Use of the wavelet theory as a tool to investigate the l-abelian complexity of a sequence

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The Hölder regularity

- A notion of regularity of a locally bounded function $f : \mathbb{R}^n \to \mathbb{R}$ at x_0 is given by the *Hölder exponent.*
  We denote $f \in \Lambda^\alpha(x_0)$ (with $\alpha \geq 0$) if there exist a constant $C > 0$ and a polynomial P of degree less than α such that $|f(x) - P(x)| < C|x - x_0|^\alpha$ in a neighborhood of x_0. We define the Hölder exponent of f at x_0 by
  $$h_f(x_0) = \sup\{\alpha \geq 0 : f \in \Lambda^\alpha(x_0)\}.$$
- A function is said *monofractal* if there exists an unique $h \geq 0$ such that $h_f(x_0) \in \{h, +\infty\}$ for any $x_0 \in \mathbb{R}$.

The wavelet leaders method[3] (WLM)

- The wavelet leader associated to the dyadic cube $\frac{k}{2^j} + [0, \frac{1}{2^{j+1}}) := \lambda$ is the quantity
  $$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$
  where $(c_\lambda)_\lambda$ are the wavelet coefficients of f.
- We set
  $$\eta_f(q) = \liminf_{j \to +\infty} \frac{\log(2^{-j} \sum_{\lambda \in \Lambda_j} d_\lambda^q)}{\log 2^{-j}}$$
  and we define the multifractal formalism associated to the wavelet leaders by,
  $$d_f^\eta(h) = \inf_q \{hq - \eta_f(q)\} + 1. \qquad (1)$$
- We have that a function f is monofractal if and only if the function η_f is linear. In this case, the slope of this last function is the unique Hölder exponent h of f.

The l-abelian complexity seen as a transducer

Already known: In [2], it is proved that

- if a function f verifies
  $$f(q^K n + \lambda) = f(q^{K_\lambda} n + r_\lambda) + t_\lambda \quad \text{for } 0 \leq \lambda < q^K$$
  with fixed $K, K_\lambda, r_\lambda \in \mathbb{Z}$, $t_\lambda \in \mathbb{R}$ and $K_\lambda < K$, then there exists a transducer $\mathcal{T}$ such that $f(n) = \mathcal{T}(n)$;
- if $\mathcal{T}$ is a complete, deterministic, subsequential transducer with input alphabet $\{0, 1, \dots, q-1\}^d$, output alphabet $\mathbb{R}$, final period p and c final components then
  $$\mathbb{E}[\mathcal{T}(n)] = e_\mathcal{T} \log_q N + \Phi_1(\log_q N) + O(N^{-\xi} \log N) \quad \text{if } N \to +\infty$$
  where $e_\mathcal{T}$ and $\xi > 0$ are some constants and Φ_1 is a p-periodic, Hölder continuous function. The Fourier coefficients of Φ_1 are known. Besides, with additional conditions on $\mathcal{T}$, we have that
  $$\mathbb{V}[\mathcal{T}(n)] = v_\mathcal{T} \log_q N - \Phi_1^2(\log_q N) + \Phi_2(\log_q N) + O(N^{-\xi} \log^2 N)$$
  where $v_\mathcal{T}$ is a real constant and Φ_2 is a p-periodic, continuous function.

In progress: In fact, we can prove that the function Φ_1 is monofractal with a Hölder exponent of 1. At the moment, we are computing the theoretical wavelet coefficients of the function Φ_1 and we hope to be more precise in the study of the function Φ_2 with this new approach. The first numerical results on Φ_2 are encouraging.

The l-abelian complexity

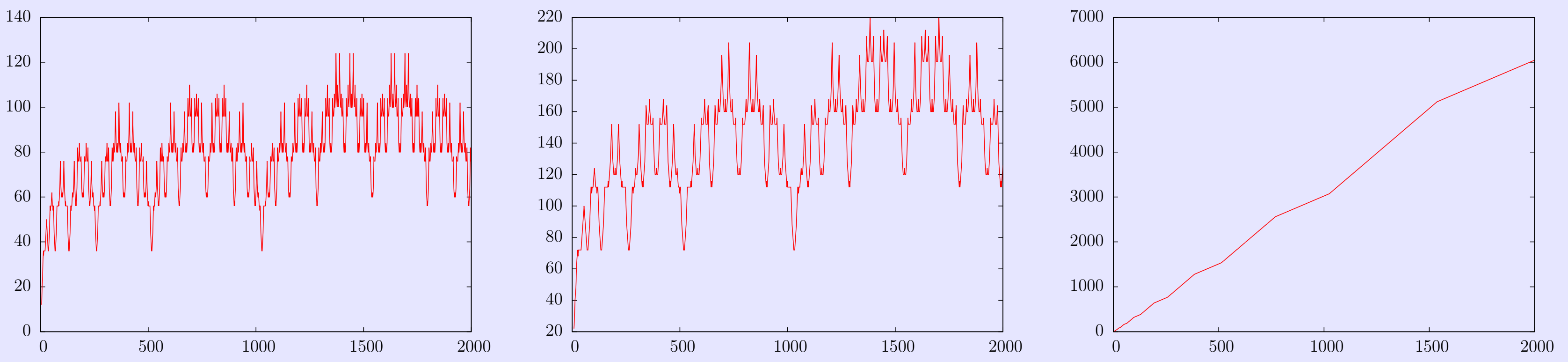
- Let u, v be two finite words on an alphabet Σ. We denote the number of occurrences of the factor x in u by $|u|_x$. We say that u and v are *l-abelian equivalent* if
  $$|u|_x = |v|_x$$
  for any word x of length $|x| \leq l$. In this case, we write $u \sim_{ab,l} v$.
- The *l-abelian complexity*[4] of an infinite word w is the function
  $$\mathcal{P}_w^{(l)} : \mathbb{N} \to \mathbb{N},\ n \mapsto \#(Fac_n(w)/\sim_{ab,l})$$
  where $Fac_n(w)$ is the set of factors of length n occuring in w.
- For any $n \in \mathbb{N}$, we have
  $$\mathcal{P}_w^{(1)}(n) \leq \mathcal{P}_w^{(2)}(n) \leq \dots \leq \mathcal{P}_w^{(l)}(n) \leq \mathcal{P}_w^{(l+1)}(n) \leq \dots \leq \mathcal{P}_w^{(\infty)}(n).$$
  where $\mathcal{P}_w^{(\infty)}$ is the factor complexity of w , i.e. the function which maps $n \in \mathbb{N}$ to the number of distinct factors of length n occurring in w.

WLM to study the l-abelian complexity

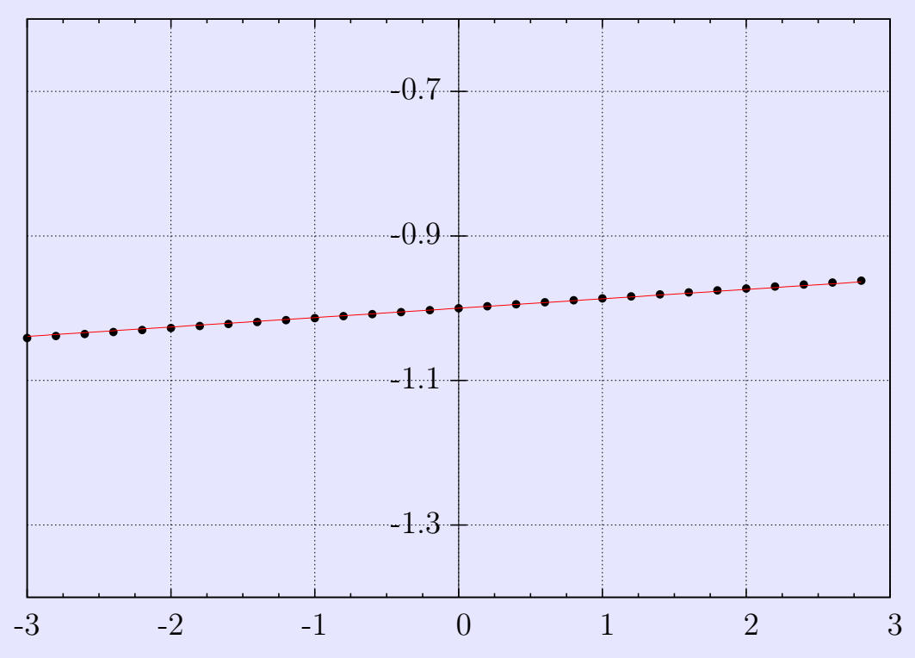
Let w be the Thue-Morse word, i.e. the word generated by the morphism

$$0 \mapsto 01$$
$$1 \mapsto 10.$$

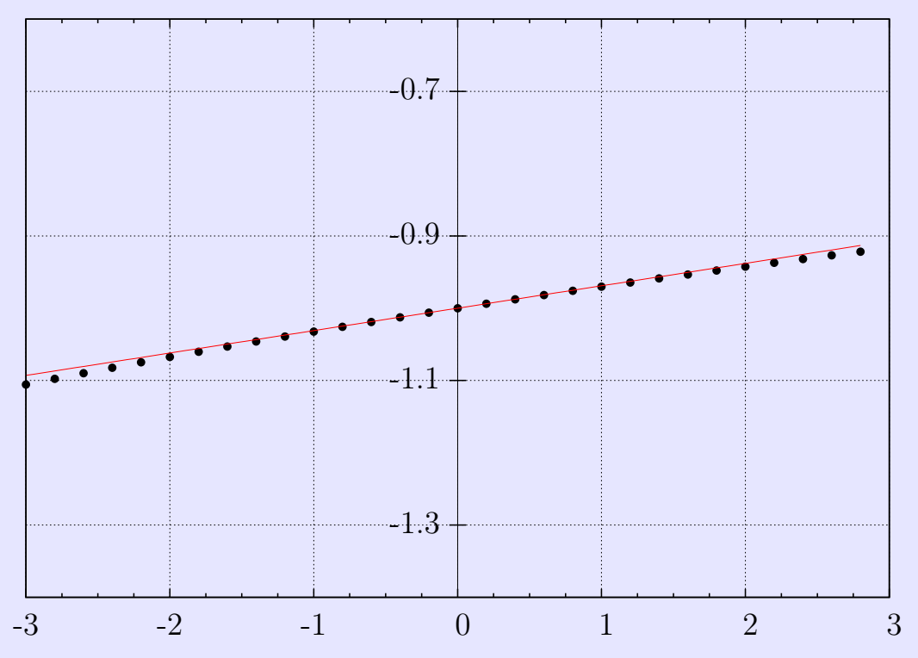
Here are the functions $\mathcal{P}_w^{(5)}$, $\mathcal{P}_w^{(8)}$ and $\mathcal{P}_w^{(\infty)}$.



Here are the functions $\eta_{\mathcal{P}_w^{(5)}}$ and $\eta_{\mathcal{P}_w^{(8)}}$.



the slope is equal to 0.013



the slope is equal to 0.031

We can see that the functions $\eta_{\mathcal{P}_w^{(l)}}$ are linear, so the functions $\mathcal{P}_w^{(l)}$ are monofractal. Besides, we see that the Hölder exponent increases if l increases too.

If we do the same method with the word generated by the morphism

$$0 \mapsto 01$$
$$1 \mapsto 00$$

we obtain a Hölder exponent equal to 0.015 (resp. 0.034) for the function $\mathcal{P}_w^{(5)}$ (resp. $\mathcal{P}_w^{(8)}$).

References :

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[5] Aline Parreau, Michel Rigo, Eric Rowland and Elise Vandomme, *A new approach to the 2-regularity of the l-abelian complexity of 2-automatic sequences*, preprint.