Multivariate ARMA Based Modal Identification of a Time-varying Beam

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January 28, 2016

The research is focused on the identification of time-varying systems

$$\boldsymbol{M}(t)\,\ddot{\boldsymbol{y}}(t) + \boldsymbol{C}(t)\,\dot{\boldsymbol{y}}(t) + \boldsymbol{K}(t)\,\boldsymbol{y}(t) = \boldsymbol{f}(t)$$

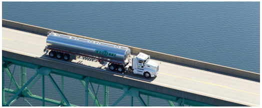
Dynamics of such systems is characterized by :

- Non-stationary time series
- Instantaneous modal properties
  - Frequencies : $\omega_r(t)$
  - Damping ratio's : $\xi_r(t)$
  - Modal deformations : $\boldsymbol{q}_r(t)$

Why time-varying behaviour can occur ?

Several possible origins :

- Structural changes



- Operating conditions

- Damage appearance

Outline of the presentation

Presentation of the multivariate time-varying ARMA model

Presentation of the experimental test setup

Linear time invariant and time-varying identifications of the system

Presentation of the results

Multivariate ARMA model in modal analysis

The ARMA model in vector form models the response signals from a structure as

$$\boldsymbol{y}[t] + \sum_{i=1}^{p} \boldsymbol{A}_i \, \boldsymbol{y}[t-i] = \boldsymbol{e}[t] + \sum_{j=1}^{q} \boldsymbol{B}_j \, \boldsymbol{e}[t-j],$$

where
$\boldsymbol{y}$ is the measurement vector $(d \times 1)$,
$\boldsymbol{A}_i$ and $\boldsymbol{B}_j$ are the AR and MA matrices $(d \times d)$,
$\boldsymbol{e}[t]$ is the innovation vector $(d \times 1)$, assumed as a zero-mean white noise process.

Multivariate ARMA model in modal analysis
Introduction of the time variation

With that kind of model, the modal parameters are obtained by the eigenvalue decomposition of the following companion matrix

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} -\boldsymbol{A}_1 & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ -\boldsymbol{A}_2 & \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\boldsymbol{A}_{p-1} & \vdots & \vdots & & \boldsymbol{I} \\ -\boldsymbol{A}_p & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix}$$

The eigenvalues of that matrix are related to the poles of the system and the first components of the eigenvectors to its mode shapes.

Multivariate ARMA model in modal analysis
Introduction of the time variation

The basis functions approach is used to manage the time variation of the parameters:

$$\boldsymbol{y}[t] + \sum_{i=1}^{p} \boldsymbol{A}_i[t]\, \boldsymbol{y}[t-i] = \boldsymbol{e}[t] + \sum_{j=1}^{q} \boldsymbol{B}_j[t]\, \boldsymbol{e}[t-j]$$

with

$$\begin{aligned} \boldsymbol{A}_i[t] &= \sum_{k=1}^{r_A} \boldsymbol{A}_{i,k}\, f_k[t] \\ \boldsymbol{B}_j[t] &= \sum_{k=1}^{r_B} \boldsymbol{B}_{j,k}\, f_k[t] \end{aligned}$$

The dynamic information about the system is contained in the AR part of the model

For any time, the companion matrix may be formed with the time-varying AR coefficients:

$$\boldsymbol{\mathcal{C}}(t) = \begin{bmatrix} -\boldsymbol{A}_1[t] & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ -\boldsymbol{A}_2[t] & \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\boldsymbol{A}_{p-1}[t] & \vdots & \vdots & & \boldsymbol{I} \\ -\boldsymbol{A}_p[t] & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix}$$

In the same way, its eigenvalue decomposition gives the poles and mode shapes for the system at time t.

Parameters identification

The parameters to identify are all the components of the $\boldsymbol{A}_{i,k}$ and $\boldsymbol{B}_{j,k}$ matrices. Let's $\boldsymbol{\Theta}$ gather them :

$$\begin{aligned}\boldsymbol{\Theta} \;=\; & [\boldsymbol{A}_{1,1},\, \boldsymbol{A}_{1,2},\, \cdots,\, \boldsymbol{A}_{1,r_A},\, \boldsymbol{A}_{2,1},\, \cdots,\, \boldsymbol{A}_{p,r_A}, \\ & \boldsymbol{B}_{1,1},\, \cdots,\, \boldsymbol{B}_{q,r_B}] \,.\end{aligned}$$

An estimate of the output signal is

$$\begin{aligned}\hat{\boldsymbol{y}}[t,\boldsymbol{\Theta}] = \; & - \sum_{i=1}^{p}\sum_{k=1}^{r_A} \boldsymbol{A}_{i,k}\, f_k[t]\, \boldsymbol{y}[t-i] \\ & - \sum_{j=1}^{q}\sum_{k=1}^{r_B} \boldsymbol{B}_{j,k}\, (-f_k[t]\, \boldsymbol{e}[t-j])\end{aligned}$$

or, in block matrix form:

$$\hat{\boldsymbol{y}}[t,\boldsymbol{\Theta}] = -\boldsymbol{\Theta} \begin{bmatrix} \boldsymbol{\phi}[t] \\ \boldsymbol{\psi}[t] \end{bmatrix}$$

The estimation is performed by minimizing the prediction error

The prediction error of the model is given by

$$\begin{aligned}\hat{\boldsymbol{e}}[t,\boldsymbol{\Theta}] &= \boldsymbol{y}[t]-\hat{\boldsymbol{y}}[t,\boldsymbol{\Theta}]\\ &= \boldsymbol{y}[t]+\boldsymbol{\Theta}\begin{bmatrix}\boldsymbol{\phi}[t]\\ \boldsymbol{\psi}[t]\end{bmatrix}\end{aligned}$$

The parameters in $\boldsymbol{\Theta}$ are obtained by minimizing the sum of the squared prediction errors:

$$\boldsymbol{\Theta}=\arg\min_{\boldsymbol{\Theta}}\underbrace{\frac{1}{N}\sum_{t=1}^{N}\hat{\boldsymbol{e}}[t,\boldsymbol{\Theta}]^{T}\,\hat{\boldsymbol{e}}[t,\boldsymbol{\Theta}]}_{V(\boldsymbol{\Theta})}$$

This nonlinear optimization problem is solved using the multi-stages least squares approach

The multi-stages least squares method decomposes a nonlinear optimization problem into a series of least squares ones.

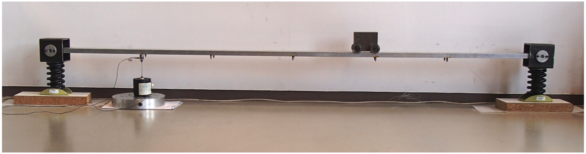
The method is initialized using a long AR model to get a first estimate of $\hat{\boldsymbol{e}}[t, \boldsymbol{\Theta}]$.

$\hat{\boldsymbol{e}}[t, \boldsymbol{\Theta}]$ is then updated at each iteration.

The experimental setup

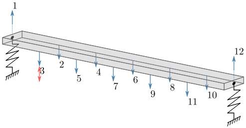
The experimental setup is an aluminum beam with a moving mass.

The whole system is supported by springs and excited by a shaker.

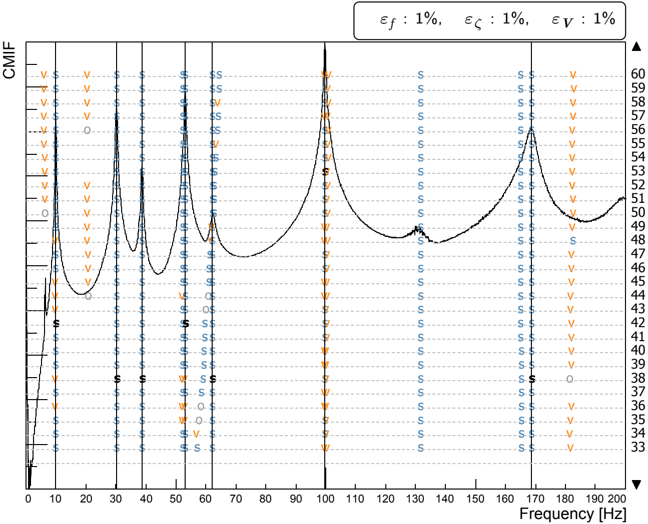


- 2.1 meter long and 8×2 cm for the cross section
- 9 kg for the beam and 3.475 kg for the moving mass (ratio of 38.6 %)

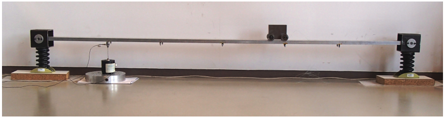
LTI modal analysis of the beam subsystem

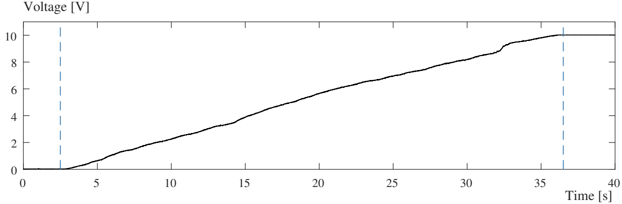


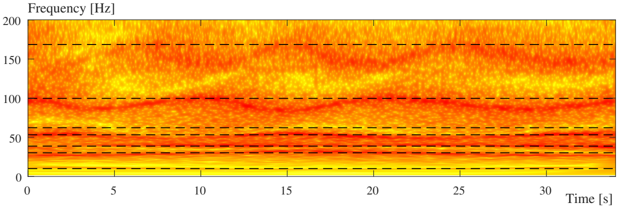
	f_r [Hz]	ζ_r [%]
1	9.86	0.32
2	30.12	0.52
3	38.6	0.65
4	53.14	0.27
5	62.17	1.56
6	99.70	0.28
7	168.60	0.99



Identification of this system with the multivariate time-varying ARMA model







Identification of this system with the multivariate time-varying ARMA model

To perform the identification, one have to fix the model structure (ARMA orders, size of the funnction bases, type of function)

For that purpose, one can use information criterion such as the Akaike Information Criteria

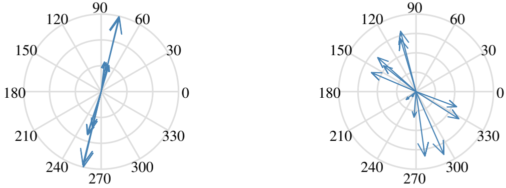
$$AIC = N \, \log \left[V(\mathbf{\Theta}) \right] + 2 \, \delta,$$

in which δ is the number of parameters to be estimated and N the number of time samples.

Identification of this system with the multivariate time-varying ARMA model

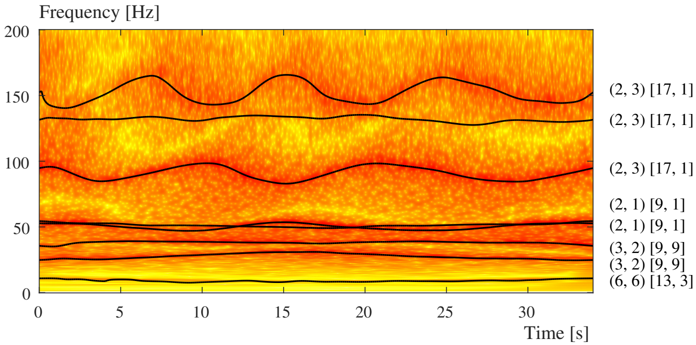
Because the size of the companion matrix, the number of spurious modes may rapidly grows! We need a tool to select physical modes from spurious ones.

The physical modes are selected using their average mean phase deviation :

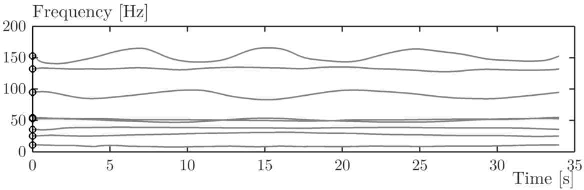


(a) Physical mode (b) Spurious mode

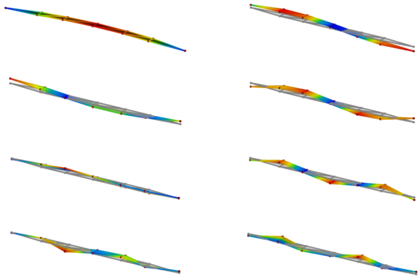
Results from several models may be selected to build the final set of identified time-varying modes



Thanks to the multivariate modeling, the mode shapes may be obtained too



	Frequency [Hz]
1	10.88
2	25.13
3	35.56
4	52.71
5	54.40
6	94.76
7	131.83
8	152.53



To conclude

- The basis functions approach is suitable to manage the time variation of the dynamic properties
- The varying modal parameters are well tracked with the ARMA model
- The multivariate modeling identifies the mode shapes too

Even if the selection of physical modes with the MPD criteria gives good results, the rising of the number of parameters to identify and the number of identified modes may become problematic.

Thank you for your attention