's take

$$\begin{aligned} \kappa &= 1 \\ \lambda &= 2 \\ \mu &= 3 \\ \nu &= 4 \\ \xi &= 5 \\ \chi &= 6 \\ \rho &= 7 \\ \sigma &= 8 \\ \tau &= 9. \end{aligned}$$

Using numerical analysis, we can now solve

$$\begin{pmatrix} 6&7&3&5&1&4&9&8&2\\ 9&2&1&3&6&8&7&5&4\\ 5&8&4&7&9&2&1&3&6\\ 8&9&6&2&3&7&4&1&5\\ 2&1&5&9&4&6&3&7&8\\ 3&4&7&1&8&5&2&6&9\\ 4&5&2&8&7&1&6&9&3\\ 1&6&9&4&5&3&8&2&7\\ 7&3&8&6&2&9&5&4&1 \end{pmatrix} \begin{pmatrix} \alpha\\ \beta\\ \gamma\\ \delta\\ \epsilon\\ \zeta\\ \eta\\ \theta\\ \iota \end{pmatrix} = \begin{pmatrix} 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{pmatrix}.$$

The result is roughly

$$10^{15} \cdot \begin{Bmatrix} -6.259323629462618\\ 3.426654756202998\\ -2.859392164516785\\ -0.213786330057330\\ 2.235038905144813\\ 6.094125101691045\\ 0.146978101914415\\ 0.070452313314347\\ -2.640747054230881 \end{Bmatrix}.$$

Let's now compute $A := A + (0.1)_9$ and $b := b + (0.1)_9$. We now want to solve

$$\begin{pmatrix} 6.1&7.1&3.1&5.1&1.1&4.1&9.1&8.1&2.1\\ 9.1&2.1&1.1&3.1&6.1&8.1&7.1&5.1&4.1\\ 5.1&8.1&4.1&7.1&9.1&2.1&1.1&3.1&6.1\\ 8.1&9.1&6.1&2.1&3.1&7.1&4.1&1.1&5.1\\ 2.1&1.1&5.1&9.1&4.1&6.1&3.1&7.1&8.1\\ 3.1&4.1&7.1&1.1&8.1&5.1&2.1&6.1&9.1\\ 4.1&5.1&2.1&8.1&7.1&1.1&6.1&9.1&3.1\\ 1.1&6.1&9.1&4.1&5.1&3.1&8.1&2.1&7.1\\ 7.1&3.1&8.1&6.1&2.1&9.1&5.1&4.1&1.1 \end{pmatrix} \begin{pmatrix} \alpha\\ \beta\\ \gamma\\ \delta\\ \epsilon\\ \zeta\\ \eta\\ \theta\\ \iota \end{pmatrix} = \begin{pmatrix} 1.1\\2.1\\3.1\\4.1\\5.1\\6.1\\7.1\\8.1\\9.1 \end{pmatrix}.$$

The result is now

$$10^{17} \cdot \begin{Bmatrix} -1.201790136856824\\ 0.657917713190976\\ -0.549003295587223\\ -0.041046975371007\\ 0.429127469787804\\ 1.170072019524681\\ 0.028219795567568\\ 0.013526844156355\\ -0.507023434412329 \end{Bmatrix}.$$

□

12 Trace

Theorem 12 (The trace of a Sudoku matrix is not constant). *The trace of a Sudoku matrix is not constant.* □

Proof. Consider

$$P = \begin{pmatrix} 6\,7\,3\,5\,1\,4\,9\,8\,2 \\ 9\,2\,1\,3\,6\,8\,7\,5\,4 \\ 5\,8\,4\,7\,9\,2\,1\,3\,6 \\ 8\,9\,6\,2\,3\,7\,4\,1\,5 \\ 2\,1\,5\,9\,4\,6\,3\,7\,8 \\ 3\,4\,7\,1\,8\,5\,2\,6\,9 \\ 4\,5\,2\,8\,7\,1\,6\,9\,3 \\ 1\,6\,9\,4\,5\,3\,8\,2\,7 \\ 7\,3\,8\,6\,2\,9\,5\,4\,1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 9\,4\,7\,2\,5\,8\,1\,3\,6 \\ 1\,2\,3\,4\,6\,7\,9\,8\,5 \\ 6\,5\,8\,1\,9\,3\,7\,2\,4 \\ 8\,9\,5\,6\,4\,2\,3\,7\,1 \\ 7\,6\,4\,9\,3\,1\,2\,5\,8 \\ 3\,1\,2\,8\,7\,5\,4\,6\,9 \\ 4\,8\,9\,7\,2\,6\,5\,1\,3 \\ 2\,3\,6\,5\,1\,9\,8\,4\,7 \\ 5\,7\,1\,3\,8\,4\,6\,9\,2 \end{pmatrix},$$

whose traces are 32 and 44, respectively. $\square$

References

1. Davis, Tom, *The Mathematics of Sudoku*, (2008). http://www.geometer.org/mathcircles/sudoku.pdf.
2. Frank, Richard, *Mathematics in Sudoku*, (2005). http://people.brandeis.edu/~kleinboc/47a/sudoku.pdf.
3. LeBoeuf, Robert, *Properties of Sudoku and Sudoku Matrices*, (2009). http://compmath.files.wordpress.com/2009/02/rlfreport.pdf.
4. ———, *Sudoku Matrices*, (2009). http://compmath.files.wordpress.com/2009/02/rlposter.pdf.