Perturbation Methods (MATH 2015-1)

Academic Year 2015-2016

V. Denoël

- *read the questions carefully*
- *write your name on each page*
- *the use of electronic devices is prohibited*
- *total duration of the examination : 4 hours*

Question 1

Provide approximation(s) of the root(s) of

$$x - 2 = \varepsilon \cosh x$$

for small ε. Make them as accurate as possible and explain why it is impossible or impractical to make better.

Question 2

Find a matched asymptotic approximation to leading order of the solution of

$$\varepsilon y''(x) + \frac{y'(x)}{x} + y(x) = 0$$

on the domain $x \in [1, 2]$, for $\varepsilon \ll 1$ and with the boundary conditions $y(1) = 0$ and $y(2) = \varepsilon$.

Question 3

A potential flow around a disk, with uniform longitudinal velocity in the far field, is characterized by the velocity field $\mathbf{u}$ such that $\mathbf{u} = \nabla\phi$ with

$$\phi = \left(r + \frac{1}{r}\right)\cos\theta$$

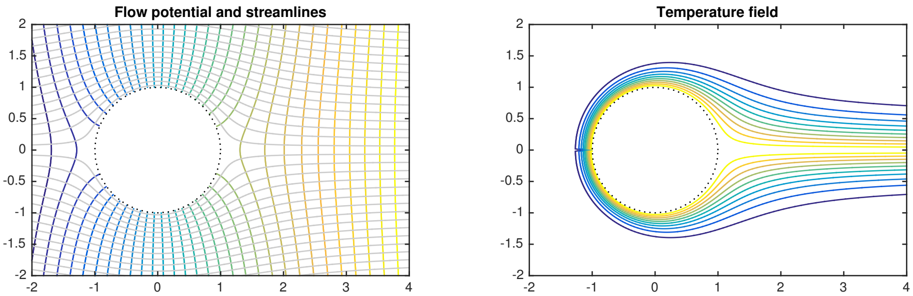
in polar coordinates, see figure below. The disk is heated at constant temperature $T = 1$ and the heat is convected in the flow by a diffusion process with small dimensionless diffusion coefficient $\varepsilon \ll 1$. The temperature field (see figure below) resulting from this heat transfer is governed by

$$\mathbf{u} \cdot \nabla T = \varepsilon \nabla^2 T \qquad \text{in } r \geq 1$$

with the boundary conditions $T = 1$ on the disk and $T \to 0$ as $r \to \infty$.

1. Justify why a boundary layer exists,
2. Find the outer solution,
3. Find the right scaling for the inner solution,
4. Determine the inner solution and seek a solution in the form of a function $f(\rho)$ of the ratio $\rho = \frac{\tilde{r}\sin\theta}{\sqrt{1+\cos\theta}}$, where $\tilde{r}$ is the stretched radial coordinate,
5. Develop a composite solution.

NB: do not focus on the regions around streamlines passing through the separation points.



Some useful relations (or not)

- $\int_0^{+\infty} e^{-\alpha u^2} du = \sqrt{\frac{\pi}{\alpha}}$ (Gauss integral)
- $\Gamma[z] = \int_0^{+\infty} t^{z-1} e^{-t} dt$ (definition of the Gamma function)
- $\nabla F = \partial_r F \, \mathbf{e}_r + \frac{1}{r} \partial_\theta F \, \mathbf{e}_\theta$ (gradient in polar coordinates)
- $\mathrm{erfc}\ x = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$ (complementary error function)

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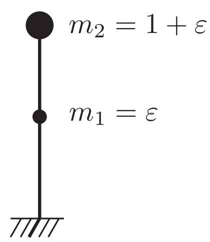
Question 1

The dimensionless stiffness and mass matrices of a lightpole with a small mass at its top are given by

$$\mathbf{K} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad ; \quad \mathbf{M} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1+\varepsilon \end{pmatrix}.$$

A sketch of this lightpole is given below. Assuming $\varepsilon \ll 1$, develop a perturbation method to determine 2nd-order-accurate approximations of the two eigen frequencies ω_1 and ω_2 of this structure. Remember eigen frequencies and mode shapes are determined by $\left(\mathbf{K} - \mathbf{M}\omega_i^2\right)\mathbf{x}_i = 0$.



Question 2

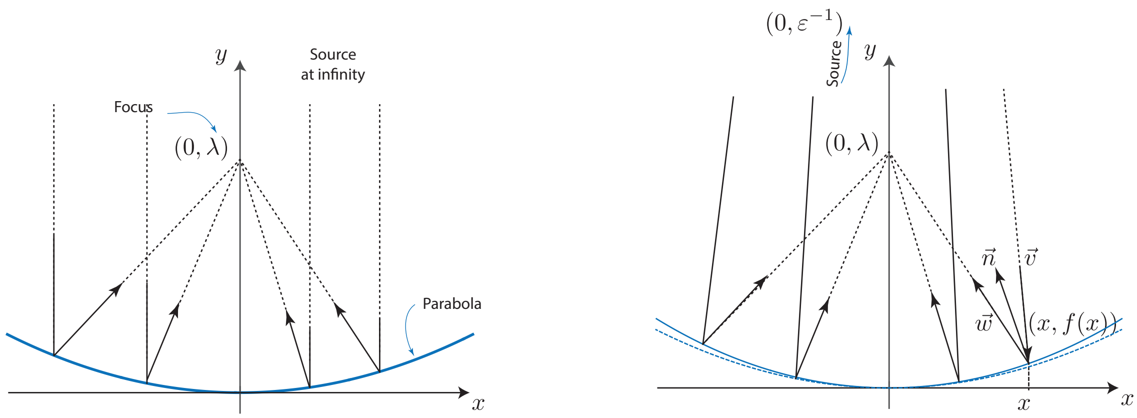
Find a matched asymptotic approximation, accurate to second order, of the solution of

$$\varepsilon y''(x) + (1-x)y'(x) - y(x) = 0$$

on the domain $x \in [0,1]$, for $\varepsilon \ll 1$ and with the boundary conditions $y(0) = 1$ and $y(1) = 0$. Sketch the solution of the problem for a value of ε of your own choice.

Question 3

A famous feature of the parabola is that it concentrates parallel rays (*viz.* the source is located at infinity) hitting its surface toward a unique locus. Using a perturbation analysis approach, determine the shape of a reflector concentrating to a focus of given position λ, the rays coming from a source located at large finite distance ε^{-1}.



Please use the coordinate system given in the above figure. You may want to express that reflection on the reflector with identical incidence and reflexion angles can be expressed by

$$\frac{\vec{v} \cdot \vec{n}}{\|v\|} + \frac{\vec{w} \cdot \vec{n}}{\|w\|} = 0.$$

It is asked to

1. determine the director $\vec{v}$ of the oncoming ray touching the reflector at abscissa x,
2. determine the director $\vec{w}$ of the reflected ray leaving the reflector at abscissa x,
3. derive the differential equation that the shape of the reflector, specified by $f(x)$, need to satisfy,
4. show that the parabola is indeed the solution of the unperturbed problem, when the source is located at infinity,
5. determine the equation in the coordinate system (x, y) of the reflector as $\varepsilon \ll 1$. (*Hint*: it can be expressed as a 4th degree polynomial)

Some useful relations (or not)

- $\int_0^{+\infty} e^{-\alpha u^2} du = \sqrt{\frac{\pi}{\alpha}}$ (Gauss integral)
- $\Gamma[z] = \int_0^{+\infty} t^{z-1} e^{-t} dt$ (definition of the Gamma function)
- $\nabla F = \partial_r F \, \mathbf{e}_r + \frac{1}{r} \partial_\theta F \, \mathbf{e}_\theta$ (gradient in polar coordinates)
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Other exercises to practice

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1. Provide approximations of the roots of

$$(1-\varepsilon)x^2 - 2x + 1 = 0$$

for small ε.

2. Provide approximations of the positive roots of

$$\tan x = \epsilon x$$

for $0 < \epsilon \ll 1$

3. The function $y(x, \varepsilon)$ satisfies

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0$$

over the domain $[0, 1]$ with the boundary conditions $y(0) = 0$ and $y(1) = 1/e$. It possesses a boundary layer in the neighborhood of $x = 0$. Find two terms in the outer approximation, applying only the boundary conditions at $x = 1$. Find two terms in the inner approximation. Match the two solutions and provide a composite solution.

4. Use the multiple scales approach to obtain an asymptotic expansion for $y(t)$ to $ord(1)$ which is valid for $t = \mathrm{ord}(\varepsilon^{-1})$ to the solution of

$$y'' + \varepsilon y'^3 + y = 0$$

with $y(0) = 1$ and $y'(0) = 0$.

5. Use the multiple scales approach to obtain an asymptotic expansion for $y(t)$ to $ord(1)$ which is valid for $t = \mathrm{ord}(\varepsilon^{-1})$ to the solution of

$$y'' + \varepsilon y^3 + y = 0$$

with $y(0) = 1$ and $y'(0) = 0$.