Permutations and negative beta-shifts

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The complexity of a dynamical system is usually measured by its entropy. For symbolic dynamical systems, the (topological) entropy is the logarithm of the exponential growth rate of the number of distinct patterns of length n. Bandt, Keller and Pompe [3] proved for piecewise monotonic maps that the entropy is also given by the number of permutations defined by consecutive elements in the trajectory of a point. Amigo, Elizalde and Kennel [1, 4] studied realizable permutations in full shifts in detail. Elizalde [5] extended this study to β-shifts (with $\beta > 1$), and he determined for each permutation the infimum of those bases β where successive elements of the β-shift are ordered according to the permutation. Archer and Elizalde [2] considered periodic patterns for full shifts with different orderings. We are interested in β-shifts with $\beta < -1$, which are ordered naturally by the alternating lexicographical order. Similarly to [5], we determine the set of $(-\beta)$-shifts allowing a given permutation. Our main results were obtained independently by Elizalde and Moore [6].

For an ordered space X, a map $f:\, X \to X$, a positive integer n, a point $x \in X$ such that $f^i(x) \neq f^j(x)$ for all $0 \le i < j < n$, and a permutation $\pi \in \mathcal{S}_n$, let

$$\mathrm{Pat}(x, f, n) = \pi \quad \text{if } \pi(i) < \pi(j) \text{ for all } 1 \le i, j \le n \text{ with } f^{i-1}(x) < f^{j-1}(x).$$

The set of allowed patterns of f is

$$\mathcal{A}(f) = \big\{\mathrm{Pat}(x, f, n) :\, x \in X, n > 0\big\}.$$

Here, we are interested in the $(-\beta)$-transformation for $\beta > 1$, which was defined by Ito and Sadahiro [7] as $x \mapsto \lfloor \frac{\beta}{\beta+1} - \beta x \rfloor - \beta x$ on the interval $[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$. It is more convenient to consider the map

$$T_{-\beta} :\, (0,1] \to (0,1], \quad x \mapsto \lfloor \beta x \rfloor + 1 - \beta x,$$

which is easily seen to be topologically conjugate to Ito and Sadahiro's one, via $x \mapsto \frac{1}{\beta+1} - x$ (which reverses the order). Theorem 1 below gives a formula for

$$B(\pi) = \inf \big\{\beta > 1 :\, \pi \in \mathcal{A}(T_{-\beta})\big\}.$$

To $\pi \in \mathcal{S}_n$, associate the circular permutation

$$\hat{\pi} = \big(\pi(1)\pi(2)\cdots\pi(n)\big) \in \mathcal{S}_n,$$

i.e., $\hat{\pi}(\pi(j)) = \pi(j+1)$ for $1 \le j < n$, $\hat{\pi}(\pi(n)) = \pi(1)$, and the sequence of digits $z_{[1,n)}$ defined by

$$z_j = \#\{1 \le i < \pi(j) : i \ne \pi(n) \ne i+1 \text{ and } \hat{\pi}(i) < \hat{\pi}(i+1), \\ \text{or } i = \pi(n)-1 \text{ and } \hat{\pi}(i) < \hat{\pi}(i+2)\}.$$

Moreover, let

$$m = \pi^{-1}(n), \quad \ell = \pi^{-1}(\pi(n)-1) \text{ if } \pi(n) \ne 1, \quad r = \pi^{-1}(\pi(n)+1) \text{ if } \pi(n) \ne n.$$

We use the abbreviation $z_{[i,j)} = z_i z_{i+1} \cdots z_{j-1}$ for $i \le j$. When

$$z_{[\ell,n)} = z_{[r,n)} z_{[r,n)} \quad \text{or} \quad z_{[r,n)} = z_{[\ell,n)} z_{[\ell,n)}, \quad \text{if } \pi(n) \notin \{1,n\}, \tag{1}$$

we also use the following digits, for $0 \le i < |r-\ell|$, $1 \le j < n$,

$$z_j^{(i)} = z_j + \begin{cases} 1 & \text{if } \pi(j) \ge \pi(r+i) \text{ and } i \text{ is even, or } \pi(j) \ge \pi(\ell+i) \text{ and } i \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Now, we can define a sequence $a = a_1 a_2 \cdots$ associated to the permuation π by

$$a = \begin{cases} z_{[m,n)}\, \overline{z_{[\ell,n)}} & \text{if } n-m \text{ is even, } \pi(n) \ne 1, \text{ and (1) does not hold,} \\ \min_{0 \le i < |r-\ell|} z^{(i)}_{[m,n)}\, \overline{z^{(i)}_{[\ell,n)}} & \text{if } n-m \text{ is even, } \pi(n) \ne 1, \text{ and (1) holds,} \\ \overline{z_{[m,n)} 0} & \text{if } n-m \text{ is even and } \pi(n) = 1, \\ z_{[m,n)}\, \overline{z_{[r,n)}} & \text{if } n-m \text{ is odd and (1) does not hold,} \\ \min_{0 \le i < |r-\ell|} z^{(i)}_{[m,n)}\, \overline{z^{(i)}_{[r,n)}} & \text{if } n-m \text{ is odd and (1) holds.} \end{cases}$$

Here and in the following, $\overline{w}$ denotes the periodic sequence with period w, and sequences are ordered by the alternating lexicographical order, i.e., $v_1 v_2 \cdots < w_1 w_2 \cdots$ if $v_1 \cdots v_k = w_1 \cdots w_k$ and $(-1)^k v_{k+1} < (-1)^k w_{k+1}$, $k \ge 0$. (Ito and Sadahiro [7] used an "alternate order", which is the inverse of our order.)

Theorem 1. *Let $\pi \in \mathcal{S}_n$. Then $B(\pi)$ is the largest positive root of* $1 + \sum_{k=1}^{\infty} \frac{a_k+1}{(-x)^k} = 0$.

Note that $B(\pi)$ is the largest positive solution of the equation

$$(-x)^{p+q} + \sum_{k=1}^{p+q} (a_k+1)\,(-x)^{p+q-k} = (-x)^q + \sum_{k=1}^{q} (a_k+1)\,(-x)^{q-k}$$

when a is eventually periodic with preperiod of length q and period of length p.

Let φ be the substitution defined by $\varphi(0) = 1$, $\varphi(1) = 100$, with the unique fixed point $u = \varphi(u)$, i.e.,

$$u = 100111001001001110011 \cdots.$$

Theorem 2. *Let $\pi \in \mathcal{S}_n$. We have $B(\pi) = 1$ if and only if $a = \overline{\varphi^k(0)}$ for some $k \ge 0$.*

If $B(\pi) > 1$, i.e., $a > u$, then $B(\pi)$ is a Perron number by [8, 9].

Instead of numbers $x \in (0,1]$, we can also consider their $(-\beta)$-expansions

$$x = -\sum_{k=1}^{\infty} \frac{d_{-\beta,k}(x)+1}{(-\beta)^k} \text{ with } d_{-\beta,k}(x) = \lfloor \beta\, T_{-\beta}^{k-1}(x) \rfloor.$$

Set $d_{-\beta}(x) = d_{-\beta,1}(x) d_{-\beta,2}(x) \cdots$. By [7], have $x < y$ if $d_{-\beta}(x) < d_{-\beta}(y)$ (w.r.t. the alternating lexicographical order), thus

$$\mathrm{Pat}(x, T_{-\beta}, n) = \mathrm{Pat}(d_{-\beta}(x), \Sigma, n),$$

where Σ denotes the shift map. For the proof of Theorem 1, infinite words w satisfying $\mathrm{Pat}(w, \Sigma, n) = \pi$ and lying in the $(-\beta)$-shift for all $\beta > B(\pi)$ are constructed. Note that, for an integer $N \geq 2$, the $(-N)$-shift is close to the full shift on N letters.

Theorem 3. *Let $\pi \in \mathcal{S}_n$. The minimal number of letters of an infinite word w satisfying $\mathrm{Pat}(w, \Sigma, n) = \pi$ (w.r.t. the alternating lexicographical order) is*

$$N(\pi) = 1 + \lfloor B(\pi) \rfloor = 1 + \mathrm{asc}(\hat{\pi}) + \begin{cases} 1 & \textit{if } (1) \textit{ holds or } a = \overline{\mathrm{asc}(\hat{\pi})0}, \\ 0 & \textit{otherwise,} \end{cases}$$

where $\mathrm{asc}(\hat{\pi})$ denotes the number of ascents in $\hat{\pi}$ with $\hat{\pi}(\pi(n)) = \pi(1)$ removed. We have $N(\pi) \leq n-1$ for all $\pi \in \mathcal{S}_n$, $n \geq 3$, with equality for $n \geq 4$ if and only if

$$\pi \in \{12\cdots n,\ 12\cdots(n-2)n(n-1),\ n(n-1)\cdots 1,\ n(n-1)\cdots 312\}.$$

In Table 1, we give the values of $B(\pi)$ for all permutations of length up to 4, and we compare them with the values obtained by [5] for the (positive) beta-shift. We see that much more permutations satisfy $B(\pi) = 1$ for the negative beta-shift than for the positive one. Some other examples, together with corresponding infinite words are below.

1. Let $\pi = 3421$. Then $\hat{\pi} = 3142$, $z_{[1,4)} = 110$, $m = 2$, $\pi(n) = 1$, $r = 3$. We obtain that $a = \overline{z_{[2,4)}0} = \overline{100} = \overline{\varphi^2(0)}$, thus $B(\pi) = 1$. Indeed, $\mathrm{Pat}(0100\,\overline{10011}, \Sigma, n) = \pi$.

2. Let $\pi = 892364157$. Then $\hat{\pi} = 536174892$, $z_{[1,9)} = 33012102$, $m = 2$, $\ell = 5$, $r = 1$, thus $a = z_{[2,9)}\,\overline{z_{[1,9)}} = \overline{30121023}$, and $B(\pi)$ is the unique root $x > 1$ of
$$x^8 - 4x^7 + x^6 - 2x^5 + 3x^4 - 2x^3 + x^2 - 3x + 4 = 1.$$
We get $B(\pi) \approx 3.831$, and we have $\mathrm{Pat}(330121023\,\overline{301210220}, \Sigma, n) = \pi$.

3. Let $\pi = 453261$. Then $\hat{\pi} = 462531$, $z_{[1,6)} = 11001$, $m = 5$, $\pi(n) = 1$, $r = 4$, thus $a = z_5\,\overline{z_4 z_5} = \overline{10}$, and $B(\pi) = 2$. We have $\mathrm{Pat}(110010\,\overline{2}, \Sigma, n) = \pi$ and $N(\pi) = 3$.

4. Let $\pi = 7325416$. Then $\hat{\pi} = 6521473$, $z_{[1,7)} = 100100$, $m = r = 1$, $\ell = 4$. Hence (1) holds, $z^{(0)}_{[1,7)} = 200100$, $z^{(1)}_{[1,7)} = 200210$, $z^{(2)}_{[1,7)} = 211210$. Since $n - m$ is even, we have
$$a = \min_{i \in \{0,1,2\}} z^{(i)}_{[1,7)}\,\overline{z^{(i)}_{[4,7)}} = \min\{200\,\overline{100}, 200\,\overline{210}, 211\,\overline{210}\} = 211\,\overline{210}.$$
Therefore, $B(\pi) \approx 2.343$ is the largest positive root of
$$\begin{aligned} 0 &= (x^6 - 3x^5 + 2x^4 - 2x^3 + 3x^2 - 2x + 1) - (-x^3 + 3x^2 - 2x + 2) \\ &= x^6 - 3x^5 + 2x^4 - x^3 - 1. \end{aligned}$$
We have $\mathrm{Pat}(211210(210)^{2k+1}\overline{2}, \Sigma, n) = \pi$ for $k \geq 0$.

$B(\pi)$	root of	π, negative beta-shift	π, positive beta-shift
1	$\beta - 1$	$12, 21$ $123, 132, 213, 231, 321$ $1324, 1342, 1432, 2134$ $2143, 2314, 2431, 3142$ $3214, 3241, 3421, 4213$	$12, 21$ $123, 231, 312$ $1234, 2341, 3412, 4123$
1.465	$\beta^3 - \beta^2 - 1$		$1342, 2413, 3124, 4231$
1.618	$\beta^2 - \beta - 1$	312 $1423, 3412, 4231$	$132, 213, 321$ $1243, 1324, 2431, 3142, 4312$
1.755	$\beta^3 - 2\beta^2 + \beta - 1$	$2341, 2413, 3124, 4123$	
1.802	$\beta^3 - 2\beta^2 - 2\beta + 1$		4213
1.839	$\beta^3 - \beta^2 - \beta - 1$	4132	$1432, 2143, 3214, 4321$
2	$\beta - 2$	1234, 1243	$2134, 3241$
2.247	$\beta^3 - 2\beta^2 - \beta + 1$	4321	4132
2.414	$\beta^2 - 2\beta - 1$		$2314, 3421$
2.618	$\beta^2 - 3\beta + 1$		1423
2.732	$\beta^2 - 2\beta - 2$	4312	

Table 1: $B(\pi)$ for the $(-\beta)$-shift and the β-shift, permutations of length up to 4.

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