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Lagrangians for Damped Linear Multi-Degree-of-Freedom Systems

This paper deals with finding Lagrangians for damped, linear multi-degree-of-freedom systems. New results for such systems are obtained using extensions of the results for single and two degree-of-freedom systems. The solution to the inverse problem for an n-degree-of-freedom linear gyroscopic system is obtained as a special case. Multi-degree-of-freedom systems that commonly arise in linear vibration theory with symmetric mass, damping, and stiffness matrices are similarly handled in a simple manner. Conservation laws for these damped multi-degree-of-freedom systems are found using the Lagrangians obtained and several examples are provided. [DOI: 10.1115/1.4023019]

1 Introduction

The inverse problem for Lagrangian dynamics—also known as the inverse problem of the calculus of variations—is to obtain, for a system described by a given set of differential equations, a Lagrangian function such that the corresponding Euler–Lagrange equations obtained using the calculus of variations yield the given set of equations that describe the system. This problem has attracted many researchers in various fields of study for its usefulness; for example, in numerical mathematics a Lagrangian function provides us with approximate solutions to nonlinear ordinary differential equations and in quantum physics, theories are based on the Hamiltonian for the system. Bolza [1] gave a general procedure for finding a Lagrangian for a single-degree-of-freedom dissipative system. This was followed by Leitmann [2] who provided some examples of nonpotential forces and the corresponding Lagrangians for which a variational principle exists. This method was extended by Udwadia et al. [3] who used a more systematic derivation to obtain several classes of nonpotential forces which could be used to obtain the equations of motion via variational calculus. Recently, the semi-inverse method [4] has been considered due to its simplicity and applicability to many cases. However, Refs. [1–4] consider only single-degree-of-freedom (SDOF) systems and the analysis of this case is relatively easy because, in the nineteenth century, Darboux [5] proved that a Lagrangian can *always* be found for the inverse problem for such SDOF systems. For multi-degree-of-freedom (MDOF) systems, the configuration variables are coupled with one another and this makes it difficult to solve the inverse problem. The general conditions for the existence of Lagrangians were apparently first obtained by Helmholtz [6,7] and are usually referred to as the Helmholtz conditions. Later, Douglas [8] analyzed in great detail the case of two degrees of freedom and obtained the necessary and sufficient conditions for their existence without utilizing these conditions. Using Douglas' results, Hojman and Ramos [9] proposed a simpler method to determine the existence of a Lagrangian for two-dimensional problems in which the potential function does not explicitly contain the generalized velocities. Mestdag et al. [10] derived the conditions under which there exists a Lagrangian and a dissipation function on the right hand side of the more general form of the Euler–Lagrange equation. They also provided some nonconservative systems to which their approach can be applied.

In the present paper, the findings obtained in Refs. [2] and [3] are extended to dissipative, constant coefficient, linear MDOF systems. The emphasis is on obtaining the Lagrangians for these MDOF systems in a simple manner, using insights obtained from our understanding of the inverse problem for the SDOF and 2-DOF systems. It is shown that the solution to a gyroscopically damped linear system is easily found as a special case of the linearly damped case. Lagrangians for special linearly damped MDOF systems with symmetric stiffness and damping matrices are also obtained along with the corresponding Jacobi integrals, which are conserved over time.

2 Lagrangians for Damped Linear Two-Degree-of-Freedom Systems

We begin with the problem of finding a Lagrangian function for a linear mass-spring-damper system in a single degree of freedom whose governing equation of motion is given by

$$m\ddot{x} + 2b\dot{x} + kx = 0, \ m > 0, \ b, \ k \geq 0 \tag{1}$$

where $x(t)$ is a generalized displacement of the mass, the dot denotes the differentiation with respect to time t, and m, b, and k are the mass, damping, and stiffness coefficients, respectively, which are all assumed to be constants. Unfortunately, Eq. (1) cannot be directly derived as the Euler–Lagrange equation from a variational principle because it does not satisfy the Helmholtz conditions [6,7] (see Eqs. (11)–(14)). However, in Refs. [2,3,11] it is shown that the following Lagrangian function results in Eq. (1)

$$L = e^{(2b/m)t}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) \tag{2}$$

More precisely, substituting Eq. (2) into the Euler–Lagrange equation of the standard form $(d/dt)(\partial L/\partial \dot{x}) - (\partial L/\partial x) = 0$ yields

$$e^{(2b/m)t}[m\ddot{x}(t) + 2b\dot{x}(t) + kx(t)] = 0 \tag{3}$$

Since the exponential factor in Eq. (3) is always positive in time, we can say that Eq. (3) is 'equivalent' to Eq. (1). This exponential

Manuscript received March 1, 2012; final manuscript received October 27, 2012; accepted manuscript posted November 19, 2012; published online May 23, 2013. Assoc. Editor: Wei-Chau Xie.

factor, however, plays a significant role, because Eq. (3) does satisfy the Helmholtz conditions.

Thinking of the Lagrangian given by Eq. (2) as representing a physical *under-damped* oscillator consisting of a 'mass' $me^{(2b/m)t}$ and 'potential energy' $(1/2)e^{(2b/m)t}kx^2$, we find that the energy of this oscillator does not decay with time as does that of the under-damped system described by Eq. (1) [11]. In fact, its energy, averaged over one period of the oscillator, remains a constant and, hence, the Lagrangian given by Eq. (2) is not a physical Lagrangian, but just a convenient mathematical tool to use the machinery of the calculus of variations to obtain Eq. (3). In a similar manner, were the Lagangian in Eq. (2) to represent an *over-damped* oscillator with 'mass' $me^{(2b/m)t}$ and 'potential energy' $(1/2)e^{(2b/m)t}kx^2$, its energy would, in general, exponentially increase with time, contrary to the behavior of the over-damped oscillator described by Eq. (1). It follows, then, that this Lagrangian given in Eq. (2) does not describe the physical linearly damped system, although the two systems are described by the same equation of motion, namely, Eq. (1). However, a *mathematical* (*not physical*) Lagrangian may yet be useful in areas such as the development of approximate solutions of differential equations and various numerical techniques.

Taking a hint from Eq. (2), we next consider a two-degree-of-freedom mass-spring-damper system using the Lagrangian given by

$$L = e^{\gamma t}\left[\frac{1}{2}\left(m_1\dot{x}_1^2 - k_1x_1^2\right) + \frac{1}{2}\left(m_2\dot{x}_2^2 - k_2x_2^2\right) + b_1\dot{x}_1x_2 + b_2x_1\dot{x}_2 + dx_1x_2\right] \tag{4}$$

where m_i, k_i, b_i, $i = 1, 2$, and γ and d are constants. Then the corresponding Euler–Lagrange equations of motion are given by

$$m_1\ddot{x}_1 + \gamma m_1\dot{x}_1 + (b_1 - b_2)\dot{x}_2 + k_1x_1 + (b_1\gamma - d)x_2 = 0 \tag{5a}$$

$$m_2\ddot{x}_2 + \gamma m_2\dot{x}_2 + (b_2 - b_1)\dot{x}_1 + k_2x_2 + (b_2\gamma - d)x_1 = 0 \tag{5b}$$

Depending upon the choice of the constants, Eqs. (5*a*) and (5*b*) can represent various systems. For example, if we choose $m_1 = m$, $m_2 = 2m$, $\gamma = 2$, $b_1 = b_2 = 0$, $k_1 = k_2 = 2k$, and $d = k$, Eq. (4) becomes

$$L = e^{2t}\left[\frac{1}{2}\left(m\dot{x}_1^2 - 2kx_1^2\right) + \left(m\dot{x}_2^2 - kx_2^2\right) + kx_1x_2\right] \tag{6}$$

and the corresponding equations of motion become

$$m\ddot{x}_1 + 2m\dot{x}_1 + 2kx_1 - kx_2 = 0 \tag{7a}$$

$$2m\ddot{x}_2 + 4m\dot{x}_2 + 2kx_2 - kx_1 = 0 \tag{7b}$$

which describe a classically damped 2-DOF system. In fact, Eqs. (7*a*) and (7*b*) describe the mechanical system shown in Fig. 1.

In order to obtain a more systematic approach to the inverse problem for a constant coefficient linear 2-DOF system we consider the following equations of motion

$$f^1 = \ddot{x}_1 + a_1\dot{x}_1 + b_1\dot{x}_2 + c_1x_1 + d_1x_2 = 0 \tag{8a}$$

$$f^2 = \ddot{x}_2 + a_2\dot{x}_1 + b_2\dot{x}_2 + c_2x_1 + d_2x_2 = 0 \tag{8b}$$

where all coefficients are constant and we have divided each equation by the corresponding masses m_1 and m_2. More generally, we consider the following set of equations

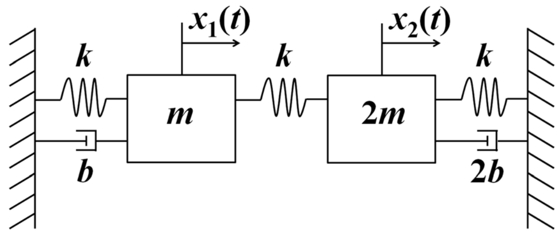

Fig. 1 Linear 2-DOF mass-spring-damper system with $b = 2m$

$$\alpha_{11}f^1 = \alpha_{11}(\ddot{x}_1 + a_1\dot{x}_1 + b_1\dot{x}_2 + c_1x_1 + d_1x_2) = 0 \tag{9a}$$

$$\alpha_{22}f^2 = \alpha_{22}(\ddot{x}_2 + a_2\dot{x}_1 + b_2\dot{x}_2 + c_2x_1 + d_2x_2) = 0 \tag{9b}$$

where $\alpha_{ii}(i = 1, 2)$ are nonzero functions of t, x_1, x_2, $\dot{x}_1$, and $\dot{x}_2$, namely, $\alpha_{11} = \alpha_{11}(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$ and $\alpha_{22} = \alpha_{22}(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$. Next, we define the functions $\beta_i(i = 1, 2)$ by

$$\beta_1 := \alpha_{11}(a_1\dot{x}_1 + b_1\dot{x}_2 + c_1x_1 + d_1x_2) \tag{10a}$$

$$\beta_2 := \alpha_{22}(a_2\dot{x}_1 + b_2\dot{x}_2 + c_2x_1 + d_2x_2) \tag{10b}$$

Now let us consider the n differential equations $\alpha_{ij}(t, \mathbf{q}, \dot{\mathbf{q}})\ddot{q}^j + \beta_i(t, \mathbf{q}, \dot{\mathbf{q}}) = 0$ $(i, j = 1, 2, \ldots, n)$, where $\mathbf{q} = [q^1\ q^2\ \cdots\ q^n]^{\mathbf{T}}$ is a generalized displacement n-vector that describes the motion of a mechanical system in an n-dimensional configuration space. The superscript "T" is used to denote the transpose of a vector (or a matrix), and the summation convention is used for repeated indices. The question of whether such a system can be obtained from a suitable Lagrangian $L(t, \mathbf{q}, \dot{\mathbf{q}})$, through the use of the Euler–Lagrange equations $(d/dt)(\partial L/\partial\dot{q}^i) - (\partial L/\partial q^i) = 0$,[1] appears to have been first investigated by Helmholtz [6,7]. The necessary and sufficient conditions for the so-called *ordered direct analytic representations* are [7]

$$\alpha_{ij} = \alpha_{ji} \tag{11}$$

$$\frac{\partial\alpha_{ij}}{\partial\dot{q}^k} = \frac{\partial\alpha_{ik}}{\partial\dot{q}^j} \tag{12}$$

$$\frac{\partial\beta_i}{\partial\dot{q}^j} + \frac{\partial\beta_j}{\partial\dot{q}^i} = 2\left(\frac{\partial}{\partial t} + \dot{q}^k\frac{\partial}{\partial q^k}\right)\alpha_{ij} \tag{13}$$

$$\frac{\partial\beta_i}{\partial q^j} - \frac{\partial\beta_j}{\partial q^i} = \frac{1}{2}\left(\frac{\partial}{\partial t} + \dot{q}^k\frac{\partial}{\partial q^k}\right)\left(\frac{\partial\beta_i}{\partial\dot{q}^j} - \frac{\partial\beta_j}{\partial\dot{q}^i}\right) \tag{14}$$

where the summation convention is applied for repeated indices. Equations (11)–(14) are a set of partial differential equations[2] that need to be satisfied by the $2n + 1$ independent variables t, $\mathbf{q}$, and

[1]Since we cannot differentiate the function $L(t, \mathbf{q}, \dot{\mathbf{q}})$, with respect to a dependent variable, say $\dot{q}^i$, by $\partial L(t, \mathbf{q}, \dot{\mathbf{q}})/\partial\dot{q}^i$ we mean $\partial L(t, \mathbf{s}, \mathbf{r})/\partial r^i|_{\substack{\mathbf{s}=\mathbf{q}\\ \mathbf{r}=\dot{\mathbf{q}}}}$, where t, $\mathbf{s}$, and $\mathbf{r}$ are considered independent variables; similarly, by $\partial L(t, \mathbf{q}, \dot{\mathbf{q}})/\partial q^i$, we mean $\partial L(t, \mathbf{s}, \mathbf{r})/\partial s^i|_{\substack{\mathbf{s}=\mathbf{q}\\ \mathbf{r}=\dot{\mathbf{q}}}}$.

[2]There is a slight abuse of notation here, since in Eqs. (11)–(14) the variables t, $\mathbf{q}$, and $\dot{\mathbf{q}}$ are considered to be independent, while in the equations of motion, $\mathbf{x}$ and $\dot{\mathbf{x}}$ (see Eq. (8)) are considered to be functions of time t.

$\dot{\mathbf{q}}$ everywhere in R^{2n+1}. Throughout this paper we shall be dealing with Lagrangians that provide the so-called ordered direct analytic representations of the equations of motion [7].

Applying these conditions to Eqs. (9*a*) and (9*b*) $i, j, k = 1, 2$, we therefore have $q^1 = x_1$, $q^2 = x_2$, and $\alpha_{12} = \alpha_{21} = 0$. Thus, Eq. (11) is automatically satisfied, and Eq. (12) yields

$$\frac{\partial \alpha_{11}}{\partial \dot{x}_2} = \frac{\partial \alpha_{12}}{\partial \dot{x}_1} = 0 \tag{15}$$

$$\frac{\partial \alpha_{21}}{\partial \dot{x}_2} = \frac{\partial \alpha_{22}}{\partial \dot{x}_1} = 0 \tag{16}$$

from which we conclude that

$$\alpha_{11} = \alpha_{11}(t, x_1, x_2, \dot{x}_1) \tag{17}$$

$$\alpha_{22} = \alpha_{22}(t, x_1, x_2, \dot{x}_2) \tag{18}$$

Next, Eq. (13) yields the following three equations

$$\frac{\partial \alpha_{11}}{\partial \dot{x}_1}(a_1\dot{x}_1 + b_1\dot{x}_2 + c_1x_1 + d_1x_2) + \alpha_{11}a_1 = \frac{\partial \alpha_{11}}{\partial t} + \dot{x}_1\frac{\partial \alpha_{11}}{\partial x_1} + \dot{x}_2\frac{\partial \alpha_{11}}{\partial x_2}, \quad (i = 1,\ j = 1) \tag{19}$$

$$\alpha_{11}b_1 + \alpha_{22}a_2 = 0, \quad (i = 1,\ j = 2 \quad \text{or} \quad i = 2,\ j = 1) \tag{20}$$

$$\frac{\partial \alpha_{22}}{\partial \dot{x}_2}(a_2\dot{x}_1 + b_2\dot{x}_2 + c_2x_1 + d_2x_2) + \alpha_{22}b_2 = \frac{\partial \alpha_{22}}{\partial t} + \dot{x}_1\frac{\partial \alpha_{22}}{\partial x_1} + \dot{x}_2\frac{\partial \alpha_{22}}{\partial x_2}, \quad (i = 2,\ j = 2) \tag{21}$$

Finally, Eq. (14) provides only one nontrivial equation for the case $i = 1, j = 2$, or $i = 2, j = 1$, which is

$$\frac{\partial \alpha_{11}}{\partial x_2}\left(a_1\dot{x}_1 + \frac{1}{2}b_1\dot{x}_2 + c_1x_1 + d_1x_2\right) + \alpha_{11}d_1 - \frac{\partial \alpha_{22}}{\partial x_1}\left(\frac{1}{2}a_2\dot{x}_1 + b_2\dot{x}_2 + c_2x_1 + d_2x_2\right) - \alpha_{22}c_2 = \frac{1}{2}\left(\frac{\partial \alpha_{11}}{\partial t}b_1 + \dot{x}_1\frac{\partial \alpha_{11}}{\partial x_1}b_1 - \frac{\partial \alpha_{22}}{\partial t}a_2 - \dot{x}_2\frac{\partial \alpha_{22}}{\partial x_2}a_2\right) \tag{22}$$

Thus, Eqs. (17)–(22) must be satisfied everywhere in R^5 if Eqs. (9*a*) and (9*b*) can be obtained from a Lagrangian $L(t, \mathbf{x}, \dot{\mathbf{x}})$, where $\mathbf{x} = [x_1 \quad x_2]^{\mathbf{T}}$, i.e., if

$$\alpha_{ij}f^j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} \tag{23}$$

We now consider two cases.

2.1 Case I: $a_2 \neq 0$.

If $a_2 \neq 0$, from Eq. (20) we have

$$\alpha_{22} = -\frac{b_1}{a_2}\alpha_{11}, \quad b_1 \neq 0 \tag{24}$$

Then, from Eqs. (17) and (18)

$$\alpha_{11} = \alpha_{11}(t, x_1, x_2), \quad \alpha_{22} = \alpha_{22}(t, x_1, x_2) \tag{25}$$

and Eqs. (19) and (21) become

$$\alpha_{11}a_1 = \frac{\partial \alpha_{11}}{\partial t} + \dot{x}_1\frac{\partial \alpha_{11}}{\partial x_1} + \dot{x}_2\frac{\partial \alpha_{11}}{\partial x_2} \tag{26}$$

$$\alpha_{22}b_2 = \frac{\partial \alpha_{22}}{\partial t} + \dot{x}_1\frac{\partial \alpha_{22}}{\partial x_1} + \dot{x}_2\frac{\partial \alpha_{22}}{\partial x_2} \tag{27}$$

However, since $\alpha_{11} = \alpha_{11}(t, x_1, x_2)$ and $\alpha_{22} = \alpha_{22}(t, x_1, x_2)$ from Eq. (25), taking the partial derivative of Eq. (26) with respect to the variable $\dot{x}_2$, we obtain $\partial\alpha_{11}/\partial x_2 = 0$. In a similar fashion, taking partial derivatives with respect to $\dot{x}_1$ yields $\partial\alpha_{11}/\partial x_1 = 0$. By doing the same with Eq. (27), we obtain $\partial\alpha_{22}/\partial x_1 = \partial\alpha_{22}/\partial x_2 = 0$ so that

$$\alpha_{11} = \alpha_{11}(t), \quad \alpha_{22} = \alpha_{22}(t) \tag{28}$$

and Eqs. (26) and (27) become

$$\alpha_{11}a_1 = \frac{\partial \alpha_{11}}{\partial t} \tag{29}$$

and

$$\alpha_{22}b_2 = \frac{\partial \alpha_{22}}{\partial t} \tag{30}$$

The general solutions to Eqs. (29) and (30) are

$$\alpha_{11}(t) = \gamma_1 e^{a_1 t} \tag{31}$$

$$\alpha_{22}(t) = \gamma_2 e^{b_2 t} \tag{32}$$

where γ_1 and γ_2 are nonzero constants. Substituting Eqs. (31) and (32) into Eq. (24) yields

$$\gamma_2 e^{b_2 t} = -\frac{b_1}{a_2}\gamma_1 e^{a_1 t} \tag{33}$$

Equation (33) should hold for all $t \geq 0$, thus

$$a_1 = b_2 \tag{34}$$

$$\gamma_2 = -\frac{b_1}{a_2}\gamma_1 \tag{35}$$

and α_{11} and α_{22} become

$$\alpha_{11}(t) = \gamma_1 e^{a_1 t} \tag{36}$$

$$\alpha_{22}(t) = -\frac{b_1}{a_2}\gamma_1 e^{a_1 t} \tag{37}$$

Finally, we use Eq. (22) to obtain

$$a_2d_1 + b_1c_2 = a_1a_2b_1 \tag{38}$$

In conclusion, we have the following possible case for the existence of a Lagrangian

$$\text{Case I: } \alpha_{11} = \gamma_1 e^{a_1 t}, \quad \alpha_{22} = -\frac{b_1}{a_2}\gamma_1 e^{a_1 t}, \quad a_1 = b_2, \quad a_2d_1 + b_1c_2 = a_1a_2b_1, \quad a_2 \neq 0, \quad b_1 \neq 0, \quad \gamma_1 \neq 0 \tag{39}$$

It should noted that Eq. (39) is symmetric; that is, if we interchange the coefficients of Eqs. (9*a*) and (9*b*) with one another, Eq. (39) still holds.

2.2 Cases II and III: $a_2 = 0$.

If $a_2 = 0$, from Eq. (20) we have

$$b_1 = 0 \tag{40}$$

Then, Eqs. (19) and (21) become

$$\frac{\partial \alpha_{11}}{\partial t} + \dot{x}_1\frac{\partial \alpha_{11}}{\partial x_1} + \dot{x}_2\frac{\partial \alpha_{11}}{\partial x_2} - \frac{\partial \alpha_{11}}{\partial \dot{x}_1}(a_1\dot{x}_1 + c_1x_1 + d_1x_2) = a_1\alpha_{11} \tag{41}$$

$$\frac{\partial \alpha_{22}}{\partial t} + \dot{x}_1 \frac{\partial \alpha_{22}}{\partial x_1} + \dot{x}_2 \frac{\partial \alpha_{22}}{\partial x_2} - \frac{\partial \alpha_{22}}{\partial \dot{x}_2}(b_2\dot{x}_2 + c_2x_1 + d_2x_2) = b_2\alpha_{22} \tag{42}$$

and Eq. (22) is now

$$\frac{\partial \alpha_{11}}{\partial x_2}(a_1\dot{x}_1 + c_1x_1 + d_1x_2) + \alpha_{11}d_1 = \frac{\partial \alpha_{22}}{\partial x_1}(b_2\dot{x}_2 + c_2x_1 + d_2x_2) + \alpha_{22}c_2 \tag{43}$$

However, $\alpha_{11} = \alpha_{11}(t, x_1, x_2, \dot{x}_1)$ and $\alpha_{22} = \alpha_{22}(t, x_1, x_2, \dot{x}_2)$ according to Eqs. (17) and (18). Hence, taking the partial derivative with respect to $\dot{x}_2$ on both sides of Eq. (41), we find that $\partial\alpha_{11}/\partial x_2 = 0$. Using a similar argument for Eq. (42), we find that $\partial\alpha_{22}/\partial x_1 = 0$, so that

$$\alpha_{11} = \alpha_{11}(t, x_1, \dot{x}_1), \quad \alpha_{22} = \alpha_{22}(t, x_2, \dot{x}_2) \tag{44}$$

and Eqs. (41) and (42) become

$$\frac{\partial \alpha_{11}}{\partial t} + \dot{x}_1 \frac{\partial \alpha_{11}}{\partial x_1} - \frac{\partial \alpha_{11}}{\partial \dot{x}_1}(a_1\dot{x}_1 + c_1x_1 + d_1x_2) = a_1\alpha_{11} \tag{45}$$

$$\frac{\partial \alpha_{22}}{\partial t} + \dot{x}_2 \frac{\partial \alpha_{22}}{\partial x_2} - \frac{\partial \alpha_{22}}{\partial \dot{x}_2}(b_2\dot{x}_2 + c_2x_1 + d_2x_2) = b_2\alpha_{22} \tag{46}$$

Having the conditions in Eq. (44), we can simplify Eq. (43) further as

$$\alpha_{11}d_1 = \alpha_{22}c_2 \tag{47}$$

where $\alpha_{11} = \alpha_{11}(t, x_1, \dot{x}_1)$ and $\alpha_{22} = \alpha_{22}(t, x_2, \dot{x}_2)$, from Eq. (44). Now let us consider two cases.

First, if $c_2 \neq 0$ and $d_1 \neq 0$, then the left hand side of Eq. (47) is a function of t, x_1, and $\dot{x}_1$, whereas the right hand side is a function of t, x_2, and $\dot{x}_2$. Hence, α_{11} and α_{22} can be functions only of time t, i.e., $\alpha_{11} = \alpha_{11}(t)$ and $\alpha_{22} = \alpha_{22}(t)$ and, accordingly, Eqs. (45) and (46) become

$$\frac{\partial \alpha_{11}}{\partial t} = a_1\alpha_{11} \tag{48}$$

$$\frac{\partial \alpha_{22}}{\partial t} = b_2\alpha_{22} \tag{49}$$

whose general solutions are, respectively,

$$\alpha_{11}(t) = \gamma_1 e^{a_1 t} \tag{50}$$

$$\alpha_{22}(t) = \gamma_2 e^{b_2 t} \tag{51}$$

where γ_1 and γ_2 are arbitrary nonzero integration constants. With Eqs. (50) and (51), Eq. (47) now reads as

$$d_1\gamma_1 e^{a_1 t} = c_2\gamma_2 e^{b_2 t} \tag{52}$$

which is required to hold for all $t \geq 0$, so that

$$a_1 = b_2, \quad \gamma_2 = \frac{d_1}{c_2}\gamma_1, \quad c_2 \neq 0, \quad d_1 \neq 0 \tag{53}$$

We thus can obtain a second possible case for the existence of a Lagrangian

$$\text{Case II: } \alpha_{11} = \gamma_1 e^{a_1 t}, \quad \alpha_{22} = \frac{d_1}{c_2}\gamma_1 e^{a_1 t}, \quad a_2 = 0, \quad b_1 = 0, \quad a_1 = b_2, \quad c_2 \neq 0, \quad d_1 \neq 0, \quad \gamma_1 \neq 0 \tag{54}$$

Finally, if $c_2 = d_1 = 0$ in Eq. (47), then from Eqs. (8*a*) and (8*b*), the equations of motion are now

$$f^1 = \ddot{x}_1 + a_1\dot{x}_1 + c_1x_1 = 0 \tag{55a}$$

$$f^2 = \ddot{x}_2 + b_2\dot{x}_2 + d_2x_2 = 0 \tag{55b}$$

and the system is not coupled anymore. Each of these uncoupled systems has the form given in Eq. (1) for which a Lagrangian and the corresponding α_{ii}'s $(i = 1, 2)$ are well-known and given in Eq. (2). Thus, this case is summarized as

$$\text{Case III: } \alpha_{11} = \gamma_1 e^{a_1 t}, \quad \alpha_{22} = \gamma_2 e^{b_2 t}, \quad a_2 = 0, \quad b_1 = 0, \quad c_2 = 0, \quad d_1 = 0, \quad \gamma_1 \neq 0, \quad \gamma_2 \neq 0 \tag{56}$$

and the 2-DOF system is uncoupled.

Table 1 summarizes the three cases of Eqs. (39), (54), and (56). Additionally, corresponding to each case, it includes one Lagrangian, which shall be obtained later. It is to be noted that there are many other possible Lagrangians which are different from the ones given in Table 1.

As a simple application, let us consider a general 2-DOF mass-spring-damper system, shown in Fig. 2. Its equations of motion are easily obtained as

$$\ddot{x}_1 + \frac{\tilde{b}_1 + \tilde{b}_2}{m_1}\dot{x}_1 - \frac{\tilde{b}_2}{m_1}\dot{x}_2 + \frac{k_1 + k_2}{m_1}x_1 - \frac{k_2}{m_1}x_2 = 0 \tag{57}$$

$$\ddot{x}_2 - \frac{\tilde{b}_2}{m_2}\dot{x}_1 + \frac{\tilde{b}_2 + \tilde{b}_3}{m_2}\dot{x}_2 - \frac{k_2}{m_2}x_1 + \frac{k_2 + k_3}{m_2}x_2 = 0 \tag{58}$$

and each equation has been normalized by each mass for the correspondence with the form given in Eqs. (8*a*) and (8*b*). The aim is to find the conditions under which a Lagrangian would exist for this system. First, by applying Case I given in Eq. (39) or Table 1, we obtain the following conditions

Table 1 Three cases when a Lagrangian function of the system described by Eqs. (8*a*) and (8*b*) exists and a corresponding Lagrangian for each case

Case	Conditions	Lagrangian
I	$\alpha_{11} = \gamma_1 e^{a_1 t}, \alpha_{22} = -\frac{b_1}{a_2}\gamma_1 e^{a_1 t}, a_1 = b_2,$ $a_2 d_1 + b_1 c_2 = a_1 a_2 b_1, a_2 \neq 0, b_1 \neq 0, \gamma_1 \neq 0$	$L = e^{a_1 t}\left(\frac{1}{2}\dot{x}_1^2 - \frac{b_1}{2a_2}\dot{x}_2^2 - \frac{c_1}{2}x_1^2 + \frac{b_1 c_2}{a_2}x_1 x_2 + \frac{b_1 d_2}{2a_2}x_2^2 + b_1\dot{x}_1 x_2\right)$
II	$\alpha_{11} = \gamma_1 e^{a_1 t}, \alpha_{22} = \frac{d_1}{c_2}\gamma_1 e^{a_1 t}, a_2 = 0, b_1 = 0,$ $a_1 = b_2, c_2 \neq 0, d_1 \neq 0, \gamma_1 \neq 0$	$L = e^{a_1 t}\left(\frac{1}{2}\dot{x}_1^2 + \frac{d_1}{2c_2}\dot{x}_2^2 - \frac{c_1}{2}x_1^2 - d_1 x_1 x_2 - \frac{d_1 d_2}{2c_2}x_2^2\right)$
III	$\alpha_{11} = \gamma_1 e^{a_1 t}, \alpha_{22} = \gamma_2 e^{b_2 t}, a_2 = 0, b_1 = 0,$ $c_2 = 0, d_1 = 0, \gamma_1 \neq 0, \gamma_2 \neq 0$ (uncoupled)	$L = e^{a_1 t}\left(\frac{1}{2}\dot{x}_1^2 - \frac{1}{2}c_1 x_1^2\right) + e^{b_2 t}\left(\frac{1}{2}\dot{x}_2^2 - \frac{1}{2}d_2 x_2^2\right)$

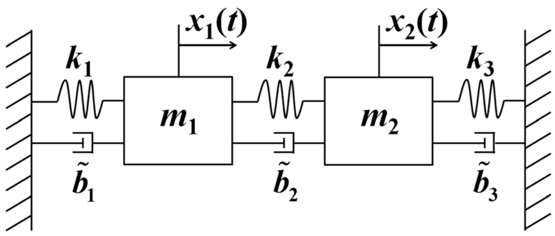

Fig. 2 General 2-DOF mass-spring-damper system

$$\frac{\tilde{b}_1+\tilde{b}_2}{m_1}=\frac{\tilde{b}_2+\tilde{b}_3}{m_2},\quad k_2=\tilde{b}_2\frac{\tilde{b}_1+\tilde{b}_2}{2m_1},\quad \tilde{b}_2\neq 0 \tag{59}$$

Moreover, Eq. (59) can be further divided into two cases whether the two masses m_1 and m_2 are equal or not

$$\tilde{b}_2=\frac{m_1m_2}{m_2-m_1}\left(\frac{\tilde{b}_3}{m_2}-\frac{\tilde{b}_1}{m_1}\right),\quad k_2=\tilde{b}_2\frac{\tilde{b}_1+\tilde{b}_2}{2m_1},\quad \tilde{b}_2\neq 0,$$
$$m_1\neq m_2,\quad \text{and}\quad \tilde{b}_1,\tilde{b}_3\quad k_1,\quad k_3\quad \text{are arbitrary} \tag{60a}$$

$$\tilde{b}_1=\tilde{b}_3,\quad k_2=\tilde{b}_2\frac{\tilde{b}_1+\tilde{b}_2}{2m}=\tilde{b}_2\frac{\tilde{b}_3+\tilde{b}_2}{2m},\quad \tilde{b}_2\neq 0,$$
$$m_1=m_2=m,\quad \text{and}\quad \tilde{b}_2,\quad k_1,\quad k_3\quad \text{are arbitrary} \tag{60b}$$

Next, we apply Case II, given by Eq. (54) or Table 1, to obtain the conditions

$$\tilde{b}_2=0,\quad \frac{\tilde{b}_1}{m_1}=\frac{\tilde{b}_3}{m_2},\quad k_2\neq 0,\quad \text{and}$$
$$m_1,\quad m_2,\quad k_1,\quad k_3\quad \text{are arbitrary} \tag{61}$$

It is straightforward to check that the mechanical system shown in Fig. 1 falls into this case.

Finally, we apply Case III, given by Eq. (56) (see Table 1), to have the conditions

$$\tilde{b}_2=0,\quad k_2=0,\quad \text{and}\quad m_1,\, m_2,\, \tilde{b}_1,\, \tilde{b}_3,\, k_1,\, k_3\quad \text{are arbitrary} \tag{62}$$

In this case, the system is uncoupled.

In brief, if a given 2-DOF mass-spring-damper system described by Fig. 2 does not satisfy at least one condition given by Eqs. (60)–(62), then it appears to be not possible to find a Lagrangian so that the Euler–Lagrange equation $(d/dt)(\partial L/\partial \dot{x}_i)-(\partial L/\partial x_i)=0$ $(i=1,2)$ yields an ordered direct analytic representation of the equation of motion of the system. In particular, from Eq. (60) we see that when the two masses m_1 and m_2 (see Fig. 2) are connected *solely* by a damper, i.e., when $\tilde{b}_2\neq 0$ and $k_2=0$, there appears to be no Lagrangian $L(t,x_1,x_2,\dot{x}_1,\dot{x}_2)$ that provides an ordered direct analytic representation of the equations of motion of the system described by Eqs. (57) and (58).

Until now we have derived the necessary and sufficient conditions for which there exists a Lagrangian function of the 2-DOF system given by Eqs. (8) and we have obtained three cases, Eqs. (39), (54), and (56), summarized in Table 1. We next address the question of finding a Lagrangian for each of these cases. Since a Lagrangian for Case III is already known in Refs. [2,3,11] (also in Eq. (2)), we focus on obtaining a Lagrangian function for Cases I and II.

First, for Case I, from Eq. (39), we know that the equations of motion are given by

$$e^{a_1t}\left(\ddot{x}_1+a_1\dot{x}_1+b_1\dot{x}_2+c_1x_1+\left(a_1b_1-\frac{b_1c_2}{a_2}\right)x_2\right)=0 \tag{63a}$$

$$-\frac{b_1}{a_2}e^{a_1t}(\ddot{x}_2+a_2\dot{x}_1+a_1\dot{x}_2+c_2x_1+d_2x_2)=0 \tag{63b}$$

where $\gamma_1=1$ is used. We next search for the conditions under which a Lagrangian $L(t,\mathbf{x},\dot{\mathbf{x}})$ exists, such that the corresponding Euler–Lagrange equations yield Eqs. (63a) and (63b), that is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)-\frac{\partial L}{\partial x_i}=\alpha_{ij}(t,\mathbf{x},\dot{\mathbf{x}})\ddot{x}_j+\beta_i(t,\mathbf{x},\dot{\mathbf{x}}) \tag{64}$$

Expanding the total time derivative, we have the following identities [7]

$$\frac{\partial^2 L}{\partial \dot{x}_i\partial \dot{x}_j}=\alpha_{ij} \tag{65a}$$

$$\frac{\partial^2 L}{\partial \dot{x}_i\partial x_j}\dot{x}_j+\frac{\partial^2 L}{\partial \dot{x}_i\partial t}-\frac{\partial L}{\partial x_i}=\beta_i \tag{65b}$$

where $i,j=1,2$. Knowing α_{ij} and β_i from Eqs. (39), (63), and (64), we obtain the following Lagrangian, using Eqs. (65a) and (65b)

$$L=e^{a_1t}\left(\frac{1}{2}\dot{x}_1^2-\frac{b_1}{2a_2}\dot{x}_2^2\right)+\dot{x}_1p(t,x_1,x_2)+\dot{x}_2q(t,x_1,x_2)$$
$$+r(t,x_1,x_2) \tag{66}$$

where $p(t,x_1,x_2)$, $q(t,x_1,x_2)$, and $r(t,x_1,x_2)$ are arbitrary functions of their arguments within the requirements that

$$\frac{\partial p}{\partial x_2}-\frac{\partial q}{\partial x_1}=b_1e^{a_1t},$$
$$\frac{\partial p}{\partial t}-\frac{\partial r}{\partial x_1}=e^{a_1t}\left(c_1x_1+\left(a_1b_1-\frac{b_1c_2}{a_2}\right)x_2\right),$$
$$\frac{\partial r}{\partial x_2}-\frac{\partial q}{\partial t}=\frac{b_1}{a_2}e^{a_1t}(c_2x_1+d_2x_2) \tag{67}$$

For example, if we choose

$$p(t,x_1,x_2)=b_1x_2e^{a_1t},\quad q(t,x_1,x_2)=0,$$
$$r(t,x_1,x_2)=e^{a_1t}\left(-\frac{c_1}{2}x_1^2+\frac{b_1c_2}{a_2}x_1x_2+\frac{b_1d_2}{2a_2}x_2^2\right) \tag{68}$$

then the Lagrangian in Eq. (66) becomes

$$L=e^{a_1t}\left(\frac{1}{2}\dot{x}_1^2-\frac{b_1}{2a_2}\dot{x}_2^2-\frac{c_1}{2}x_1^2+\frac{b_1c_2}{a_2}x_1x_2+\frac{b_1d_2}{2a_2}x_2^2+b_1\dot{x}_1x_2\right) \tag{69}$$

which is shown in Table 1.

For Case II, following the same procedure shown in the previous Case I, we can again obtain Lagrangian functions and one possible Lagrangian is

$$L=e^{a_1t}\left(\frac{1}{2}\dot{x}_1^2+\frac{d_1}{2c_2}\dot{x}_2^2\right)+\dot{x}_1u(t,x_1,x_2)+\dot{x}_2v(t,x_1,x_2)$$
$$+w(t,x_1,x_2) \tag{70}$$

where $u(t,x_1,x_2)$, $v(t,x_1,x_2)$, and $w(t,x_1,x_2)$ are arbitrary functions of their arguments within the requirements that

$$\frac{\partial u}{\partial x_2} = \frac{\partial v}{\partial x_1}, \quad \frac{\partial u}{\partial t} - \frac{\partial w}{\partial x_1} = e^{a_1 t}(c_1 x_1 + d_1 x_2),$$
$$\frac{\partial v}{\partial t} - \frac{\partial w}{\partial x_2} = e^{a_1 t}\left(d_1 x_1 + \frac{d_1 d_2}{c_2} x_2\right) \tag{71}$$

For example, if we choose

$$u(t, x_1, x_2) = 0, \quad v(t, x_1, x_2) = 0,$$
$$w(t, x_1, x_2) = -e^{a_1 t}\left(\frac{c_1}{2} x_1^2 + d_1 x_1 x_2 + \frac{d_1 d_2}{2c_2} x_2^2\right) \tag{72}$$

then the Lagrangian Eq. (70) becomes

$$L = e^{a_1 t}\left(\frac{1}{2}\dot{x}_1^2 + \frac{d_1}{2c_2}\dot{x}_2^2 - \frac{c_1}{2}x_1^2 - d_1 x_1 x_2 - \frac{d_1 d_2}{2c_2}x_2^2\right) \tag{73}$$

which is listed in Table 1. Comparing Eqs. (7*a*) and (7*b*), which are the equations of motion of the system shown in Fig. 1, with Case II, we see that

$$a_2 = 0, \quad b_1 = 0, \quad a_1 = b_2 = 2, \quad c_2 = -\frac{k}{2m},$$
$$d_1 = -\frac{k}{m}, \quad \gamma_1 = 1, \quad c_1 = \frac{2k}{m}, \quad d_2 = \frac{k}{m} \tag{74}$$

and the Lagrangian given in Eq. (73) then reduces to the one given in Eq. (6). The obtained Lagrangians are summarized in Table 1.

3 Lagrangians for Special Constant Coefficient Multi-Degree-of-Freedom Linear Systems

If the number of degrees of freedom of a system is much greater than two, then solving the Helmholtz conditions becomes quite complex and, hence obtaining Lagrangians for ordered direct analytic representations of *n*-degree-of-freedom systems by solving these conditions becomes, in general, extremely difficult, if not nearly impossible. In this section, we therefore use ideas from SDOF and 2-DOF systems to expand our thinking to MDOF systems and, hence, altogether bypass the need for solving the Helmholtz conditions. For two-degree-of-freedom systems, we used the Lagrangian given by Eq. (4), which we now generalize for an MDOF system as

$$L = e^{\gamma t}\left(\frac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{K}\mathbf{x} - \mathbf{x}^{\mathbf{T}}\mathbf{B}\dot{\mathbf{x}}\right) \tag{75}$$

where $\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]^{\mathbf{T}}$ is a generalized displacement *n*-vector, and $\mathbf{M}$ and $\mathbf{K}$ are the *n* by *n* symmetric mass and stiffness matrices, respectively. In addition, the scalar γ and the *n* by *n* matrix $\mathbf{B}$ are determined according to the requirements of the problem, as shall be shown later. Then the equation of motion obtained via the Euler–Lagrange equation is

$$\mathbf{M}\ddot{\mathbf{x}} + \left(\gamma\mathbf{M} + \mathbf{B} - \mathbf{B}^{\mathbf{T}}\right)\dot{\mathbf{x}} + \left(\mathbf{K} - \gamma\mathbf{B}^{\mathbf{T}}\right)\mathbf{x} = \mathbf{0} \tag{76}$$

First, if we choose $\gamma = 0$ and $\mathbf{B}^{\mathbf{T}} = -\mathbf{B}$ (skew-symmetric), Eq. (76) becomes

$$\mathbf{M}\ddot{\mathbf{x}} + 2\mathbf{B}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{77}$$

which is the general equation of motion for an *n*-degree-of-freedom linear mechanical system with *gyroscopic* damping. The corresponding Lagrangian is

$$L = \frac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{K}\mathbf{x} - \mathbf{x}^{\mathbf{T}}\mathbf{B}\dot{\mathbf{x}} = \frac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{K}\mathbf{x} + \dot{\mathbf{x}}^{\mathbf{T}}\mathbf{B}\mathbf{x} \tag{78}$$

where $\mathbf{M}$ and $\mathbf{K}$ are symmetric matrices and $\mathbf{B}$ is a skew-symmetric matrix. We note that since $\gamma = 0$, the Lagrangian is a physical Lagrangian (see the remarks that follow Eq. (3)).

As previously stated, Eq. (77) has a skew-symmetric damping matrix $\mathbf{B}$. However, in many practical applications and, especially in the theory of linear vibrations, the equations of motion have symmetric stiffness and damping matrices. In order to encompass such systems, we choose $\gamma \neq 0$ and $\mathbf{B}^{\mathbf{T}} = -\mathbf{B}$. Then, with the Lagrangian given in Eq. (75), the Euler–Lagrange equation yields

$$\mathbf{M}\ddot{\mathbf{x}} + (\gamma\mathbf{M} + 2\mathbf{B})\dot{\mathbf{x}} + (\gamma\mathbf{B} + \mathbf{K})\mathbf{x} = \mathbf{0} \tag{79}$$

When the skew-symmetric matrix $\mathbf{B} = \mathbf{0}$, Eq. (79) reduces to the equation

$$\mathbf{M}\ddot{\mathbf{x}} + \gamma\mathbf{M}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{80}$$

which describes a proportionally damped system. The Lagrangian from which this equation is obtainable is simply given, using Eq. (75), by

$$L = e^{\gamma t}\left(\frac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{K}\mathbf{x}\right) \tag{81}$$

for any matrix $\mathbf{M} > \mathbf{0}$ and any symmetric matrix $\mathbf{K}$. Having disposed of the case $\mathbf{B} = \mathbf{0}$, from this point on we shall then concentrate on the case when the skew-symmetric matrix is $\mathbf{B} \neq \mathbf{0}$.

We would thus want the matrices $\gamma\mathbf{M} + 2\mathbf{B}$ and $\gamma\mathbf{B} + \mathbf{K}$ in Eq. (79) to be symmetric, where $\mathbf{B}(\neq \mathbf{0})$ is a skew-symmetric matrix. The required conditions are not obvious, therefore, let us consider a 3-DOF system, which, by extension, will help us to adduce the general procedure for handling linearly damped MDOF systems. We start by considering diagonal mass matrices. If we have

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix} \tag{82}$$

then Eq. (79) becomes

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} \\ -2b_{12} & \gamma m_2 & 2b_{23} \\ -2b_{13} & -2b_{23} & \gamma m_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$+ \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} \\ -\gamma b_{12} & k_2 & \gamma b_{23} \\ -\gamma b_{13} & -\gamma b_{23} & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{83}$$

and the damping and stiffness matrices of this system are not symmetric. However, noting the negative sign in the term that involves $\dot{x}_2^2$ in Eq. (69), if we choose to use the same $\mathbf{B}$ matrix as in Eq. (82) and

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & -m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & -k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \tag{84}$$

in our Lagrangian given in Eq. (75), then the equations of motion that we obtain are

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & -m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} \\ -2b_{12} & -\gamma m_2 & 2b_{23} \\ -2b_{13} & -2b_{23} & \gamma m_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} \\ -\gamma b_{12} & -k_2 & \gamma b_{23} \\ -\gamma b_{13} & -\gamma b_{23} & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (85)$$

or, equivalently

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} \\ 2b_{12} & \gamma m_2 & -2b_{23} \\ -2b_{13} & -2b_{23} & \gamma m_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} \\ \gamma b_{12} & k_2 & -\gamma b_{23} \\ -\gamma b_{13} & -\gamma b_{23} & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (86)$$

in which the damping and stiffness matrices are now both symmetric if we set $b_{13} = 0$.

We now generalize this observation to general n-DOF systems. Our aim is to find the appropriate matrices **M**, **K**, and **B** which, when inserted into the Lagrangian given in Eq. (75), will result in the Euler–Lagrange equation given in Eq. (79) that is of the form

$$\bar{\mathbf{M}}\ddot{\mathbf{x}} + \bar{\mathbf{B}}\dot{\mathbf{x}} + \bar{\mathbf{K}}\mathbf{x} = \mathbf{0} \quad (87)$$

where $\bar{\mathbf{M}}$ is a positive definite diagonal matrix and the matrices $\bar{\mathbf{B}}$ and $\bar{\mathbf{K}}$ are both symmetric matrices, such as in Eq. (86).

Consider the diagonal matrices $\mathbf{M} = \text{diag}(m_1, m_2, \ldots, m_n)$ and $\mathbf{K} = \text{diag}(k_1, k_2, \ldots, k_n)$, where $n \geq 2$. We propose to: (i) change the signs of some of the elements of these matrices, and (ii) provide a procedure to make the matrix $\bar{\mathbf{M}}$ positive definite and the matrices $\bar{\mathbf{B}}$ and $\bar{\mathbf{K}}$ symmetric. We do this in the following manner. If we place negative signs on m_i, m_j, $m_k, \ldots, k_i$, k_j, $k_k, \ldots$ $(i, j, k, \ldots = 1, 2, \ldots, n$, and $i < j < k < \cdots)$, then the elements of the skew-symmetric matrix **B**, which is given by

$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1n} \\ -b_{12} & 0 & b_{23} & \cdots & b_{2n} \\ -b_{13} & -b_{23} & 0 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{1n} & -b_{2n} & -b_{3n} & \cdots & 0 \end{bmatrix} \quad (88)$$

should have its elements altered by the following rule:

(1) the elements are set so that $b_{ij} = 0$, $b_{ik} = 0$, $b_{jk} = 0$, … and
(2) after deleting the ith, jth, kth, … rows and ith, jth, kth, … columns of the **B** matrix in Eq. (88), the remaining elements of **B** are set to zero

Clearly, if we want to place only one negative sign, say, on m_i and k_i $(i = 1, 2, \ldots, n)$, then only the second rule (2) applies, since **B** is skew-symmetric. Additionally, changing the signs of all the m_i's and k_i's $(i = 1, 2, \ldots, n)$, i.e., $m_1, m_2, \ldots, m_n$ and $k_1, k_2, \ldots, k_n$, will result in $\mathbf{B} = \mathbf{0}$, which is a case already considered in Eqs. (80) and (81), although in a more general manner.

For example, in a 4-DOF system, if we have

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & -m_2 & 0 & 0 \\ 0 & 0 & -m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \\ 0 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{bmatrix} \quad (89)$$

that is, $i = 2$ and $j = 3$ in this case, then we should choose the elements of the **B** matrix by the rule:

(1) $b_{23} = 0$, and
(2) after we delete the 2nd and 3rd rows and columns, the remaining elements should be set to zero, i.e., set $b_{14} = 0$

In brief, we should choose the following **B** matrix

$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 \\ -b_{12} & 0 & 0 & b_{24} \\ -b_{13} & 0 & 0 & b_{34} \\ 0 & -b_{24} & -b_{34} & 0 \end{bmatrix} \quad (90)$$

Now using the Lagrangian given by Eq. (75), with **M** and **K** defined in Eq. (89) and **B** defined in Eq. (90), the equations of motion given by Eq. (79) become

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & -m_2 & 0 & 0 \\ 0 & 0 & -m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} & 0 \\ -2b_{12} & -\gamma m_2 & 0 & 2b_{24} \\ -2b_{13} & 0 & -\gamma m_3 & 2b_{34} \\ 0 & -2b_{24} & -2b_{34} & \gamma m_4 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} & 0 \\ -\gamma b_{12} & -k_2 & 0 & \gamma b_{24} \\ -\gamma b_{13} & 0 & -k_3 & \gamma b_{34} \\ 0 & -\gamma b_{24} & -\gamma b_{34} & k_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (91)$$

or

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} & 0 \\ 2b_{12} & \gamma m_2 & 0 & -2b_{24} \\ 2b_{13} & 0 & \gamma m_3 & -2b_{34} \\ 0 & -2b_{24} & -2b_{34} & \gamma m_4 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} & 0 \\ \gamma b_{12} & k_2 & 0 & -\gamma b_{24} \\ \gamma b_{13} & 0 & k_3 & -\gamma b_{34} \\ 0 & -\gamma b_{24} & -\gamma b_{34} & k_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (92)$$

and Eq. (92) has symmetric damping and stiffness matrices along with a positive definite mass matrix. The corresponding Lagrangian is given by Eq. (75).

More generally, when placing negative signs on m_i, m_j, $m_k, \ldots,$ k_i, k_j, $k_k, \ldots,$ (and following the procedure described earlier) we have 2^n different choices, each of which yields the corresponding equations of motion in the form of Eq. (87). However, by symmetry, half of them are equivalent. We can extend this symmetry to the situation where we consider placing *no* negative signs on any of the $m_i, k_i, i = 1, 2, \ldots, n$ (and following the procedure) to be equivalent to placing negative signs on *every* $m_i, k_i, i = 1, 2, \ldots, n$, which yields, as previously mentioned $\mathbf{B} = \mathbf{0}$. Due to this symmetry, we therefore have 2^{n-1} differently 'structured' systems of equations of motion of the form of Eq. (87) (each with differently structured matrices $\bar{\mathbf{B}}$ and $\bar{\mathbf{K}}$) for which the corresponding Lagrangians can be obtained by using Eq. (75). Furthermore, for each system structure one can use any (permissible) parameter values.

Thus, for large n, one generates Lagrangians for numerous ordered direct representations of different n-DOF damped linear systems.

As a special case, in an n-DOF system ($n \geq 2$), if one chooses

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -m_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -m_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{n+1} m_n \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -k_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & k_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{n+1} k_n \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & b_1 & 0 & 0 & \cdots & 0 \\ -b_1 & 0 & b_2 & 0 & \cdots & 0 \\ 0 & -b_2 & 0 & b_3 & \cdots & 0 \\ 0 & 0 & -b_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{93}$$

using the matrices given by Eq. (93) in the Lagrangian given by Eq. (75), the Euler–Lagrange equation of motion given by Eq. (79) yields

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & m_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & m_n \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \vdots \\ \ddot{x}_n \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_1 & 0 & 0 & \cdots & 0 \\ 2b_1 & \gamma m_2 & -2b_2 & 0 & \cdots & 0 \\ 0 & -2b_2 & \gamma m_3 & 2b_3 & \cdots & 0 \\ 0 & 0 & 2b_3 & \gamma m_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma m_n \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_1 & 0 & 0 & \cdots & 0 \\ \gamma b_1 & k_2 & -\gamma b_2 & 0 & \cdots & 0 \\ 0 & -\gamma b_2 & k_3 & \gamma b_3 & \cdots & 0 \\ 0 & 0 & \gamma b_3 & k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{94}$$

which is the equation for an MDOF system with *tridiagonal* symmetric damping and stiffness matrices of the form of that in Eq. (87). It should be noted that because the Lagrangian has negative quantities in the mass matrix, it is not a physical Lagrangian. Yet, it can give the equations of motion of certain systems consisting of a chain of masses in which each mass is connected to its neighbors by linear dashpots and springs. The strength of the procedure introduced in this section is that it totally bypasses the Helmholtz conditions, which are near-impossible to solve for arbitrarily large finite values of n.

To exemplify what has been discussed thus far, we consider a 3-DOF system described by Eq. (86) with $b_{13} = 0$, where $\mathbf{M}$, $\mathbf{B}$, and $\mathbf{K}$ are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & -m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & b_{12} & 0 \\ -b_{12} & 0 & b_{23} \\ 0 & -b_{23} & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & -k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \tag{95}$$

Using the matrices given in Eq. (95) in the Lagrangian function given by Eq. (75), the resulting Euler–Lagrange equations of motion are given by Eq. (86), which is of the form of Eq. (87), where

$$\bar{\mathbf{M}} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} \gamma m_1 & 2b_{12} & 0 \\ 2b_{12} & \gamma m_2 & -2b_{23} \\ 0 & -2b_{23} & \gamma m_3 \end{bmatrix}, \quad \bar{\mathbf{K}} = \begin{bmatrix} k_1 & \gamma b_{12} & 0 \\ \gamma b_{12} & k_2 & -\gamma b_{23} \\ 0 & -\gamma b_{23} & k_3 \end{bmatrix} \tag{96}$$

In the theory of linear vibrations, one often considers proportionally damped systems. One can then particularize the damping matrix $\bar{\mathbf{B}}$ of this system to have proportional damping so that

$$\bar{\mathbf{B}} = \alpha \bar{\mathbf{M}} + \beta \bar{\mathbf{K}} \tag{97}$$

where α and β are some constants. Substituting Eq. (96) into Eq. (97), we have the following relationships

$$\alpha = \gamma - \frac{2}{\gamma}\frac{k_i}{m_i}, \quad \beta = \frac{2}{\gamma}, \quad i = 1, 2, 3 \tag{98}$$

that is, for proportional damping to be possible, each ratio k_i/m_i (see Eq. (95)) should be the same. It is straightforward to verify that Eq. (98) still holds for general n-DOF ($i = 1, 2, \ldots, n$) tridiagonal systems whose equations of motion are described by Eq. (94). Now, consider a mass-spring-damper system of the type shown in Fig. 3 that commonly appears in the theory of linear vibrations.

The equations of motion of the system are

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + b \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{99}$$

with $b \neq 0$. (When $b = 0$, the forces can be derived from a potential and a Lagrangian for the system can be easily found.) We note the tridiagonal form of the damping and stiffness matrices and Eq. (99) describes a proportionally damped system with $\alpha = 0$ and $\beta = b/k$. For this system to be described by the Lagrangian given in Eq. (75), in which the matrices $\mathbf{M}$, $\mathbf{B}$, and $\mathbf{K}$ are specified

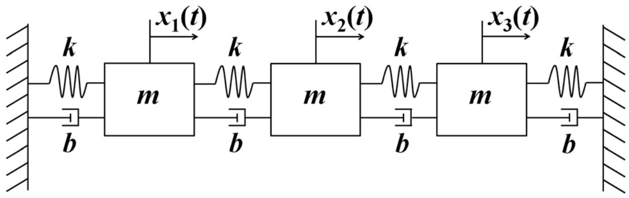

Fig. 3 General 3-DOF mass-spring-damper system

by Eq. (95), we require α and β to also satisfy the conditions given in Eq. (98). Hence $\gamma = 2k/b$ and noting that $m_i = m$ and $k_i = 2k$ $(i = 1, 2, 3)$, we require

$$\frac{k}{b} = \frac{b}{m} \tag{100}$$

The application of this result to n-degree-of-freedom tridiagonal systems easily follows by extension. In fact, the condition Eq. (100), along with $\gamma = 2k/b$, should also be met for general n-DOF tridiagonal systems of the type given in Eq. (99) to be proportionally damped.

To illustrate this result, we apply it to a proportionally damped system described by Eq. (99) with the parameters $m = 0.64$ kg, $b = 0.8$ Ns/m, and $k = 1$ N/m. Noting that Eq. (100) holds, we find

$$\gamma = \frac{2k}{b} = 2.5\ \text{s}^{-1}, \quad \alpha = 0\ \text{s}^{-1}, \quad \beta = \frac{b}{k} = 0.8\ \text{s}, \\ m_1 = m_2 = m_3 = m = 0.64\ \text{kg} \tag{101a}$$

$$b_{12} = -\frac{b}{2} = -0.4\ \text{Ns/m}, \quad b_{23} = \frac{b}{2} = 0.4\ \text{Ns/m}, \\ k_1 = k_2 = k_3 = 2k = 2\ \text{N/m} \tag{101b}$$

by comparing Eq. (99) with Eqs. (87) and (96). The Lagrangian for this dissipative 3-DOF system given by

$$\begin{aligned} L &= e^{\gamma t}\left(\frac{1}{2}\dot{\mathbf{x}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x} - \mathbf{x}^{\mathrm{T}}\mathbf{B}\dot{\mathbf{x}}\right) \\ &= e^{2.5t}\left[0.32\left(\dot{x}_1^2 - \dot{x}_2^2 + \dot{x}_3^2\right) - \left(x_1^2 - x_2^2 + x_3^2\right)\right. \\ &\quad \left. + 0.4(x_1\dot{x}_2 + \dot{x}_2 x_3 - \dot{x}_1 x_2 - x_2\dot{x}_3)\right] \end{aligned} \tag{102}$$

then yields the equations of motion given in Eq. (99) for the specific values of m, k, and b, chosen in this example.

As the last application of the results obtained thus far, we propose a Jacobi integral that is conserved at all times for the types of linear multi-degree-of-freedom systems considered here (see Eq. (79)). When the Lagrangian does not explicitly contain time (and the actual velocity is a virtual velocity), i.e., when $\partial L(t, \mathbf{q}, \dot{\mathbf{q}})/\partial t = 0$, the Jacobi integral I given by Ref. [12]

$$I := \dot{\mathbf{q}}^{\mathrm{T}}\frac{\partial L}{\partial \dot{\mathbf{q}}} - L \tag{103}$$

is conserved. The Lagrangian in Eq. (75), however, explicitly contains time, however, by using the transformation

$$\mathbf{y} = \mathbf{x}e^{(\gamma/2)t} \tag{104}$$

it becomes

$$L = \frac{1}{2}\left(\dot{\mathbf{y}} - \frac{\gamma}{2}\mathbf{y}\right)^{\mathrm{T}}\mathbf{M}\left(\dot{\mathbf{y}} - \frac{\gamma}{2}\mathbf{y}\right) - \frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{K}\mathbf{y} - \mathbf{y}^{\mathrm{T}}\mathbf{B}\left(\dot{\mathbf{y}} - \frac{\gamma}{2}\mathbf{y}\right) \tag{105}$$

Hence, the Jacobi integral is given by Eq. (103) as

$$I = \frac{1}{2}\left(\dot{\mathbf{y}} + \frac{\gamma}{2}\mathbf{y}\right)^{\mathrm{T}}\mathbf{M}\left(\dot{\mathbf{y}} - \frac{\gamma}{2}\mathbf{y}\right) - \dot{\mathbf{y}}^{\mathrm{T}}\left(\mathbf{B} + \mathbf{B}^{\mathrm{T}}\right)\mathbf{y} + \frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{K}\mathbf{y} - \frac{\gamma}{2}\mathbf{y}^{\mathrm{T}}\mathbf{B}\mathbf{y} \tag{106}$$

and since the matrix $\mathbf{B}$ is assumed to be skew-symmetric in this paper, Eq. (106) simplifies to

$$I = \frac{1}{2}\left(\dot{\mathbf{y}} + \frac{\gamma}{2}\mathbf{y}\right)^{\mathrm{T}}\mathbf{M}\left(\dot{\mathbf{y}} - \frac{\gamma}{2}\mathbf{y}\right) + \frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{K}\mathbf{y} \tag{107}$$

Rewriting this Jacobi integral in terms of $\mathbf{x}$ and $\dot{\mathbf{x}}$ by using Eq. (104), we obtain the conservation law

$$I(t, \mathbf{x}, \dot{\mathbf{x}}) = e^{\gamma t}\left[\frac{1}{2}(\dot{\mathbf{x}} + \gamma\mathbf{x})^{\mathrm{T}}\mathbf{M}\dot{\mathbf{x}} + \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x}\right] = \text{constant} \tag{108}$$

Thus, for an MDOF linear system whose equation of motion is given by Eq. (79), it is guaranteed that the function $I(t, \mathbf{x}, \dot{\mathbf{x}})$ in Eq. (108) is conserved at all times. For example, corresponding to the Lagrangian given in Eq. (102) for the dissipative proportionally damped 3-DOF system, shown in Fig. 3, with the parameters given in Eqs. (101a) and (101b), we obtain the conservation law

$$\begin{aligned} I(t, \mathbf{x}, \dot{\mathbf{x}}) &= e^{2.5t}\left[0.32\left(\dot{x}_1^2 - \dot{x}_2^2 + \dot{x}_3^2\right) + 0.8(x_1\dot{x}_1 - x_2\dot{x}_2 + x_3\dot{x}_3)\right. \\ &\quad \left. + \left(x_1^2 - x_2^2 + x_3^2\right)\right] = \text{constant} \end{aligned} \tag{109}$$

4 Conclusions

Here we have discussed extensions of the inverse problem of the calculus of variations for nonpotential forces to multi-degree-of-freedom systems. For a two-degree-of-freedom linear system with linear damping, the conditions for the existence of a Lagrangian are explicitly obtained by solving the Helmholtz conditions. Three general cases for when such Lagrangians are guaranteed to exist are obtained, depending on the parameter values of the coupled linear systems. The Helmholtz conditions are near-impossible to solve for general n-degree-of-freedom systems and, although they are explicit, from a practical standpoint they provide little assistance in solving the inverse problem for such systems. By using and generalizing the results for single-degree-of-freedom systems, a simple procedure that does not require the use of the Helmholtz conditions and that is easily extended to n-degree-of-freedom linear systems is developed. We specifically include systems that commonly arise in the theory of linear vibrations—systems with positive definite mass matrices and symmetric stiffness and damping matrices. The method yields several new Lagrangians for linear multi-degree-of-freedom systems. Conservation laws for such dissipative MDOF systems are also obtained by finding the corresponding Jacobi integrals. Although the approach employed herein is simple and easy to apply to other examples, a more rigorous and systematic way for arriving at general solutions of the inverse problem for multi-degree-of-freedom systems needs much more work. At present, it remains an open problem. Our discussion herein is restricted to linear damping and more general forms of nonpotential forces will be considered in future work.

Acknowledgment

The authors appreciate the comments on a very early draft of this paper by Professor G. Leitmann at UC Berkeley.

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