Parallel Finite Element Assembly of High Order Whitney Forms

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***Abstract*—This paper presents an efficient method for the finite element assembly of high order Whitney elements. We start by reviewing the classical assembly technique and by highlighting the most time consuming part. This classical approach can be reformulated into a computationally efficient matrix – matrix product. We conclude by presenting numerical results on a wave guide problem.**

***Index Terms*—Finite element analysis, high order assembly, high performance computing, Whitney elements.**

I. Introduction

There is a growing consensus that state of the art finite element technology requires, and will continue to require, too extensive computational resources to provide the necessary resolution for complex high-frequency electromagnetic compatibility simulations, even at the rate of computational power increase. This leads us to consider methods with a higher order of grid convergence than the classical second order.

II. Classic finite element assembly

By applying the classical Galerkin finite element (FE) scheme with a curl-conforming basis, the solution of the time harmonic propagation of an electrical wave is computed using elementary integrals $\mathcal{T}_{i,j}^{e}$, as developed in [1]. Each $\mathcal{T}_{i,j}^{e}$ is giving the contribution of the degrees of freedom (DOF) i and j of the mesh element e.

The classical finite element assembly algorithm consists in iterating on every element. For a given element, the $\mathcal{T}_{i,j}^{e}$ terms are computed for every pair of DOF i and j. These values are assembled in the FE linear system matrix.

It is worth noticing that increasing the basis order will have two impacts on the computation time: *a*) each element will have more DOFs, thus increasing the number of $\mathcal{T}_{i,j}^{e}$ to compute; *b*) the numerical quadrature will require more points, thus slowing down the computation of each $\mathcal{T}_{i,j}^{e}$. These two phenomena will substantially increase the assembly time, as shown in Figure 1.

III. Efficient assembly

The key idea of a fast assembly procedure is to compute all the $\mathcal{T}_{i,j}^{e}$ terms with matrix – matrix products, as proposed by [2], [3] for standard nodal Lagrange finite elements. Indeed, this operation exhibits an excellent cache reuse, and is standardized in the Basic Linear Algebra Subprograms (BLAS) library, which can be highly optimized for modern multi-core architectures.

The $\mathcal{T}_{i,j}^{e}$ terms can be computed by the product of two matrices. The first matrix will be composed of the Jacobian matrices and potentially non linear terms. The second matrix will be composed only of the basis functions defined over the reference element, and is thus geometrically invariant.

It is worth noticing that depending on the mesh elements orientation, the curl-conforming basis functions cannot simply be reordered, as for classical H^1 Lagrange bases. This situation may be overcome by considering more than one reference element, as proposed by [4], and by taking transformations into account in the Jacobian matrices.

IV. Numerical results

Figure 1 presents the assembly times of the classical and efficient assembly procedures for an increasing basis order. The FE matrix is assembled for a propagation problem into a wave guide, meshed with 5585 curved tetrahedra. The tests were done on a Intel Core i7 960 using OpenBLAS with 4 threads. It is worth mentioning that the classical implementation also uses 4 threads for the assembly.

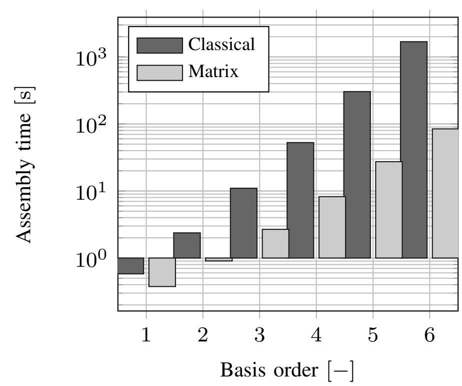

Fig. 1. Assembly time for the classical and fast procedures

It can be seen from Figure 1 that the matrix procedure is much faster than the classical one for high order interpolations. For instance, the speedup on an order 5 problem, with around 500000 unknowns, is around 11.

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