Perturbation Methods (MATH 2015-1)

Academic Year 2016-2017

V. Denoël

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Read the question carefully, write your name on each page, the use of electronic devices is not allowed (except personal calculator)
Total duration of the examination : 1 hour

Provide approximation(s) of the **real** root(s) of

$$\varepsilon x^{2n} - x^2 + 1 = 0$$

for $\varepsilon \ll 1$.

Make them accurate up to three terms. If required, discuss as a function of the value of the strictly positive natural number $n \in \mathbb{N}_0$.

Some useful relations (or not)

- $(x+\alpha)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k \alpha^{n-k}$ (binomial theorem)
- $\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$ (complementary error function)
- the number of positive roots of a polynomial $\mathcal{P}[x]$ is smaller or equal to the number of sign changes in the coefficients of successive powers of x (the rule of sign)
- $\nabla F = \partial_r F \, \mathbf{e}_r + \frac{1}{r} \partial_\theta F \, \mathbf{e}_\theta$ (gradient in polar coordinates)

Solution

This is a singularly perturbed problem. In the unperturbed problem, there are two roots located at $x = \pm 1$. In the perturbed problem, there are at most $2n$ roots. The rule of sign actually indicates that there are at most two positive roots to this equation, and at most two negative roots.

Let us first focus on the roots in the neighborhood of $x = \pm 1$. To do so the regular power series ansatz

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + ...$$

yields

$$1 - x_0^2 - 2x_0 x_1 \varepsilon + \varepsilon \left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2\right)^{2n} - \left(x_1^2 + 2x_0 x_2\right) \varepsilon^2 + \text{ord}(\varepsilon^3) = 0.$$

No matter the value of $n > 0$, the leading order solution is $x_0 = \pm 1$, as expected.

Application of the binomial theorem yields,

$$\begin{aligned}\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2\right)^{2n} &= \begin{pmatrix} 2n \\ 0 \end{pmatrix} x_0^{2n} + \begin{pmatrix} 2n \\ 1 \end{pmatrix} \left(\varepsilon x_1 + \varepsilon^2 x_2\right) x_0^{2n-1} + \begin{pmatrix} 2n \\ 2 \end{pmatrix} \left(\varepsilon x_1 + \varepsilon^2 x_2\right)^2 x_0^{2n-2} \\ &= x_0^{2n} + 2n\varepsilon x_0^{2n-1} x_1 + \varepsilon^2 \left(2n x_2 x_0^{2n-1} + n(2n-1) x_1^2 x_0^{2n-2}\right) + ...\end{aligned}$$

The governing equation therefore reads

$$1 - x_0^2 + x_0 \left(x_0^{2n-1} - 2x_1\right) \varepsilon - \left(x_1^2 - 2n x_0^{2n-1} x_1 + 2x_0 x_2\right) \varepsilon^2 + \text{ord}(\varepsilon^3) = 0$$

and matching at the different orders yields

$$\begin{aligned}\text{ord}\left(\varepsilon^0\right) &: \quad 1 - x_0^2 = 0 \\ \text{ord}\left(\varepsilon^1\right) &: \quad x_0 \left(x_0^{2n-1} - 2x_1\right) = 0 \\ \text{ord}\left(\varepsilon^2\right) &: \quad x_1^2 - 2n x_0^{2n-1} x_1 + 2x_0 x_2 = 0\end{aligned}$$

which gives the two solutions

$$x_{\pm} = \pm \left(1 + \frac{\varepsilon}{2} + \frac{4n-1}{8} \varepsilon^2 + ...\right)$$

If $n = 1$, these two roots are the only roots of the problem (because this is a 2nd degree polynomial).

If $n > 1$, the problem is singularly perturbed and there might be another positive root and another negative root (remember two positive and two negative roots at most). These are large roots corresponding to the balance of εx^{2n} and x^2 in the governing equation, i.e. roots of order $\varepsilon^{\frac{1}{2-2n}}$. It is therefore natural to introduce the rescaling

$$X = \frac{x}{\varepsilon^{\frac{1}{2-2n}}} \qquad \to \qquad x = \varepsilon^{\frac{1}{2-2n}} X$$

so that the original equation becomes

$$\varepsilon x^{2n} + 1 = x^2$$

$$\varepsilon^{\frac{1}{1-n}} X^{2n} - \varepsilon^{\frac{1}{1-n}} X^2 + 1 = 0 \qquad \to \qquad X^{2n} - X^2 + \varepsilon^{\frac{1}{n-1}} = 0.$$

Because $n > 1$, the last term $\varepsilon^{\frac{2}{n-2}}$ is another small number of this problem (not as small as ε but it is still smaller than 1). In fact, it is as small as n is small and unfortunately tends toward unity as $n \to +\infty$. We therefore redefine $\tilde{\varepsilon} = \varepsilon^{\frac{1}{n-1}}$ and naturally seek a solution in the form

$$X = X_0 + \tilde{\varepsilon} X_1 + \tilde{\varepsilon}^2 X_2 + ...$$

Substitution into the re-scaled governing equation yields

$$\begin{aligned} &X_0^{2n} - X_0^2 + \tilde{\varepsilon}\left(2nX_0^{2n-1}X_1 - 2X_0X_1 + 1\right) + ... \\ &\qquad \tilde{\varepsilon}^2\left(2nX_2X_0^{2n-1} + n(2n-1)X_1^2X_0^{2n-2} - 2X_0X_2 - X_1^2\right) + \mathrm{ord}(\tilde{\varepsilon}^3) = 0 \end{aligned}$$

At successive order, we have

$$\begin{aligned} \mathrm{ord}\left(\tilde{\varepsilon}^0\right) &: \quad X_0^{2n} - X_0^2 = 0 \\ \mathrm{ord}\left(\tilde{\varepsilon}^1\right) &: \quad 2nX_0^{2n-1}X_1 - 2X_0X_1 + 1 = 0 \\ \mathrm{ord}\left(\tilde{\varepsilon}^2\right) &: \quad \left(2nX_2X_0^{2n-1} + n(2n-1)X_1^2X_0^{2n-2} - 2X_0X_2 - X_1^2\right) = 0 \end{aligned}$$

At leading order, $X_0^{2n} - X_0^2 = 0$, so that $X_0 = 0$ or $X_0 = \pm 1$. The solution X_0 is not interesting (bad scaling otherwise) and is discarded. At order $\tilde{\varepsilon}$, we find$X_1 = \frac{-X_0}{2(n-1)}$, $X_2 = \frac{-X_0(2n+1)}{8(n-1)^2}$, and then

$$x_{\pm} = \pm\varepsilon^{\frac{1}{2-2n}}\left(1 - \frac{1}{2(n-1)}\varepsilon^{\frac{1}{n-1}} - \frac{2n+1}{8(n-1)^2}\varepsilon^{\frac{2}{n-1}} + ...\right).$$

In $n \gg 1$, $\tilde{\varepsilon}$ is not small and the expansion above fails to be efficient (the asymptoticness of the series is not really broken, but it is just inefficient). In fact, as $\tilde{\varepsilon} \to 0$, the governing equation is composed of three terms which are all of order 1, and the former balance between the different terms does not work anymore. The solution should include all three terms. Because there is no analytical solution to $X^{2n} - X^2 + \tilde{\varepsilon} = 0$, out of the scope of perturbation methods, one may want to find an iterative scheme. Noticing that the leading term might be considered to be x^{2n} , a possible iterative scheme is

$$\varepsilon x^{2n} - x^2 + 1 = 0 \to x^{(n+1)} = \pm\sqrt[2n]{\frac{x^{(n)2} - 1}{\varepsilon}}$$

Initiating the sequence with $x^{(0)} = 1 + \sqrt[2n]{\varepsilon}$, we have

$$x^{(1)} = \sqrt[2n]{\frac{\left(1 + \sqrt[2n]{\varepsilon}\right)^2 - 1}{\varepsilon}} = \sqrt[2n]{\frac{2 + \sqrt[2n]{\varepsilon}}{\varepsilon^{\frac{2n-1}{2n}}}} = \varepsilon^{-\frac{2n-1}{(2n)^2}}\sqrt[2n]{2 + \sqrt[2n]{\varepsilon}}$$

and further iterations might be developed (in fact, they are required to make it accurate because we have a balance of 3 terms in the limit case, which means that x^{2n} is not necessarily leading; this was just a personal choice).

In summary:
The equation has two roots for $n = 1$. They are given by

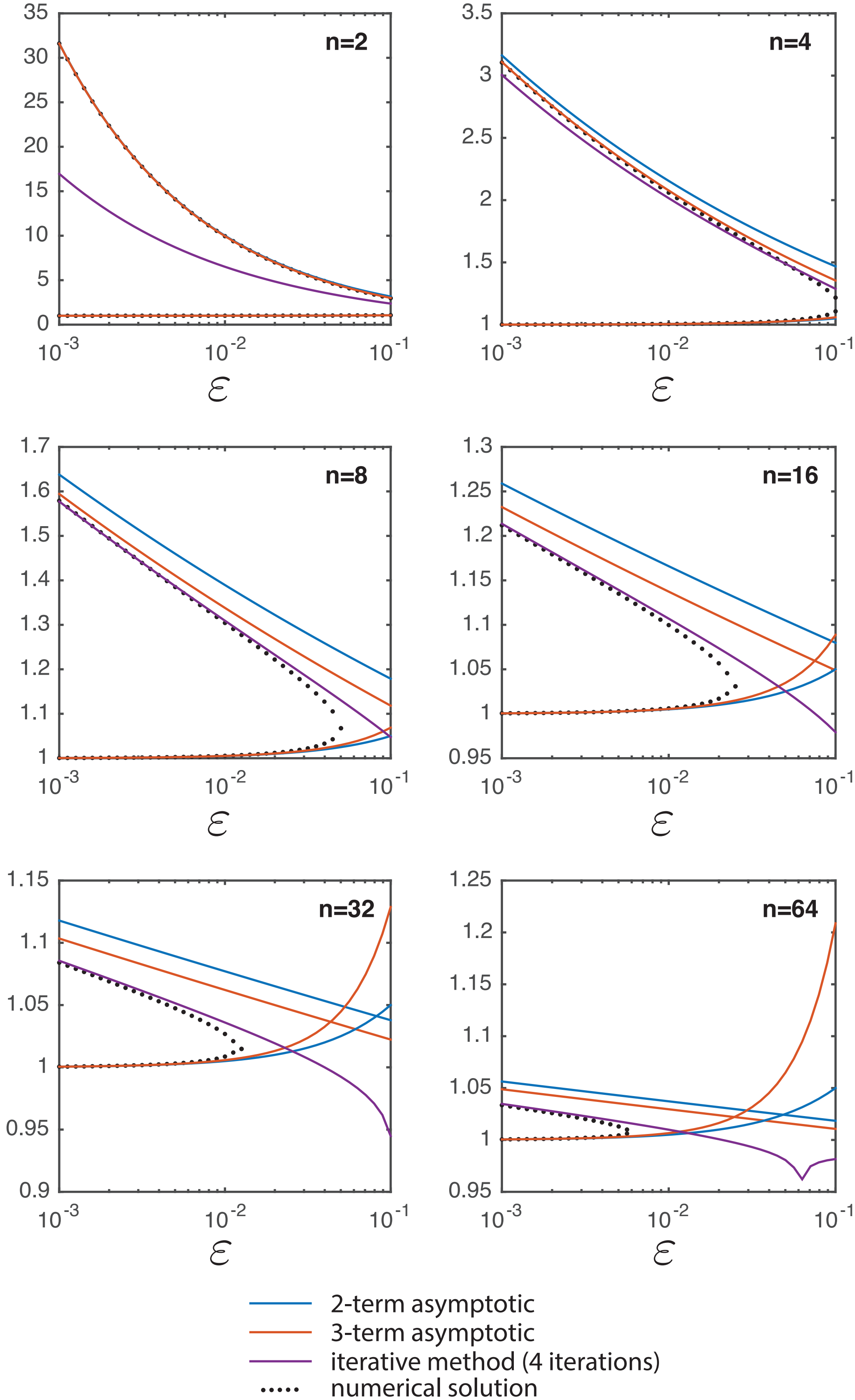
$$x_\pm = \pm 1 \pm \frac{\varepsilon}{2} \pm \frac{2n-1}{8}\varepsilon^2 + ...$$

The equation has two more roots for $n > 1$. If n is moderate, say $1 < n \lesssim 10$, they are given by

$$x_\pm = \pm\varepsilon^{\frac{1}{2-2n}}\left(1 - \frac{1}{2(n-1)}\varepsilon^{\frac{1}{n-1}} - \frac{2n+1}{8(n-1)^2}\varepsilon^{\frac{2}{n-1}} + ...\right)$$

If n is very large, on top of the regularly perturbed root, there is another root, close to 1, which can be captured with an iterative scheme. Notice that, when $n \to +\infty$, the equation does not possess any root any longer.

The figure below shows a comparison between the expansion, the iterative scheme and the numerical solutions.


n=2
n=4
n=8
n=16
n=32
n=64
ε
10^{-3}
10^{-2}
10^{-1}
2-term asymptotic
3-term asymptotic
iterative method (4 iterations)
numerical solution