Spatial autocorrelation: robustness of measures and tests

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University of Liege

London, December 14, 2015

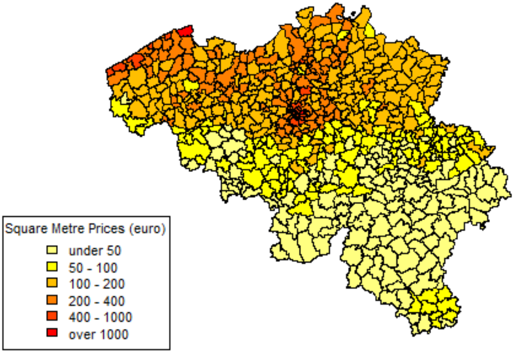
Spatial Data

Spatial data :

- geographical positions
- non spatial attributes

Example[a]

Mean price of land for sales (€/m^2) in Belgian municipalities



[a]Statistics from Belgian Federal Government - `http://statbel.fgov.be`

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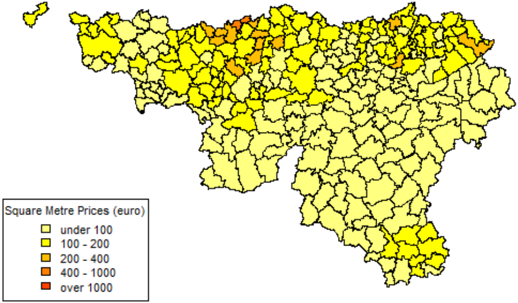

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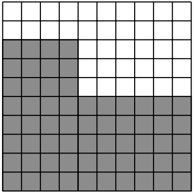
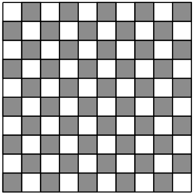
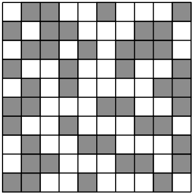
Spatial autocorrelation



Positive spatial autocorrelation



Negative spatial autocorrelation



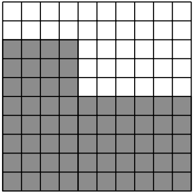
No spatial autocorrelation

Weighting matrix W

- Binary weights,
- Row-standardized,
- Globally standardized, . . .

Convention: zero diagonal and $S_0 = \sum_i \sum_j W_{ij}$

Spatial autocorrelation



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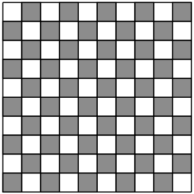

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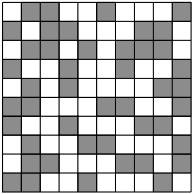

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The values $z_1, \ldots, z_n$ are observed at locations $s_1, \ldots, s_n$

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$$I(z) = \frac{n}{S_0} \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(z_i - \bar{z})(z_j - \bar{z})}{\sum_{i=1}^{n} (z_i - \bar{z})^2}$$

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Tests based on asymptotic normality

Without spatial autocorrelation, I, c and G are **asymptotically Gaussian** under normality (N) and/or randomisation (R) assumption.

Package spdep in R:

- `moran.test`: under N and R assumption
- `geary.test`: under N and R assumption
- `globalG.test`: under R assumption

Example

Statistic	Expected value	Variance	p-value
$I = 0.53$	-0.0038	0.0006	$< 2 \times 10^{-16}$
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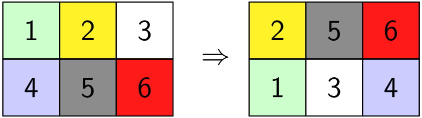

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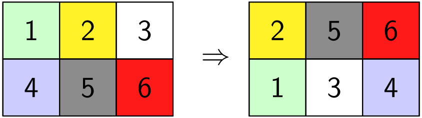

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1	2	3
4	5	6

$\Rightarrow$

2	5	6
1	3	4

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- **Benefits**: No assumption on the distribution
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Example (nsim=5000)

Statistic	Observed rank	p-value	Package `spdep` in R:
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Inference

Dray's test based on I

Decomposition of positive / negative influence on autocorrelation

$$I(z) = \underbrace{\sum_{I(u_k)<E[I]} I(u_k)cor^2(u_k, z)}_{S_I^-(z)} + \underbrace{\sum_{I(u_k)>E[I]} I(u_k)cor^2(u_k, z)}_{S_I^+(z)}$$

where u_k are eigenvector of HWH, $H = I_n - \frac{1}{n}\mathbb{1}\mathbb{1}'$.

- Simultaneous positive and negative spatial autocorrelation
- Permutation test

Example (nsim=5000)

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Moran's I	**Geary's c**	**Getis and Ord's G**
Test under R	Test under R	Test under R
Test under N	Test under N	
Permutation test	Permutation test	Permutation test
Dray's test		

Robustness of the tests

How contamination influences the result of the tests?

Impact on the decision of the test

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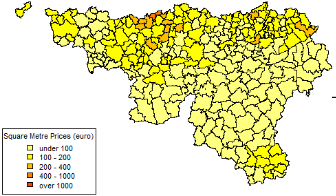

Test	p-value
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I Permutation	$< 2 \times 10^{-4}$
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c under R	$< 2 \times 10^{-16}$
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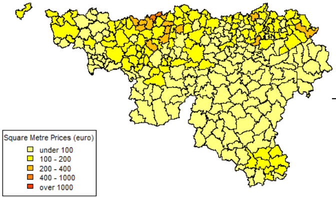

Test	Decision
I under R	RH_0
I under N	RH_0
I Permutation	RH_0
Dray test	RH_0
c under R	RH_0
c under N	RH_0
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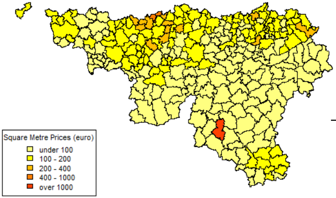

Paliseul: 690€/ m^2

Test	Decision
I under R	RH_0
I under N	RH_0
I Permutation	RH_0
Dray test	RH_0
c under R	RH_0
c under N	RH_0
c Permutation	FTR
G under R	RH_0
G Permutation	RH_0

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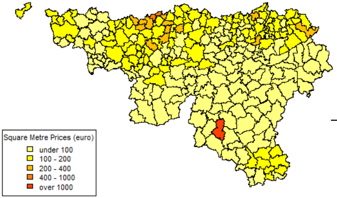

Paliseul: 730€/ m^2

Test	Decision
I under R	RH_0
I under N	RH_0
I Permutation	RH_0
Dray test	RH_0
c under R	FTR
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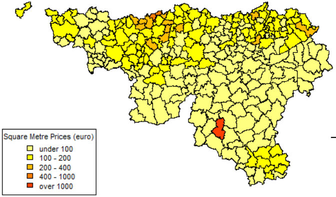

Paliseul $\approx$ max(Belgium) (1200€)

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I under N	RH_0
I Permutation	RH_0
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c under R	FTR
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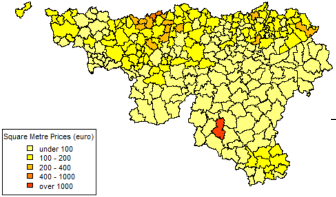

Paliseul: 1610€/ m^2

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I under N	FTR
I Permutation	FTR
Dray test	FTR
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Robustness of the tests

Resistance of a test (Ylvisaker, 1977)

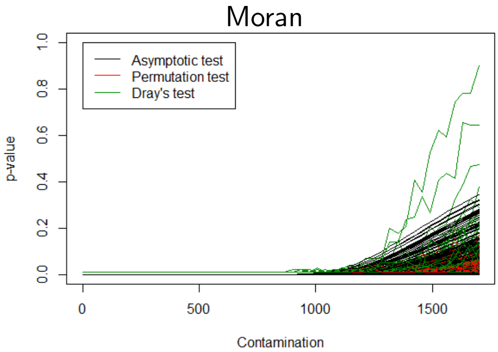
- **Resistance to acceptance**: smallest proportion of fixed observations which will always implies the *non rejection* of H_0.
- **Resistance to rejection**: smallest proportion m_0 of fixed observations which will always implies the *rejection* of H_0.

Result: Resistance to rejection of considered tests: $\frac{1}{n}$

Robustness of the tests

Impact on the p-value

Hairplot: $\xi \mapsto$ p-value$(z + \xi e_i)$ for $i = 1, \ldots, n$



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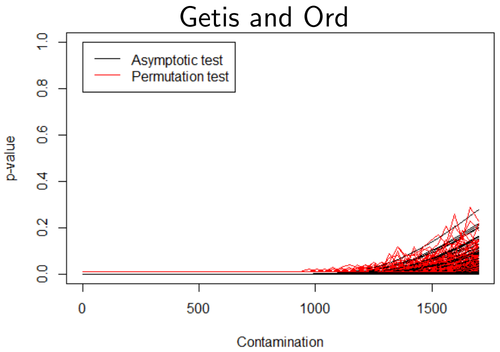

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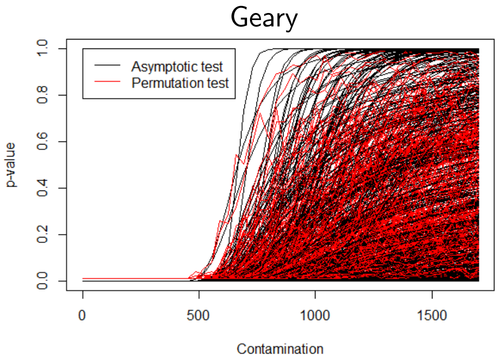

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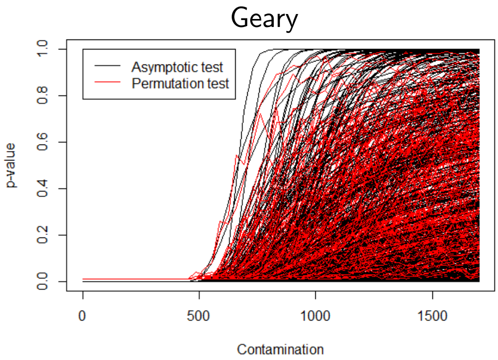
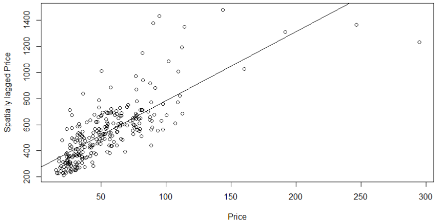


$\Rightarrow$ **robust alternative must be considered**

Moran scatterplot

Moran can be interpreted as the slope in a LS regression of Wz over z



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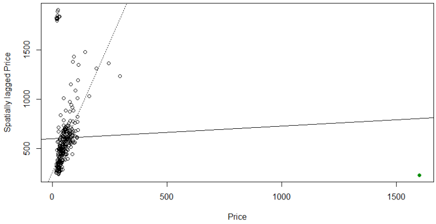

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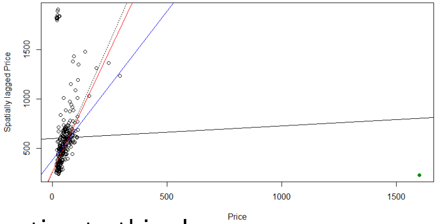

Idea: Robustly estimate this slope

Robust techniques

- Least Trimmed Squares (Rousseeuw, 1984)
- M-estimator (Huber, 1973)

Robust regression

Regression: $y = ax + b$ and residuals: $r_i = y_i - \hat{y}_i$

Least squares regression

Minimisation of $\sum_{i=1}^{n} r_i^2$

Least Trimmed Squares (Rousseeuw, 1984)

Minimisation of the trimmed sum: $\sum_{i=1}^{n-h} r_{(i)}^2$

M-estimator (Huber, 1973)

Minimisation of another function: $\sum_{i=1}^{n} \rho\left(\frac{r_i}{\hat{\sigma}}\right)$
e.g., $\rho(r) = r^2$, $\rho(r) = |r|$

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Robust inference based on Moran

Impact on the decision of the test: resistance

Linked to the Breakdown point of the robust regression

Least Trimmed Squares	M-estimator
h/n	≤ 0.5

Impact on the p-value: hairplot

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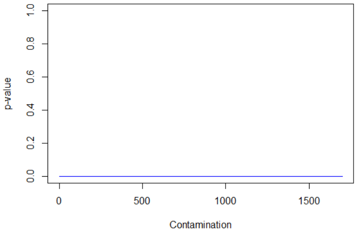

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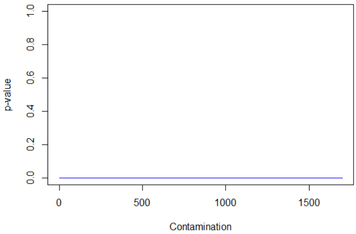

What about level and power?

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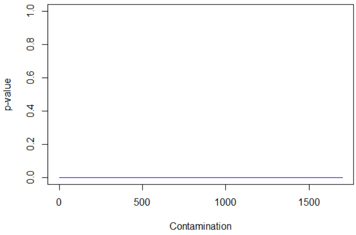
	h/n	≤ 0.5

Impact on the p-value: hairplot



What about level and power? $\Rightarrow$ simulations

Simulations

Model

The Spatial autoregressive (SAR) model: $Z = \rho W Z + \varepsilon$

Simulation settings (Holmberg and Lundevaller, 2015)

- Spatial domain: grid ($n = 100, 400, 900, 1600$)
- Weighting matrix: queen and rook contiguity

- Spatial correlation coefficient: $\rho = 0, 0.1, 0.2, 0.3$
- Distribution of ε: $N(0, 1)$, $\mathcal{P}(1)$, Bern(0.5)

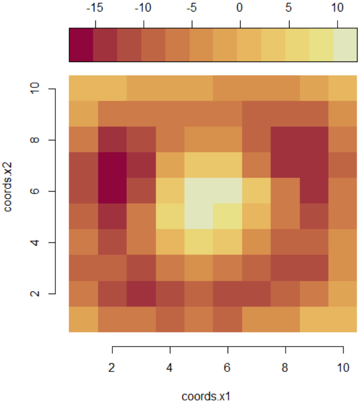
Contamination process

A number δ of observations are contaminated

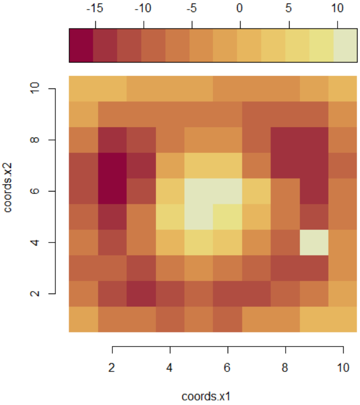
Example

$n = 100$, queen contiguity, $\rho = 0.2$ and $\varepsilon \sim N(0, 1)$

$\delta = 0$



$\delta = 1$



Comparison

1. Without contamination ($\delta = 0$)

Similar results are expected for classical and robust tests

2. With contamination ($\delta > 0$)

Robust tests should resist to contamination and be powerful

Results ($\delta = 0$)

Robust tests vs initial tests

Good correlation between the p-value of robust and classical Moran's tests ($\geq 90\%$)

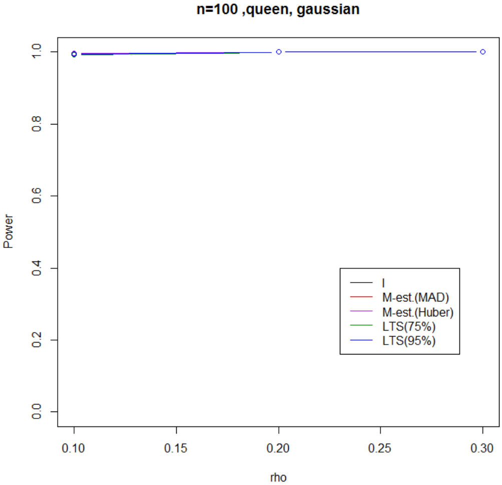
Level ($n = 100$)

ε	W	Moran	M-estimator		LTS	
			MAD	Huber	75%	95%
N(0,1)	Queen	0.055	0.061	0.062	0.072	0.073
N(0,1)	Rook	0.051	0.051	0.050	0.053	0.054
$\mathcal{P}(1)$	Queen	0.044	0.054	0.053	0.080	0.063
$\mathcal{P}(1)$	Rook	0.052	0.056	0.055	0.057	0.056
Bern(0.5)	Queen	0.067	0.086	0.087	0.091	0.091
Bern(0.5)	Rook	0.041	0.061	0.050	0.076	0.076

Results ($\delta = 0$)

Power

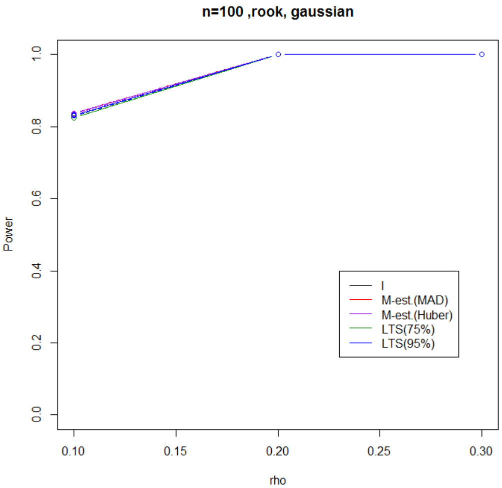
proportion of rejected null hypothesis with $\rho = 0.1$, 0.2 or 0.3



Results ($\delta = 0$)

Power

proportion of rejected null hypothesis with $\rho = 0.1$, 0.2 or 0.3



Results ($\delta > 0$)

500 simulations for $n = 100$, $\varepsilon \sim N(0,1)$

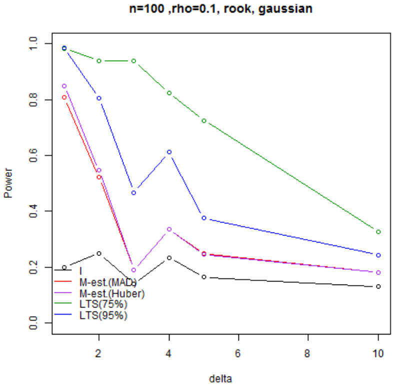
Level

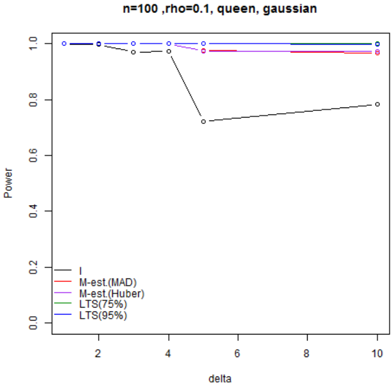
δ	W	Moran	M-estimator		LTS	
			MAD	Huber	75%	95%
1	Queen	0.064	0.072	0.072	0.140	0.126
1	Rook	0.026	0.030	0.030	0.094	0.046
2	Queen	0.050	0.068	0.068	0.134	0.090
2	Rook	0.040	0.052	0.052	0.096	0.064
3	Queen	0.054	0.062	0.060	0.104	0.090
3	Rook	0.054	0.076	0.076	0.086	0.098
4	Queen	0.026	0.048	0.048	0.092	0.064
4	Rook	0.050	0.064	0.062	0.090	0.080
5	Queen	0.058	0.072	0.070	0.124	0.126
5	Rook	0.040	0.058	0.058	0.062	0.054
10	Queen	0.052	0.060	0.060	0.072	0.070
10	Rook	0.068	0.074	0.070	0.078	0.068

Results ($\delta > 0$)

500 simulations for $n = 100$, $\varepsilon \sim N(0,1)$

Power

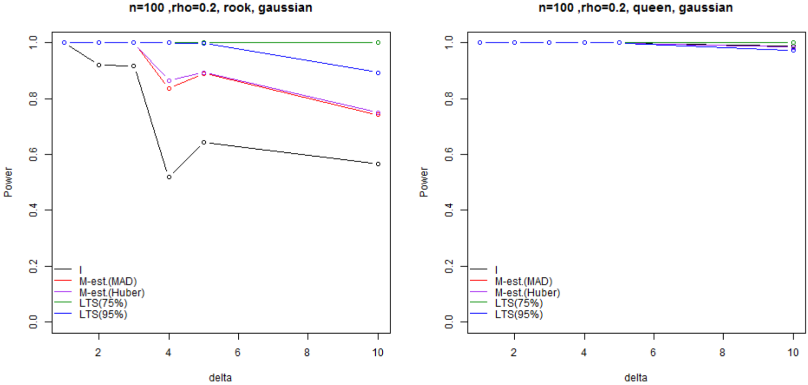




Results ($\delta > 0$)

500 simulations for $n = 100$, $\varepsilon \sim N(0, 1)$

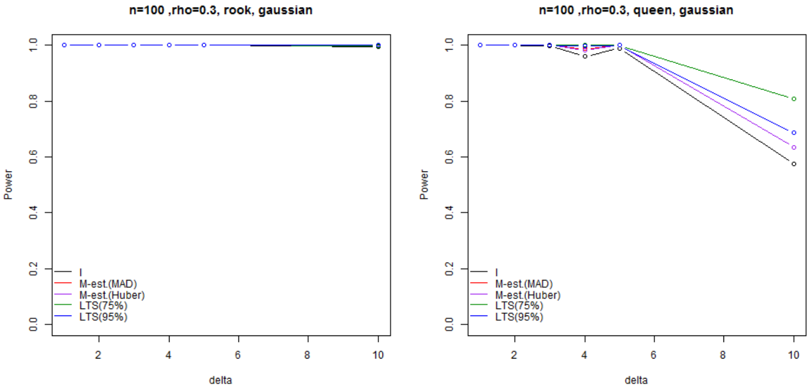
Power



Results ($\delta > 0$)

500 simulations for $n = 100$, $\varepsilon \sim N(0, 1)$

Power



Results ($\delta > 0$)

500 simulations for $n = 100$, $\varepsilon \sim N(0, 1)$

Preliminary observations

- Robust test using Least Trimmed Square gives good results
- Robust test using M-estimator is more sensitive to leverage points

Perspective

In progress

- Simulations and comparison need to pursue
- Proposition of a robust test based on the others statistics

Perspective

- Study robustness of local measures and local tests

In progress

- Simulations and comparison need to pursue
- Proposition of a robust test based on the others statistics

  For Geary's c:
  - $\hat{\gamma}(h) = S_z^2 c$ for $\hat{\gamma}(h)$ the empirical variogram
  - Robust estimation of the variogram using Q_n
  - $\hat{c} \propto \frac{1}{2}\left(\frac{Q_n(\mathcal{V}_n)}{Q_n(z)}\right)^2$

Perspective

- Study robustness of local measures and local tests

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