A Refined Method for Estimating the Global Hölder Exponent

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Abstract. In this paper, we recall basic results we have obtained about generalized Hölder spaces and present a wavelet characterization that holds under more general hypothesis than previously stated. This theoretical tool gives rise to a method for estimating the global Hölder exponent which seems to be more precise than other wavelet-based approaches. This work should prove helpful for estimating long range correlations.

Keywords: wavelets, uniform Hölder exponent, generalized Hölder spaces, long range correlations, Brownian motion.

1 Introduction

A continuous function $f \in L^\infty(\mathbf{R}^d)$ belongs to the uniform Hölder space $\Lambda^\alpha(\mathbf{R}^d) = \Lambda^\alpha$ with $\alpha > 0$ if there exists a constant C such that for each $x \in \mathbf{R}^d$, there exists a polynomial P_x of degree at most α for which

$$|f(x+h) - P_x(h)| \leq C|h|^\alpha. \tag{1}$$

Since these spaces are embedded, one can define the Hölder exponent as follows,

$$H_f = \sup\{\alpha : f \in \Lambda^\alpha\}.$$

The Hölder exponent of a function is a notion of global regularity: the larger the Hölder exponent, the more regular the corresponding function. In particular, if f is n times continuously differentiable then f belongs to Λ^α for any $\alpha \leq n$. For example, a sample path of a Brownian motion W belongs to $\Lambda^{1/2}$ almost surely and $H_W = 1/2$ almost surely. More generally, the sample path of a fractional Brownian motion of Hurst index H ($H \in (0,1)$) belongs to Λ^H almost surely and the associated Hölder exponent is equal to H almost surely (a Brownian motion is a fractional Brownian motion with Hurst index $H = 1/2$) [11]. Figure 1 clearly illustrates the fact that the regularity increases with the Hölder exponent.

Among methods for estimating the Hölder exponent, the wavelet-based approach [9] is both fast and relatively efficient. It thus allows to test if whether or

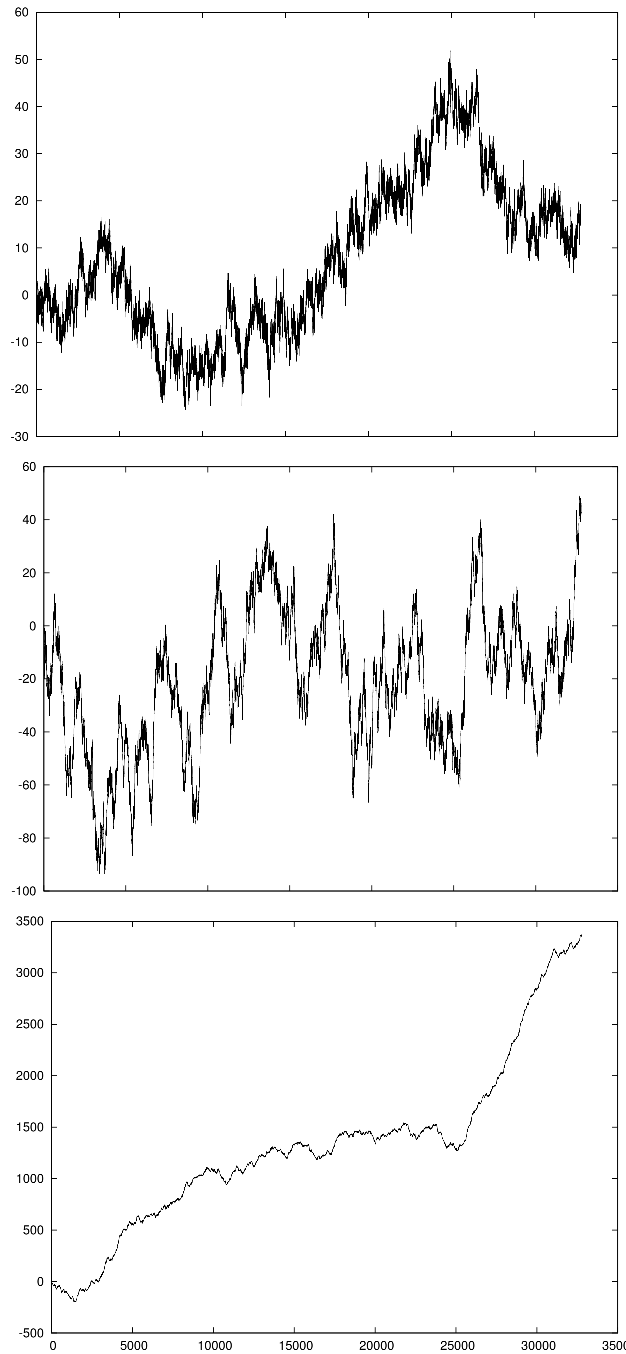

Fig. 1. A fractional Brownian motion with Hurst index $H = 0.3$ (top), a Brownian motion, i.e. a fractional Brownian motion with Hurst index $H = 0.5$ (middle) and a fractional Brownian motion with Hurst index $H = 0.7$ (bottom). One clearly sees that the regularity of the walk increases with the Hurst index.

not a fractional Brownian motion displays long range correlations (i.e. is associated to a Hurst exponent $H > 1/2$). However, such a method for estimating the Hölder exponent cannot allow to make the distinction between say a Brownian motion and a process displaying the same Hölder exponent. Yet, the Brownian motion W exhibits a very specific behavior, since one has

$$|W(t+h) - W(t)| \leq C\sqrt{|h| \log|\log|h||}, \tag{2}$$

almost surely, for a constant $C > \sqrt{2}$ [5].

The purpose of this paper is to provide a method that could help to make the distinction between a process satisfying inequality (2) and a process belonging to $\Lambda^{1/2}$ for which inequality (2) is not satisfied. The basic idea is to generalize the spaces Λ^α by replacing the the expression $|h|^\alpha$ appearing in the right-hand side of (1) with something more general, covering the usual case. Such spaces are both inspired by generalized Besov spaces (see e.g. [10]) and moduli of continuity for Hölder spaces [4].

This paper is organized as follows: we first review the notion of generalized Hölder space based on admissible sequences, as introduced in [6, 7]. Next, we give a wavelet characterization of these spaces. Finally, we propose an application for better estimating the Hölder exponent of a Brownian motion, which should help to make the distinction between a Brownian motion and another process associated to the same Hölder exponent. This theorem as well as the application are new and generalize previous results (see [9, 7]).

Throughout the paper, B denotes the open unit ball and we use the letter C for generic positive constant whose value may be different at each occurrence.

2 Definitions

All the definitions and results presented in this section are taken from [6, 7]; the reader is referred to these articles for further details.

2.1 Admissible Sequences

The notion of generalized Hölder space we want to introduce is based on the following definition.

Definition 1. A sequence $\sigma = (\sigma_j)_{j\in\mathbf{N}}$ of real positive numbers is called admissible if there exists a positive constant C such that

$$C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j,$$

for any $j \in \mathbf{N}$.

If σ is such a sequence, we set

$$\underline{\Theta}_j = \inf_{k\in\mathbf{N}} \frac{\sigma_{j+k}}{\sigma_k} \qquad \text{and} \qquad \overline{\Theta}_j = \sup_{k\in\mathbf{N}} \frac{\sigma_{j+k}}{\sigma_k}$$

and define the lower and upper Boyd indices as follows,

$$\underline{s}(\sigma) = \lim_j \frac{\log_2 \underline{\Theta}_j}{j} \qquad \text{and} \qquad \overline{s}(\sigma) = \lim_j \frac{\log_2 \overline{\Theta}_j}{j}.$$

Since $(\log \overline{\Theta}_j)_{j\in\mathbf{N}}$ is a subadditive sequence, such limits always exist. If σ is an admissible sequence, let $\varepsilon > 0$; there exists a positive constant C such that

$$C^{-1} 2^{j(\underline{s}(\sigma)-\varepsilon)} \leq \underline{\Theta}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\Theta}_j \leq C 2^{j(\overline{s}(\sigma)+\varepsilon)},$$

for any $j, k \in \mathbf{N}$.

The following result allows to generate admissible sequences from existing admissible sequences:

Proposition 1. If τ and υ are two admissible sequences and α is a real number, then

- The sequence $\sigma = (2^{-j\alpha})_{j\in\mathbf{N}}$ is an admissible sequence with $\underline{s}(\sigma) = \overline{s}(\sigma) = -\alpha$.
- Let $\varphi : [0,1] \to (0,\infty)$ be a weakly varying function, i.e. a function satisfying

  $$\lim_{x\to 0} \frac{\varphi(rx)}{\varphi(x)} = 1,$$

  for any $r > 0$. The sequence $\sigma = (2^{-\alpha j}\varphi(2^j))_{j\in\mathbf{N}}$ is an admissible sequence such that $\underline{s}(\sigma) = \overline{s}(\sigma) = -\alpha$.
- The sequences $\tau + \upsilon$ and $\tau\upsilon$ are admissible sequences (with $\underline{s}(\tau\upsilon) \geq \underline{s}(\tau) + \underline{s}(\upsilon)$ and $\overline{s}(\tau\upsilon) \leq \overline{s}(\tau) + \overline{s}(\upsilon)$)
- If $\alpha > 0$, $\alpha\tau$ is an admissible sequence (with $\underline{s}(\alpha\tau) = \underline{s}(\tau)$ and $\overline{s}(\alpha\tau) = \overline{s}(\tau)$).
- If $\alpha > 0$, τ^α is an admissible sequence such that $\underline{s}(\tau^\alpha) = \alpha\underline{s}(\tau)$ and $\overline{s}(\tau^\alpha) = \alpha\overline{s}(\tau)$.
- If $\alpha < 0$, τ^α is an admissible sequence such that $\underline{s}(\tau^\alpha) = \alpha\overline{s}(\tau)$ and $\overline{s}(\tau^\alpha) = \alpha\underline{s}(\tau)$.

For example, $\varphi = |\log|$ is a weakly varying function.

The following result gives informations about the convergence of the series associated to an admissible sequence.

Proposition 2. Let σ be an admissible sequence.

- If $\underline{s}(\sigma^{-1}) > 0$, there exists a positive constant C such that for any $J \in \mathbf{N}$,

  $$\sum_{j\geq J} \sigma_j \leq C\sigma_J.$$

- If $\overline{s}(\sigma^{-1}) < n$ with $n \in \mathbf{N}$, there exists a positive constant C such that for any $J \in \mathbf{N}$,

  $$\sum_{j=1}^{J} 2^{jn}\sigma_j \leq C 2^{Jn}\sigma_J.$$

2.2 Generalized Hölder spaces

We can now introduce a definition of generalized Hölder space.

As usual, $[\alpha]$ will denote the greatest integer lower than α,

$$[\alpha] = \sup\{p \in \mathbf{Z} : p \leq \alpha\}$$

and $\Delta_h^n f$ will stand for the finite difference of order n: given a function f defined on $\mathbf{R}^d$ and $x, h \in \mathbf{R}^d$, $\Delta_h^1 f(x) = f(x+h) - f(x)$ and

$$\Delta_h^{n+1} f(x) = \Delta_h^1 \Delta_h^n f(x),$$

for any $n \in \mathbf{N}$.

Definition 2. Let $\alpha > 0$ and σ be an admissible sequence; a function $f \in L^\infty(\mathbf{R}^d)$ belongs to the space $\Lambda^{\sigma,\alpha}(\mathbf{R}^d) = \Lambda^{\sigma,\alpha}$ if there exists $C > 0$ such that

$$\sup_{|h| \leq 2^{-j}} \|\Delta_h^{[\alpha]+1} f\|_\infty \leq C\sigma_j,$$

for any $j \in \mathbf{N}$.

One sets $\Lambda^\sigma = \Lambda^{\sigma,\overline{s}(\sigma^{-1})}$. The application

$$|f|_{\Lambda^{\sigma,\alpha}} = \sup_j (\sigma_j^{-1} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty})$$

defines a semi-norm on $\Lambda^{\sigma,\alpha}$ and therefore $\|f\|_{\Lambda^{\sigma,\alpha}} = \|f\|_{L^\infty} + |f|_{\Lambda^{\sigma,\alpha}}$ is a norm on this space. We have the following result.

Theorem 3. Let $\alpha > 0$ and σ be an admissible sequence; the space $\Lambda^{\sigma,\alpha}$ equipped with the norm $\|\cdot\|_{\Lambda^{\sigma,\alpha}}$ is a Banach space.

Many equivalent definitions of theses spaces can be given. In particular, they can be written using polynomials. Let us denote the set of polynomials of degree at most n by $\mathbf{P}_n$.

Theorem 4. Let σ be an admissible sequence, $\alpha > 0$ and $f \in L^\infty(\mathbf{R}^d)$; f belongs to $\Lambda^{\sigma,\alpha}$ if and only if there exists a constant $C > 0$ such that

$$\inf_{P \in \mathbf{P}_{[\alpha]}} \|f - P\|_{L^\infty(2^{-j}B + x_0)} \leq C\sigma_j, \tag{3}$$

for any $x_0 \in \mathbf{R}^d$ and any $j \in \mathbf{N}$.

For example a sample path of a Brownian motion belongs to Λ^σ with

$$\sigma = (2^{-j/2}\sqrt{\log j})_j \tag{4}$$

almost surely.

One also can show that these spaces are related to the regularity of their elements. For example, we have the following result:

Theorem 5. Let σ be an admissible sequence and N, M be two positive integers such that

$$N < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < M.$$

Any element of Λ^σ is equal almost everywhere to a function $f \in C^N(\mathbf{R}^d)$ satisfying $D^\beta f \in L^\infty(\mathbf{R}^d)$ for any multi-index β such that $|\beta| \leq N$ and

$$\sup_{|h| \leq 2^{-j}} \|\Delta_h^{M-|\beta|} D^\beta f\|_\infty \leq C 2^{j|\beta|} \sigma_j, \tag{5}$$

for any $j \in \mathbf{N}$ and $|\beta| \leq N$. Conversely, if $f \in L^\infty(\mathbf{R}^d) \cap C^N(\mathbf{R}^d)$ satisfies inequality (5) for $|\beta| = N$, then $f \in \Lambda^\sigma$.

3 Wavelet characterization

3.1 Definitions

Under some general assumptions (see e.g. [9, 2]), there exists a function φ and $2^d - 1$ functions $(\psi^{(i)})_{1 \leq i < 2^d}$ called wavelets such that

$$\{\varphi(x-k)\}_{k \in \mathbf{Z}^d} \cup \{\psi^{(i)}(2^j x - k) : 1 \leq i < 2^d, k \in \mathbf{Z}^d, j \in \mathbf{N}_0\}$$

form an orthogonal basis of $L^2(\mathbf{R}^d)$. Any function $f \in L^2(\mathbf{R}^d)$ can be decomposed as follows,

$$f(x) = \sum_{k \in \mathbf{Z}^d} C_k \varphi(x-k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}^d} \sum_{1 \leq i < 2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int_{\mathbf{R}^d} f(x) \psi^{(i)}(2^j x - k)\, dx$$

and

$$C_k = \int_{\mathbf{R}^d} f(x) \varphi(x-k)\, dx.$$

The above formulas are still valid in more general settings; they have to be interpreted as a duality product between regular functions (the wavelets) and distributions [9, 3]. In what follows, we will suppose that the wavelets are the Daubechies wavelets [2] (which are compactly supported and can be chosen arbitrarily regular, let us say r-regular with $r > \overline{s}(\sigma^{-1})$).

3.2 The characterization

We aim at showing the following result.

Theorem 6. Let σ be an admissible such that $\underline{s}(\sigma^{-1}) > 0$. If f belongs to Λ^σ, there exists $C > 0$ such that

$$|C_k| \le C \qquad \text{and} \qquad |c_{j,k}^{(i)}| \le C\sigma_j, \tag{6}$$

for any $j \in \mathbf{N}$, any $k \in \mathbf{Z}^d$ and any $i \in \{1, \ldots, 2^d - 1\}$.

Conversely, if $f \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$ and (6) holds, then $f \in \Lambda^\sigma$.

Other hypothesis are given in [7], but the admissible sequence is requested to be strong.

Proof. Let $f \in \Lambda^\sigma$ and let us prove that (6) holds. Let $M \in \mathbf{N}$ and $j_0 \in \mathbf{N}$ be such that $M > \overline{s}(\sigma^{-1})$ and $\mathrm{supp}\psi^{(i)} \subset 2^{j_0}B$ for any i. We have

$$|C_k| = |\int f(x)\varphi(x-k)\, dx| \le C\|f\|_{L^\infty}.$$

Using Theorem 4, for $k \in \mathbf{Z}^d$ and $j \ge j_0$ one can find a polynomial P of degree less or equal to $M-1$ such that

$$\|f(\cdot) - P(\cdot - k/2^j)\|_{L^\infty(B')} \le C\sigma_{j-j_0},$$

where $B' = k2^{-j} + 2^{-(j-j_0)}B$. One gets

$$\begin{aligned}
|c_{j,k}^{(i)}| &= 2^{jd}|\int f(x)\psi^{(i)}(2^j x - k)\, dx| \\
&= 2^{jd}|\int_{B'} \left(f(x) - P(x - k/2^j)\right)\, \psi^{(i)}(2^j x - k)\, dx| \\
&\le C\sigma_j \sup_{i \in \{1,\ldots,2^d-1\}} \|\psi^{(i)}\|_{L^1(\mathbf{R}^d)}.
\end{aligned}$$

Let us now suppose that (6) holds. We need to show that the hypothesis of Proposition 5 are satisfied. Let $N, M \in \mathbf{N}_0$ be such that

$$N < \underline{s}(\sigma^{-1}) \le \overline{s}(\sigma^{-1}) < M \le r$$

and let us define

$$f_{-1}(x) = \sum_{k \in \mathbf{Z}^d} C_k \varphi(x-k) \quad \text{and} \quad f_j(x) = \sum_{i=1}^{2^d-1} \sum_{k \in \mathbf{Z}^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

for any $j \in \mathbf{N}_0$. One easily checks that f_j converges uniformly on every compact set, so that f_j has the same regularity as $\psi^{(i)}$. For every j, we have $|f_j| \le C\sigma_j$ so that $g = \sum_{j=-1}^\infty f_j$ uniformly converges to a function belonging to $L^\infty(\mathbf{R}^d)$. One thus gets $f = g$. Using the properties of φ and $\psi^{(i)}$, one gets

$$|D^\beta f_j| \le C2^{|\beta| j}\sigma_j,$$

for any multi-index β such that $|\beta| \leq M$. Consequently, g is differentiable up to order N, which shows that we have $f \in C^N(\mathbf{R}^d)$ and $|D^\beta f| \leq C$ for any β such that $|\beta| \leq N$. Let α be a multi-index such that $|\alpha| = N$ and let $h \in \mathbf{R}^d$ and $j_0 \in \mathbf{N}$ be such that $|h| < 2^{-j_0}$. We have

$$\begin{aligned}
&\|\Delta_h^{M-N} D^\alpha f\|_{L^\infty} \\
&\leq \sum_{j \leq j_0} \|\Delta_h^{M-N} D^\alpha f_j\|_{L^\infty} + \sum_{j > j_0} 2^{M-N} \|D^\alpha f_j\|_{L^\infty} \\
&\leq C \sum_{j \leq j_0} |h|^{M-N} \sup_{|\beta| = M-N} \|D^{\alpha+\beta} f_j\|_{L^\infty} + C \sum_{j > j_0} 2^{Nj} \sigma_j \\
&\leq C|h|^{M-N} 2^{Mj_0} \sigma_{j_0} + C 2^{N(j_0+1)} \sigma_{j_0+1} \\
&\leq C 2^{Nj_0} \sigma_{j_0},
\end{aligned}$$

hence the conclusion.

3.3 Usefulness in the Estimation of the Hölder Exponent

The usual version of Theorem 6 theoretically allows to estimate the uniform Hölder regularity of a function f by looking at the behavior of the wavelet coefficients versus the scales j. For the sake of simplicity, let us suppose that f belongs to $L^2([0,1])$ (this is by no mean a restriction, see e.g. [1]) and let Ψ_j denotes the set of wavelet coefficients at scale j. Let us suppose that $f \in \Lambda^H$; at each scale j, one computes the wavelet power spectrum of f (see e.g. [12]):

$$S_f(j) = \sqrt{\frac{1}{\#\Psi_j} \sum_{i,k} |c_{j,k}^{(i)}|^2}.$$

Following the standard wavelet characterization [9], one should have

$$S_f(j) \sim C\omega^{(H)}(2^{-j})$$

with $\omega^{(H)}(r) = r^H$. This implies

$$\log_2 S_f(j) \sim -Hj + C,$$

so that a log-log plot can be used to estimate the slope H. Another method consists in fitting a parametric curve $C\omega^{(H)}(2^{-j})$ to the function S_f. In this case, one can also apply Theorem 6 and modify $\omega^{(H)}$ to obtain a better fit. For the Brownian motion, having (2) or (4) in mind, one should choose

$$\omega_W^{(H)}(r) = (r \log|\log r|)^H \tag{7}$$

in order to get a sharper estimation and help to discern between two models.

As an illustration, the wavelet power spectrum of a Brownian motion W (2^{15} points) is represented in Figure 2. When trying to fit the curve $C2^{-jh}$ to S_W

using the Levenberg-Marquardt algorithm [8], one gets $h_0 = 0.48$ with (asymptotic) standard deviation $5\,10^{-2}$ (see Figure 2). The same computation with the curve $C\omega_W^{(H)}(2^{-j})$ gives $h_0 = 0.499$ with (asymptotic) standard deviation $3\,10^{-2}$, which is closer to the expected value $1/2$. Of course, additional work has to be done in order to suitably validate this method.

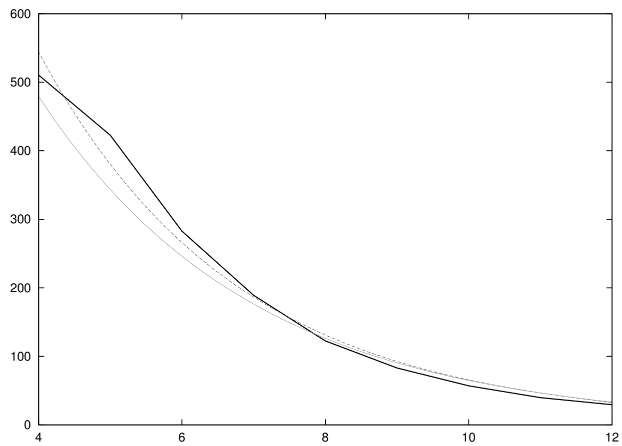

Fig. 2. The function S_W (thick black), with the curves $j \mapsto C2^{-jh}$ (grey) and $j \mapsto C\omega_W^{(H)}(2^{-j})$ (dashed lines) defined by equality (7). Both curves are obtained with the Levenberg-Marquardt algorithm.

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