Anticoherence and entanglement of spin states

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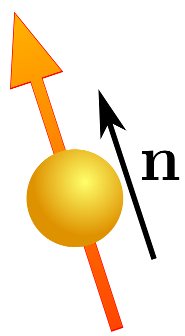
Outline

- Anticoherence (definition)
- Anticoherence and entanglement
- All anticoherent states for 2, 3 and 4 spin-$\frac{1}{2}$ (or qubits)
- Anticoherence and symmetry in the Majorana representation

Coherent vs anticoherent spin states

Coherent state



$\langle \mathbf{J} \rangle = j\hbar\, \mathbf{n}$: "most classical"

Definition

A spin$-j$ state $|\psi_j\rangle$ is coherent if it is eigenstate of $\mathbf{J} \cdot \mathbf{n}$ for some unit vector $\mathbf{n}$ with the highest eigenvalue $j\hbar$

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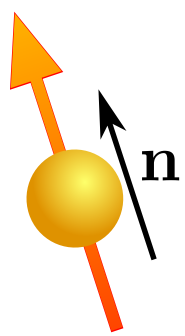

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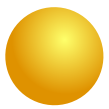
Anticoherent state



$\langle \mathbf{J} \rangle = 0$: "the spin points nowhere" [2]

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Definition

A spin$-j$ state $|\psi_j\rangle$ is anticoherent to order t or t-anticoherent if $\langle\psi_j|(\mathbf{J}\cdot\mathbf{n})^k|\psi_j\rangle$ does not depend on $\mathbf{n}$ for $k = 1, \ldots, t$ [1, 2]

[1] J. Zimba, EJTP **3**, 143 (2006)

[2] E. Bannai, M. Tagami, J. Phys. A **44**, 342002 (2011)

Conditions for anticoherence

Examples : [standard basis : $\mathbf{J}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle,\ J_z|j,m\rangle = m\hbar|j,m\rangle$]

- For all j, the state $|\psi_j^{\text{cat}}\rangle = \frac{1}{\sqrt{2}}(|j,j\rangle + |j,-j\rangle)$ is 1-anticoherent
- For integer j, the state $|j,0\rangle$ is 1-anticoherent

Conditions: [3]

- A spin$-j$ state $|\psi_j\rangle$ is t-anticoherent if:

$$\boxed{\langle\psi_j|J_+^r J_z^q|\psi_j\rangle = \frac{\mathrm{Tr}(J_z^q)}{2j+1}\,\delta_{r0} \qquad \begin{array}{l} r = 0,\ldots,t \\ q = 0,\ldots,t-r \end{array}}$$

$(t+1)^2 - 1$ conditions

[3] D. Baguette, F. Damanet, O. Giraud, JM, PRA **92**, 052333 (2015).

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$\Rightarrow$ a spin$-j$ state is 1-anticoherent if $\langle\mathbf{J}\rangle = 0$

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$(t+1)^2-1$ conditions

$\Rightarrow$ a spin$-j$ state is 1-anticoherent if $\langle\mathbf{J}\rangle = 0$

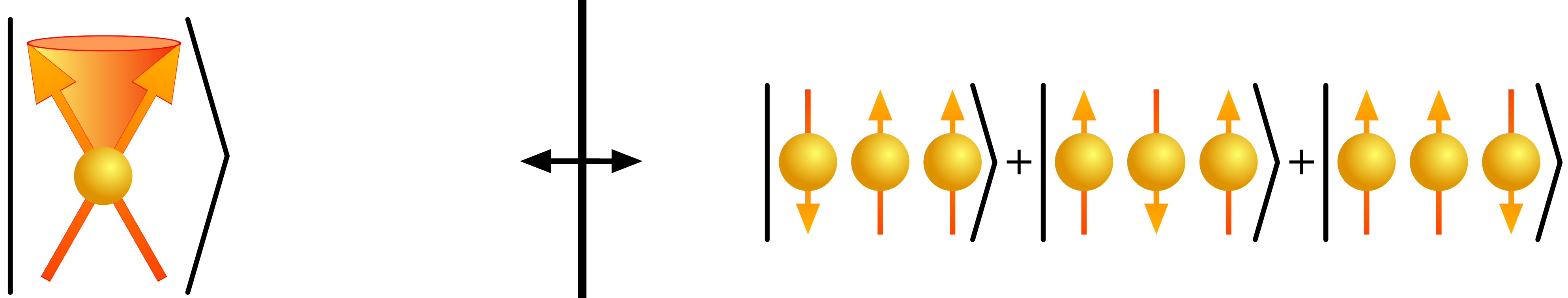
$\Rightarrow$ a spin$-j$ state is 2-anticoherent if $\langle\mathbf{J}\rangle = 0$ and if

$$\langle J_k J_\ell\rangle = 0 \quad \forall\, k \neq \ell, \qquad \langle J_k^2\rangle = \frac{j(j+1)}{3}\hbar^2 \quad \forall\, k$$

[3] D. Baguette, F. Damanet, O. Giraud, JM, PRA **92**, 052333 (2015).

One-to-one mapping

Single spin-j state $\|\psi_j\rangle$	$N \equiv 2j$-qubit symmetric state $\|\psi_S\rangle$
spin operators $\mathbf{J}^2, J_z$	collective spin operators $\mathbf{S}^2, S_z$
standard basis $\{\|j,m\rangle\}$	symmetric Dicke basis $\{\|D_N^{(j-m)}\rangle\}$
full Hilbert space $\mathcal{H}$	symmetric subspace $\mathcal{H}_S$



spin-3/2, $|\frac{3}{2},\frac{1}{2}\rangle$ state

3 spin-$\frac{1}{2}$ or qubits, $|D_3^{(1)}\rangle$ state

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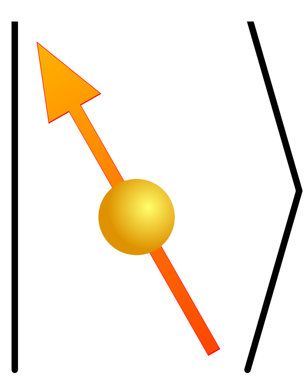
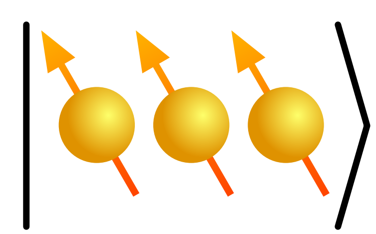
coherent state	**symmetric separable state**



spin-3/2

$\longleftrightarrow$



3 spin-$\frac{1}{2}$ or qubits

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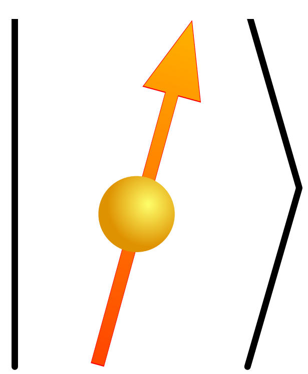
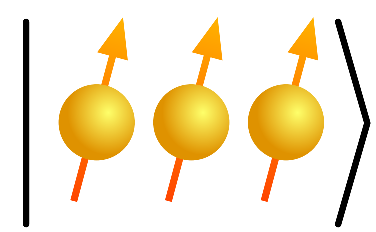
coherent state	symmetric separable state
rotation	**local unitary transf.** $U^{\otimes N}$





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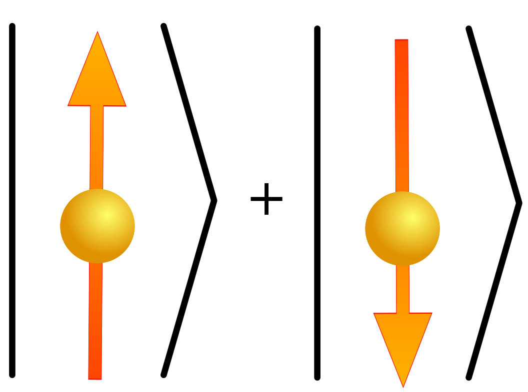
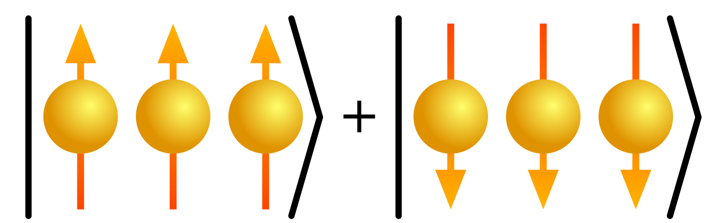
coherent state	symmetric separable state
rotation	local unitary transf. $U^{\otimes N}$
anticoherent state	**maximally entangled symmetric state**



spin-3/2, $\vert\frac{3}{2},\frac{3}{2}\rangle + \vert\frac{3}{2},-\frac{3}{2}\rangle$ state

$\longleftrightarrow$



3 spin-$\frac{1}{2}$ or qubits, $\vert\mathrm{GHZ}\rangle$ state

D. Baguette, T. Bastin, JM, PRA **89**, 032118 (2014)

Coherent vs anticoherent symmetric state $|\psi_S\rangle$

coherent	anticoherent to order *t*
composite system fully characterised by the pure state $|\psi_S\rangle = |\phi, \ldots, \phi\rangle$	composite system fully characterised by the pure state $|\psi_S\rangle$

O. Giraud, D. Braun, D. Baguette, T. Bastin, JM, PRL **114**, 080401 (2015)

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each subsystem fully characterised	no information about the state of subsystems of $k \leqslant t$ qubits
most classical	less classical

O. Giraud, D. Braun, D. Baguette, T. Bastin, JM, PRL **114**, 080401 (2015)

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t-qubit reduced density matrix

$$\rho_t = \mathrm{Tr}_{\neg t}(|\psi_S\rangle\langle\psi_S|)$$

O. Giraud, D. Braun, D. Baguette, T. Bastin, JM, PRL **114**, 080401 (2015)

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Coherent:

$$\rho_t = |\Phi\rangle\langle\Phi|$$

with $|\Phi\rangle = |\underbrace{\phi, \cdots, \phi}_{t}\rangle$

Anticoherent:

$$\rho_t = \frac{\mathbb{1}_{t+1}}{t+1}$$

maximally mixed state in the symmetric subspace

O. Giraud, D. Braun, D. Baguette, T. Bastin, JM, PRL **114**, 080401 (2015)

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with $|\Phi\rangle = |\underbrace{\phi, \cdots, \phi}_{t}\rangle$

Anticoherent:

$$\rho_t = \frac{\mathbb{1}_{t+1}}{t+1}$$

$\Rightarrow$ *maximal entropy of entanglement for any* $(k, N-k)$ *bipartitions with* $k = 1, \ldots, t$

O. Giraud, D. Braun, D. Baguette, T. Bastin, JM, PRL **114**, 080401 (2015)

Interests of anticoherent states

Anticoherent states, i.e. (symmetric) states with maximally mixed one-qubit reduced density matrices, have a wide range of interests :

- valuable resource for two-party communication tasks (*teleportation, super-dense coding, transmission of classical information in QC with noise, . . .*)
- maximize any entanglement monotone based on linear homogenous positive functions of pure state within their SLOCC class of states
- play an important role in LU-equivalence of multiqubit states
- maximally fragile and ideal candidates for ultra-sensitive sensors, . . .

Aim: identify all *inequivalent** anticoherent states for $j = 1, \frac{3}{2}, 2, \ldots$

* *not connected by a rotation (symmetric LU)*

SLOCC classes and 1-anticoherent states

- $|\psi\rangle, |\phi\rangle \in$ same SLOCC* entanglement class $\Leftrightarrow$ $|\psi\rangle = \underbrace{A_1 \otimes \ldots \otimes A_N}_{\text{ILO}^{**}} |\phi\rangle$

* Stochastic Local Operations and Classical Communication

** Invertible Local Operation

SLOCC classes and 1-anticoherent states

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- SLOCC classes provide a partition of a Hilbert space

- In almost every SLOCC class there exists, up to local unitaries, a unique state $|\psi_c\rangle$ (normal form/critical state) characterised by

$$\rho_1 = \mathrm{Tr}_{(N-1)\ \text{qubits}}\big(|\psi_c\rangle\langle\psi_c|\big) = \frac{\mathbb{1}}{2}$$

F. Verstraete, J. Dehaene, B. De Moor, PRA **68**, 012103 (2003)

G. Gour and N. R. Wallach, NJP **13**, 073013 (2011)

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$|\psi_c\rangle$ is 1-anticoherent and unique up to a rotation

F. Verstraete, J. Dehaene, B. De Moor, PRA **68**, 012103 (2003)

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SLOCC classes and 1-anticoherent states

- $|\psi\rangle, |\phi\rangle \in$ same SLOCC entanglement class $\Leftrightarrow$ $|\psi\rangle = \underbrace{A_1 \otimes \ldots \otimes A_N}_{\text{ILO}} |\phi\rangle$

- SLOCC classes provide a partition of a Hilbert space

- In almost every SLOCC class there exists, up to local unitaries, a unique state $|\psi_c\rangle$ (normal form/critical state) characterised by

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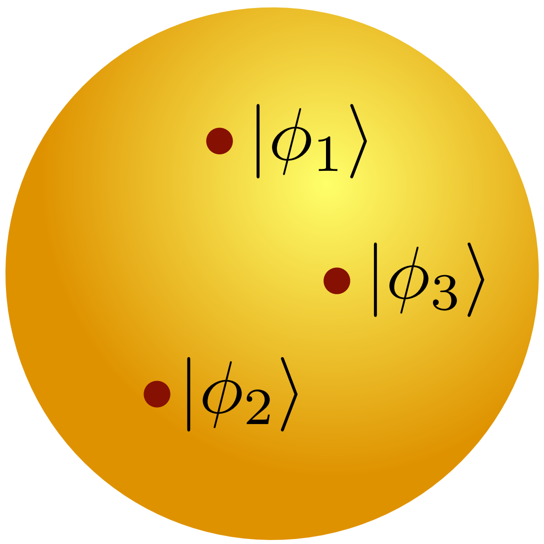
$|\psi_c\rangle$ is 1-anticoherent and unique up to a rotation

- $|\psi_c\rangle$ is a natural representative of its SLOCC class

F. Verstraete, J. Dehaene, B. De Moor, PRA **68**, 012103 (2003)

G. Gour and N. R. Wallach, NJP **13**, 073013 (2011)

Majorana's representation



Bloch sphere

$$\leftrightarrow \quad |\psi_S\rangle = \mathcal{N} \sum_{\pi} |\phi_{\pi(1)}, \cdots, \phi_{\pi(N)}\rangle$$

$\leftrightarrow$ N single-qubit states $|\phi_i\rangle$

$\leftrightarrow$ N points (θ_i, ϕ_i)

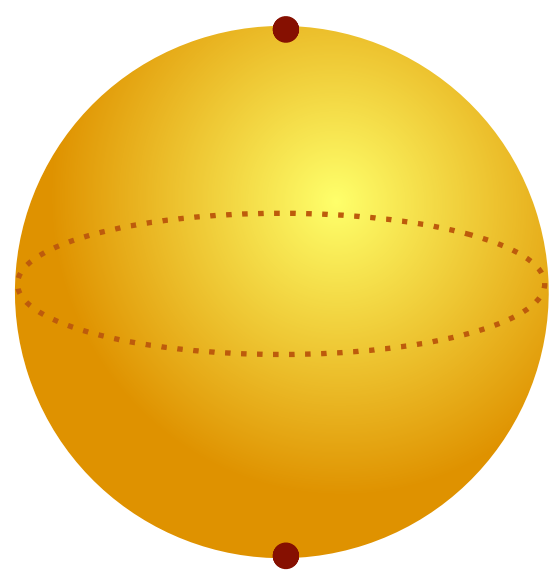
Ettore Majorana
(1906-1938?)

- {Symmetric separable / Coherent} state $|\phi, \ldots, \phi\rangle \leftrightarrow N$ degenerate points
- Symmetric entangled state $\leftrightarrow$ at least two distinct points

2-qubit anticoherent symmetric states

Order t	# SLOCC classes	**Representative state**
1	1	$\|\Psi^+\rangle$

$$|\Psi^+\rangle \equiv |D_2^{(1)}\rangle = \frac{|01\rangle+|10\rangle}{\sqrt{2}}$$



$$\langle \mathbf{S} \rangle = 0$$

$$\Updownarrow$$

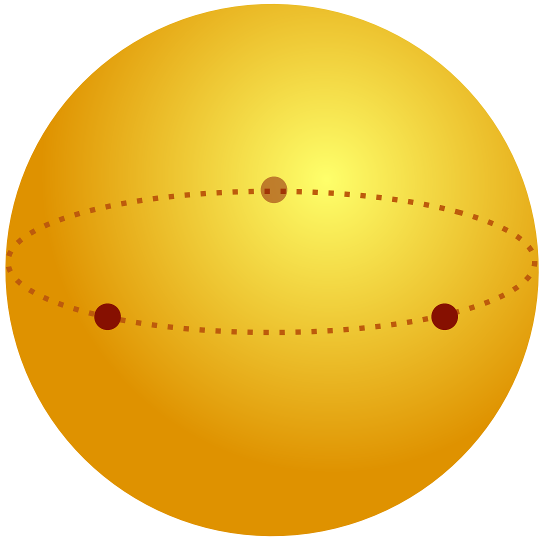
$$\rho_1 = \frac{\mathbb{1}}{2}$$

W. Dür, G. Vidal, J. I. Cirac, PRA **62**, 062314 (2000)

3-qubit anticoherent symmetric states

Order t	# SLOCC classes	**Representative state**
1	1	$\|\mathrm{GHZ}_3\rangle$

$$|\mathrm{GHZ}_3\rangle = \frac{|000\rangle+|111\rangle}{\sqrt{2}}$$



$$\langle \mathbf{S} \rangle = 0$$

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W. Dür, G. Vidal, J. I. Cirac, PRA **62**, 062314 (2000)

T. Bastin *et al.*, PRL **103**, 070503 (2009)

All 4-qubit anticoherent symmetric states

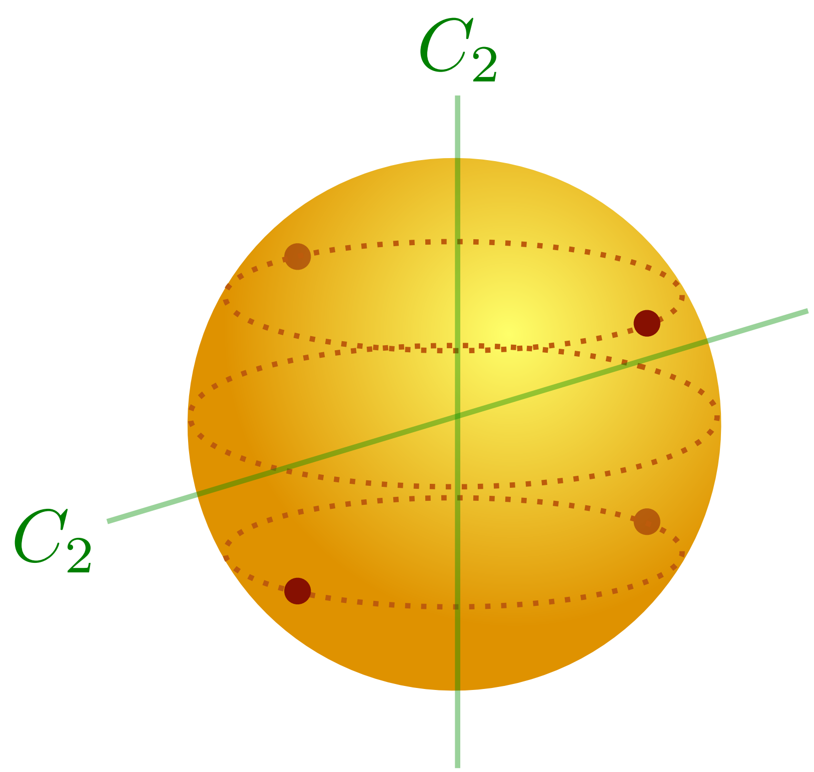
Order t	**# SLOCC classes**	**Representative state**
1	∞	$\lvert\psi_\mu\rangle = \mathcal{N}(\lvert D_4^{(0)}\rangle + \mu\lvert D_4^{(2)}\rangle + \lvert D_4^{(4)}\rangle), \quad \mu \in \mathbb{C}$

anticoherent states

$\Updownarrow$

dihedral

D_2 symmetry



$\langle \mathbf{S} \rangle = 0$

$\Updownarrow$

$\rho_1 = \frac{\mathbb{1}}{2}$

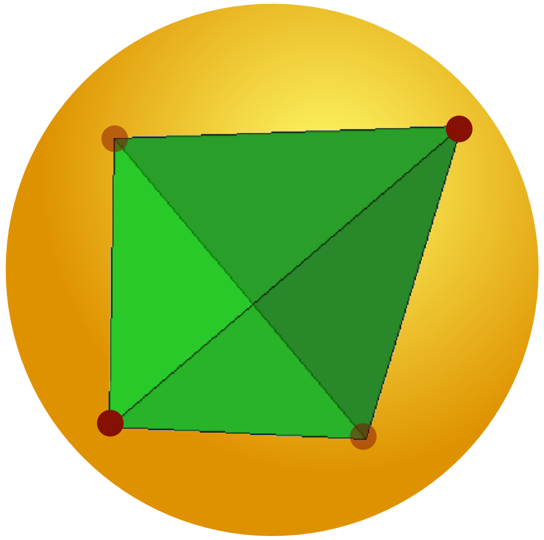
In the computational basis :

$$\lvert\psi_\mu\rangle = \frac{1}{\sqrt{2+|\mu|^2}}\left[\lvert 0000\rangle + \lvert 1111\rangle + \frac{\mu}{\sqrt{6}}(\lvert 0011\rangle + \lvert 0101\rangle + \lvert 0110\rangle + \lvert 1001\rangle + \lvert 1010\rangle + \lvert 1100\rangle)\right]$$

D. Baguette, T. Bastin, JM, PRA **89**, 032118 (2014)

All 4-qubit anticoherent symmetric states

Order t	# SLOCC classes	Representative state
1	∞	$\|\psi_\mu\rangle = \mathcal{N}(\|D_4^{(0)}\rangle + \mu\|D_4^{(2)}\rangle + \|D_4^{(4)}\rangle), \quad \mu \in \mathbb{C}$
2	1	$\left\|\psi_{\mu=i\sqrt{2}}\right\rangle \equiv \|\text{tetrahedron}\rangle$



$$\langle \mathbf{S} \rangle = 0$$

$$\Delta S_{\mathbf{n}}^2 \neq f(\mathbf{n})$$

In the computational basis :

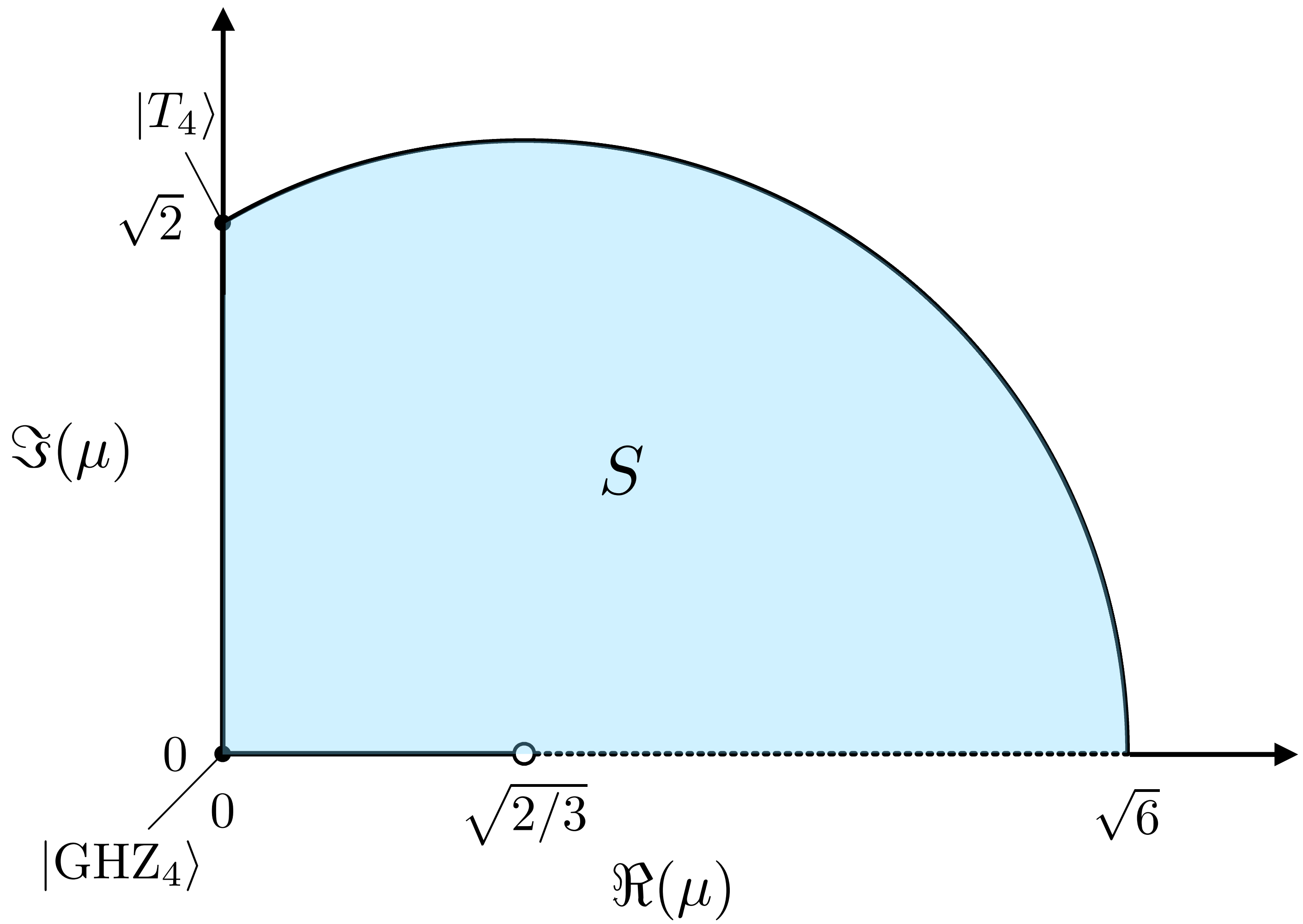
$$|\text{tetrahedron}\rangle = \frac{1}{2}\Big[|0000\rangle + |1111\rangle + \frac{i}{\sqrt{3}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle)\Big]$$

D. Baguette, T. Bastin, JM, PRA **89**, 032118 (2014)

All *inequivalent* 4-qubit anticoherent symmetric states

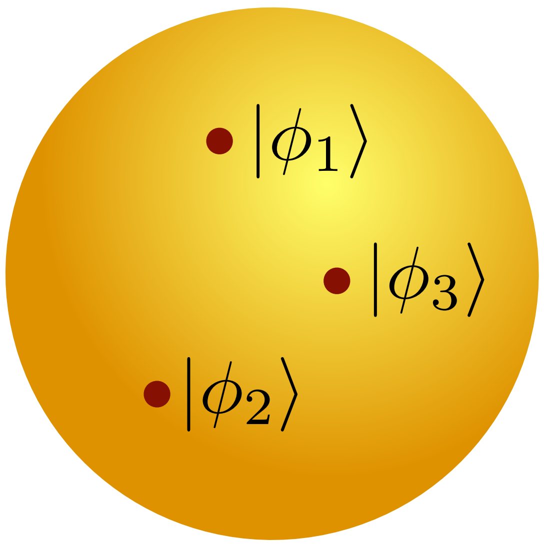
The states $|\psi_\mu\rangle = \mathcal{N}\left(|D_4^{(0)}\rangle + \mu|D_4^{(2)}\rangle + |D_4^{(4)}\rangle\right)$ with $\mu \in S \subset \mathbb{C}$:

- are 1-anticoherent
- belong to different SLOCC classes for different values of μ



D. Baguette, T. Bastin, JM, PRA **89**, 032118 (2014)

Majorana's stellar representation



Bloch sphere

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$\leftrightarrow \quad N$ single-qubit states $|\phi_i\rangle$

$\leftrightarrow \quad N$ points (θ_i, ϕ_i)

Ettore Majorana
(1906-1938?)

- Symmetric separable state $|\phi, \ldots, \phi\rangle \;\leftrightarrow\; N$ degenerate points
- Symmetric entangled state $\leftrightarrow$ at least two distinct points
- Symmetric maximally entangled state $\overset{?}{\leftrightarrow}$ points distributed symmetrically on the Bloch sphere

Symmetry $\overset{?}{\Rightarrow}$ anticoherence

Symmetry group of Majorana points	*Anticoherence to order 1*
Dihedral (D_n)	✔
Rotation-reflection (S_{2n})	✔
Inversion (i)	✔
Cyclic — C_n, C_{nv}	? (symmetry is not sufficient)
Cyclic — C_{nh}	✔

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More info : see poster Q 42.4 today 16:30 Empore Lichthof

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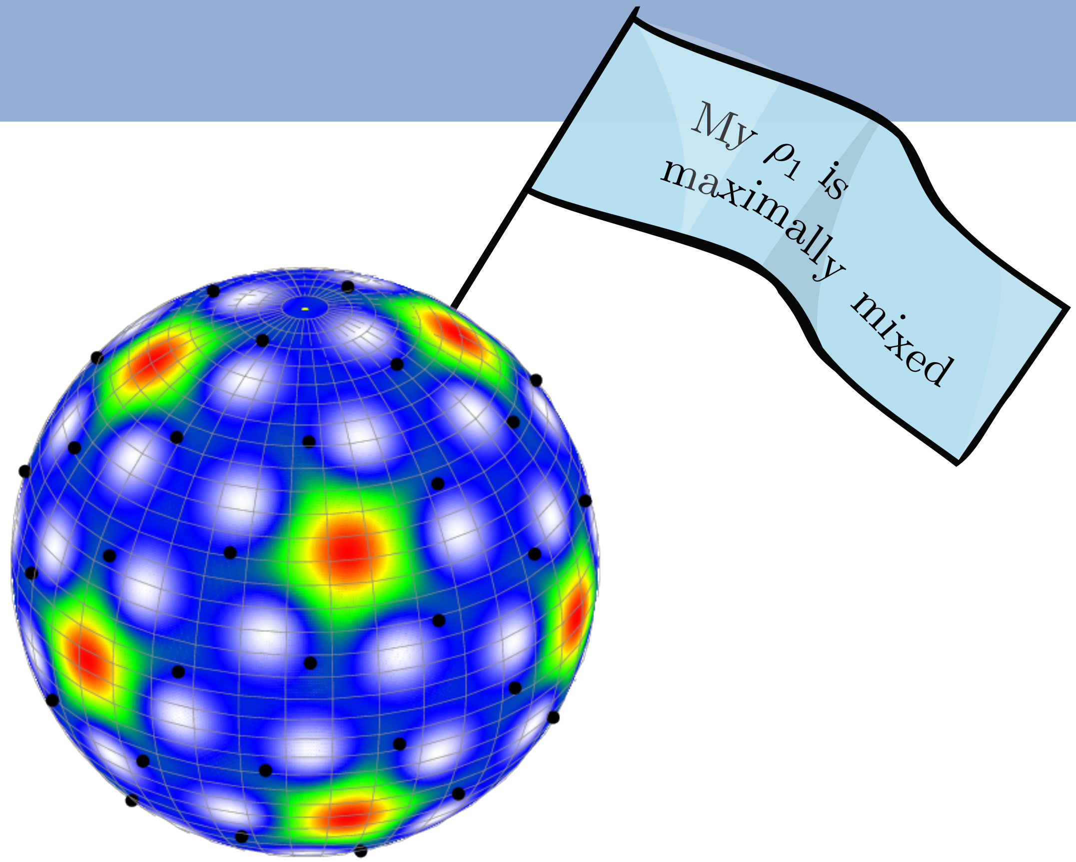

$$N = 60, t = 5$$

References:

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