Some examples and counter-examples about the $\boldsymbol{S}$-adic conjecture

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1 Introduction

A usual tool in the study of sequences (or infinite words) over a finite alphabet A is the complexity function p that counts the number of factors of each length n occurring in the sequence (see Chapter 4 of [6] for a survey on this function). This function is clearly bounded by $d^n, n \in \mathbb{N}$, where d is the number of letters in A but not all functions bounded by d^n are complexity functions. As an example, it is well known (see [23]) that either the sequence is ultimately periodic (and then $p(n)$ is ultimately constant), or its complexity function grows at least like $n+1$. Non-periodic sequences with minimal complexity $p(n) = n+1$ for all n exist: they are called *Sturmian sequences* and a large bibliography is devoted to them (see Chapter 2 of [22] and Chapter 6 of [17] for surveys on these sequences).

There is a huge literature about sequences with a low complexity. Indeed, see for instance [1,2,3,4,7,8,14,16,18,19,25]. By "low complexity" we usually mean that "the complexity is bounded by a linear function". Moreover, most of these sequences can also be obtained by a finite number of morphisms. Formally, an *S-adic sequence* is defined as follows. Let $\mathbf{w}$ be a sequence over a finite alphabet A. If S is a set of morphisms, an *S-adic representation* of $\mathbf{w}$ is given by a sequence $(\sigma_n : A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ of morphisms in S and a sequence $(a_n)_{n\in\mathbb{N}}$ of letters, $a_i \in A_i$ for all i such that $A_0 = A$ and $\mathbf{w} = \lim_{n\to+\infty} \sigma_0\sigma_1\cdots\sigma_n(a_{n+1})$. The sequence $(\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ is the *directive word* of the representation. In the sequel, we will say that a sequence $\mathbf{w}$ is *S-adic* if there exists a set S of morphisms such that $\mathbf{w}$ admits an S-adic representation.

An open problem is to determine the link between being an S-adic sequence and having a sub-linear complexity (see [4,15,21]). This problem is called the *S-adic conjecture* and states that *one can find a condition C such that a sequence has a sub-linear complexity if and only if it is an S-adic sequence satisfying Condition C*. It is clear that we cannot avoid considering a particular condition since there exist some purely substitutive sequences with a quadratic complexity.

In this paper, we present some examples either that illustrate some interesting properties or that are counter-examples to what could be believed to be "a good Condition C". Observe that, due to lack of space, all proofs are omitted.

In all what follows, we consider that alphabets are finite subsets of $\mathbb{N}$ and if $\sigma : A^* \to B^*$ is a morphism with $A = \{0, 1, \ldots, k\}$, we write $\sigma = [\sigma(0), \ldots, \sigma(k)]$. The following example is classical when considering S-adic sequences.

Example 1.1. Let us define the four morphisms R_0, R_1, L_0 and L_1 over $\{0, 1\}$ by $R_0 = [0, 10]$, $R_1 = [01, 1]$, $L_0 = [0, 01]$ and $L_1 = [10, 1]$. It is well known (see for instance [5]) that for any Sturmian sequence $\mathbf{w}$, there is a sequence $(k_n)_{n\in\mathbb{N}}$ of integers such that

$$\mathbf{w} = \lim_{n\to+\infty} L_0^{k_0} R_0^{k_1} L_1^{k_2} R_1^{k_3} L_0^{k_4} R_0^{k_5} \cdots L_1^{k_{4n+2}} R_1^{k_{4n+3}}(0). \tag{1}$$

It is important to notice that, when we talk about an S-adic sequence, the corresponding directive word $(\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ is always implicit (even when it is not unique). Indeed, for a given set S of morphisms, we will see that two distinct S-adic sequences can have different properties depending on their respective directive word.

2 Naive ideas

A natural idea to try to understand the conjecture is to consider examples composed of well-known morphisms. For instance, one could consider the *Fibonacci morphism* $\varphi = [01, 0]$ whose fixed point is a Sturmian sequence and the *Thue-Morse morphism* $\mu = [01, 10]$ whose fixed points both have a sub-linear complexity. We have:

Proposition 2.1. *If $S = \{\varphi, \mu\}$ where φ and μ are defined above, any S-adic sequence is uniformly recurrent and has an at most linear complexity.*

Then, one could ask if a sufficient condition to get a sub-linear complexity could simply be to consider a set S of morphisms such that all morphisms in S admits only fixed points with sub-linear complexity. The following example negatively answers this question.

Example 2.2. Let $\alpha = [001, 1]$ and $E = [1, 0]$ be morphisms over $\{0, 1\}$. Observe that both $\alpha E = [1, 001]$ and $E\alpha = [110, 0]$ are *primitive*, i.e., there is a power of them such that all letters occur in all images. It is well known that all fixed points of such morphisms are uniformly recurrent and have a sub-linear complexity. We consider the sequence $\mathbf{w}_{\alpha,E} = \lim_{n\to+\infty} \alpha E \alpha^2 E \alpha^3 E \cdots \alpha^{n-1} E \alpha^n(0)$.

Proposition 2.3 (Cassaigne). *The sequence $\mathbf{w}_{\alpha,E}$ is S-adic for $S = \{\alpha E, E\alpha\}$, is uniformly recurrent and has a quadratic complexity.*

Remark 2.4. Previous result is even stronger than just considering sets S of morphisms with fixed points of sub-linear complexity. Indeed, the sequence also has *bounded partial quotients*, i.e., all morphisms occur with bounded gaps in the directive word.

An opposite question of the previous one is to ask whether S-adic sequences can have a sub-linear complexity when S contains a morphism that admits a fixed point that does not have a sub-linear complexity. The next example positively answers that question.

Example 2.5. Let us consider the morphism α previously defined. From [24] (see also Proposition 3.2) we know that $\alpha^\omega(0)$ has a quadratic complexity. We have:

Proposition 2.6 ([21]). *Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of non-negative integers. The sequence $\mathbf{w} = \lim_{n\to+\infty} \alpha^{k_0}\mu\alpha^{k_1}\mu\alpha^{k_2}\mu\cdots\alpha^{k_n}\mu(0)$ is uniformly recurrent. Moreover, $\mathbf{w}$ has an at most linear complexity if and only if the sequence $(k_n)_{n\in\mathbb{N}}$ is bounded.*

As a first conclusion, Proposition 2.6 shows that it is not enough to put some conditions on the morphisms in S to determine the condition of the conjecture and that we also have to take care of the directive word. Moreover, Propositions 2.3 and 2.6 show that considering only "good morphisms" can provide too much complexity and considering "bad morphisms" does not ensure that the complexity will not be sub-linear.

3 Comparisons with substitutive sequences

When $\mathrm{Card}(S) = 1$, the complexity functions that can occur has been completely determined by Pansiot in [24]. Indeed, he proved that for purely substitutive sequences $\mathbf{w} = \sigma^\omega(a)$, the complexity function $p_\mathbf{w}$ can only have five asymptotic behaviors that are $\Theta(1)$, $\Theta(n)$, $\Theta(n\log n)$, $\Theta(n\log\log n)$ and $\Theta(n^2)$. Moreover, when $\mathbf{w}$ is aperiodic, the complexity function cannot be ultimately constant (due to a result of Morse and Hedlund [23]), i.e., $p_\mathbf{w} \neq \Theta(1)$ and the class of complexity of the sequence only depends on the growth rate of images.

The aim of this section is to discuss about which results of Pansiot can be transposed to the case $\mathrm{Card}(S) > 1$. First, let us introduce some notations. Let $\sigma : A^* \to A^*$ be a morphism. We denote by $A_\mathfrak{B}$ the set of *bounded letters*, i.e., the set of letters $a \in A$ such that $\lim_{n\to\infty} |\sigma^n(a)| < \infty$. For a directive word $(\sigma_n)_{n\in\mathbb{N}}$, we denote by $\mathcal{A}_\mathfrak{B}$ the set of sequences $(a_i)_{i\in\mathbb{N}} \in \prod_{i=0}^{\infty} A_i$ such that $\lim_{n\to\infty} |\sigma_0\sigma_1\cdots\sigma_n(a_{n+1})| < \infty$.

3.1 Growth of images

Pansiot's work implies in particular that any purely substitutive sequence has a *reasonably low* complexity (recall that a sequence can have an exponential complexity). Therefore, one could ask if there exists an equivalent result for S-adic sequences. Proposition 3.1 below negatively answers this question since it implies that one can get any complexity with an S-adic sequence.

Proposition 3.1 (Cassaigne [9]). *Let A be an alphabet. There is a finite set S of morphisms over $A' = A\cup\{\ell\}$, $\ell \notin A$, such that any sequence over A is S-adic and we have $\mathcal{A}_\mathfrak{B} = A'^{\mathbb{N}} \backslash A^*\ell^\omega$.*

However, there is something interesting to observe in that result: the fact that $\mathcal{A}_\mathfrak{B}$ is non-empty. This is to be compared with the case of purely substitutive sequences.

Proposition 3.2 (Pansiot [24]). *Let $\mathbf{w} = \sigma^\omega(a)$ be a purely substitutive sequence over A. Then its complexity function satisfies $p_\mathbf{w}(n) = \Theta(n^2)$ if and only if $A_\mathfrak{B} \neq \emptyset$ and there are some arbitrary large factors of $\mathbf{w}$ in $A_\mathfrak{B}^*$.*

A natural open question is therefore to ask whether any high complexity can be reached by S-adic sequences satisfying the *ω-growth Property*, i.e., by S-adic sequences such that $\mathcal{A}_\mathfrak{B} = \emptyset$. The following result is a partial answer to that question since it deals with *everywhere growing* S-adic sequences, i.e., S-adic sequences satisfying the ω-growth Property and such that $|\sigma(a)| \geq 2$ for all morphisms σ in S and all letters a.

Proposition 3.3 ([21]). *If $\mathbf{w}$ is an everywhere growing S-adic sequence such that $\mathrm{Card}(S) < \infty$, there is a constant C such that $p_\mathbf{w}(n) \leq Cn\log n$ for all integers $n \geq 1$. Moreover, there exists an everywhere growing S-adic sequence with complexity $p(n) = \Theta(n\log n)$.*

3.2 Comparable lengths of images

For purely substitutive sequences $\mathbf{w} = \sigma^\omega(a)$ over A, Pansiot proved that when $p_{\mathbf{w}}(n) \neq \Theta(n^2)$ and $A_{\mathfrak{B}} \neq \emptyset$, there exist two morphisms $f : B^* \to B^*$ and $g : B \to A$ such that $\mathbf{w} = g(f^\omega(b))$ for $b \in B$ and $B_{\mathfrak{B}} = \emptyset$. This is to be compared with the following result that can be found in [15] (see also [21]).

Proposition 3.4 ([15,21]). *Any uniformly recurrent sequence with an at most linear complexity is S-adic and satisfies the ω-growth Property.*

Pansiot also proved the following result and Durand generalized it.

Proposition 3.5 (Pansiot [24]). *Let $\mathbf{w} = \sigma^\omega(a)$ be an aperiodic purely substitutive sequence over A. If $A_{\mathfrak{B}} = \emptyset$, we have $p_{\mathbf{w}}(n) = \Theta(n) \Leftrightarrow \exists K > 0 : \forall n, \max_{a,b\in A} \frac{|\sigma^n(a)|}{|\sigma^n(b)|} \leq K$.*

Proposition 3.6 (Durand [11]). *If $\mathbf{w}$ is an S-adic sequence satisfying the ω-growth Property with $\mathrm{Card}(S) < \infty$ and if there is constant K such that $\forall n, \max_{a,b\in A_{n+1}} \frac{|\sigma_0\cdots\sigma_n(a)|}{|\sigma_0\cdots\sigma_n(b)|} \leq K$, then $\mathbf{w}$ has an at most linear complexity.*

In particular, this implies that all primitive S-adic sequences with $\mathrm{Card}(S) < \infty$ have an at most linear complexity, where *primitive S-adic* means that there is a constant r such that for all $s \geq 0$, all letters in A_s occur in all images $\sigma_s \cdots \sigma_{s+r}(a)$, $a \in A_{s+r+1}$. These conditions are too restrictive for the conjecture since there exist some Sturmian sequences that do not satisfy $\max_{a,b\in A_{n+1}} \frac{|\sigma_0\cdots\sigma_n(a)|}{|\sigma_0\cdots\sigma_n(b)|} \leq K$ (see Example 1.1 with $(k_n)_{n\in\mathbb{N}}$ unbounded).

However, we can observe in this example that this condition is still infinitely often satisfied. Indeed, let us consider the directive word $(\tau_n)_{n\in\mathbb{N}}$ defined by $\tau_0 = L_0^{k_0} R_0^{k_1} L_1$, $\tau_{2n+1} = L_1^{k_{4n+2}-1} R_1^{k_{4n+3}} L_0$ for $n \geq 0$ and $\tau_{2n} = L_0^{k_{4n}-1} R_0^{k_{4n+1}} L_1$ for $n \geq 1$. For all n, there exist some integers i and j such that either $\tau_n = [0^i 1 0^{j+1}, 0^i 1 0^j]$ or $\tau_n = [1^i 0 1^j, 1^i 0 1^{j+1}]$. With these morphisms, we have

$$\tau_0 \tau_1 \cdots \tau_n \cdots = L_0^{k_0} R_0^{k_1} L_1^{k_2} R_1^{k_3} \cdots L_1^{k_{4n+2}} R_1^{k_{4n+3}} \cdots \tag{2}$$

and $\max_{a,b\in A_{n+1}} \frac{|\tau_0\cdots\tau_n(a)|}{|\tau_0\cdots\tau_n(b)|} \leq K$. The sequence $(\tau_n)_{n\in\mathbb{N}}$ is called a *contraction* of the directive word $(L_0^{k_0} R_0^{k_1} \cdots)$. Observe that the set $S = \{\tau_n \mid n \in \mathbb{N}\}$ of morphisms might be infinite (when $(k_n)_{n\in\mathbb{N}}$ is unbounded). Consequently, it may be interesting to work either with infinite set of morphisms or with contractions. But, Example 2.5 shows that Proposition 3.6 is not true anymore when $\mathrm{Card}(S) = \infty$. Indeed, if we consider the contraction $(\sigma_n)_{n\in\mathbb{N}}$ of the directive word of Proposition 2.6 defined for all $n \geq 0$ by

$$\sigma_n = \alpha^{k_n} \mu, \tag{3}$$

we have $|\sigma_0 \cdots \sigma_n(0)| = |\sigma_0 \cdots \sigma_n(1)|$ for all n although the complexity is not sub-linear as soon as the sequence $(k_n)_{n\in\mathbb{N}}$ is unbounded.

3.3 Number of different powers and their size

There are still significant differences between the two contractions in Equations (2) and (3): the number of different powers and their size. First, let us introduce some notations. Let $\sigma : A^* \to B^*$ be a morphism and b a letter in B. We let $\mathrm{Pow}_\sigma(b)$ denote the set of integers

i such that there is a letter $a \in A$ and two letters $c \neq b$ and $d \neq b$ in B such that $\sigma(a) \in \{B^*cb^idB^*, b^idB^*, B^*cb^i\}$.

With the same notations of (2) and (3), for all n we have $\mathrm{Card}(\mathrm{Pow}_{\tau_n}(0)) \in \{1,2\}$, $\mathrm{Card}(\mathrm{Pow}_{\tau_n}(1)) \in \{1,2\}$ although $\mathrm{Card}\left(\mathrm{Pow}_{\alpha^{k_n}\mu}(0)\right) = 1$ and $\mathrm{Card}\left(\mathrm{Pow}_{\alpha^{k_n}\mu}(1)\right) = k_n + 1$.

For purely substitutive sequences $\mathbf{w} \in A^{\mathbb{N}}$, it is known (see [12]) that if $\mathbf{w}$ is k-power-free (i.e., there is no factor of $\mathbf{w}$ that can be written u^k with $u \neq \varepsilon$), then $p_{\mathbf{w}}(n) \leq Cn \log n$ for all $n \geq 1$. Moreover, if $\mathrm{Card}(A) = 2$ and if $k \geq 3$, then we even have $p_{\mathbf{w}}(n) \leq Cn$ for all $n \geq 1$ (see [13]). Our approach is a little bit different since we do not count the maximal power k that one can observe in a sequence but the number of distinct powers. Now let us give some examples with interesting behaviors: all of them are uniformly recurrent and have a sub-linear complexity but the number of powers of letters in images is sometimes unbounded. In that case, the thing to observe is that the size of the different powers grows exponentially. Indeed, in Example 3.9 we have $\max\left\{\frac{i}{j} \;\middle|\; i,j \in \mathrm{Pow}_{\beta^n}(1)\right\} = \max\left\{\frac{i}{j} \;\middle|\; i,j \in \mathrm{Pow}_{\gamma^n}(0)\right\} = 3^n$ although we have $\max\left\{\frac{i}{j} \;\middle|\; i,j \in \mathrm{Pow}_{\alpha^{k_n}\mu}(1)\right\} = k_n + 1$.

Example 3.7. For all positive integers k, let us consider the morphism χ_k over $\{1,2,3\}$ defined by $\chi_k = [2, 31^k, 31^{k-1}]$. Cassaigne proved the following result that can be found in Chapter 4 of [6].

Proposition 3.8 (Cassaigne). *If $S = \{\chi_k \mid k \geq 1\}$, then for any S-adic sequence $\mathbf{w}$, we have*

1. *for all $n \geq 0$, $p_{\mathbf{w}}(n+1) - p_{\mathbf{w}}(n) \in \{2,3\}$, the value 2 being assumed infinitely often;*
2. *for all $n \geq 1$, $p_{\mathbf{w}}(n) \leq 3n$ and $\liminf_{n\to+\infty} p_{\mathbf{w}}(n) - 3n = -\infty$.*

Moreover, we can easily see that $\mathrm{Card}(\mathrm{Pow}_{\chi_k}(i)) \in \{1,2\}$ for all i in $\{1,2,3\}$.

Example 3.9. Let us define the morphisms β and γ over $\{0,1\}$ by $\beta = [010, 111]$ and $\gamma = [000, 101]$. We consider the sequence $\mathbf{w}_{\beta,\gamma} = \lim_{n\to+\infty} \beta\gamma\beta^2\gamma^2\beta^3\gamma^3 \cdots \beta^n\gamma^n(0)$.

Proposition 3.10. *The sequence $\mathbf{w}_{\beta,\gamma}$ is uniformly recurrent and has an at most linear complexity. Moreover, for all n we have $\mathrm{Card}(\mathrm{Pow}_{\beta^n}(1)) = \mathrm{Card}(\mathrm{Pow}_{\gamma^n}(0)) = n$ and $\mathrm{Card}(\mathrm{Pow}_{\beta^n}(0)) = \mathrm{Card}(\mathrm{Pow}_{\gamma^n}(1)) = 1$.*

As a kind of generalization of these examples, we can prove the following.

Proposition 3.11. *Let $\mathbf{w}$ be a sequence over A. If there is a constant $C \geq 1$, a sequence $(u_n)_{n\in\mathbb{N}}$ of factors of $\mathbf{w}$ and an increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers such that $(|u_n|)_{n\in\mathbb{N}}$ is increasing and for all n, the word u_n^i is a factor of $\mathbf{w}$ for all integers i such that $\frac{k_n}{C} \leq i \leq k_n$, then $\mathbf{w}$ does not have a sub-linear complexity.*

4 Conclusion

From all considered examples in the paper, a first remark is that there is no obvious property that we can find to determine a statement of the conjecture. However, we think that it could be interesting to deepen the reflexion about powers introduced in Section 3.3. Indeed, another similar result to all considered examples is the case of uniformly recurrent sequences with complexity $2n$. For those sequences, we can prove that there is a set S of morphisms over $\{0,1,2\}$ with $\mathrm{Card}(S) = 5$ such that any such sequences is S-adic (see [20]). We can also prove that there is a contraction of its directive word such that all contracted morphisms σ_n are strongly primitive and that $\mathrm{Card}(\mathrm{Pow}_{\sigma_n}(i)) \leq K$ for a constant K. Furthermore, the set S of contracted morphisms σ_n can be partitioned into a finite number of set S_i such that for all i, all morphisms in S_i are equals excepts for the values in $\mathrm{Pow}_{\sigma_n}(i)$.

A nice thing in that result is that the employed techniques to obtain the S-adic representation are similar to those used to prove Proposition 3.4. Consequently, one could hope finding something similar for the general case, but this question seems to be very difficult. However, some other methods related to return words might be fruitful in that direction. They also could yield to other interesting questions.

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