Multivariate ARMA Based Modal Identification of a Time-varying Beam

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Abstract

The present paper addresses the problem of modal identification of time-varying systems. The identification is based on a multivariate autoregressive moving-average model in which the time variability of the system is caught using a basis functions approach. In this approach, the time-varying regressive coefficients in the model are expended in the chosen basis functions and only the projection coefficients have to be identified. In that way, the initial time-varying problem then becomes a time-invariant one that can be solved. Because a multivariate model is used, in addition to the time-varying poles, the time-varying mode shapes may be identified too. The method is first presented and then applied on an experimental demonstration structure. The experimental structure consists of a supported beam on which a mass is travelling. The mass is chosen sufficiently large to have a significant influence on the dynamics of the primary system. This kind of problem is a classical example commonly used by many authors to test time-varying identification methods.

Keywords: Modal identification, Time-varying systems, Multivariate identification, Basis functions, Autoregressive Moving-average model.

1 Introduction

Modal analysis plays a major role in the field of structure dynamics either numerically or experimentally. Concerning the analysis of linear time invariant structures, both computational and identification methods are pretty complete and robust and new challenges appear now to the extension to more complicated behaviours such as nonlinear and time-varying. This paper deals with this latter case. Some analyses of time-varying dynamic structure have already been performed. We can find for example in [1] a double identification based on wavelet transform and on a modified version of the Stochastic Subspace Identification method (SSI) applied on short successive time windows (short-time SSI). Another way to handle the time variability is to expand it on a set of Basis Functions (BF) such as explained in [2] for general time-varying processes. The method of basis function (also named Functional Series in [3], [4]) is also used to create time-varying Auto Regressive Moving-Average identification models for modal analysis purposes. A recurrent setup to test such methods consist in a simple beam on which a heavy mass is moving. A similar structure is used in this paper.

The paper is organized as follows. Section 2 presents the way to perform modal analysis using multivariate ARMA models in the field of linear time invariant structure dynamics along with its adaptation to the analysis in linear time-varying behaviour. In Sections 3 and 4 the setup used in this study and its time invariant and time-varying analyses are presented with their corresponding results.

2 Presentation of the ARMAV method for LTI modal analysis and its extension to time-varying systems

The AutoRegrassive Moving-Average (ARMA) model can be used in the field of experimental modal analysis. In its vector, or multivariate, form (ARMAV) it is possible to identify a complete modal model i.e. that the mode shapes of the system can be identified in addition to the poles (eigenfrequencies and damping ratio's). If we denote by $\boldsymbol{y}[t]$ the $d \times 1$ vector of the output measurements the ARMAV model of order p and q for the AR and MA parts, respectively, is given as follows:

$$\boldsymbol{y}[t] + \sum_{i=1}^{p} \boldsymbol{A}_i\, \boldsymbol{y}[t-i] = \boldsymbol{e}[t] + \sum_{j=1}^{q} \boldsymbol{B}_j\, \boldsymbol{e}[t-j], \tag{1}$$

in which $\boldsymbol{e}[t]$ is the innovation, the part of the signal that may not be predicted by the past values of $\boldsymbol{y}$. If we are faced to a dynamical system of order n, driven by its motion equation

$$\boldsymbol{M}\, \ddot{\boldsymbol{y}}(t) + \boldsymbol{C}\, \dot{\boldsymbol{y}}(t) + \boldsymbol{K}\, \boldsymbol{y}(t) = \boldsymbol{f}(t), \tag{2}$$

it may be shown that it can be represented by an ARMAV(2, 1) or (2, 2) in noise-free or noisy identification, respectively [5, 6]. If the output dimension d is not equal to the dimension n of the system, an order $2m$ should be considered with m equals the rounded up value of n/d.

Let us now consider the system (1) in a time-varying framework. In that case, the structural properties of the system (through the $\boldsymbol{M}$, $\boldsymbol{K}$ and $\boldsymbol{C}$ matrices) are no more constant but may exhibit a variation over time (or any other varying parameter). The motion equation then becomes

$$\boldsymbol{M}(t)\, \ddot{\boldsymbol{y}}(t) + \underbrace{\dot{\boldsymbol{M}}(t)\, \dot{\boldsymbol{y}}(t)}_{\text{Negligible}} + \boldsymbol{C}(t)\, \dot{\boldsymbol{y}}(t) + \boldsymbol{K}(t)\, \boldsymbol{y}(t) = \boldsymbol{f}(t) \tag{3}$$

in which the mass variation rate is assumed to be slow with respect to the dynamics of the system.

If the dynamic properties of the system vary with time, it will be the same for the parameters used in the identification model. The parameters in the ARMAV model (1) will then be time-varying too what will lead to the next model:

$$\boldsymbol{y}[t] + \sum_{i=1}^{p} \boldsymbol{A}_i[t]\, \boldsymbol{y}[t-i] = \boldsymbol{e}[t] + \sum_{j=1}^{q} \boldsymbol{B}_j[t]\, \boldsymbol{e}[t-j]. \tag{4}$$

The time-varying $\boldsymbol{A}_i[t]$ and $\boldsymbol{B}_j[t]$ are the parameters to be identified which is not an easy task due to their variation. A method to manage this variation is the basis functions approach [2] also named functional series approach in [3, 4]. This method does the assumption that the varying parameters can be projected on a set of preselected basis functions (polynomials, sine/cosine, etc.) and the aim of the method is to identify those projection coefficients:

$$\boldsymbol{A}_i[t] = \sum_{k=1}^{r_A} \boldsymbol{A}_{i,k}\, f_k[t] \tag{5}$$

$$\boldsymbol{B}_j[t] = \sum_{k=1}^{r_B} \boldsymbol{B}_{j,k}\, f_k[t], \tag{6}$$

in which the $\boldsymbol{A}_{i,k}$ and $\boldsymbol{B}_{j,k}$ are these projection coefficients and $f_k[t]$ are the known basis functions. Note that at this stage the identification problem becomes time invariant and the system is now modelled as

$$\boldsymbol{y}[t] + \sum_{i=1}^{p} \sum_{k=1}^{r_A} \boldsymbol{A}_{i,k}\, f_k[t]\, \boldsymbol{y}[t-i] = \boldsymbol{e}[t] + \sum_{j=1}^{q} \sum_{k=1}^{r_B} \boldsymbol{B}_{j,k}\, f_k[t]\, \boldsymbol{e}[t-j]. \tag{7}$$

Introducing the following generalised regressors:

$$\phi[t]^T = \left[f_1[t]\boldsymbol{y}[t-1]^T,\ f_2[t]\boldsymbol{y}[t-1]^T,\ \cdots,\ f_{r_a}[t]\boldsymbol{y}[t-p]^T\right] \tag{8}$$

$$\psi[t]^T = \left[-f_1[t]\boldsymbol{e}[t-1]^T,\ -f_2[t]\boldsymbol{e}[t-1]^T,\ \cdots,\ -f_{r_b}[t]\boldsymbol{e}[t-q]^T\right] \tag{9}$$

and the matrix of identification parameters

$$\boldsymbol{\Theta} = \left[\boldsymbol{A}_{1,1},\ \boldsymbol{A}_{1,2},\ \cdots,\ \boldsymbol{A}_{1,r_A},\ \boldsymbol{A}_{2,1},\ \cdots,\ \boldsymbol{A}_{p,r_A},\ \boldsymbol{B}_{1,1},\ \cdots,\ \boldsymbol{B}_{q,r_B}\right], \tag{10}$$

the prediction error of the model can be expressed as

$$\begin{aligned}\hat{\boldsymbol{e}}[t,\boldsymbol{\Theta}] &= \boldsymbol{y}[t] - \hat{\boldsymbol{y}}[t,\boldsymbol{\Theta}] \\ &= \boldsymbol{y}[t] + \boldsymbol{\Theta}\begin{bmatrix}\phi[t] \\ \psi[t]\end{bmatrix}.\end{aligned} \tag{11}$$

The latter prediction error is then used to build a parameter-dependent criterion to be minimized. A commonly used function is the Sum of Squared Errors (SSE) and is defined as

$$V(\boldsymbol{\Theta}) = \frac{1}{N}\sum_{t=1}^{N}\hat{\boldsymbol{e}}[t,\boldsymbol{\Theta}]^T\,\hat{\boldsymbol{e}}[t,\boldsymbol{\Theta}]. \tag{12}$$

The parameters to identify will be those that minimize this cost function, i.e.:

$$\boldsymbol{\Theta} = \arg\min_{\boldsymbol{\Theta}}\frac{1}{N}\sum_{t=1}^{N}\hat{\boldsymbol{e}}[t,\boldsymbol{\Theta}]^T\,\hat{\boldsymbol{e}}[t,\boldsymbol{\Theta}]. \tag{13}$$

The latter minimization cannot be performed using a usual least square approach because of the nonlinear dependence of the prediction error in $\boldsymbol{\Theta}$ because the ψ contains time-lagged values of the unknown innovation. This should require nonlinear optimization schemes but they often require the evaluation of the gradient of the objective function (sometimes also the evaluation of its Hessian) that may be difficult or very time consuming if a finite difference approach is chosen. A way to bypass that difficulty is to apply another iterative method such as the Multi Stage Least Suares (MSLS) method [7]. The latter assumes that, at the iteration k, an estimate of the prediction error is known such as in Equation (11). Then, the estimate of $\boldsymbol{\Theta}$ at iteration $k+1$ can be obtained via a least squares step. It remains to initialize the iterative procedure with a first estimate of the prediction error. This can be done assuming a least square estimate of a long AR only model of the system (the MA order q is put to zero).

Once the (time-dependent) Autoregressive and Moving-average matrices ara identified, the modal parameters of the system can be obtained through a modal decomposition of the companion matrix made up with the Autoregressive part of the model [7, 6]

$$\boldsymbol{\mathcal{C}}(t) = \begin{bmatrix} -\boldsymbol{A}_1[t] & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ -\boldsymbol{A}_2[t] & \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\boldsymbol{A}_{p-1}[t] & \vdots & \vdots & & \boldsymbol{I} \\ -\boldsymbol{A}_p[t] & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} \tag{14}$$

3 Presentation of the experimental setup

The problem under study in that work consist in a simple beam supported at its both ends by springs. On this beam moves a steel block having a mass which is not negligible with respect to the mass of the beam. In that way, this moving mass will affect the dynamic properties of the system depending of its position along the beam. The system is excited by a random force with a shaker and twelve accelerometers (five pairs along the beam and two at the springs levels) record it time response. The whole system is show in Figure 1. In addition, the position of the mass is monitored using a laser sensor.

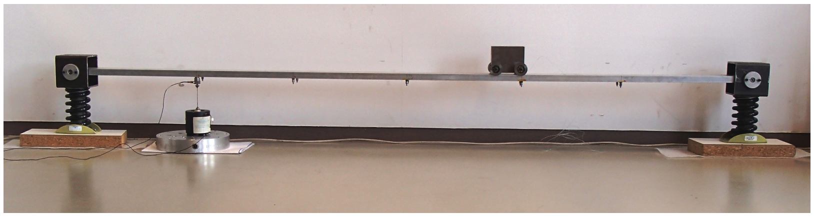


Figure 1: Picture of the time-varying experimental setup.

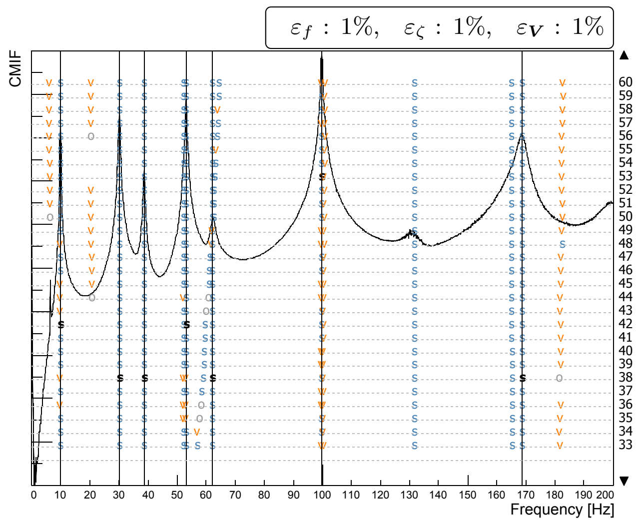


Figure 2: Stabilization diagram of the time invariant subsystem. Bold black **s** represent the user-selected poles.

Mode #	f_r [Hz]	ζ_r [%]	Type of mode
1	9.86	0.32	First bending
2	30.12	0.52	Opposite phase spring motion with small second bending
3	38.6	0.65	In phase spring motion with small first bending
4	53.14	0.27	Second bending
5	62.17	1.56	Rotation around the beam axis
6	99.70	0.28	Third bending
7	168.60	0.99	Fourth bending

Table 1: Experimental modal parameters of the supported beam.

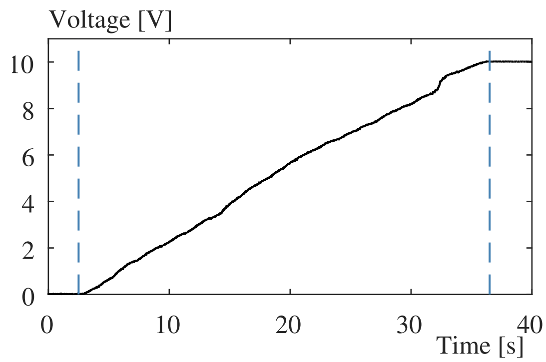
3.1 Linear time invariant identification

Before testing the structure in its time-varying behaviour, it is interesting to first test only the subsystem of the supported beam to get its modal properties. These results will further serve for comparison when the moving mass will be introduced in the system.

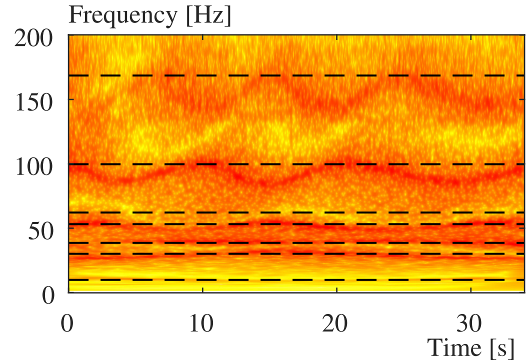
This LTI modal analysis is performed in LMS Test.Lab using the embedded PolyMAX method [8]. The obtained stabilization diagram is shown in Figure 2 and the related results are listed in Table 1. The mode shapes of the subsystem are quite usual for a supported beam, let us just note the presence of stable column of poles close to 130 Hz we suspect to be a horizontal motion of the beam.

4 Time-varying identification

Let us now move the mass on the beam while the latter is randomly excited by the shaker. In Figure 3(b), the wavelet transform of the third sensor (where the shaker is attached) is shown. This represents the evolution of the dynamics of the system through the variation of its resonance frequencies when the mass is pulled along the beam as shown in Figure 3(a). The interesting thing is that the resonance frequencies oscillate in ranges whose the upper value correspond to the eigenfrequencies identified in the previous LTI analysis. This is explained by the fact that when the mass is located at a node of vibration, its effect on that mode is reduced and conversely, the additional inertial effects are maximum when the mass is located at antinodes of vibration. The only exception concerns the fifth mode which is a global rotation of the beam along its axis. Because the rotation is globally the same along the beam, all its frequency line is just decreased and remains approximately constant along the time recording. Unfortunately, due



(a) Time varying position of the mass. The dashed lines show the selected time span for the identification.



(b) Wavelet transform of the third response sensor. The dashed lines represent the eigenfrequencies of the beam subsystem.

Figure 3: Illustration of the time varying behaviour of the system.

the the inertia properties of the moving mass, the frequency of that mode is located at approximately the same value than the fourth one. Some frequency crossings are even observed, because of the oscillation of the fourth frequency.

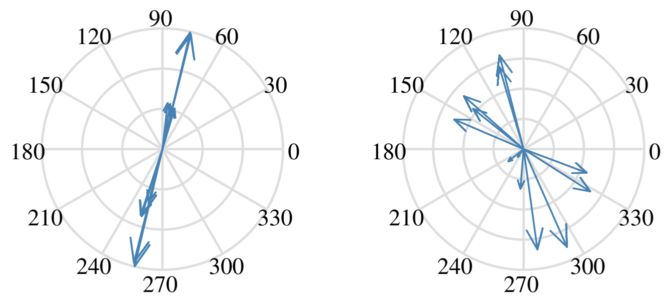
The identification of the time-varying modal properties of the system is not an easy task. As usual in most of modal identification methods, we have to select a proper structure of the model. In general the problem is solved for a series of increasing model orders and stabilization plots are used to select a good set of results. Obviously, in our case, a usual stabilization plot is not very practical due to the time dependence of our results. A second difficulty is that we do not have only a single model order but two (one for each of the AR and MA parts) and we also have to deal with the sizes of the functions bases related to the expansions of the AR and MA coefficients.

Another way to select good models rely on using some tools balancing the accuracy and complexity of identification models. The reason why the complexity of the model is took into account comes from the principle of parsimony [2] stating that the fewer the number of parameters used to explain a model, the better it is. For example, the Akaike Information Criterion (AIC) [7, 6] given by

$$AIC = N \log\left[V(\mathbf{\Theta})\right] + 2\,\delta, \tag{15}$$

in which δ is the number of parameters to be estimated $(d^2\,p\,r_A + d^2\,q\,r_B)$ and N the number of time samples. We will then look to models close to the minimum of that criteria.

A second property of that method is that the number of modes that can be identified is related to the AR model order p but also to the number of measurement sensors d indeed, the number of identified poles (in complex-conjugated pairs) is given by the size of the companion matrix $\mathcal{C}$ in (14). In our case, because we want the estimation of at least seven modes, an AR model order $p = 1$ is not sufficient because with our twelve sensors, this will lead to the identification of six pairs of complex-conjugated modes. The minimal AR model order should then be two, leading to the estimation of at least twelve modes (in which spurious modes are present). Each increasing value of p will then add six modes in the identification process and the number of spurious modes dramatically increases with each increment of p because the number of structural modes remains the same. That fact adds another difficulty in the selection of the proper set of results. However this can be managed using some properties of the physical modes that are not shared with the spurious modes. In general, and it is also a property used in the selection of modes through the stabilization diagram, the physical modes of a dynamic structure exhibit low damping ratio's conversely to spurious modes that may take any value, even negative. A simple method explained in [9] that we also used in [10], is to look to the trajectory of each identified poles in the complex plane. The physical poles will then be selected by only keeping the ones whose their trajectory is the closer to the unit circle. This may give a good selection of the physical modes but in our case, we also have additional information about the time-varying modes. Because multivariate models are used, the identification of the mode shapes is also performed and the latter can also bring information about the physical or spurious behaviour of any mode at any time. It is known that even if they are complex, physical mode shapes are aligned in the complex plane (each of their component are close to be in phase or antiphase with each other) conversely to spurious modes that have a more random dispersion of their components. This information can be used to select the physical modes using the Mode Phase Discrepancy (MPD) criterion that measure the spread of the components of the modes in the complex plane.



(a) Physical mode (low MPD) (b) Spurious mode (high MPD)

Figure 4: Illustration of the mode phase dispersion in the complex plane. (a) and (b) represent mode shapes taken at $t = 4$ s. (a) is a physical mode at a frequency of 84.79 Hz and (b) is classified as a spurious one. Its frequency is 65.71 Hz.

To build our set of time-varying modal parameters, we have to compute the eigenvalue decomposition of the companion matrix for preselected time instants. Because the basis functions coefficients are calculated the results (eigenvalues/eigenvector) may be obtained for any time, not necessarily a sampled time step in the data. The eigenvalues and mode shapes are then calculated for closely spaced time instants and the mode tracking between two time instants is performed by the Modal Assurance Criterion (MAC). Once each mode trajectory is computed, their related MPD values are calculated and a preselected number of modes are retained by selecting the mode trajectories having the lowest average MPD values.

Unfortunately, there is no model (p, q, r_A, r_B) that is able to identify all the modes properly. Especially, the first mode at the lowest frequency is the most difficult to identify even if it looks not vary a lot. To perform the full identification, several models are used and the best occurrence with respect to its best average MPD value of each mode in each model is retained. This operation can be compared to the picking of modes in several model orders in stabilization diagrams.

Once this last step is performed, we obtain the result shown in Figure 5.

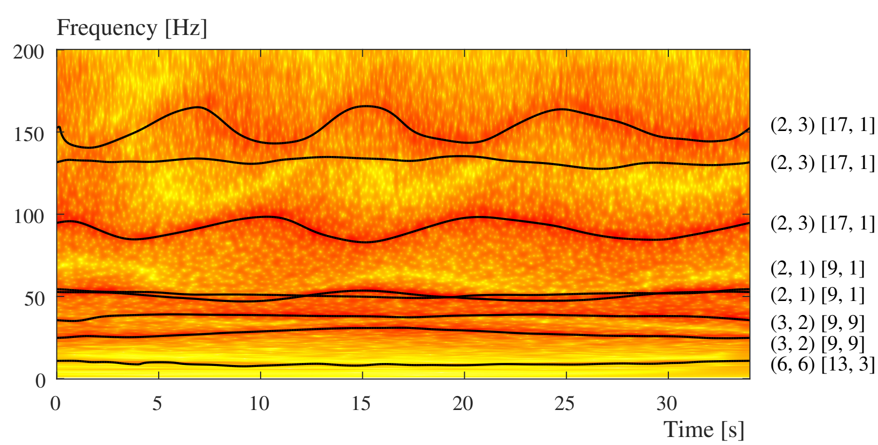
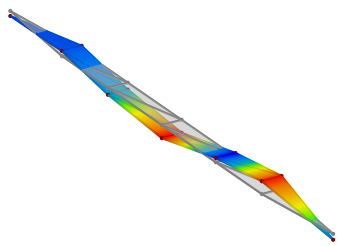
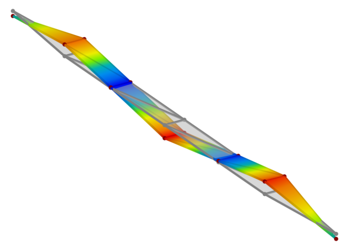


Figure 5: Final set of results. Each mode comes from a TV-ARMAV model $(p,\, q)\, [r_A,\, r_B]$.

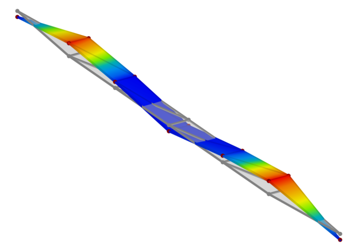
Because a written format is not suitable to show time-varying results, let us use as example a single mode at few time instants to illustrate the identification of the time-varying mode shapes. Let us look at the sixth mode (close to 100 Hz) at five time samples corresponding to some positions of the mass close to nodes or antinodes of the sixth mode shape. It is clearly visible that when the mass is located at antinodes of vibration, its amplitude decreases with respect to the other two antinodes (Figures 6(a), 6(c) and 6(e)). Conversely, Figures 6(b) and 6(d) show that this mode shape is no more perturbed when the mass lies at nodes of vibration and then does not add additional inertia force on that particular mode.



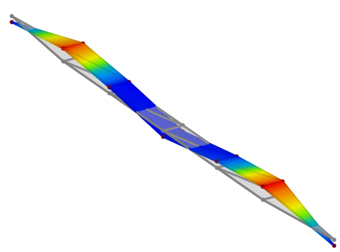
(a) Time : $t = 5$ s



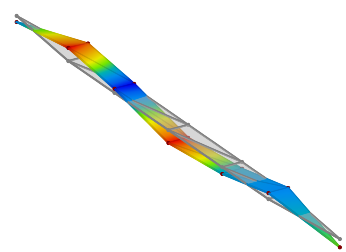
(b) Time : $t = 10$ s



(c) Time : $t = 15$ s



(d) Time : $t = 20$ s



(e) Time : $t = 30$ s

Figure 6: Time-varying shape of the sixth mode at particular time spots.

5 Conclusion

All along this paper, a method able to perform the modal identification of time-varying systems was presented. The combination of a multivariate model and an expansion of its parameters on a set of basis functions lead to the estimation of all the modal parameters (poles and modes) with their time dependence. The results obtained with that method are in good accordance with the expected behaviour of the test structure. Because only the mass is modified in a positive manner the additional inertia forces cause the eigenfrequencies to decrease more or less depending where the mass is located. The examination of the time-varying mode shapes shows the same relation.

The selection of the final set of the modal trajectories using the MPD criteria seems to perform well even if the number of spurious modes increases.

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