# Eulerian Formulation of Spatially Constrained Elastic Rods

*PhD Dissertation Defense*

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Université de Liège

FREEDOM TO RESEARCH

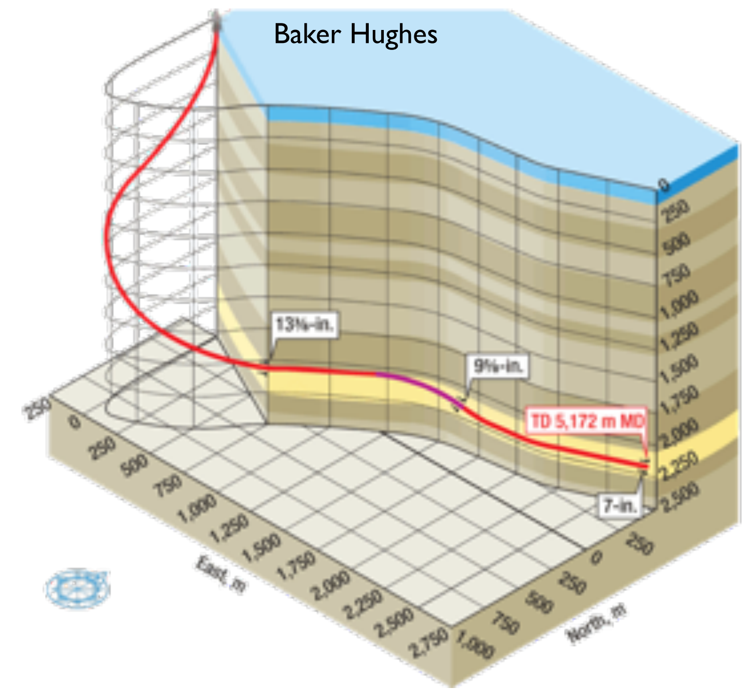
Friday December 18 2015

1. Introduction
2. Eulerian formulation of elastic rods deforming in space
3. Surface constrained elastic rods
4. Eulerian formulation adapted to normal ringed surface
5. Applications
6. Conclusion

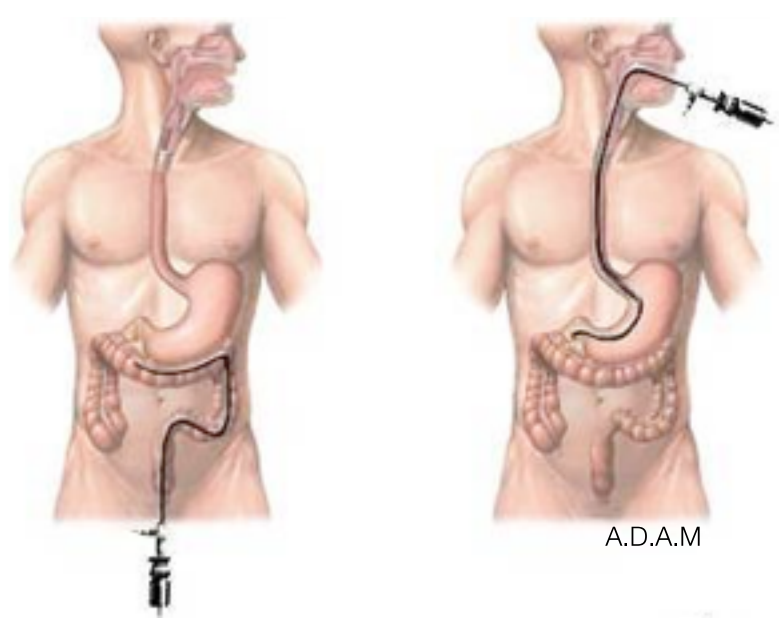
# Constrained Rod - Inside

- Engineering applications
    - Petroleum, mining, gas, geothermal, etc.

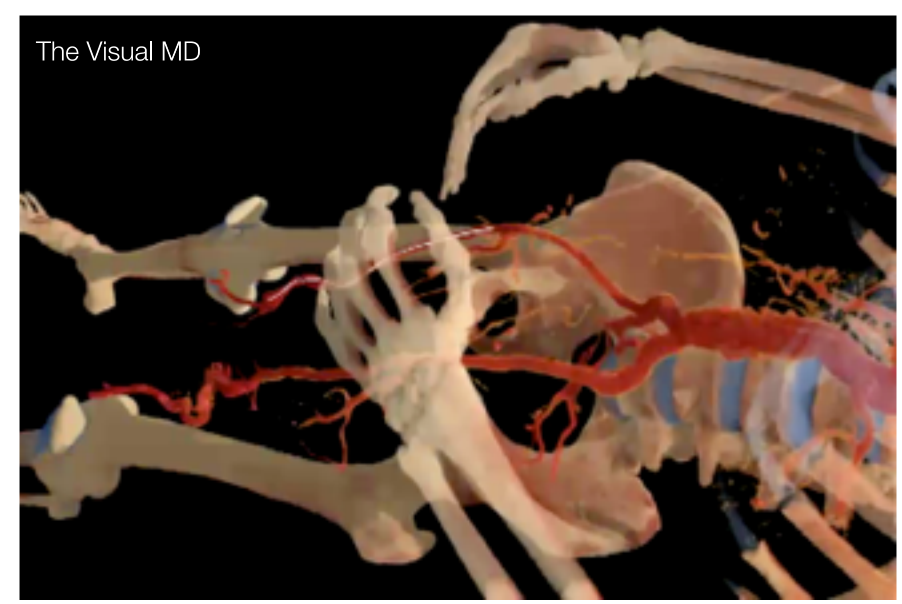

Drillstring length
~ 5 km




- Medical applications
    - Endoscopic examination of internal organs
    - Endovascular procedures







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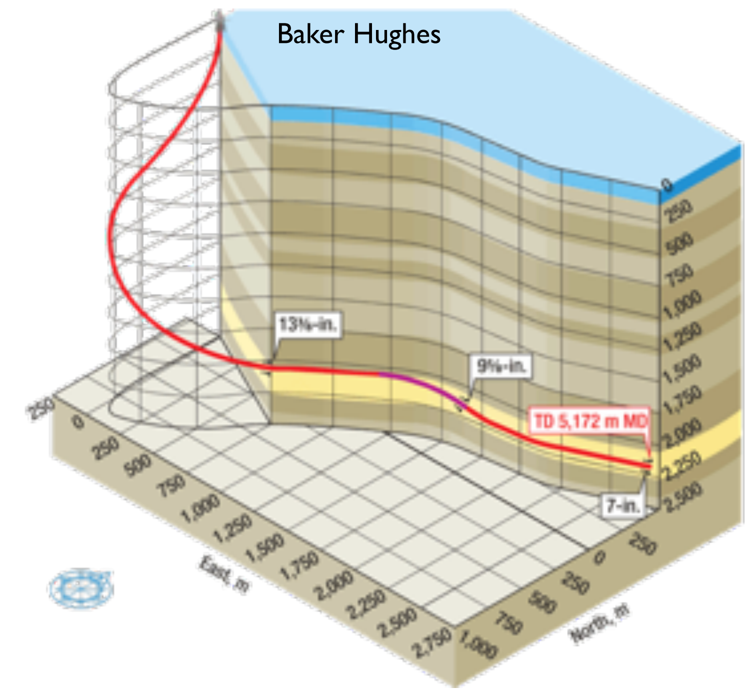


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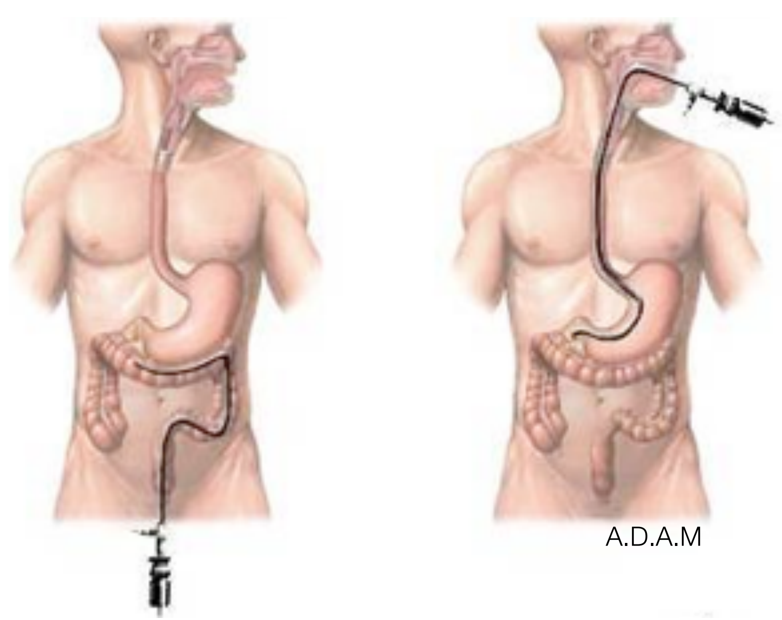
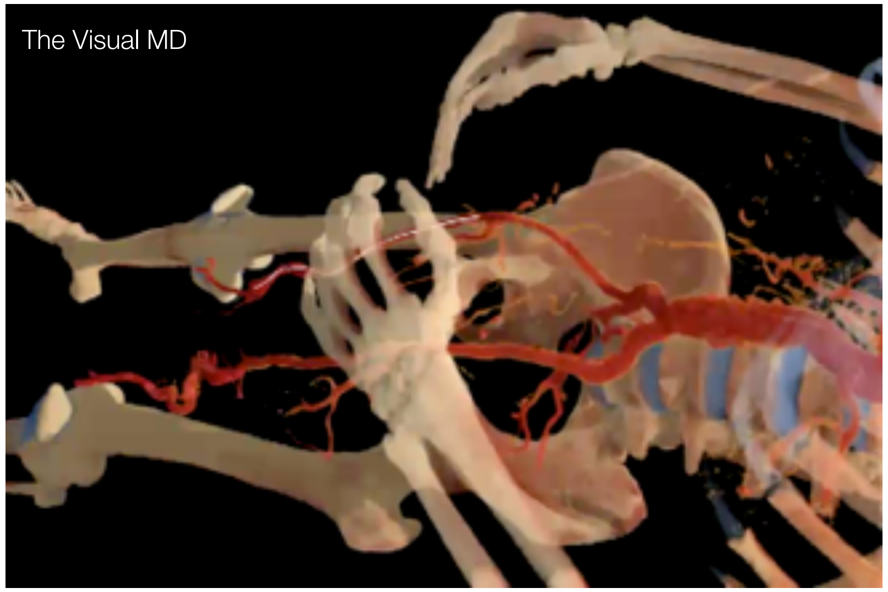
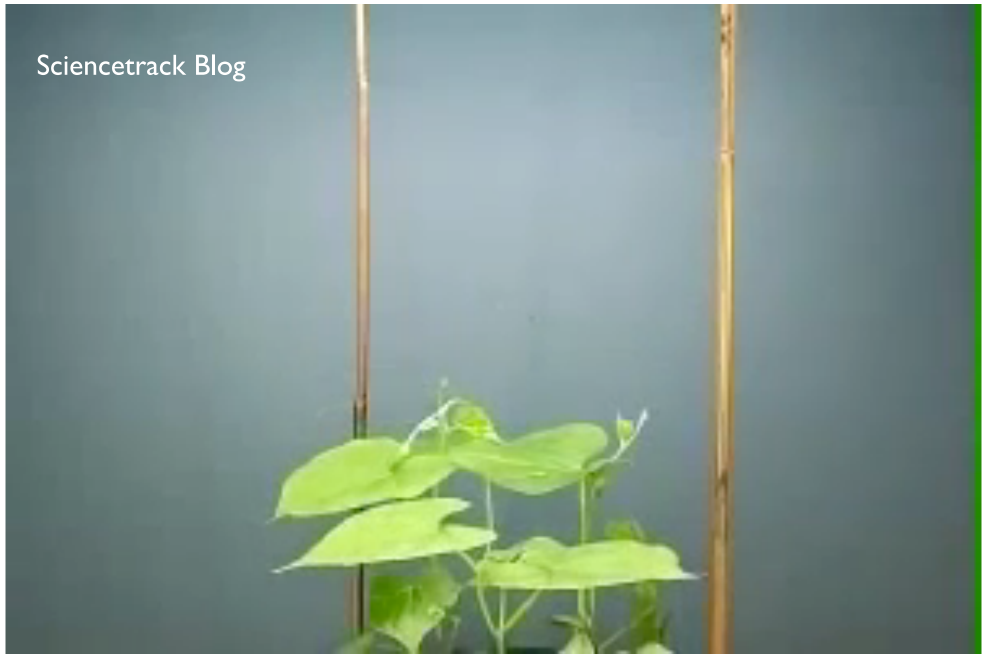




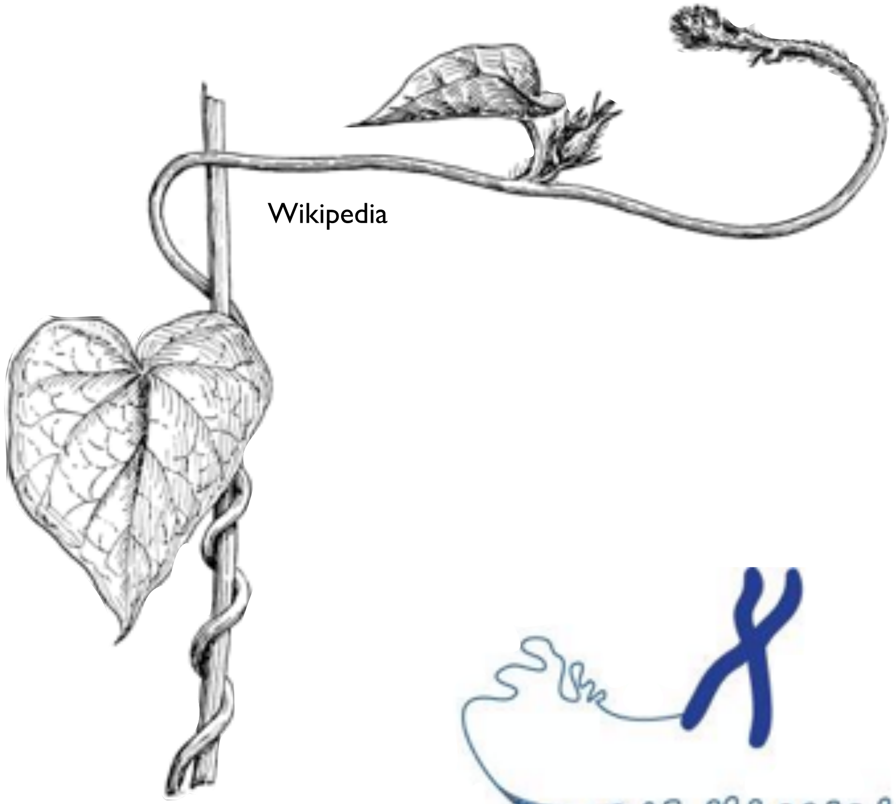


# Constrained Rod - Outside

- Twining plants

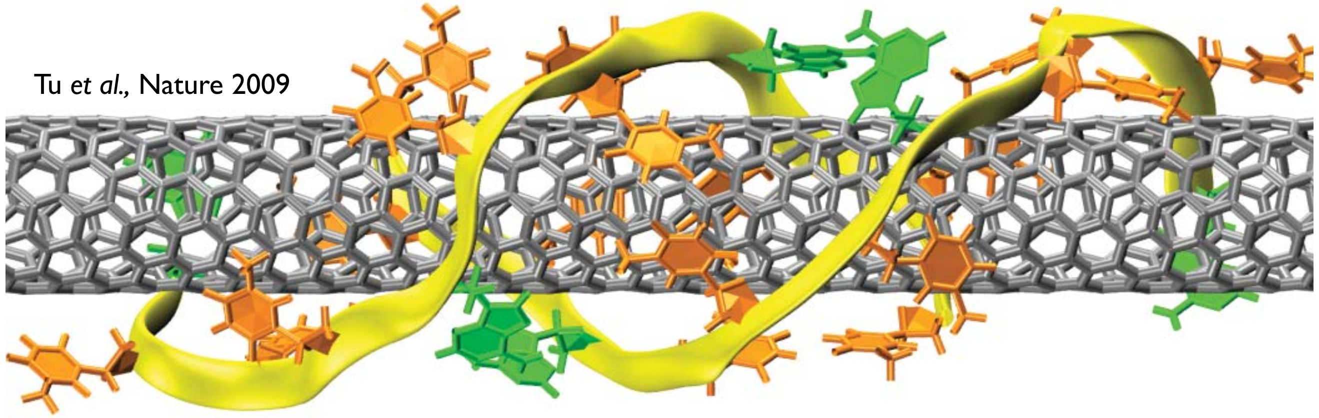


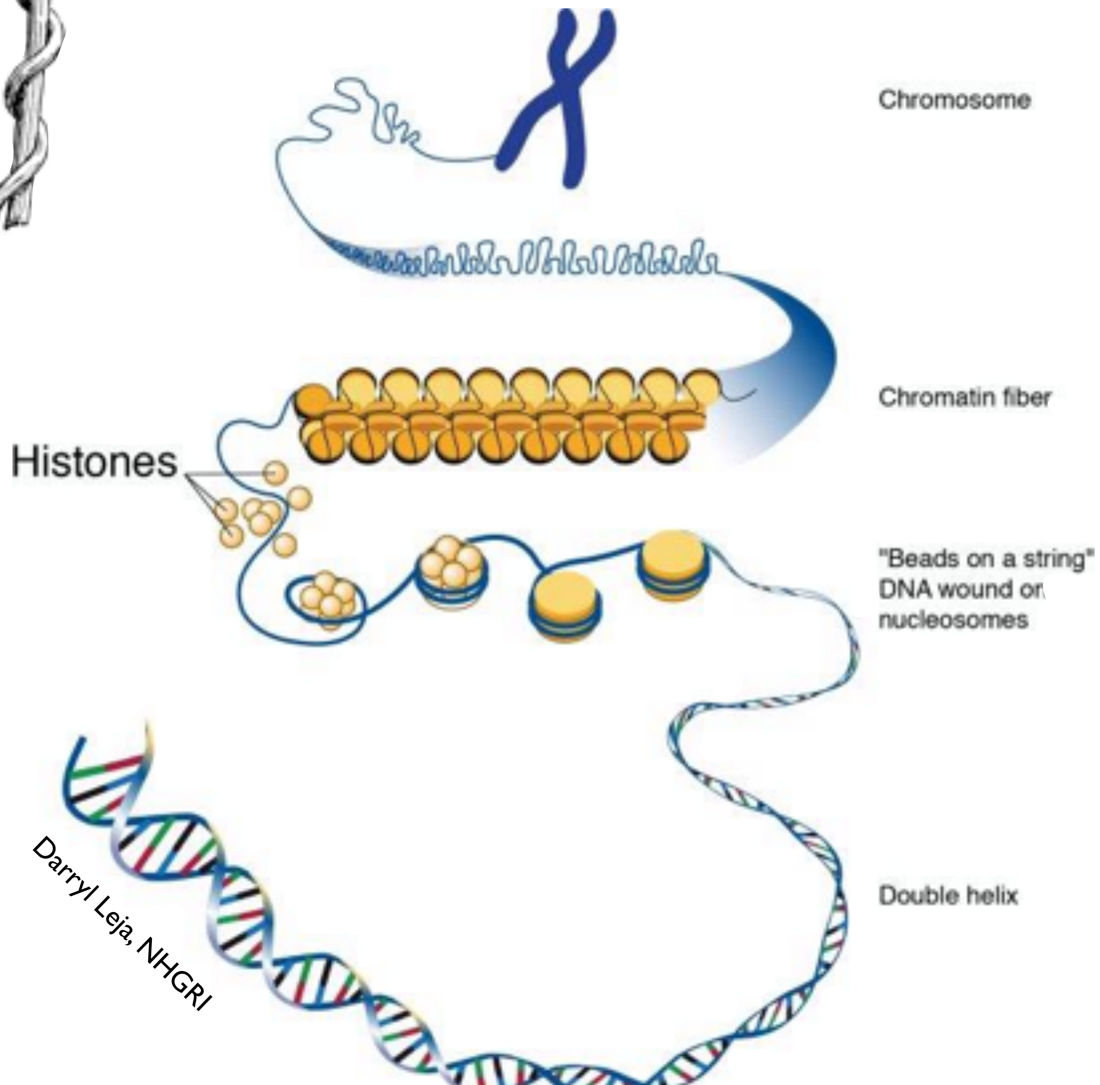

1 sec. = 4 h




- DNA wrapping

Tu *et al.*, Nature 2009





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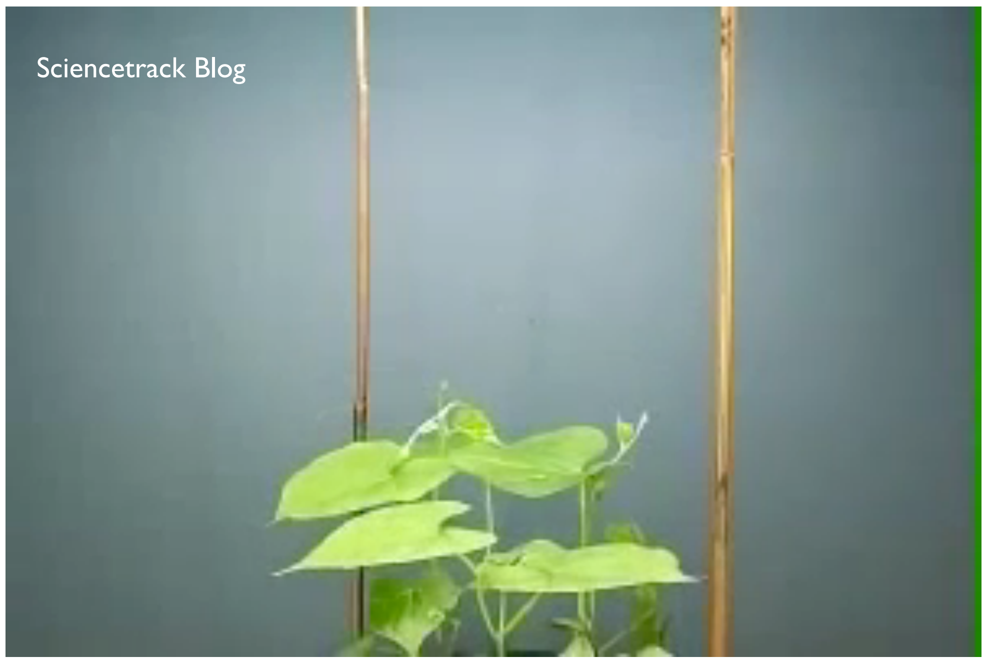


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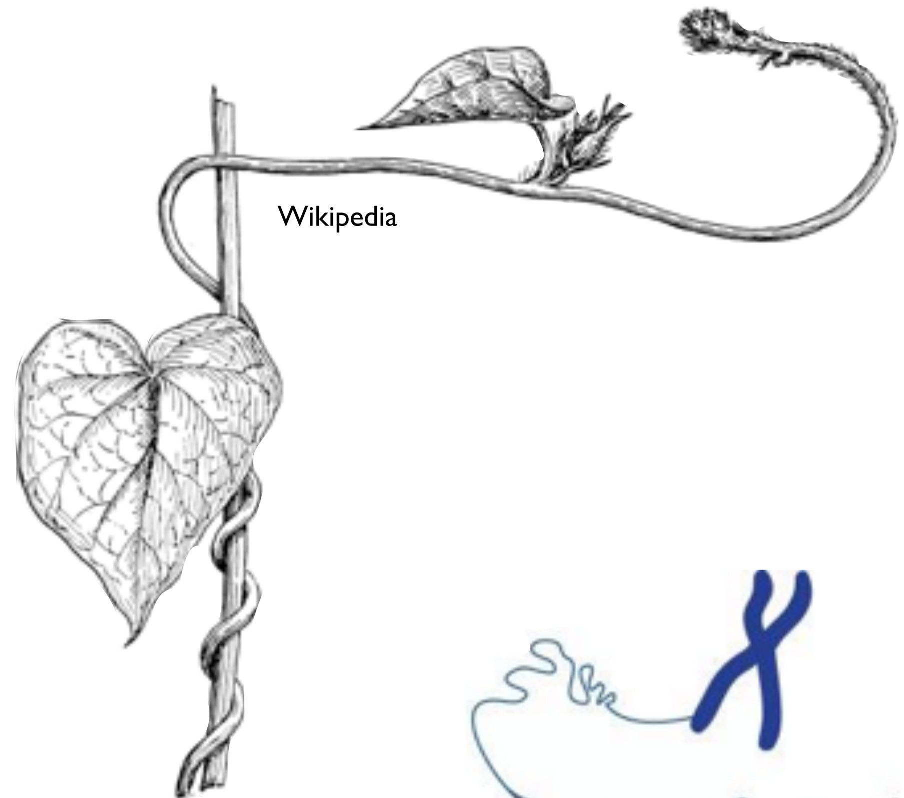


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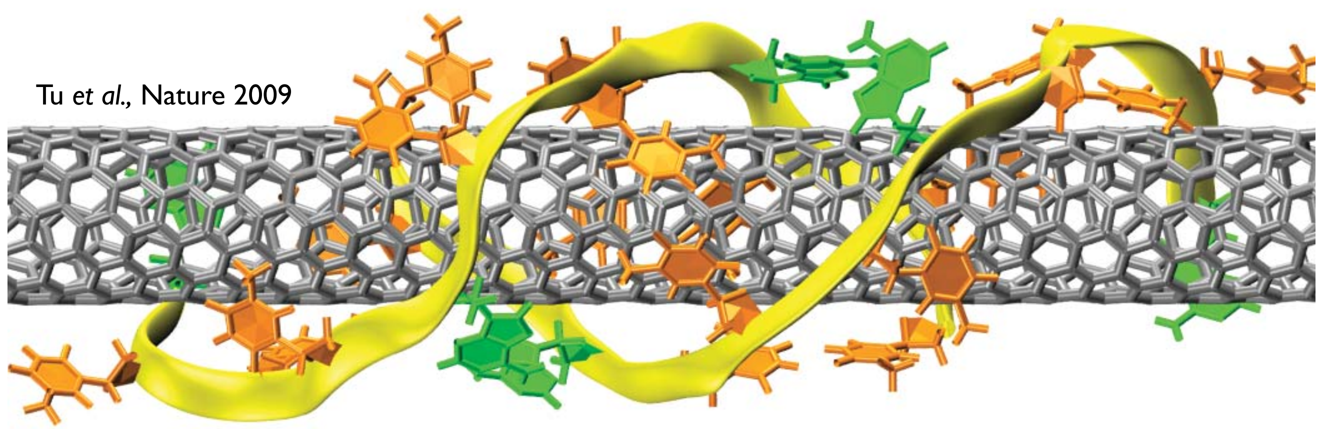
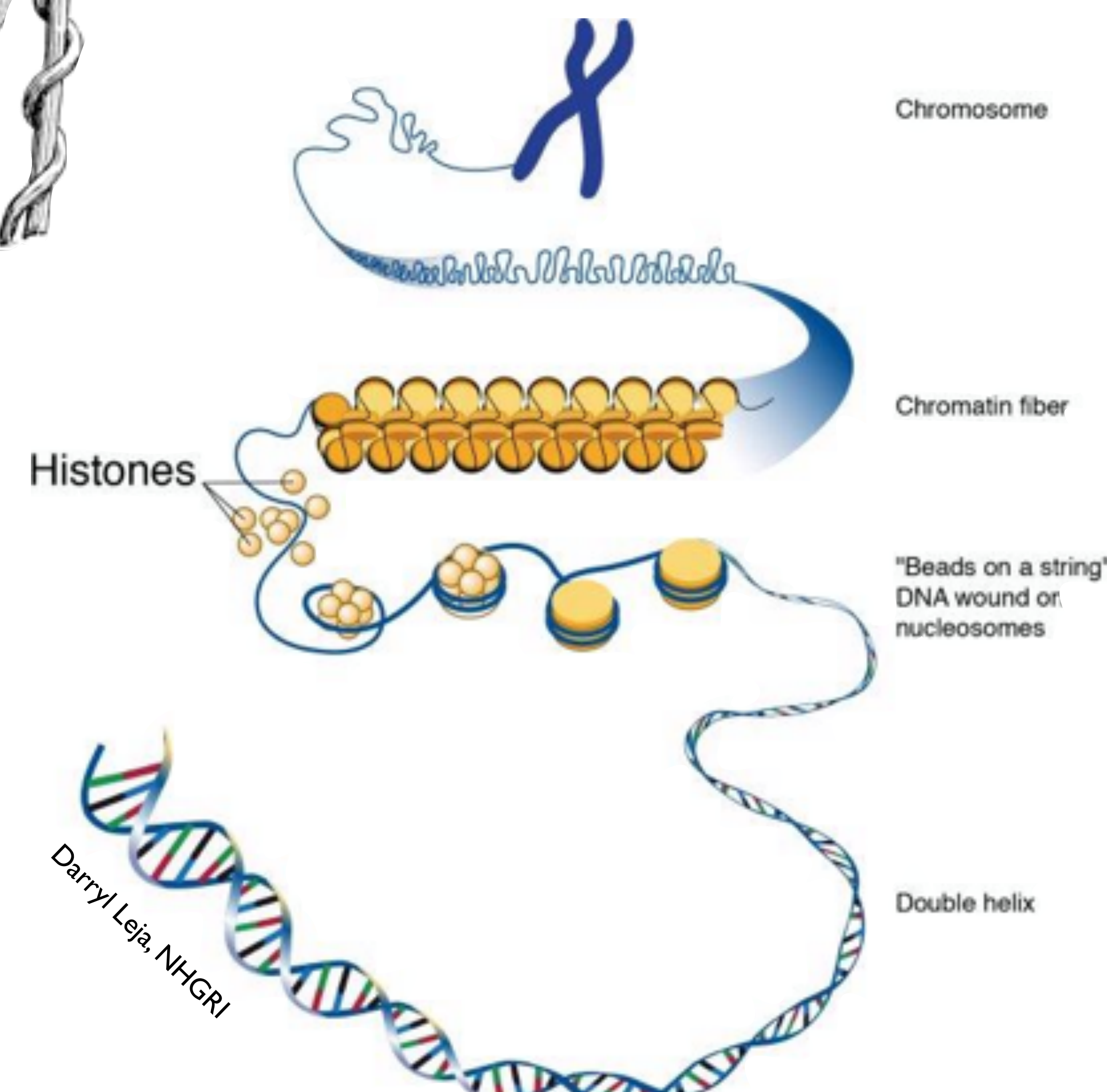
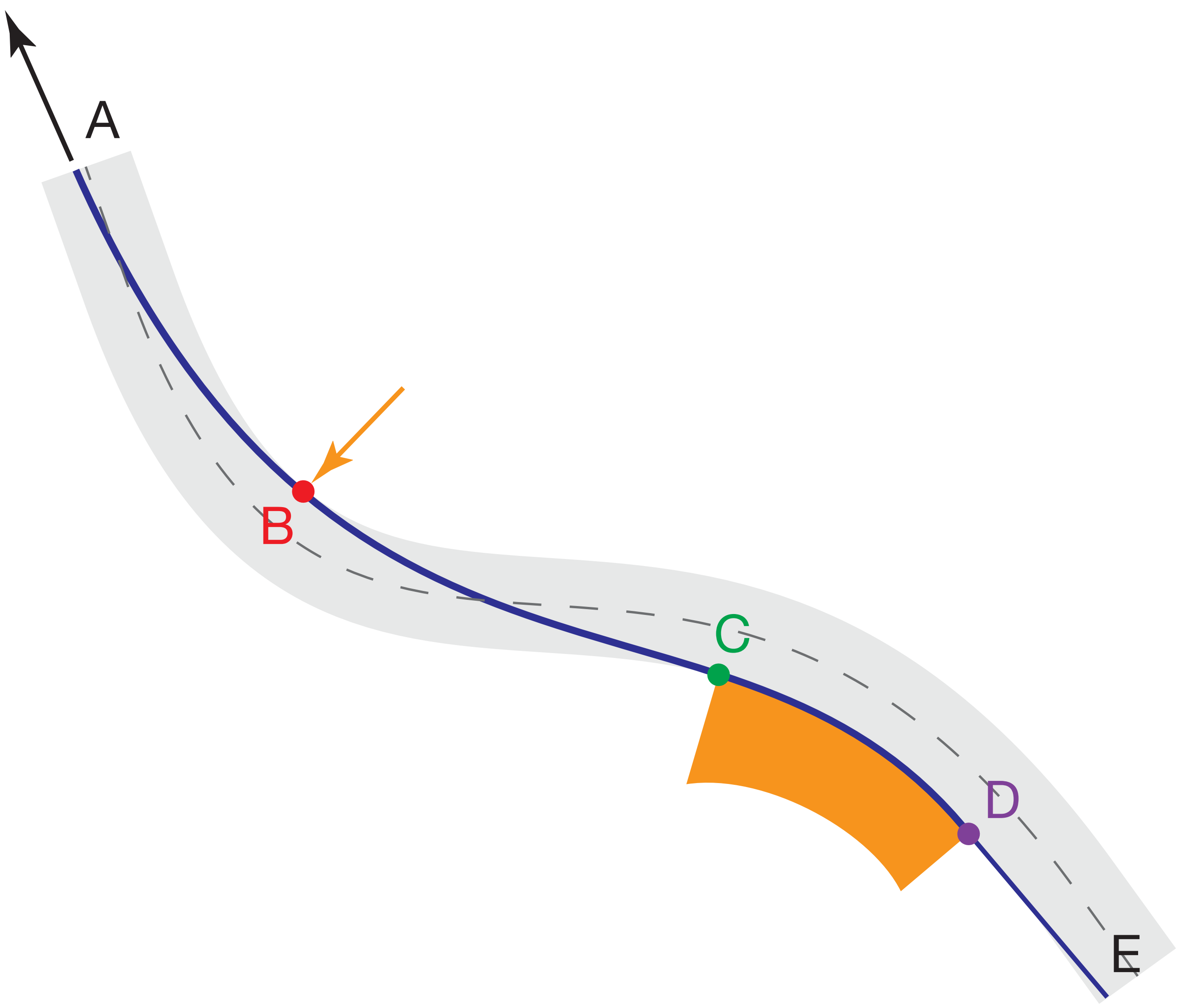


# Segmentation Strategy (Chen & Li 2007, Denoël 2008)

1. Consider a contact pattern and the associated contact positions
2. Tune the contact positions to ensure the rod integrity
3. Check the validity of the contact pattern



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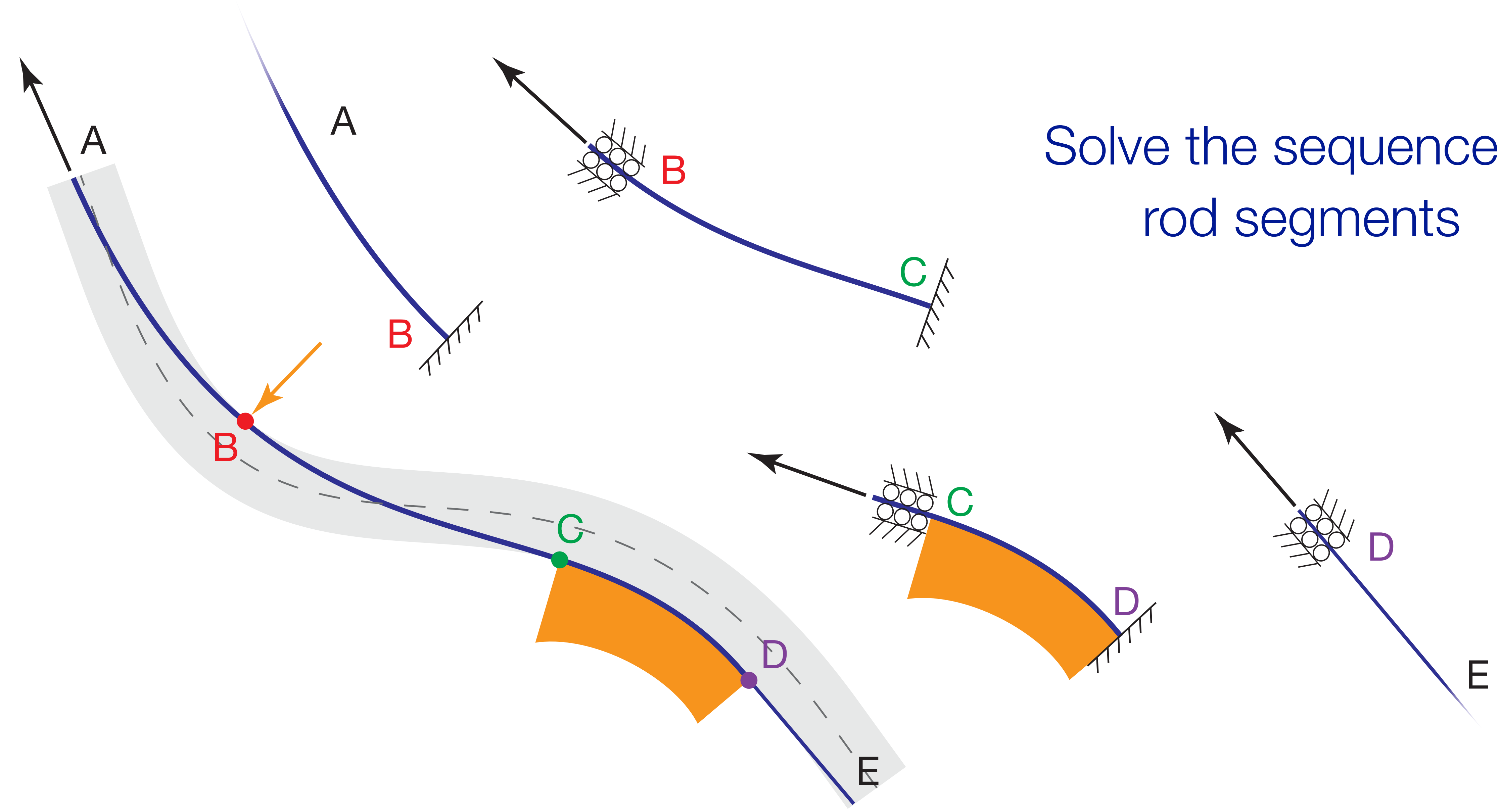


Solve the sequence of rod segments

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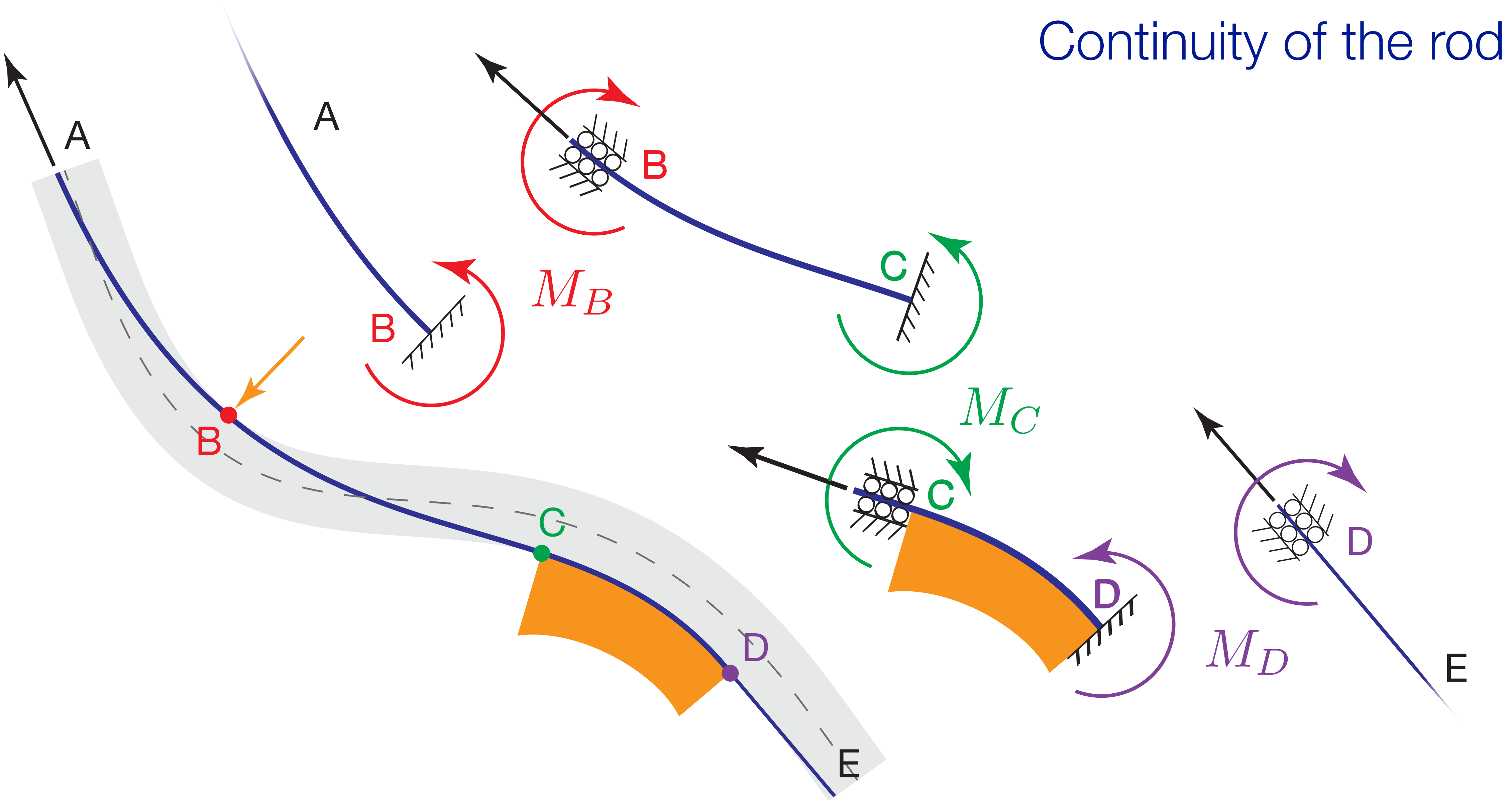

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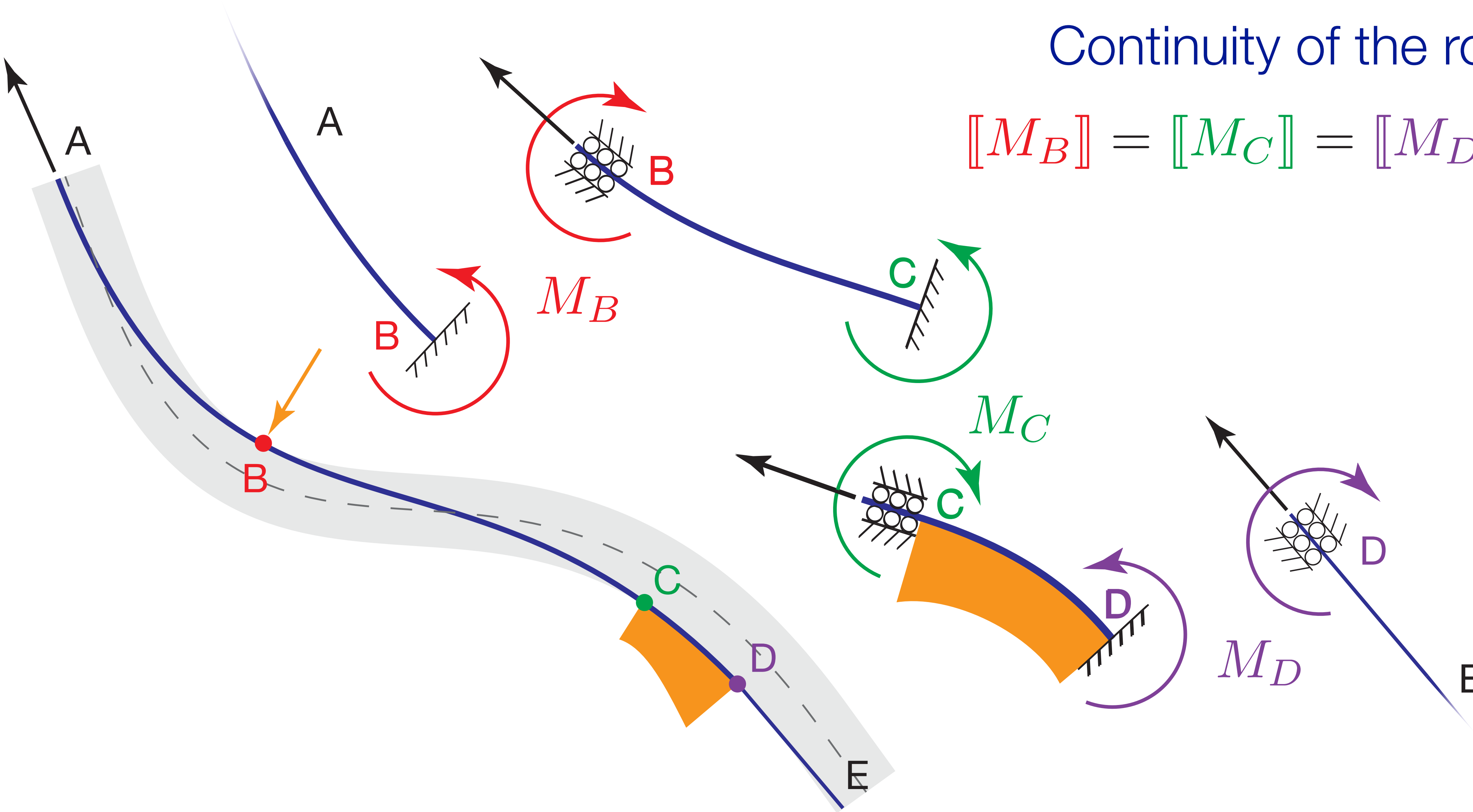


Continuity of the rod

$$[\![M_B]\!] = [\![M_C]\!] = [\![M_D]\!] = 0$$

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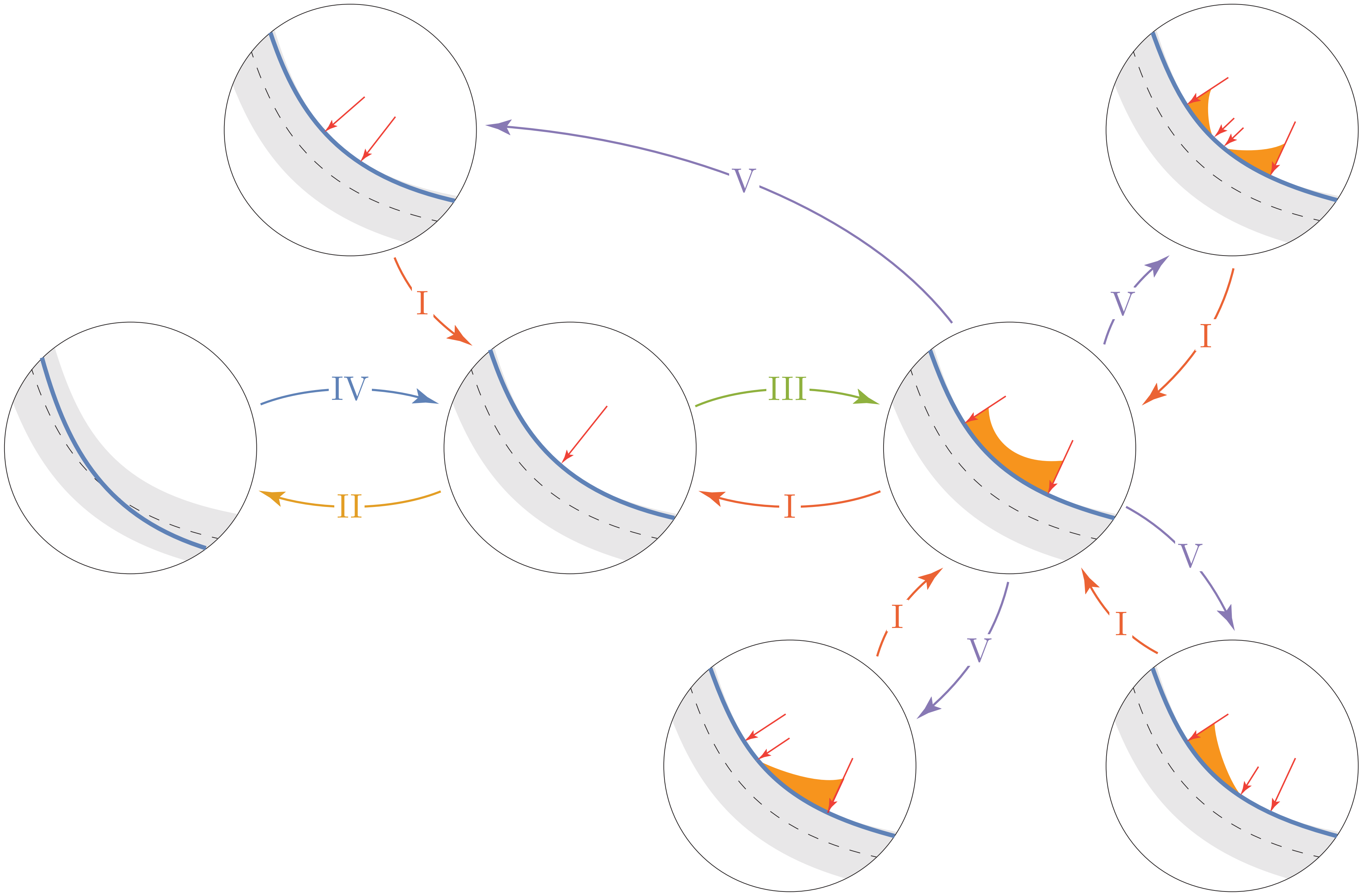
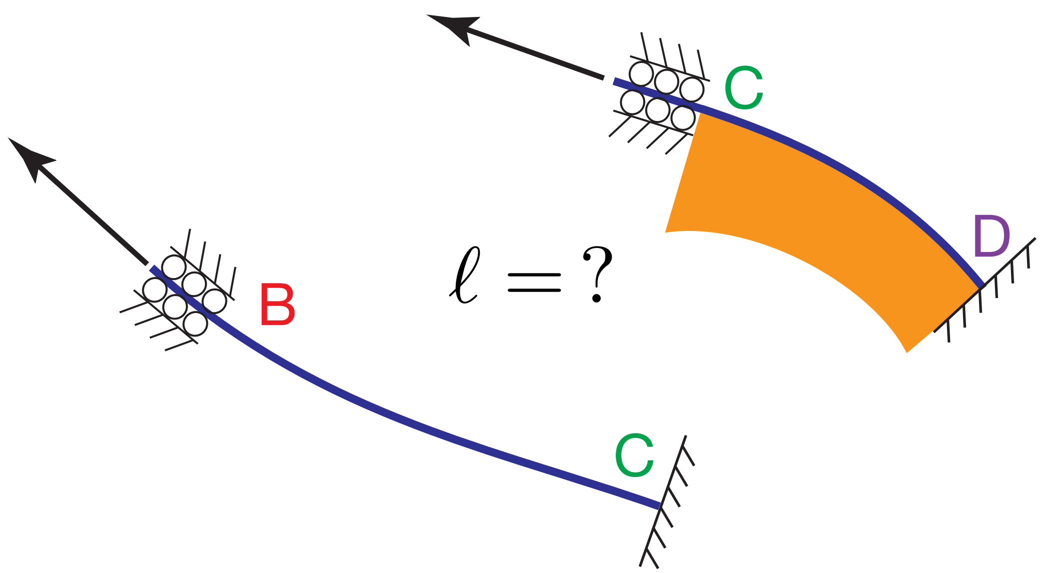


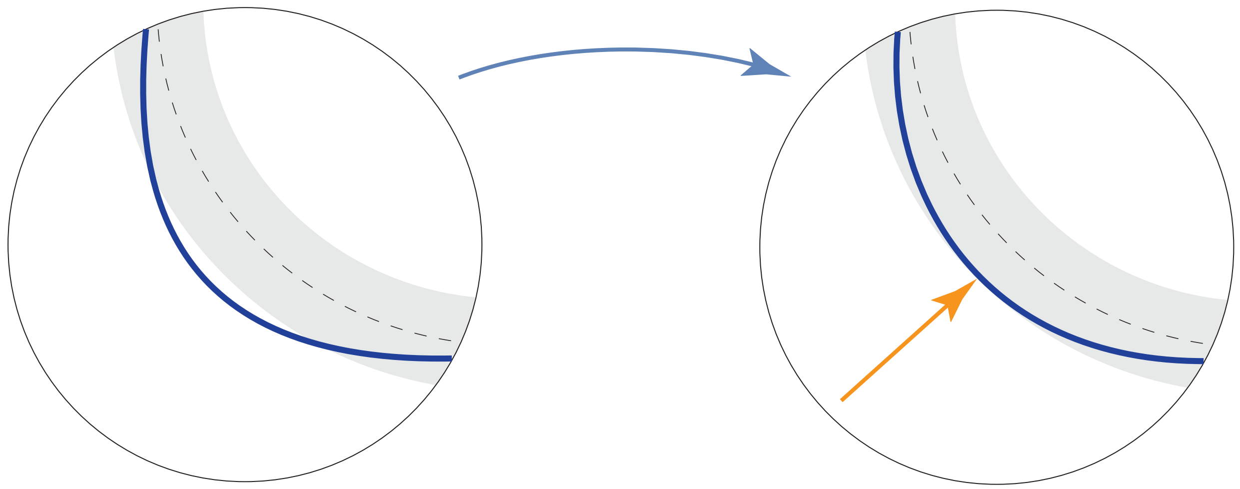
# Lagrangian

- Segmentation drawbacks
  - Initially unknown domain

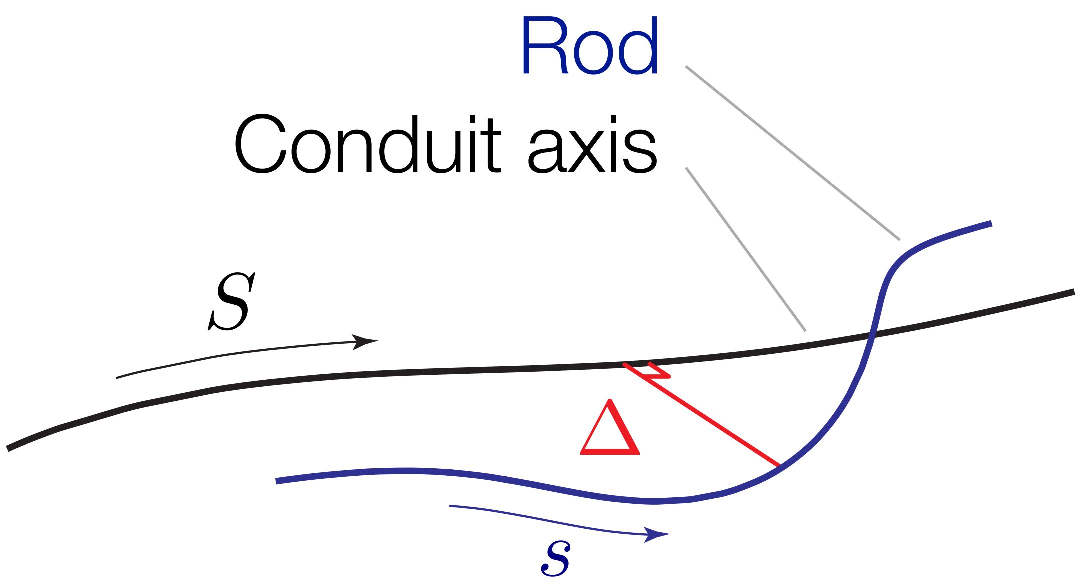



Free boundary problem

  - Evaluation of the distance rod/conduit axis

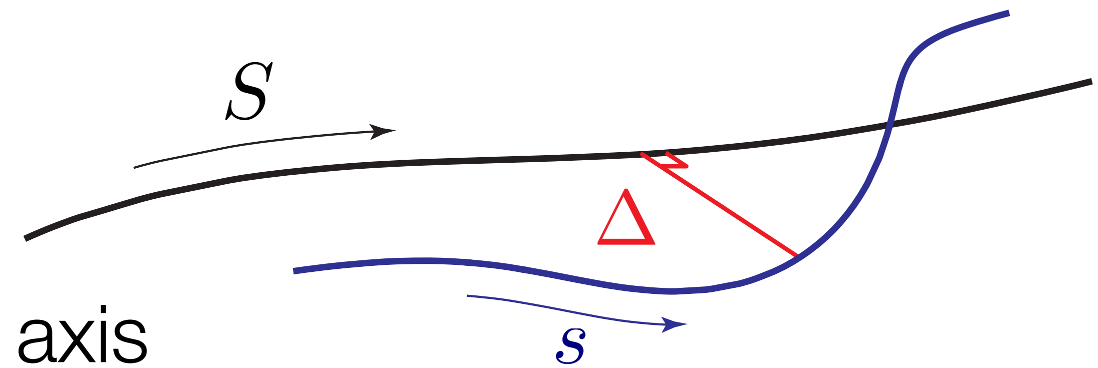


Contact detection

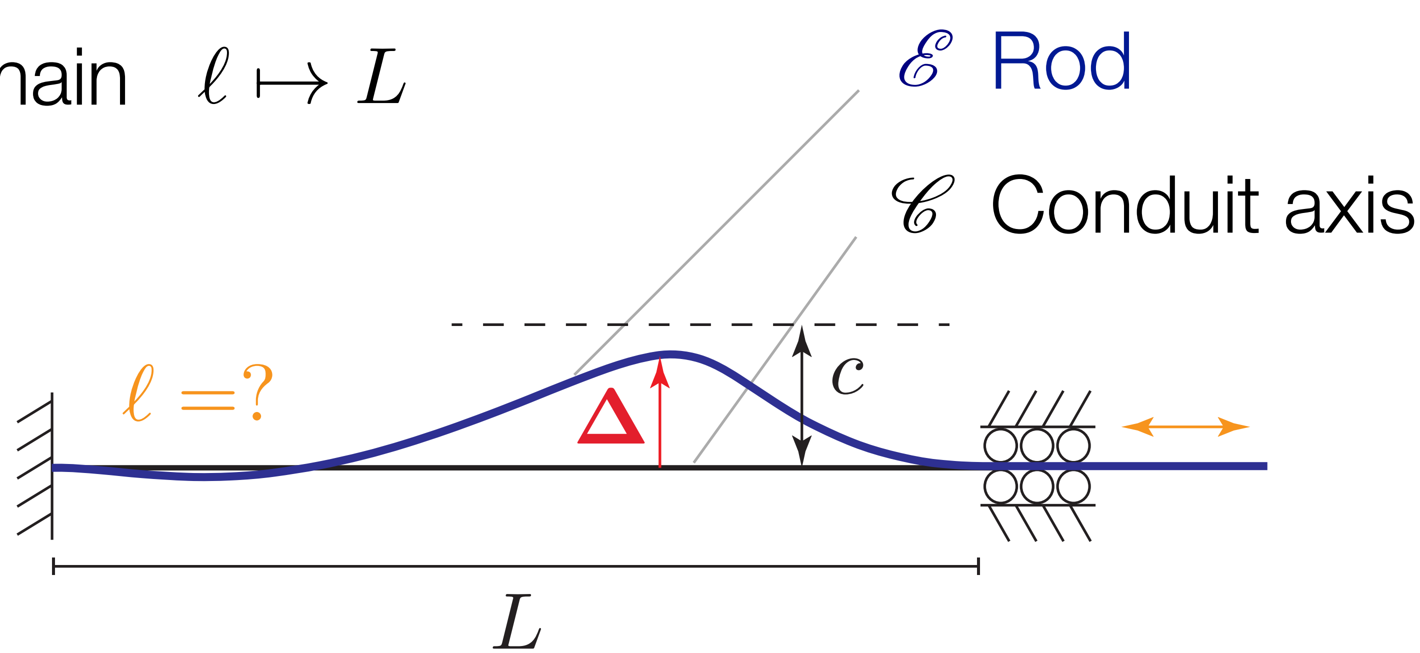


# Lagrangian vs. Eulerian

- Segmentation drawbacks (Lagrangian)
  - Initially unknown domain
  - Evaluation of the distance rod/conduit axis

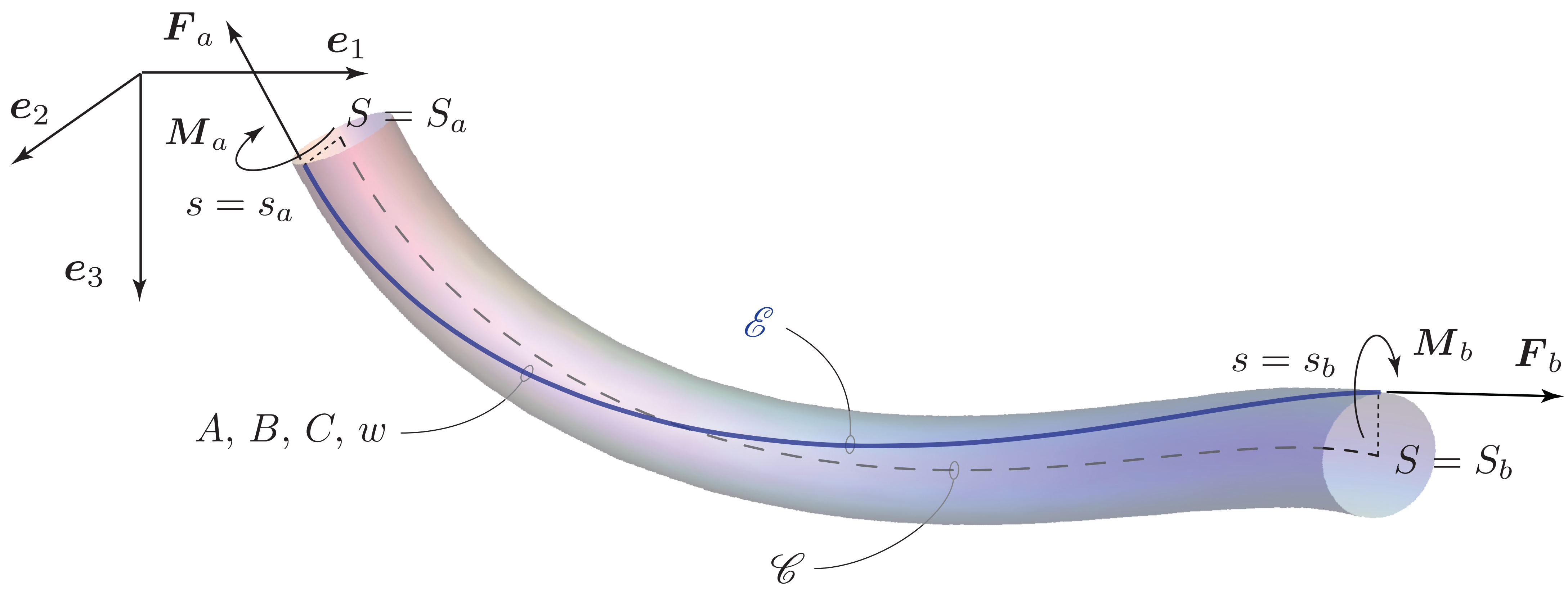



- Eulerian formulation (Denoël & Detournay, 2011)
  - Rod relative deflection $\boldsymbol{\Delta}\,(S)$
  - Coordinate $s \mapsto S$
  - Domain $\ell \mapsto L$




Self-feeding

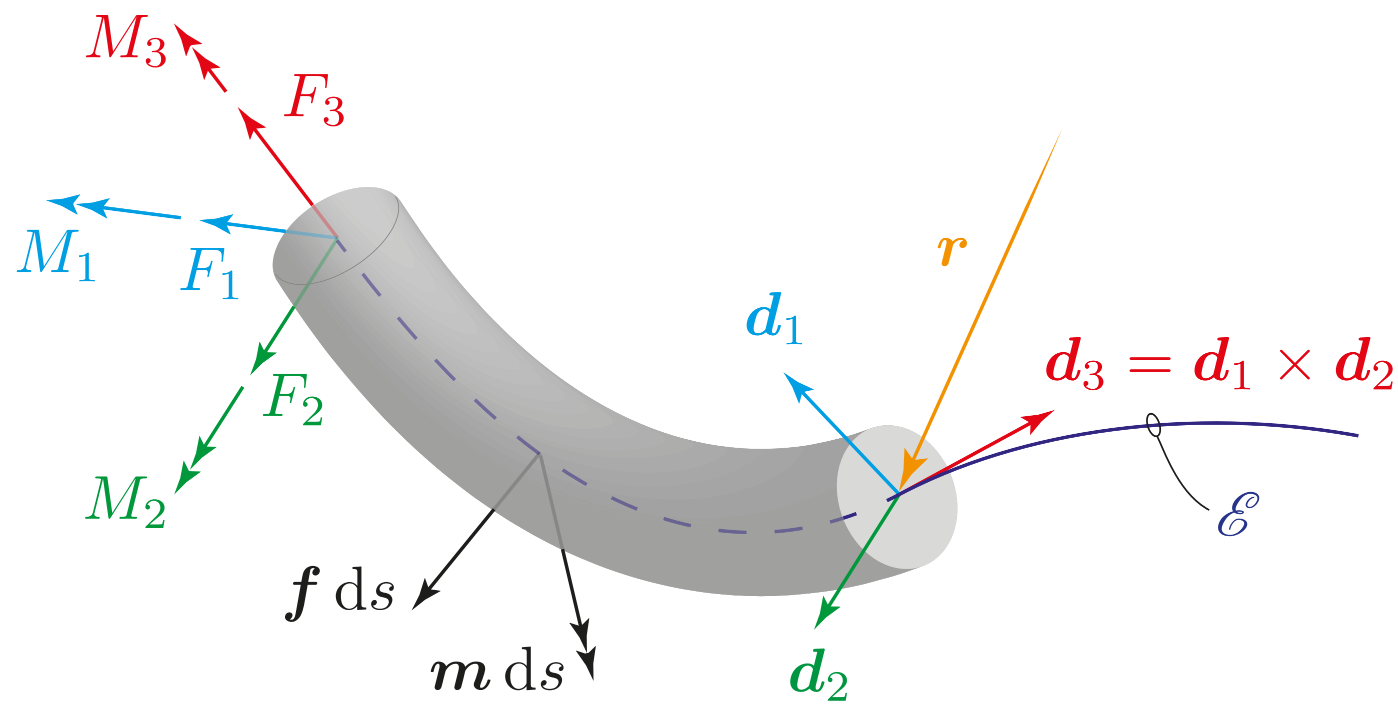
# Canonical Problem




- Rod configuration between contacts $(s \in [s_a, s_b])$
  - Known extremities positions and inclinations $x_j\left(s_{a,b}\right), x_j'\left(s_{a,b}\right)$
  - Known axial force $\boldsymbol{F}_b$ and torque $\boldsymbol{M}_b$
- Unknowns
  - Rod length $\ell = s_b - s_a$, axial force $\boldsymbol{F}_a$ and torque $\boldsymbol{M}_a$

# Lagrangian Formulation (Antman, 2005)

- Rod definition
  - Centroid $\boldsymbol{r}(s) = x_k\,\boldsymbol{e}_k$
    → Space curve $\mathscr{E}$
  - Directors $\{\boldsymbol{d}_k(s)\}$
    → Section orientation




- Equilibrium

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}s} + \boldsymbol{f} = 0$$

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}s} + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} \times \boldsymbol{F} + \boldsymbol{m} = 0$$

- Kinematics

$$\frac{\mathrm{d}\boldsymbol{d}_k}{\mathrm{d}s} = \boldsymbol{u} \times \boldsymbol{d}_k$$

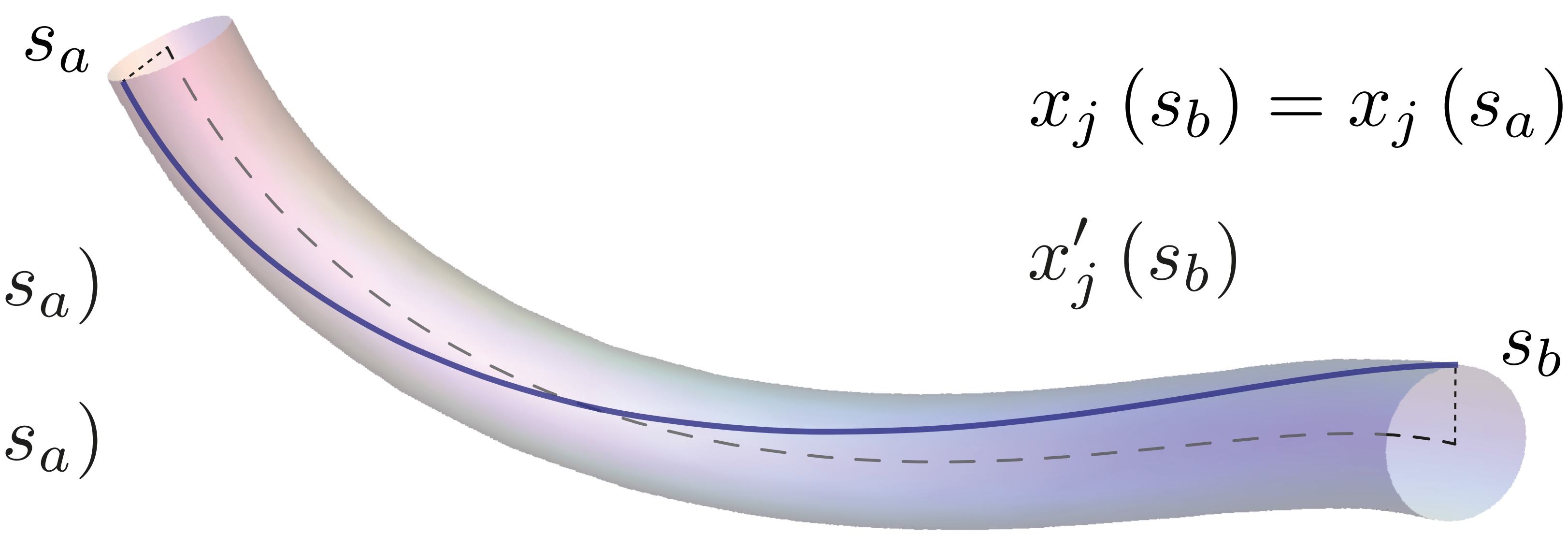
$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \alpha\,\boldsymbol{d}_3$$

- Constitutive equations

$$F_3 = A\,(\alpha - 1) \qquad M_{1,2} = B\,u_{1,2} \qquad M_3 = C\,u_3$$

# Issues with Lagrangian Formulation (Chen & Li, 2007)

- Isoperimetric constraints (boundary conditions)




$x_j(s_a)$

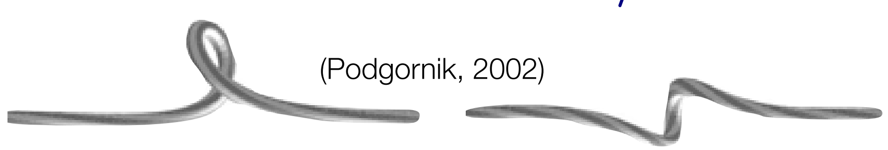
$x'_j(s_a)$

$$x_j(s_b) = x_j(s_a) + \int_{s_a}^{s_b} \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} \cdot \boldsymbol{e}_j \, \mathrm{d}s$$

$x'_j(s_b)$

→ Integral constraints on the *unknown length* $\ell = s_b - s_a$ of the rod

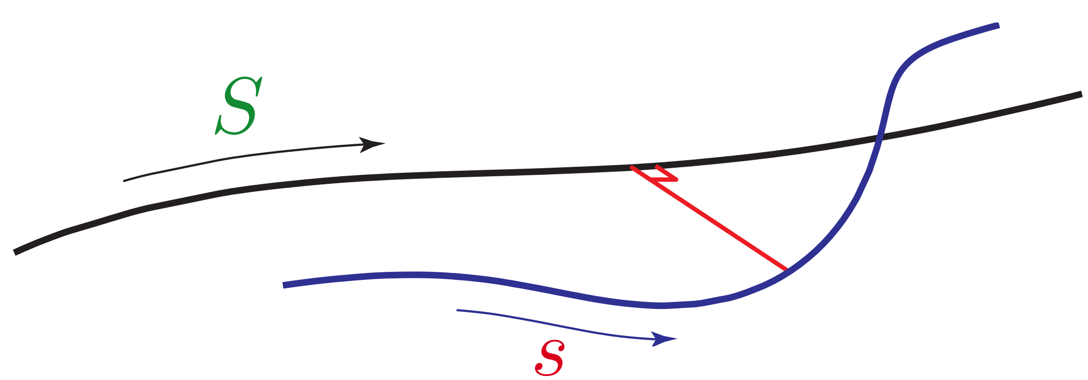
- Ill-conditioning of the governing equations when $B/w\,\ell^3 \ll 1$
- Parasitic solutions with curling


(Podgornik, 2002)

- Contact detection (Lagrangian vs. Eulerian)

Constraint axis $X_j(S)$

Rod centerline $x_j(s)$



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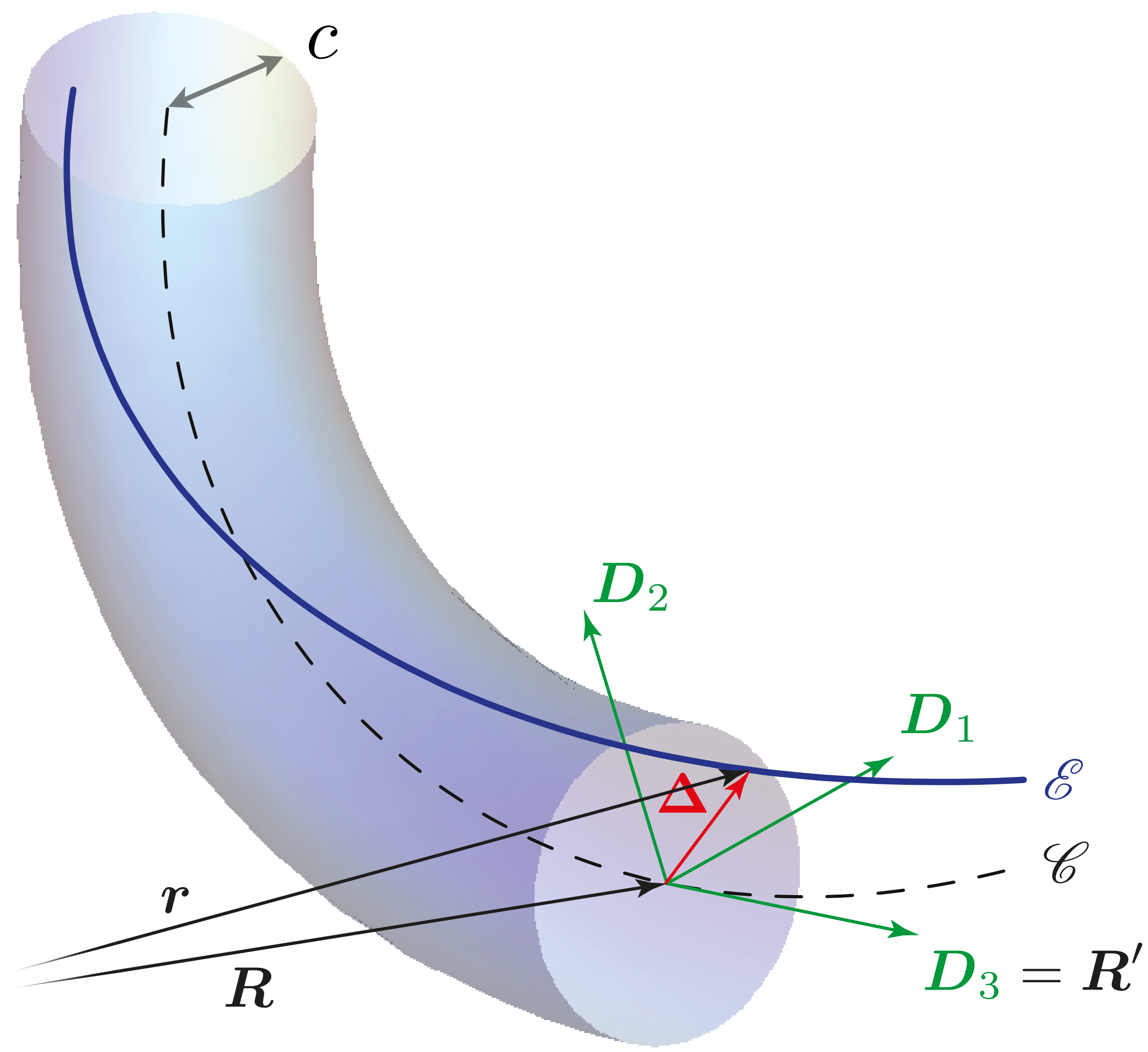
# Eulerian Formulation

- Orthonormal frame $\{\boldsymbol{D}_j(S)\}$ attached to the reference curve $\mathscr{C}$

- Eccentricity vector

$$\begin{cases} \boldsymbol{r}(s) = \boldsymbol{R}(S) + \boldsymbol{\Delta}(S) \\ \dfrac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}S} \cdot \boldsymbol{\Delta} = 0 \end{cases}$$

→ Contact detection $\|\boldsymbol{\Delta}\| \leq c$




- Jacobian of the mapping

$$S(s) \longrightarrow \frac{\mathrm{d}\cdot}{\mathrm{d}s} = \boxed{\frac{\mathrm{d}S}{\mathrm{d}s}} \frac{\mathrm{d}\cdot}{\mathrm{d}S}$$

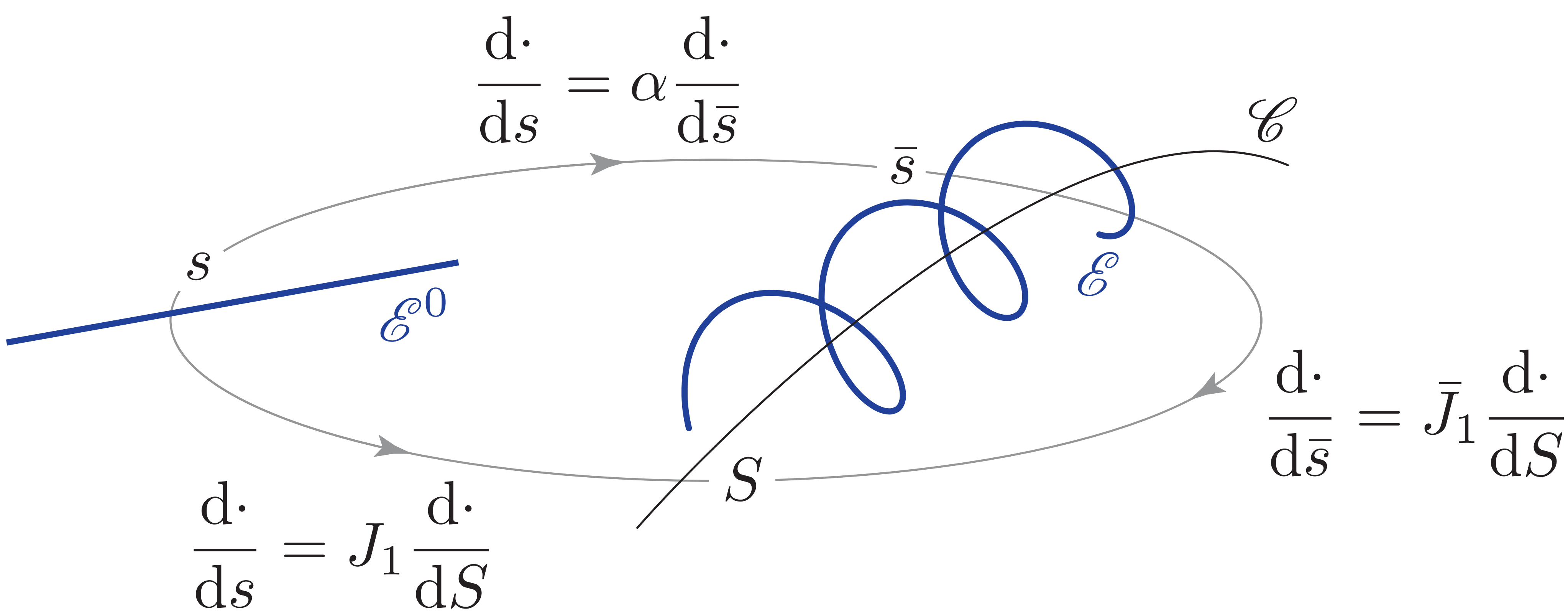
# Mappings: 3 curvilinear coordinates

**Lagrangian**

Unstressed config.

**Stretched**

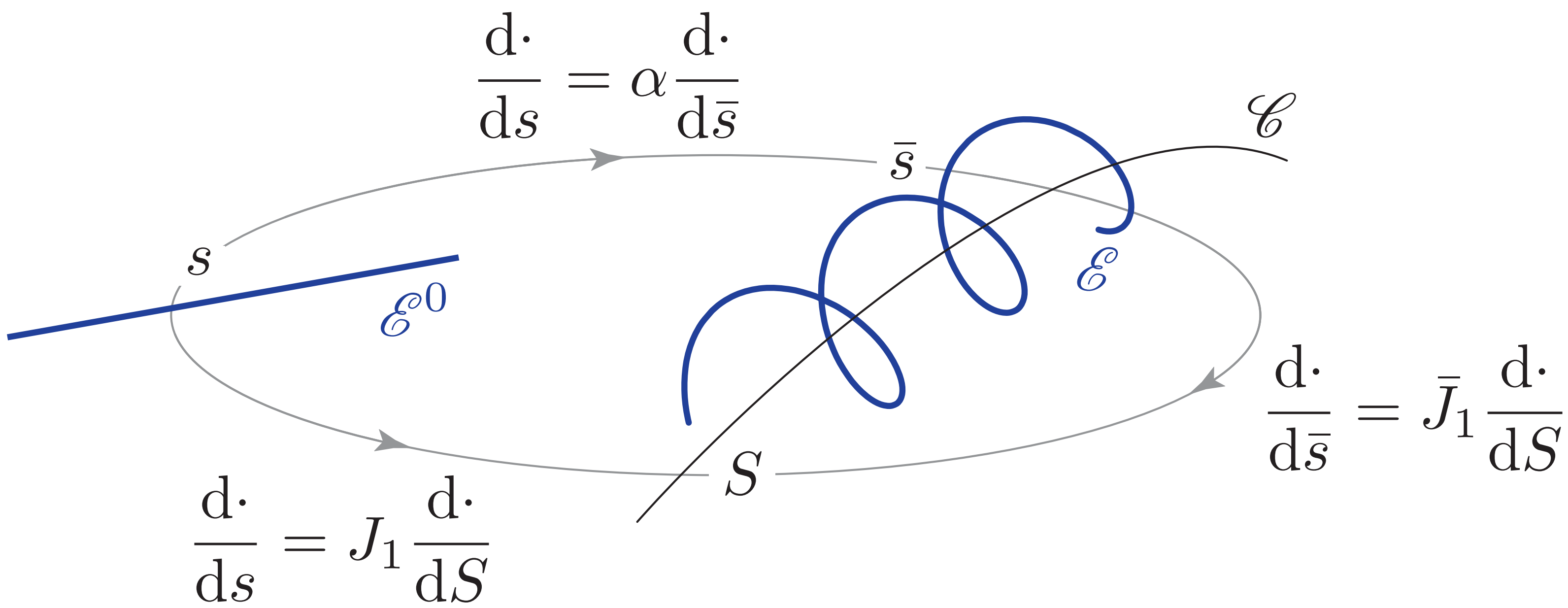
Deformed config.

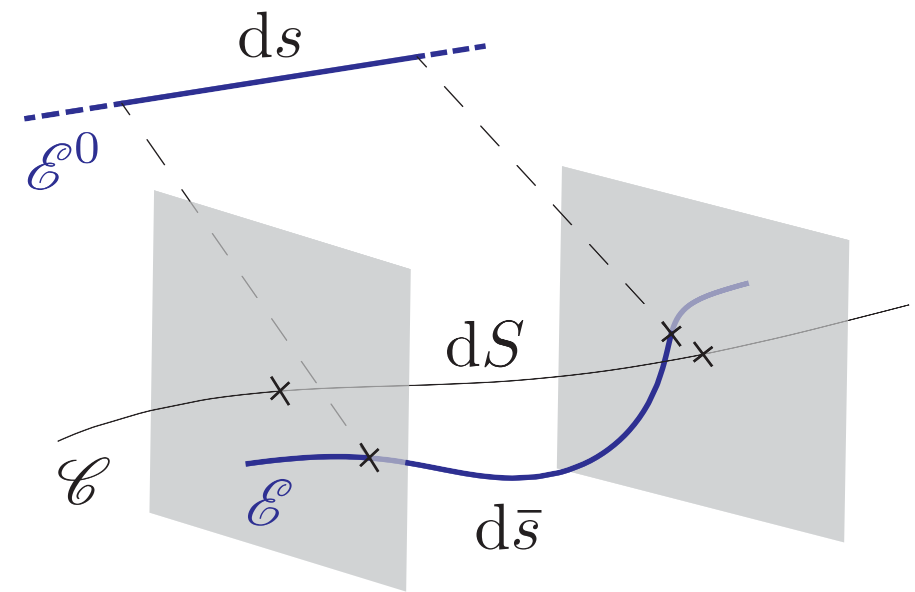



**Eulerian**

Reference curve

# Jacobian of the Mapping



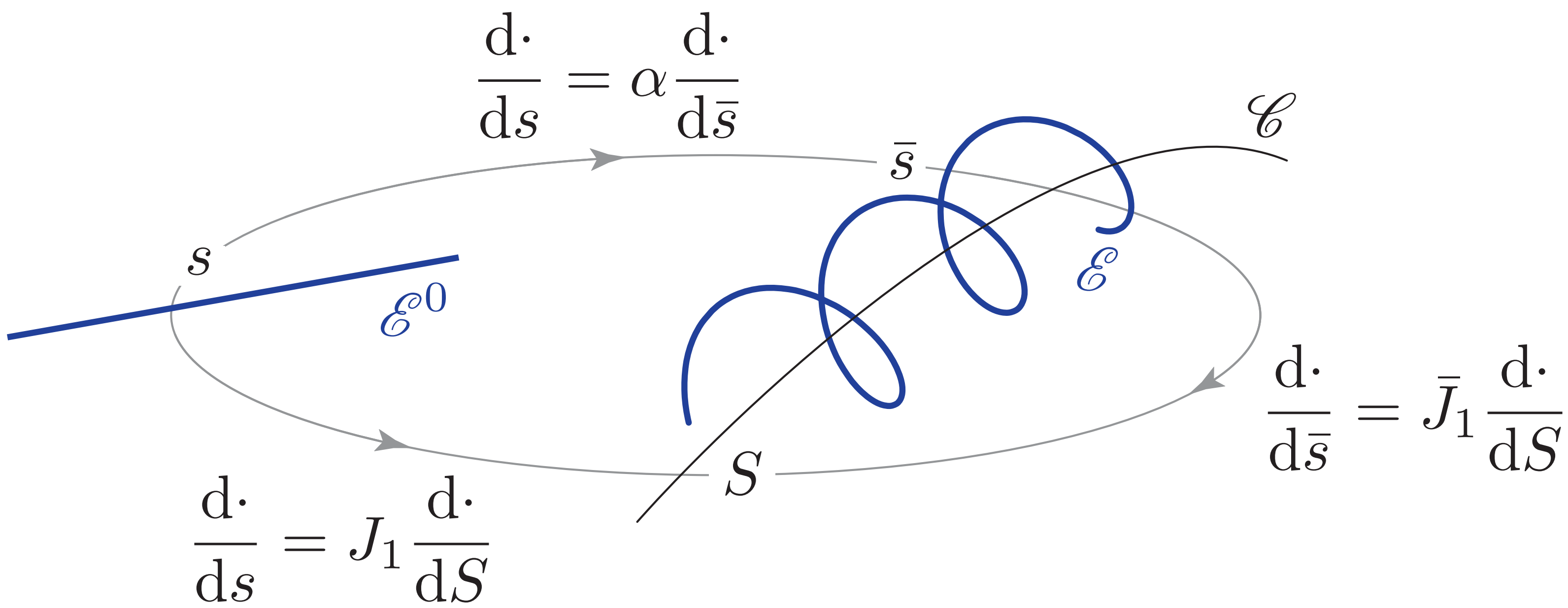



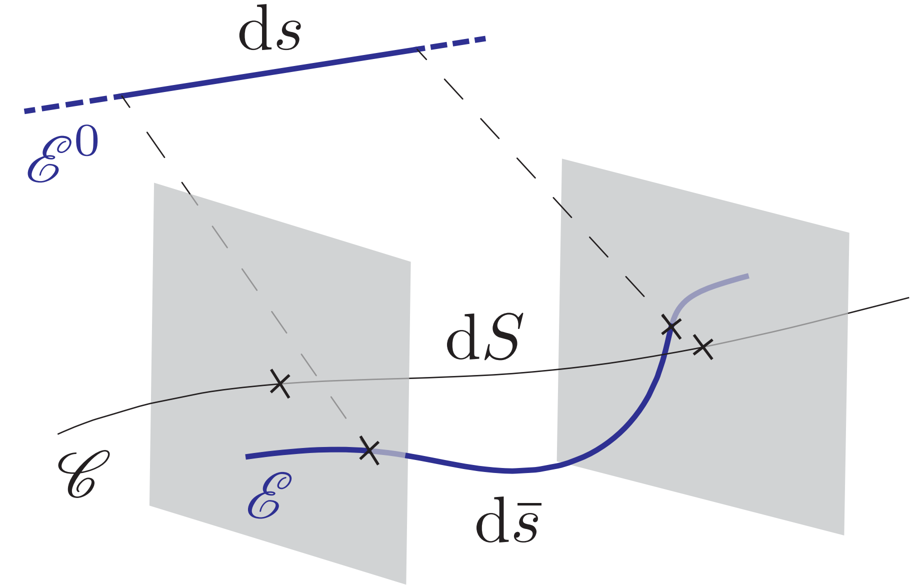

$$\left.\begin{aligned} &\boldsymbol{r}\left(s\left(S\right)\right) = \boldsymbol{R}\left(S\right) + \boldsymbol{\Delta}\left(S\right) \\ &\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \alpha\, \boldsymbol{d}_3 \end{aligned}\right\} \longrightarrow \quad \alpha\, \boldsymbol{d}_3 = \left(\boldsymbol{R}' + \boldsymbol{\Delta}'\right) \frac{\mathrm{d}S}{\mathrm{d}s}$$

→ Drift between $S$ and $s$:

- Eccentricity between the rod and the reference curve
- Stretch of the rod

# Jacobian of the Mapping




$$\left.\begin{aligned} \boldsymbol{r}\left(s\left(S\right)\right) &= \boldsymbol{R}\left(S\right) + \boldsymbol{\Delta}\left(S\right) \\ \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} &= \alpha\, \boldsymbol{d}_3 \end{aligned}\right\} \longrightarrow \quad \boxed{J_1 = \frac{\mathrm{d}S}{\mathrm{d}s} = \pm \frac{\alpha}{\left\|\boldsymbol{R}' + \boldsymbol{\Delta}'\right\|}}$$

→ Drift between $S$ and $s$:

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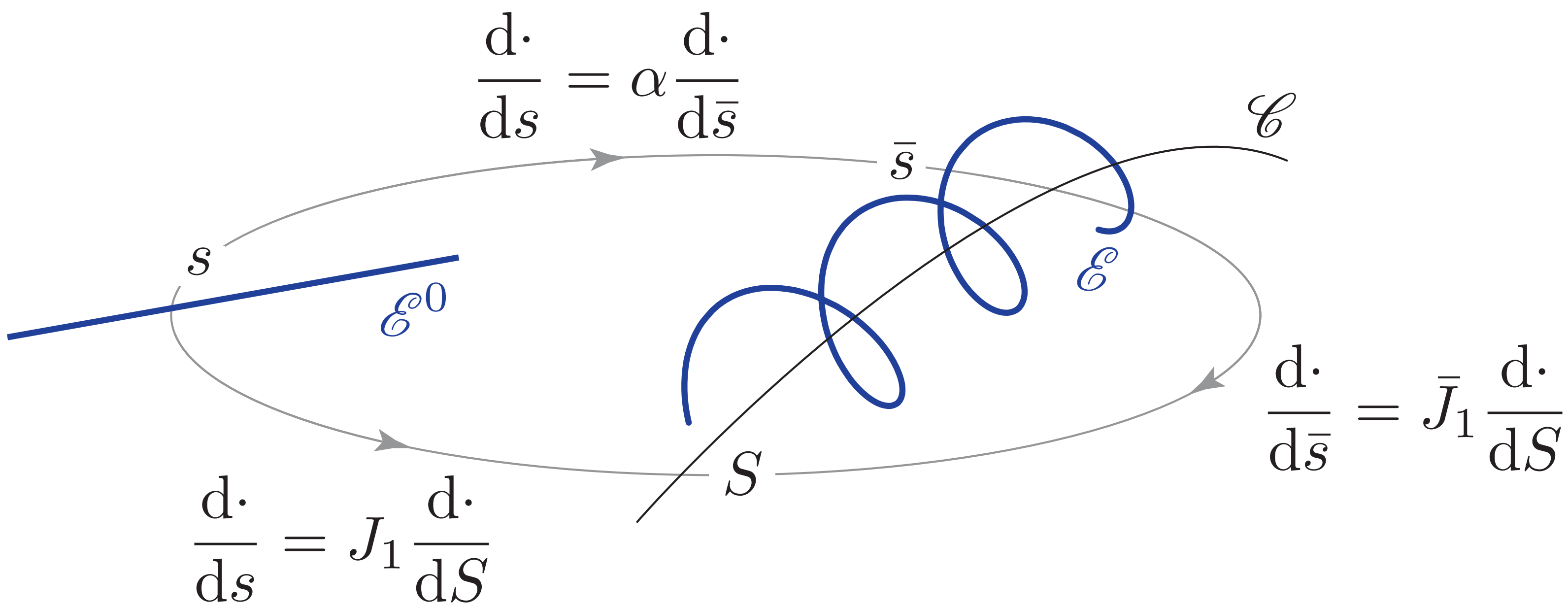
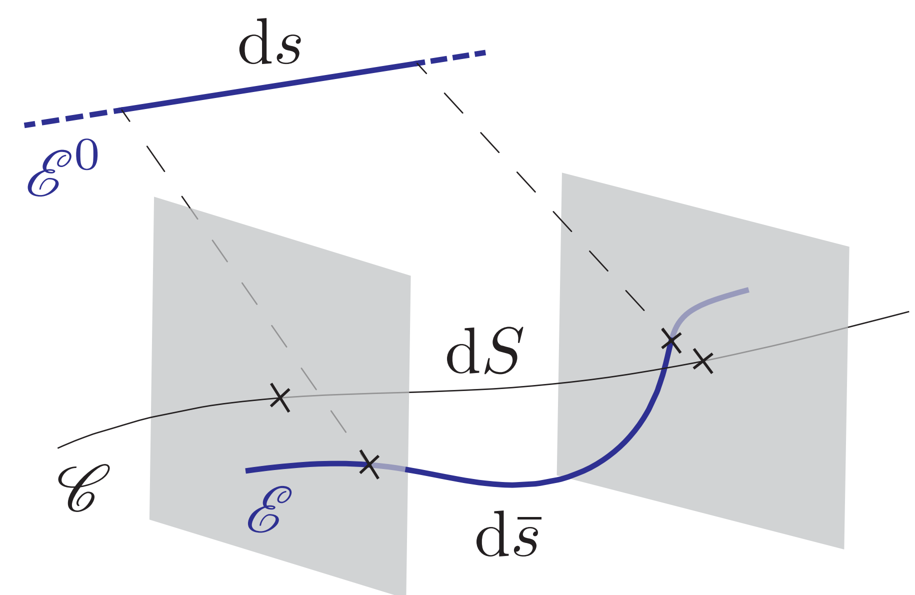
# Jacobian of the Mapping

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(Podgornik, 2002)

→ Drift between $S$ and $s$:

- Eccentricity between the rod and the reference curve
- Stretch of the rod

# Rod Attitude

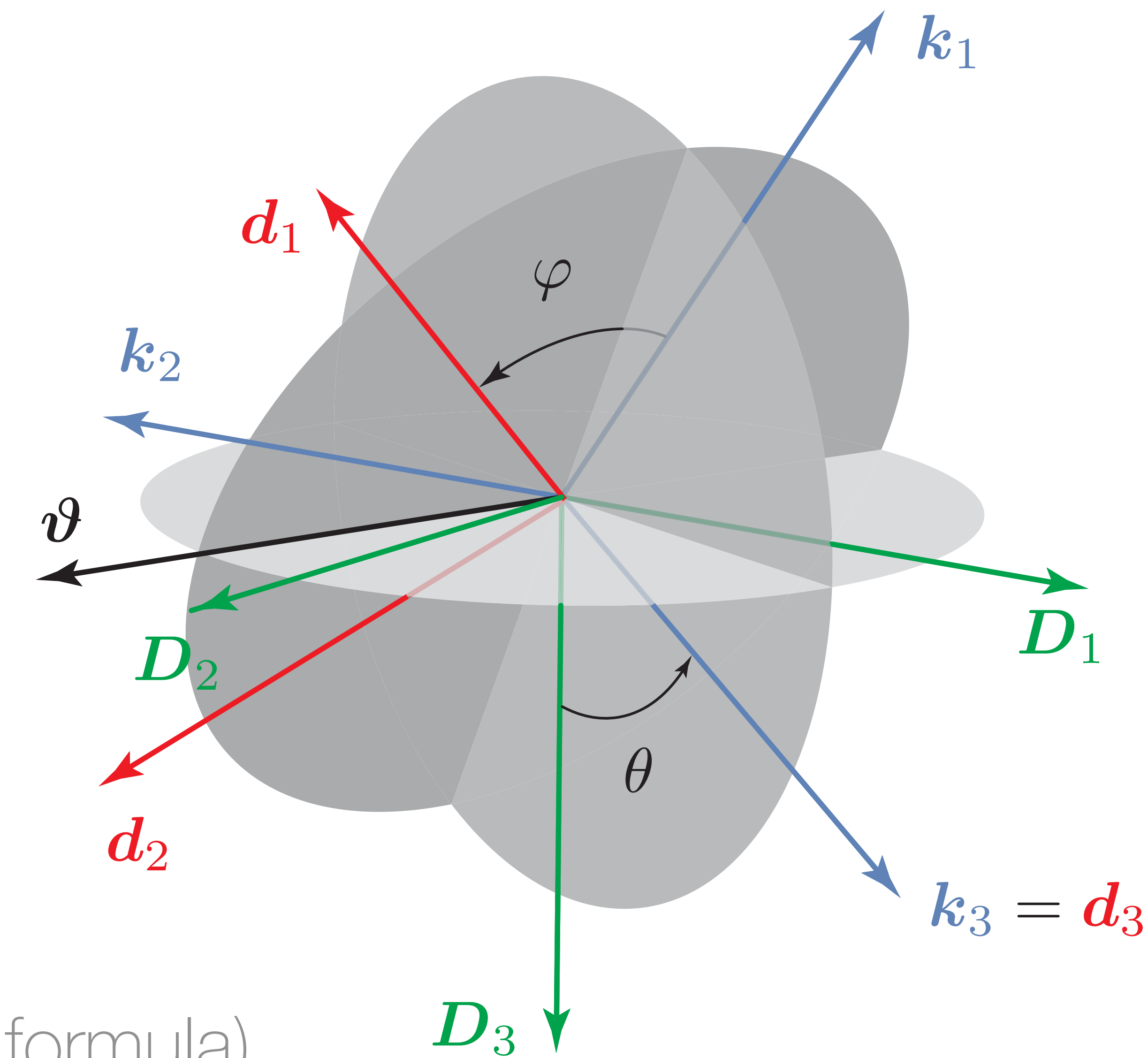
- Orientation of the rod directors $\{\boldsymbol{d}_j\}$

$$\boldsymbol{d}_1(S) = \cos\varphi\,\boldsymbol{k}_1 - \sin\varphi\,\boldsymbol{k}_2$$

$$\boldsymbol{d}_2(S) = \sin\varphi\,\boldsymbol{k}_1 + \cos\varphi\,\boldsymbol{k}_2$$

$$\boldsymbol{d}_3(S) = J_1\left(\boldsymbol{D}_3 + \boldsymbol{\Delta}'\right)/\alpha$$

where $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the images of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ through the rotation mapping $\boldsymbol{D}_3$ on $\boldsymbol{d}_3$




- Intermediate $\{\boldsymbol{k}_j\}$-basis (Rodrigues formula)

$$\boldsymbol{k}_j = \boldsymbol{D}_j + \boldsymbol{\vartheta}\times\boldsymbol{D}_j + \boldsymbol{\vartheta}\times(\boldsymbol{\vartheta}\times\boldsymbol{D}_j)\,/\,(1+\cos\theta)$$

with $\boldsymbol{\vartheta} = \boldsymbol{D}_3 \times \boldsymbol{d}_3$

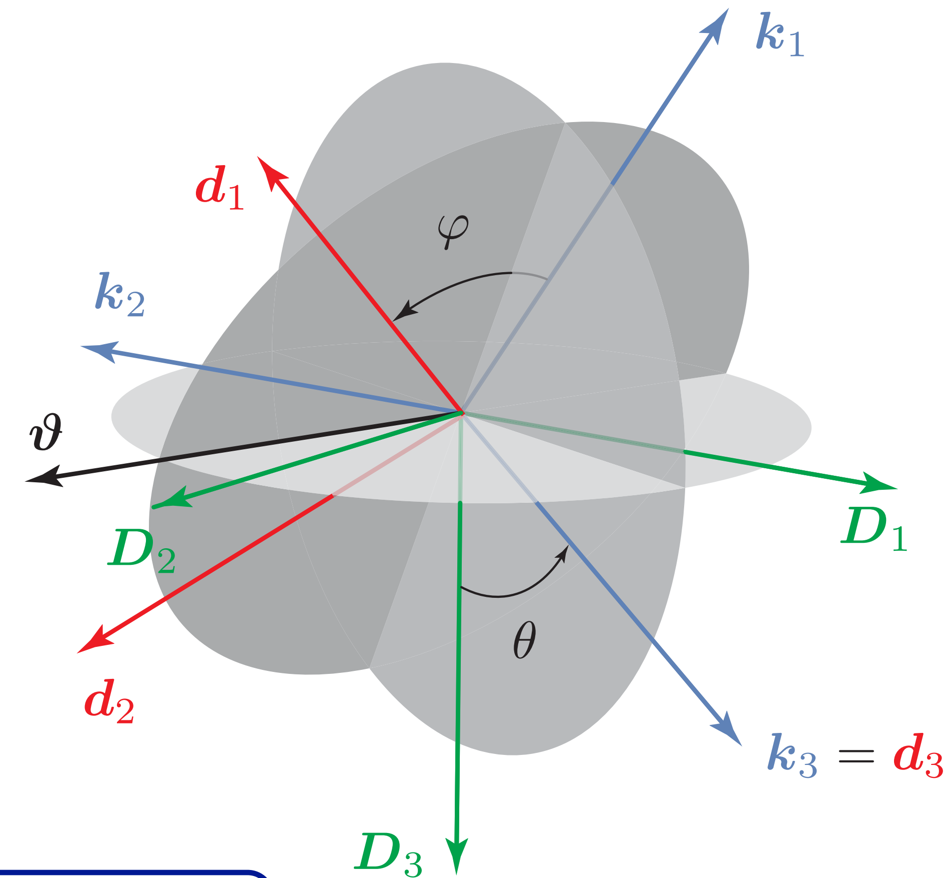
# Strain Variables

- Kinematic

$$\frac{\mathrm{d}\boldsymbol{D}_j}{\mathrm{d}S} = \boldsymbol{U} \times \boldsymbol{D}_j \longleftarrow \text{Known}$$

$$\frac{\mathrm{d}\boldsymbol{d}_j}{\mathrm{d}s} = \boldsymbol{u} \times \boldsymbol{d}_j$$

$$\frac{\mathrm{d}\boldsymbol{k}_j}{\mathrm{d}s} = \boldsymbol{w} \times \boldsymbol{k}_j \longrightarrow \boxed{\boldsymbol{u} = \boldsymbol{w}\,(\alpha, \boldsymbol{\Delta}, \boldsymbol{U}) + J_1\,\varphi'\,\boldsymbol{k}_3}$$




- Constitutive relations (circular cross section)

$$\boldsymbol{F} = \widetilde{F}_1\,\boldsymbol{k}_1 + \widetilde{F}_2\,\boldsymbol{k}_2 + A\,(\alpha - 1)\,\boldsymbol{k}_3$$

$$\boldsymbol{M} = \underbrace{B\,(\widetilde{w}_1\,\boldsymbol{k}_1 + \widetilde{w}_2\,\boldsymbol{k}_2)}_{\text{Bending}} + \underbrace{C\,(\widetilde{w}_3 + J_1\,\varphi')\,\boldsymbol{k}_3}_{\text{Twisting}}$$

$$\widetilde{F}_j = \boldsymbol{F} \cdot \boldsymbol{k}_j$$

$$\widetilde{w}_j = \boldsymbol{w} \cdot \boldsymbol{k}_j$$

# Governing Equations

- Mixed order nonlinear BVP

$$
\begin{aligned}
J_1\,\widetilde{F}_1' + A\,(\alpha-1)\,\widetilde{w}_2 - \widetilde{w}_3\,\widetilde{F}_2 + \widetilde{f}_1 &= 0\\
J_1\,\widetilde{F}_2' + \widetilde{w}_3\,\widetilde{F}_1 - A\,(\alpha-1)\,\widetilde{w}_1 + \widetilde{f}_2 &= 0\\
J_1\,\alpha'\,A + \widetilde{w}_1\,\widetilde{F}_2 - \widetilde{w}_2\,\widetilde{F}_1 + \widetilde{f}_3 &= 0\\
J_1\,\widetilde{w}_1'\,B - \widetilde{w}_2\,\widetilde{w}_3\,B + C\,\widetilde{w}_2\,(\widetilde{w}_3 + J_1\,\varphi') - \alpha\,\widetilde{F}_2 + \widetilde{m}_1 &= 0\\
J_1\,\widetilde{w}_2'\,B + \widetilde{w}_1\,\widetilde{w}_3\,B - C\,\widetilde{w}_1\,(\widetilde{w}_3 + J_1\,\varphi') + \alpha\,\widetilde{F}_1 + \widetilde{m}_2 &= 0\\
\left(J_1^2\,\varphi'' + J_2\,\varphi' + J_1\,\widetilde{w}_3'\right) C + \widetilde{m}_3 &= 0
\end{aligned}
$$

with 11 boundary conditions

$$
\begin{aligned}
\{\varphi\,(S_a)\,,\Delta_i\,(S_a)\,,\Delta_i'\,(S_a)\} &\qquad i=1,2\\
\{\alpha\,(S_b)\,,\varphi'\,(S_b)\,,\Delta_i\,(S_b)\,,\Delta_i'\,(S_b)\} &
\end{aligned}
$$

# Governing Equations

- Mixed order nonlinear BVP

$$J_1\,\widetilde{F}_1' + \mathcal{G}_1\left[\alpha, \boldsymbol{\Delta}, \boldsymbol{F}, \boldsymbol{U}\right] + \widetilde{f}_1 = 0$$

$$J_1\,\widetilde{F}_2' + \mathcal{G}_2\left[\alpha, \boldsymbol{\Delta}, \boldsymbol{F}, \boldsymbol{U}\right] + \widetilde{f}_2 = 0$$

$$J_1\,\alpha'\,A + \mathcal{G}_3\left[\alpha, \boldsymbol{\Delta}, \boldsymbol{F}, \boldsymbol{U}\right] + \widetilde{f}_3 = 0$$

$$J_1^3\,\Delta_1'''\,B + \mathcal{H}_1\left[\alpha, \boldsymbol{\Delta}, \boldsymbol{U}\right] - \alpha\,\widetilde{F}_2 + \widetilde{m}_1 = 0$$

$$J_1^3\,\Delta_2'''\,B + \mathcal{H}_2\left[\alpha, \boldsymbol{\Delta}, \boldsymbol{U}\right] + \alpha\,\widetilde{F}_1 + \widetilde{m}_2 = 0$$

$$J_1^2\,\varphi''\,C + \mathcal{H}_3\left[\alpha, \boldsymbol{\Delta}, \boldsymbol{U}\right] + \widetilde{m}_3 = 0$$

with 11 boundary conditions

$$\left\{\varphi\left(S_a\right), \Delta_i\left(S_a\right), \Delta_i'\left(S_a\right)\right\} \qquad i = 1, 2$$

$$\left\{\alpha\left(S_b\right), \varphi'\left(S_b\right), \Delta_i\left(S_b\right), \Delta_i'\left(S_b\right)\right\}$$

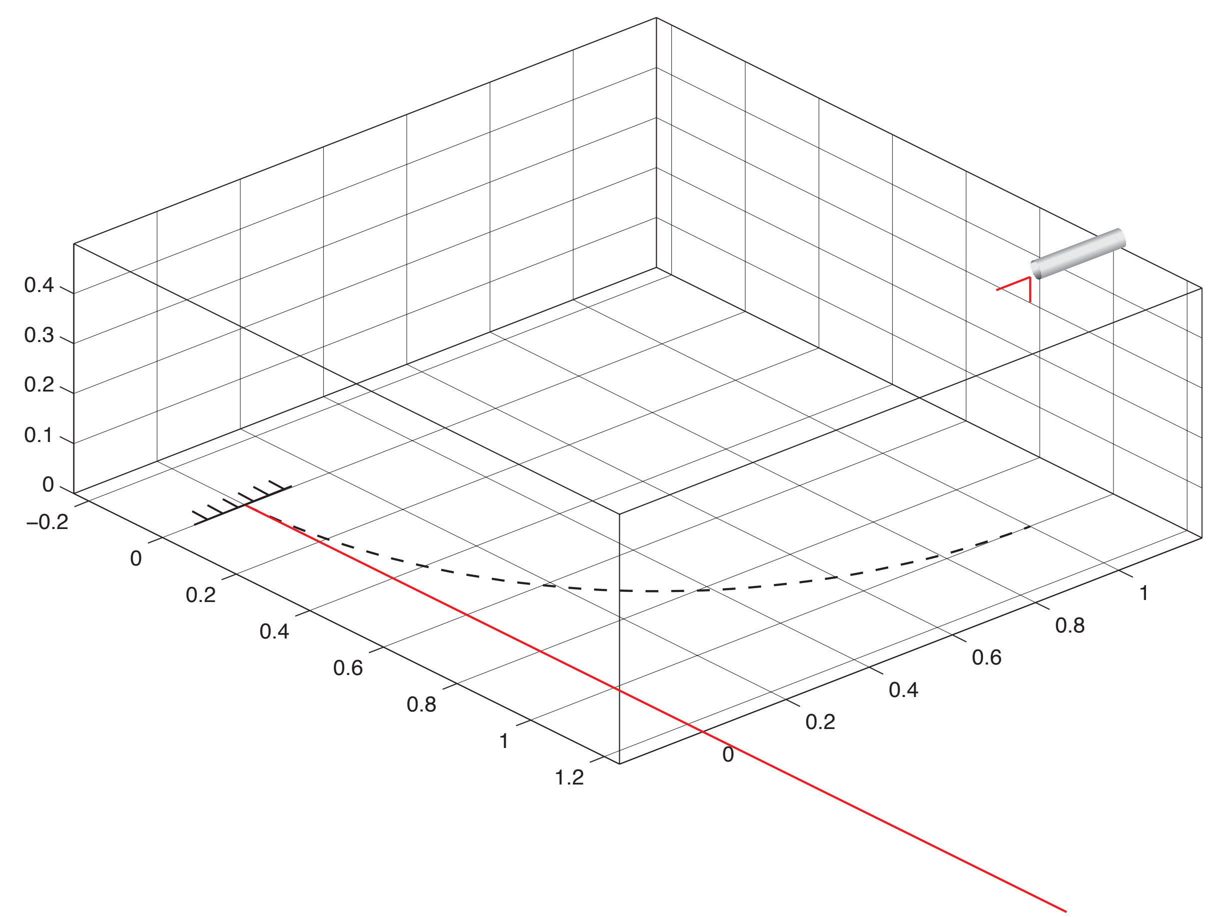
# Application

- Collocation method (Ascher et al., 1979)

$$\boldsymbol{\Delta}^* \in \mathcal{P}_{k+3,\pi} \cap C^2\left[S_a, S_b\right]$$
$$\varphi^* \in \mathcal{P}_{k+2,\pi} \cap C^1\left[S_a, S_b\right]$$
$$\boldsymbol{F}^* \in \mathcal{P}_{k+1,\pi} \cap C^0\left[S_a, S_b\right]$$

where $k \geq 3$ is the number of collocation points per subinterval and $\mathcal{P}_{n,\pi}$ is the set of all piecewise polynomial functions (*B*-splines) of order $n$

$$L = \frac{\pi}{4}$$
$$\widetilde{f}_1 = \widetilde{f}_2 = \widetilde{f}_3 = 0$$
$$F_3(L) = 0$$



# Application

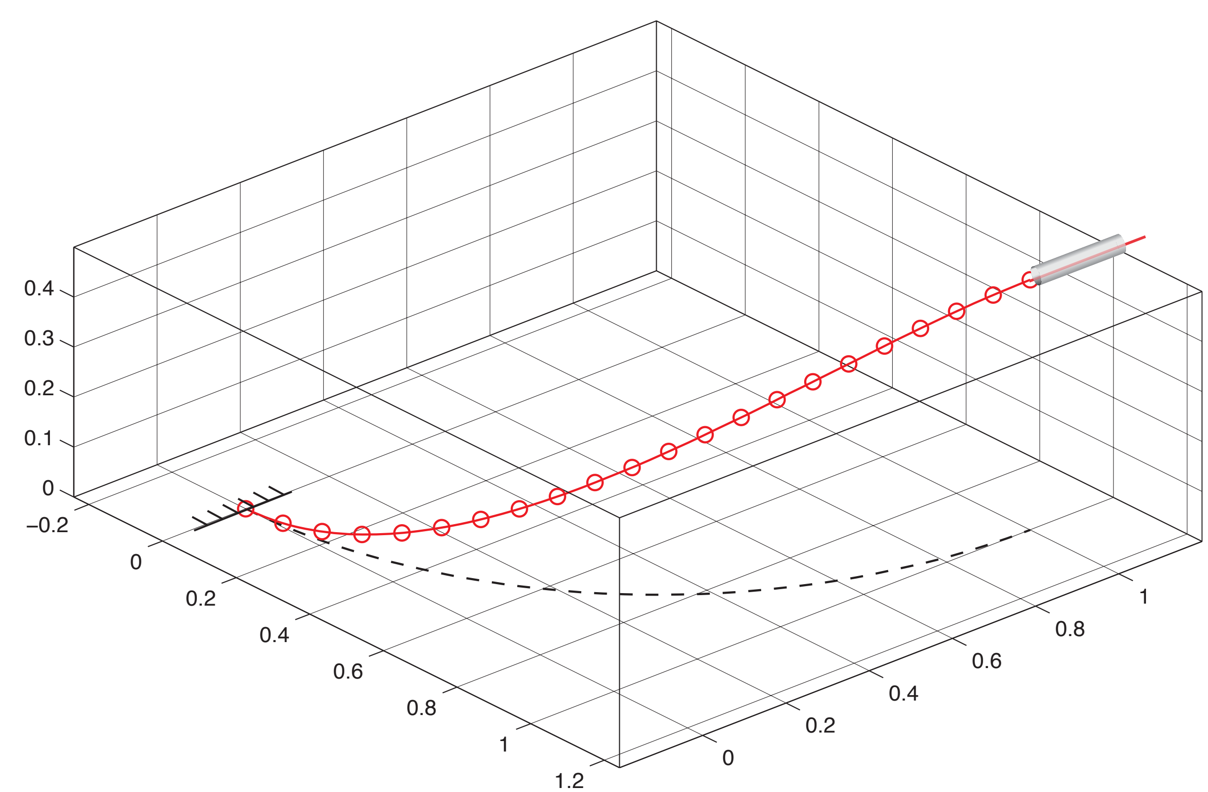
- Collocation method (Ascher et al., 1979)

$$\boldsymbol{\Delta}^* \in \mathcal{P}_{k+3,\pi} \cap C^2\left[S_a, S_b\right]$$
$$\varphi^* \in \mathcal{P}_{k+2,\pi} \cap C^1\left[S_a, S_b\right]$$
$$\boldsymbol{F}^* \in \mathcal{P}_{k+1,\pi} \cap C^0\left[S_a, S_b\right]$$

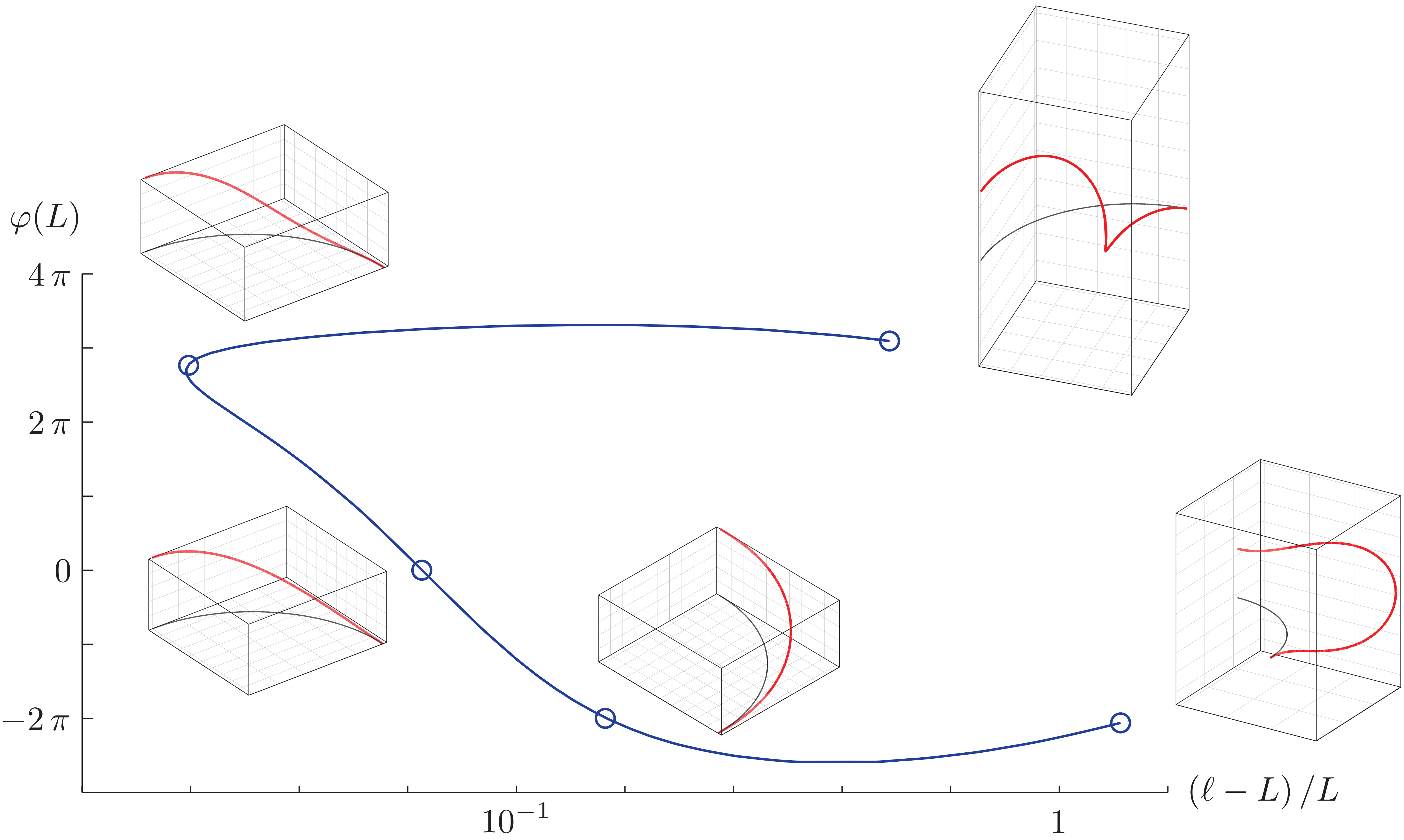
where $k \geq 3$ is the number of collocation points per subinterval and $\mathcal{P}_{n,\pi}$ is the set of all piecewise polynomial functions (*B*-splines) of order $n$

o o o Lagrangian
—— Eulerian

$$L = \frac{\pi}{4}$$
$$\widetilde{f}_1 = \widetilde{f}_2 = \widetilde{f}_3 = 0$$
$$F_3\left(L\right) = 0$$



# Self-feeding



1. Introduction
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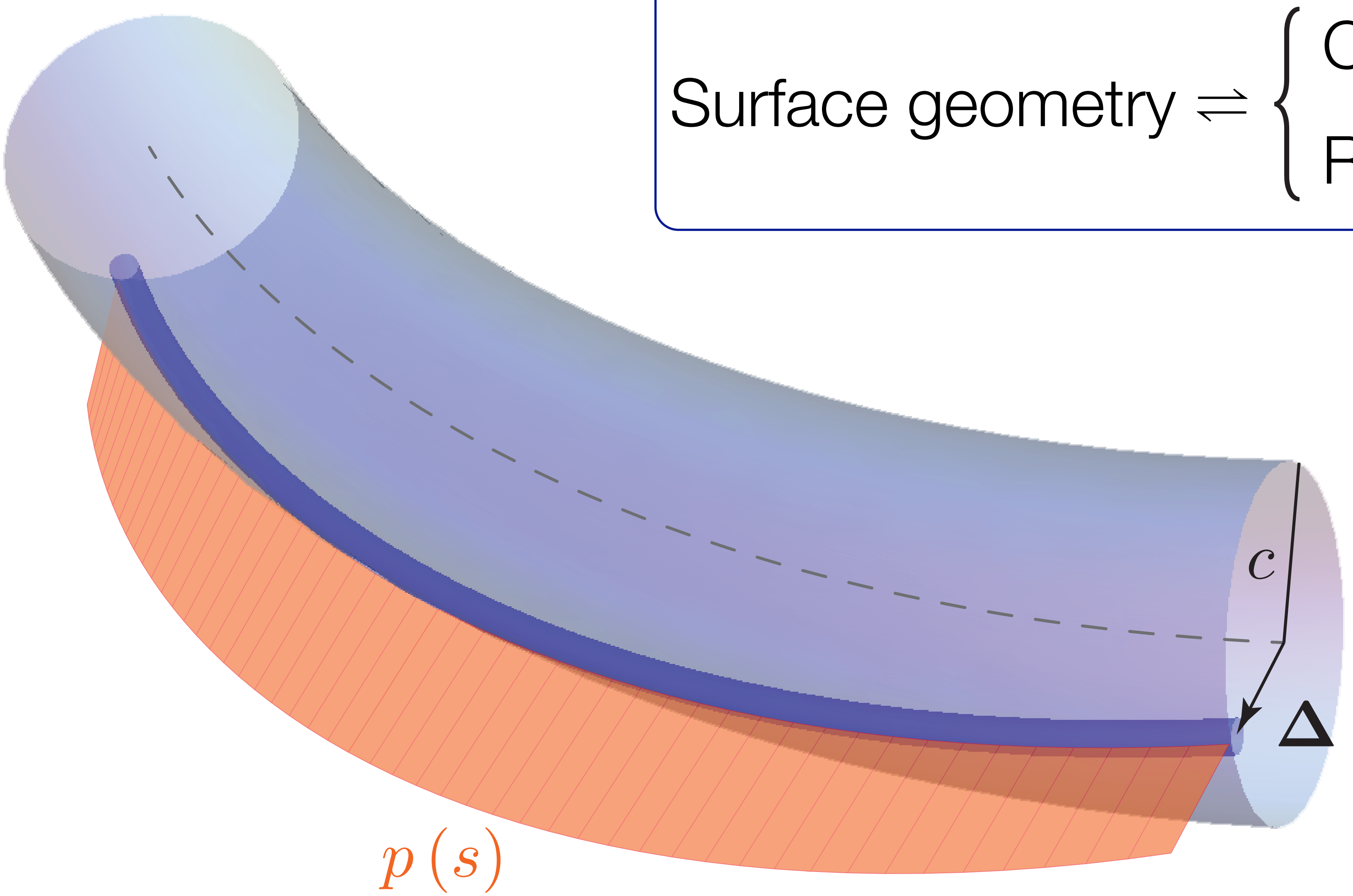
# Continuous Contact (frictionless - stiff)

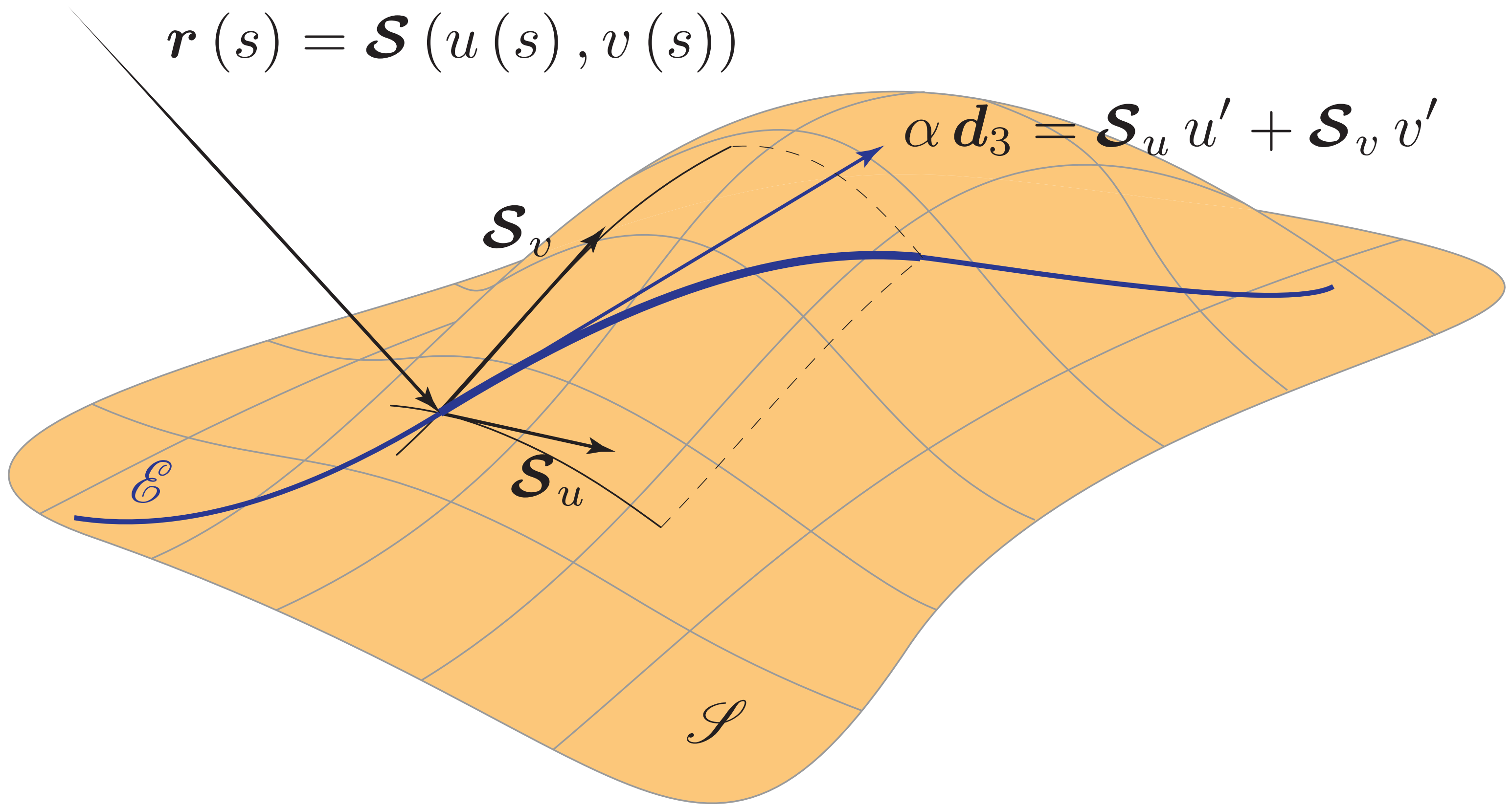
- Unknown reaction pressure $p(s)$ — +1 unknown
- Restriction of the solution to the surface — +1 equation

$$\text{Surface geometry} \rightleftharpoons \begin{cases} \text{Contact geometry} \\ \text{Reaction pressure} \end{cases}$$



# Surface Bound Rods

- Surface parameterization $\boldsymbol{\mathcal{S}}(u, v)$
- Rod axis re-parameterization

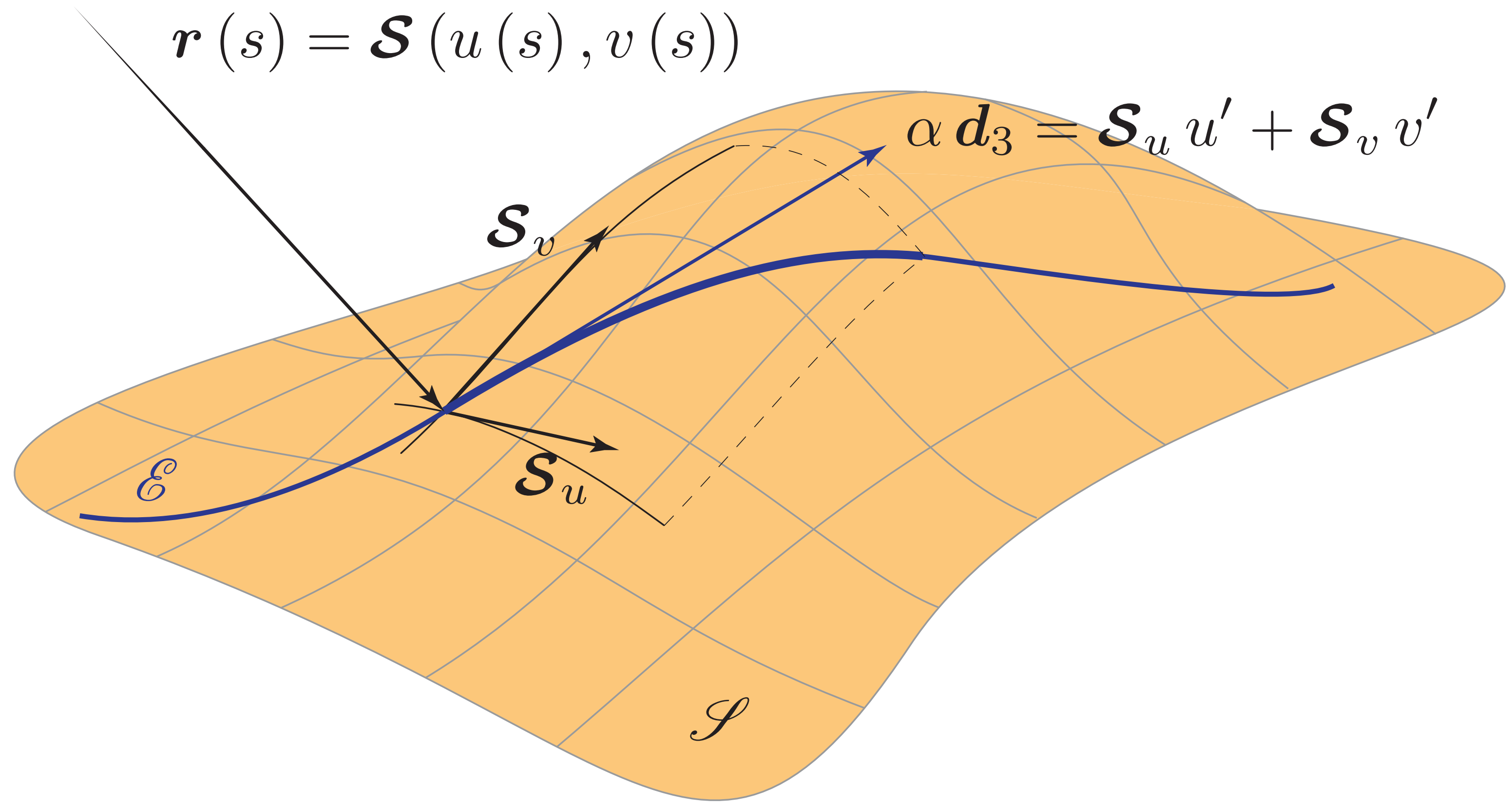



- Stretch $\alpha(s) = \|\boldsymbol{\mathcal{S}}_u u' + \boldsymbol{\mathcal{S}}_v v'\|$

  - Skew coordinate system $\{\boldsymbol{\mathcal{S}}_u, \boldsymbol{\mathcal{S}}_v\}$

# Surface Bound Rods

- Surface parameterization $\boldsymbol{\mathcal{S}}(u, v)$
- Rod axis re-parameterization




- Stretch $\alpha(s) = \sqrt{E\, u'^2 + 2\, F\, u'\, v' + G\, v'^2}$ (first fundamental form)$^{1/2}$

$$E(u, v) = \boldsymbol{\mathcal{S}}_u \cdot \boldsymbol{\mathcal{S}}_u \qquad F(u, v) = \boldsymbol{\mathcal{S}}_u \cdot \boldsymbol{\mathcal{S}}_v \qquad G(u, v) = \boldsymbol{\mathcal{S}}_v \cdot \boldsymbol{\mathcal{S}}_v$$

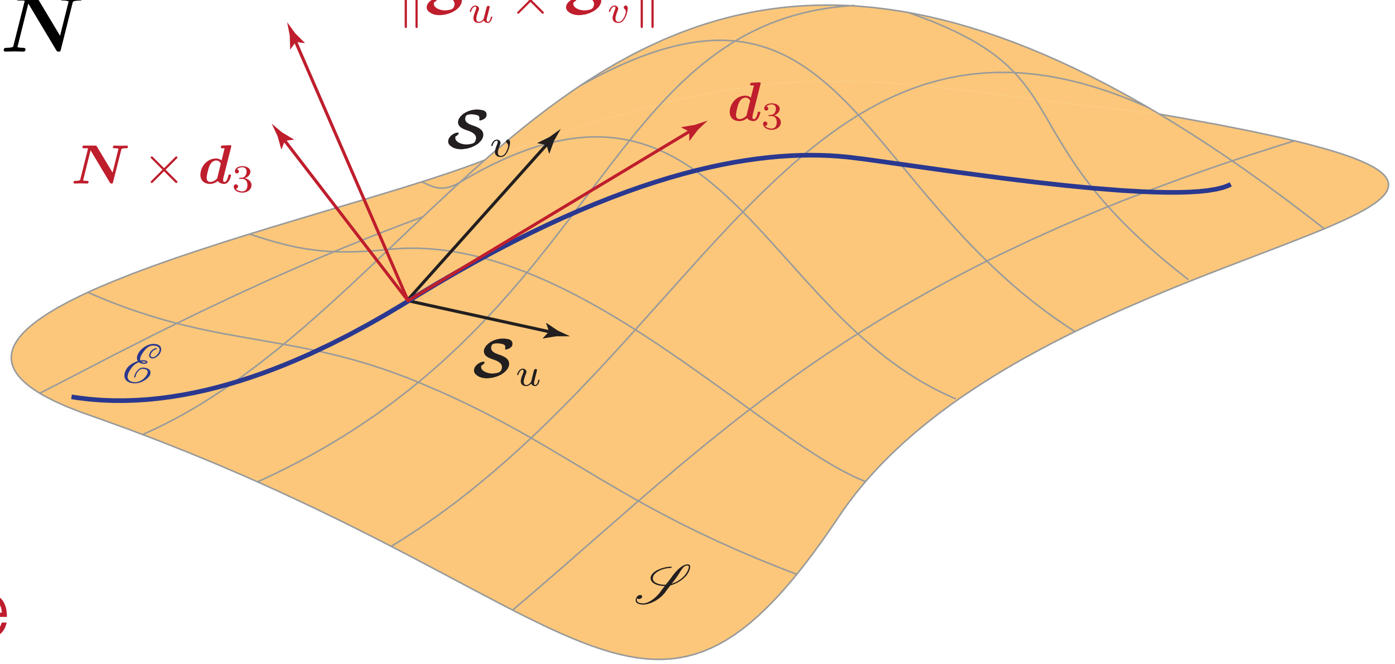
# Kinematics

- Orientation of the rod directors $\{\boldsymbol{d}_j(s)\}$

$$\boldsymbol{d}_1(s) = \cos\psi\,\boldsymbol{N}\times\boldsymbol{d}_3 + \sin\psi\,\boldsymbol{N}$$

$$\boldsymbol{d}_2(s) = -\sin\psi\,\boldsymbol{N}\times\boldsymbol{d}_3 + \cos\psi\,\boldsymbol{N}$$

$$\boldsymbol{d}_3(s) = \alpha^{-1}\left(\boldsymbol{\mathcal{S}}_u\,u' + \boldsymbol{\mathcal{S}}_v\,v'\right)$$




- Kinematics of the Darboux frame

$$\alpha^{-1}\frac{\mathrm{d}\boldsymbol{d}_3}{\mathrm{d}s} = \kappa_g\,\boldsymbol{N}\times\boldsymbol{d}_3 + \kappa_n\,\boldsymbol{N} \qquad \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \kappa\,\boldsymbol{n}$$

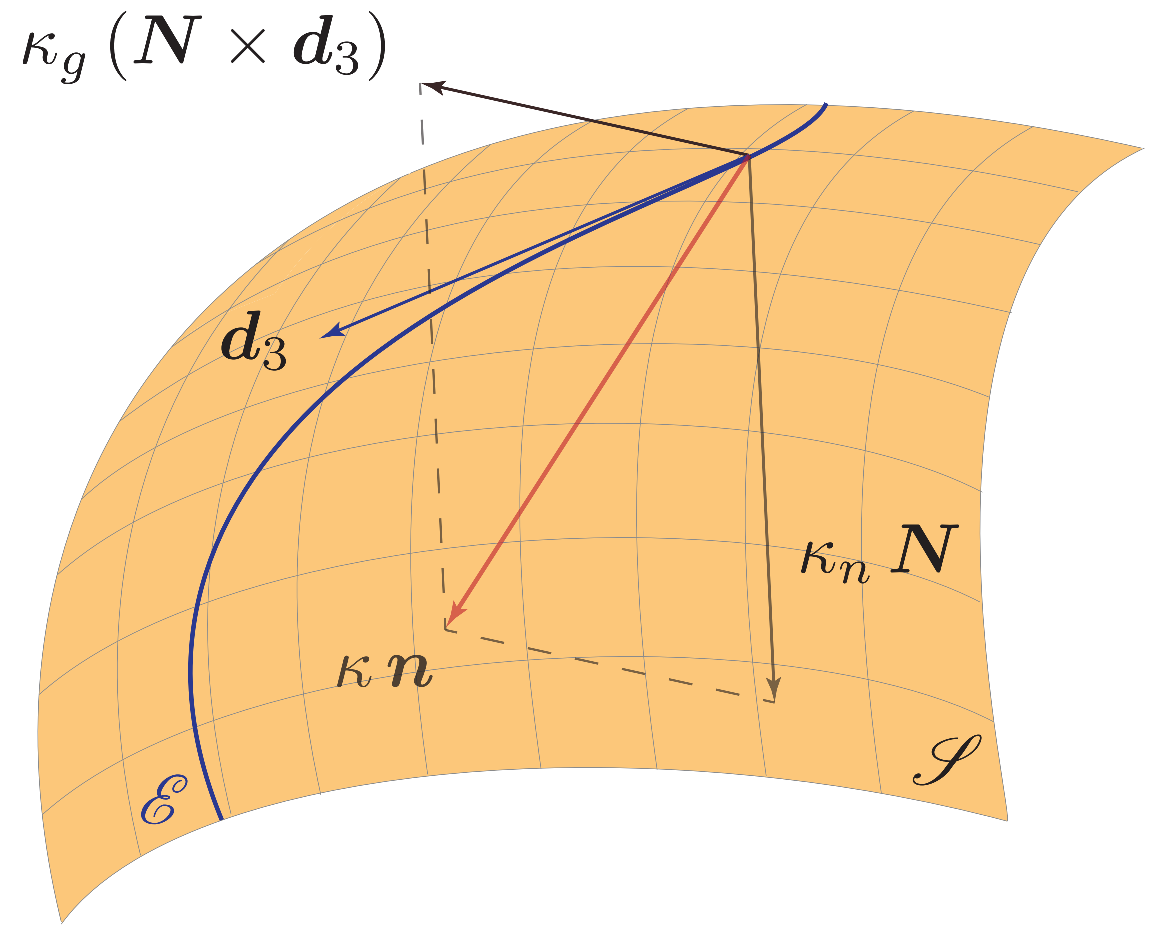
$$\alpha^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\left(\boldsymbol{N}\times\boldsymbol{d}_3\right) = -\kappa_g\,\boldsymbol{d}_3 + \tau_g\,\boldsymbol{N} \qquad \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}s} = -\tau\,\boldsymbol{n}$$

$$\alpha^{-1}\frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}s} = -\kappa_n\,\boldsymbol{d}_3 - \tau_g\,\boldsymbol{N}\times\boldsymbol{d}_3 \qquad \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s} = -\kappa\,\boldsymbol{t} + \tau\,\boldsymbol{b}$$

# Geometric Invariants

$$\alpha^{-1}\frac{\mathrm{d}\boldsymbol{d}_3}{\mathrm{d}s} = \kappa_g\,\boldsymbol{N}\times\boldsymbol{d}_3 + \kappa_n\,\boldsymbol{N}$$

$\kappa\,\boldsymbol{n}$




- Normal curvature (extrinsic)

$$\kappa_n = \frac{e\,u'^2 + 2\,f\,u'\,v' + g\,v'^2}{E\,u'^2 + 2\,F\,u'\,v' + G\,v'^2}$$

$$e\,(u,v) = \boldsymbol{\mathcal{S}}_{uu}\cdot\boldsymbol{N} \qquad f\,(u,v) = \boldsymbol{\mathcal{S}}_{uv}\cdot\boldsymbol{N} \qquad g\,(u,v) = \boldsymbol{\mathcal{S}}_{vv}\cdot\boldsymbol{N}$$

- Geodesic curvature (intrinsic)

$$\kappa_g = \sqrt{\frac{E\,G - F^2}{\left(E\,u'^2 + 2\,F\,u'\,v' + G\,v'^2\right)^3}}\,\Big[\Gamma^2_{11}u'^3 - \Gamma^1_{22}v'^3 + \left(2\,\Gamma^2_{12} - \Gamma^1_{11}\right)u'^2v' - \left(2\,\Gamma^1_{12} - \Gamma^2_{22}\right)u'v'^2 + u'\,v'' - u''\,v'\Big]$$

# Geometric Invariants

$$\alpha^{-1}\,\frac{\mathrm{d}}{\mathrm{d}s}\,(\boldsymbol{N}\times\boldsymbol{d}_3) = -\kappa_g\,\boldsymbol{d}_3 + \tau_g\,\boldsymbol{N}$$

$$\alpha^{-1}\,\frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}s} = -\kappa_n\,\boldsymbol{d}_3 - \tau_g\,\boldsymbol{N}\times\boldsymbol{d}_3$$

- Geodesic torsion (extrinsic)

$$\tau_g = \frac{(f\,E - e\,F)\,u'^2 + (g\,F - f\,G)\,v'^2 + (g\,E - e\,G)\,u'\,v'}{(E\,u'^2 + 2\,F\,u'\,v' + G\,v'^2)\,\sqrt{E\,G - F^2}}$$

- Constitutive relations (circular cross section)

$$\boldsymbol{F} = F_g\,\boldsymbol{N}\times\boldsymbol{d}_3 + F_n\,\boldsymbol{N} + A\,(\alpha - 1)\,\boldsymbol{d}_3$$

$$\boldsymbol{M} = \underbrace{\alpha\,B\,(\kappa_g\,\boldsymbol{N} - \kappa_n\,\boldsymbol{N}\times\boldsymbol{d}_3)}_{\text{Bending}} + \underbrace{C\,(\alpha\,\tau_g + \psi')\,\boldsymbol{d}_3}_{\text{Twisting}}$$

# Governing Equations

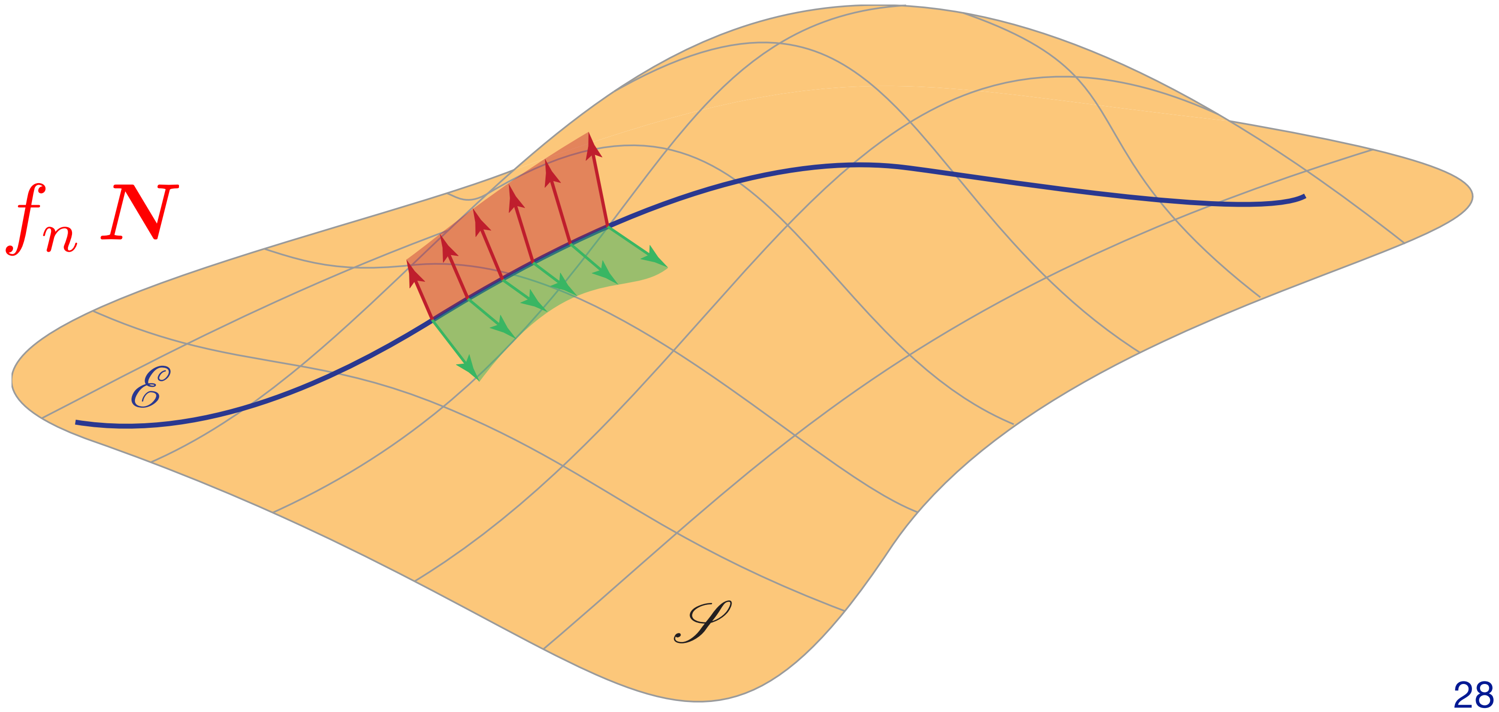
- Nonlinear differential-algebraic equations

$$
\begin{aligned}
F_g' + \alpha\,(\kappa_g\,F_3 - \tau_g\,F_n) + f_g &= 0\\
F_n' + \alpha\,(\kappa_n\,F_3 + \tau_g F_g) + f_n &= 0\\
A\,\alpha' - \alpha\,(\kappa_g\,F_g + \kappa_n\,F_n) + f_3 &= 0\\
B\,(\alpha\,\kappa_n)' + \alpha\,\kappa_g\,(B\,\alpha\,\tau_g - C\,u_3) + \alpha\,F_n &= 0\\
B\,(\alpha\,\kappa_g)' + \alpha\,\kappa_n\,(C\,u_3 - B\,\alpha\,\tau_g) + \alpha\,F_g &= 0\\
\psi'' + (\alpha\,\tau_g)' &= 0
\end{aligned}
$$

with

- Normal reaction pressure $f_n\,\boldsymbol{N}$
- Tangential force

  $f_g\,\boldsymbol{N} \times \boldsymbol{d}_3 + f_3\,\boldsymbol{d}_3$



# Weightless Elastica on a Sphere $(\alpha = 1)$

- Sphere of radius $R$

$$\boldsymbol{\mathcal{S}}\,(u,v) = R\,(\cos u\,\cos v\,\boldsymbol{e}_1 + \cos u\,\sin v\,\boldsymbol{e}_2 + \sin u\,\boldsymbol{e}_3)$$

  - Constant normal curvature and geodesic torsion

$$\kappa_n\,(u,v) = \frac{1}{R} \qquad \tau_g\,(u,v) = 0$$

- Scaling: $\ell^\star = R\,,\ F^\star = \dfrac{B}{R^2}$

$$\mathcal{F}_g = -\mathcal{M}_3 - \varkappa_g' \qquad \mathcal{F}_n = \varkappa_g\,\mathcal{M}_3 \qquad \mathcal{F}_3 = \frac{\varkappa_0^2 - \varkappa_g^2}{2}$$

- Governing equation for $\varkappa_g = \kappa_g\,R$

$$2\,\varkappa_g'' + \left(\varkappa_g^2 - \varkappa_0^2\right)\varkappa_g = 0$$

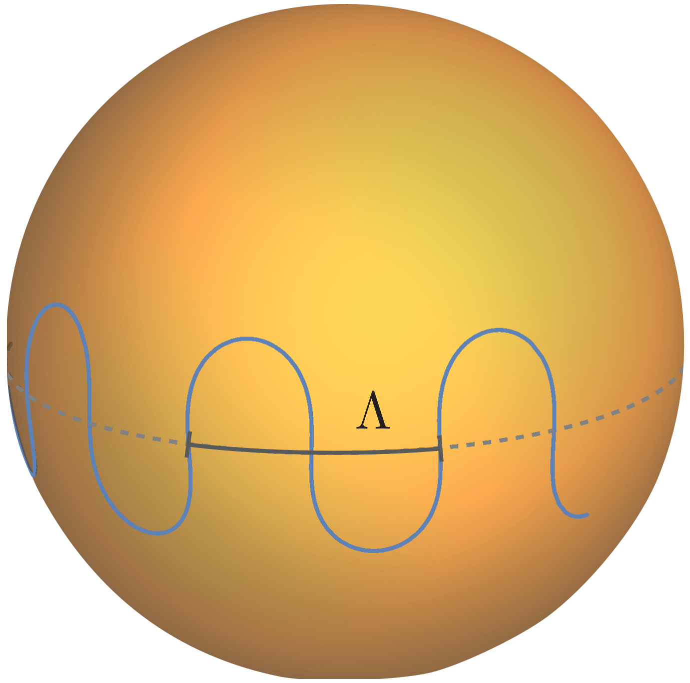
Independent of $\mathcal{M}_3$

# Two Families of Solutions (Love, 1927; Langer *et al.*, 1984)

- Inflexional (or *wavelike*) $\varkappa_m^2 > 2\,\varkappa_0^2$

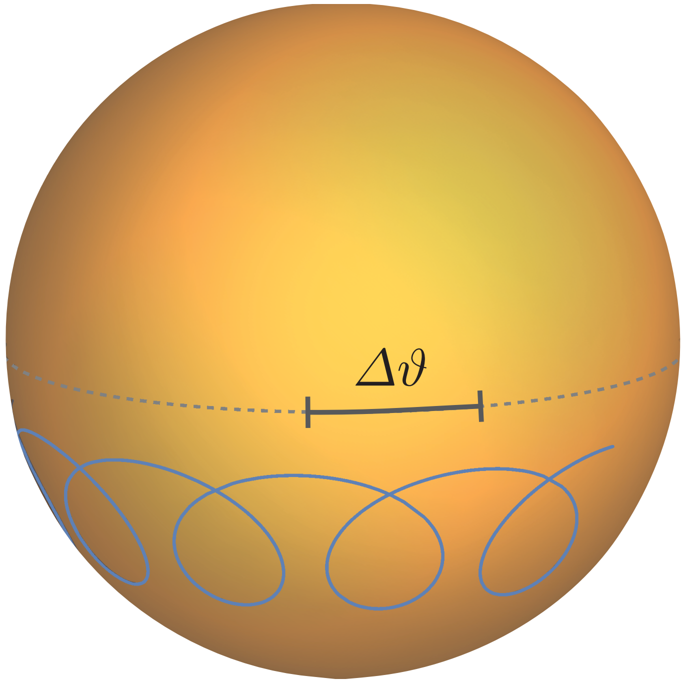
$$\varkappa_g(\chi) = \varkappa_m \,\mathrm{cn}\left(\frac{\chi - \chi_m}{2\,k}\varkappa_m, k^2\right)$$

with $$k^2 = \frac{\varkappa_m^2}{2\left(\varkappa_m^2 - \varkappa_0^2\right)}$$




- Non-inflexional (or *orbitlike*) $\varkappa_m^2 \leq 2\,\varkappa_0^2$

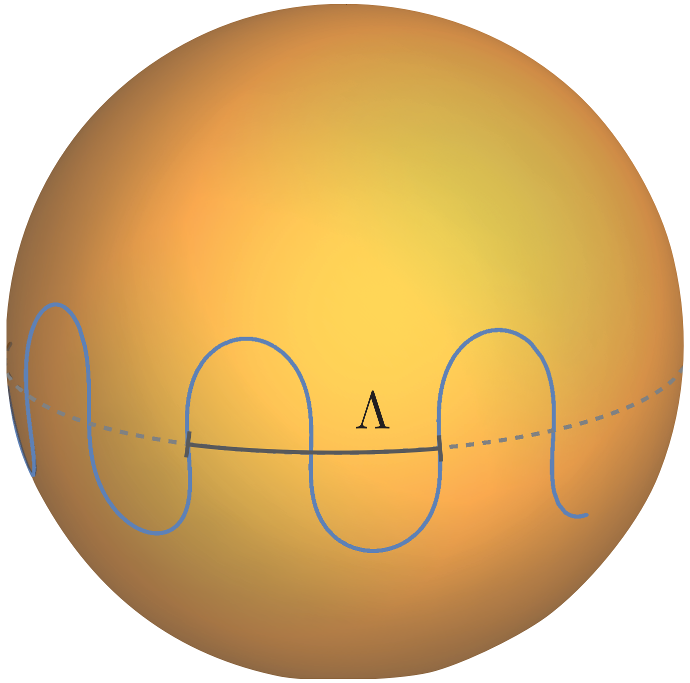
$$\varkappa_g(\chi) = \varkappa_m \,\mathrm{dn}\left(\frac{\chi - \chi_m}{2}\varkappa_m, \frac{1}{k^2}\right)$$




- Reaction pressure $$\rho(\chi) = \frac{\varkappa_g^2 - \varkappa_0^2}{2} - \varkappa_g' \,\mathcal{M}_3$$

# Closed Solutions

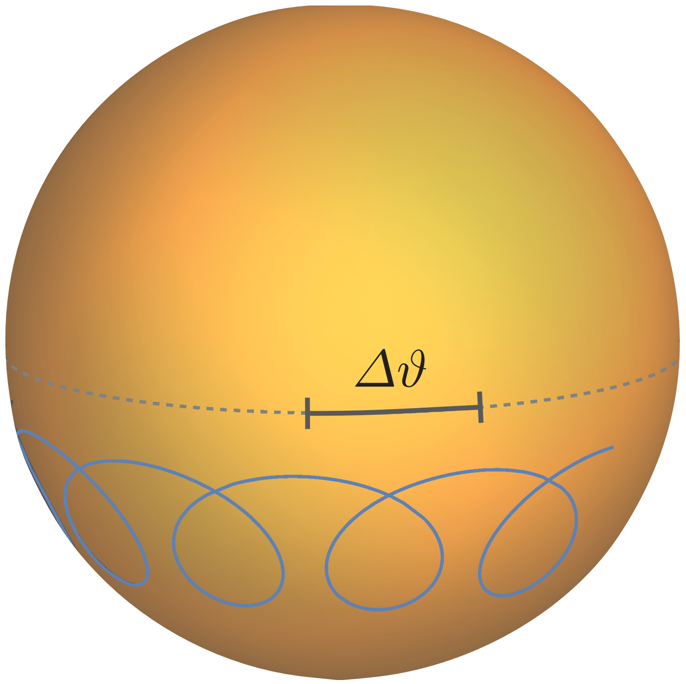
$$q = \begin{cases} \frac{\Lambda}{2\,\pi} & (\varkappa_m^2 > 2\,\varkappa_0^2) \\ \frac{\Delta\vartheta}{2\,\pi} & (\varkappa_m^2 \leq 2\,\varkappa_0^2) \end{cases}$$

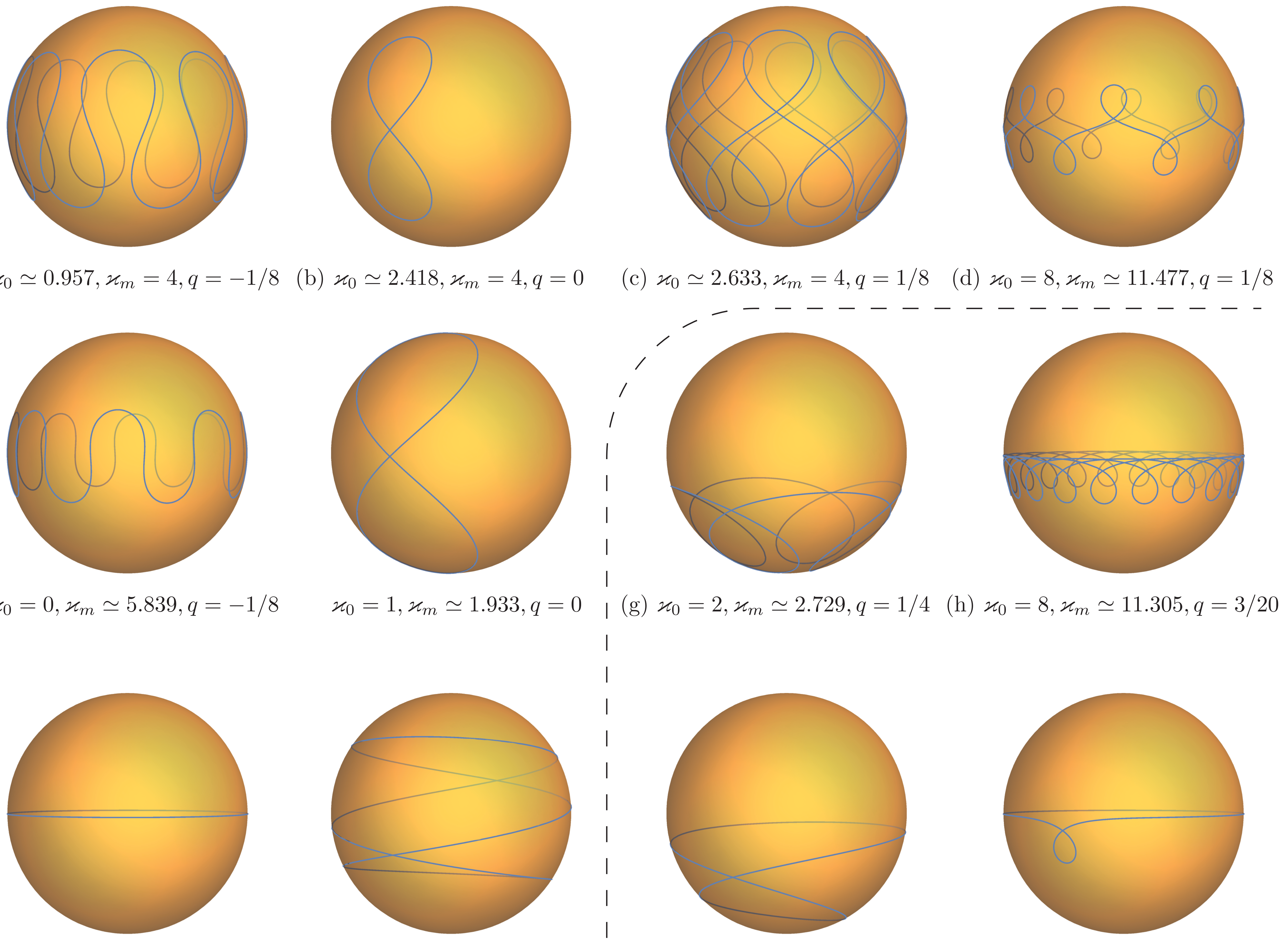



If $\boxed{q = \frac{n}{m} \in \mathbb{Q}}$

The elastica closes after

- $m$ periods of $\varkappa_g$
- going $n$ times around the poles





(a) $\varkappa_0 \simeq 0.957, \varkappa_m = 4, q = -1/8$ (b) $\varkappa_0 \simeq 2.418, \varkappa_m = 4, q = 0$ (c) $\varkappa_0 \simeq 2.633, \varkappa_m = 4, q = 1/8$ (d) $\varkappa_0 = 8, \varkappa_m \simeq 11.477, q = 1/8$

(e) $\varkappa_0 = 0, \varkappa_m \simeq 5.839, q = -1/8$ $\varkappa_0 = 1, \varkappa_m \simeq 1.933, q = 0$ (g) $\varkappa_0 = 2, \varkappa_m \simeq 2.729, q = 1/4$ (h) $\varkappa_0 = 8, \varkappa_m \simeq 11.305, q = 3/20$

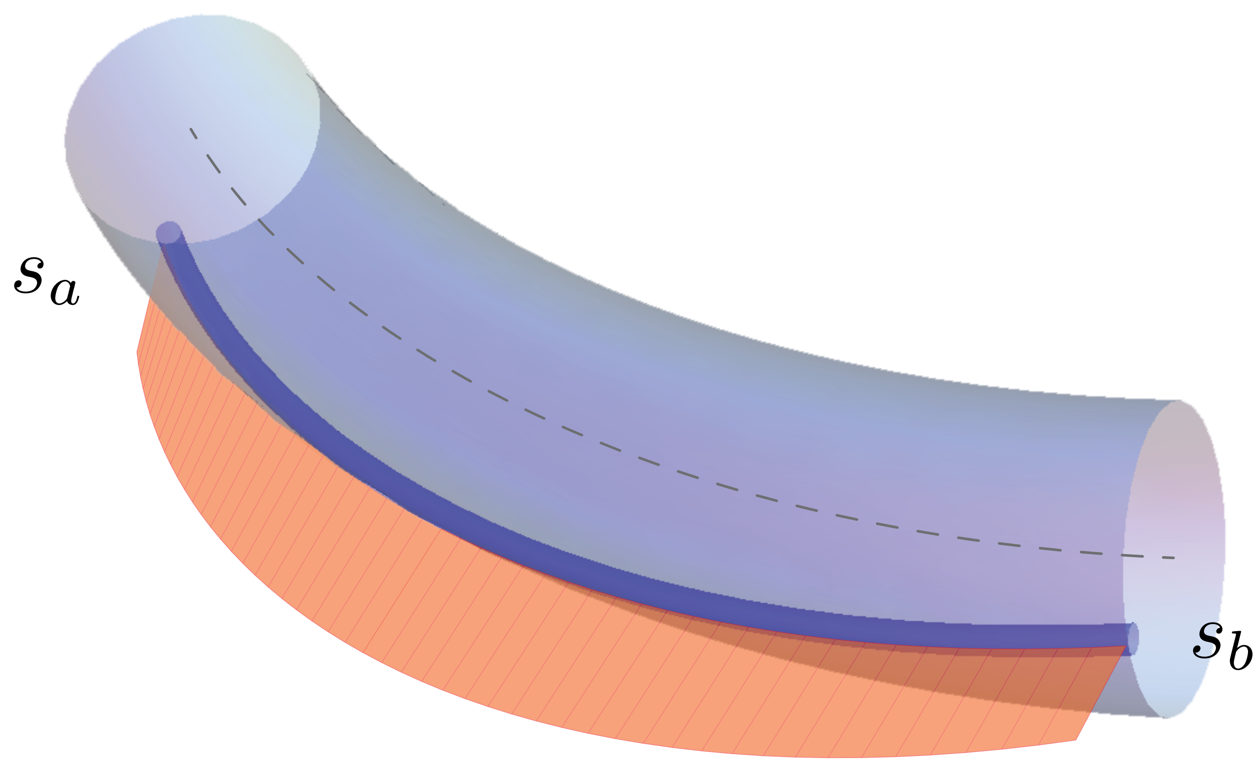
(i) $\varkappa_m = 0$ (j) $\varkappa_0 = 0, \varkappa_m \simeq 0.606, q = 3$ (k) $\varkappa_0 = 1, \varkappa_m \simeq 0.164, q = 2$ (l) $\varkappa_0 = 8, \varkappa_m \simeq 1.2 \times 10^{-7}, q = 1$

# Issues

- Lagrangian formulation $\boldsymbol{r}(s) = \boldsymbol{\mathcal{S}}(u(s), v(s))$
- Isoperimetric constraints (boundary conditions)

$x_j(s_a)$

$x'_j(s_a)$




$$x_j(s_b) = x_j(s_a) + \int_{s_a}^{s_b} \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} \cdot \boldsymbol{e}_j \, \mathrm{d}s$$

$x'_j(s_b)$

→ Integral constraints on the *unknown length* $\ell = s_b - s_a$ of the rod

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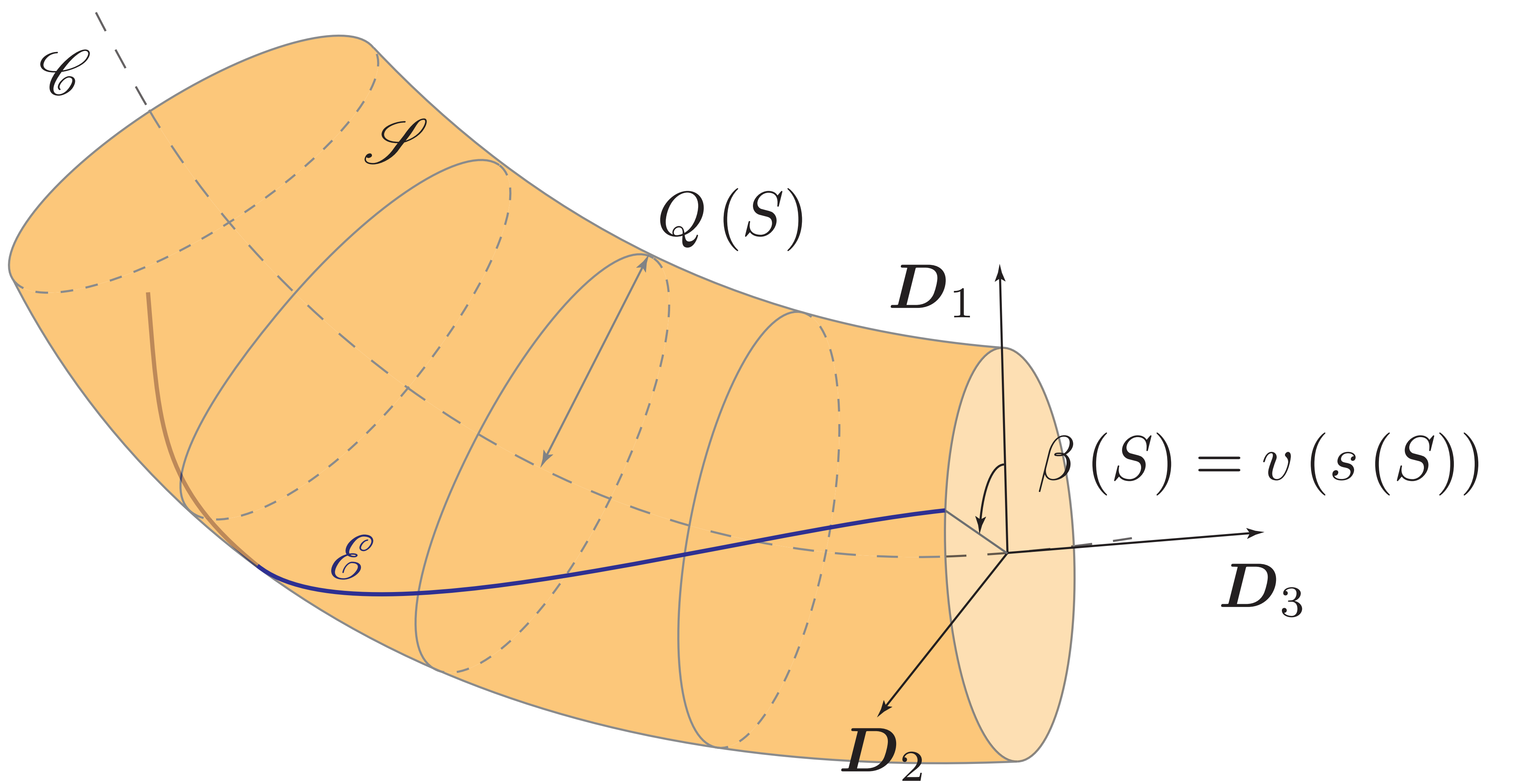
# Eulerian Formulation for Surface Bound Rods

- Normal ringed surface $\left(\boldsymbol{R}' \cdot \boldsymbol{D}_1 = \boldsymbol{R}' \cdot \boldsymbol{D}_2 = 0\right)$

$$\boldsymbol{\mathcal{S}}\left(u, v\right) = \boldsymbol{R}\left(u\right) + Q\left(u\right)\left(\cos v\, \boldsymbol{D}_1\left(u\right) + \sin v\, \boldsymbol{D}_2\left(u\right)\right)$$

- Lagrangian formulation $\boldsymbol{r}\left(s\right) = \boldsymbol{\mathcal{S}}\left(u\left(s\right), v\left(s\right)\right)$ (parametric)
- Eulerian formulation $\boldsymbol{r}\left(s\left(S\right)\right) = \boldsymbol{\mathcal{S}}\left(S, \beta\left(S\right)\right)$ (explicit)



# Governing Equations

- Mixed order nonlinear BVP (differential-algebraic equations)

$$J_1\,F_g' + \alpha\,(\kappa_g\,F_3 - \tau_g\,F_n) + f_g = 0$$
$$J_1\,F_n' + \alpha\,(\kappa_n\,F_3 + \tau_g F_g) + f_n + p = 0$$
$$J_1\,\alpha'\,A - \alpha\,(\kappa_g\,F_g + \kappa_n\,F_n) + f_3 = 0$$
$$J_1\;(\alpha\,\kappa_g)'\,B + \alpha\,\kappa_n\,(C\,u_3 - B\,\alpha\,\tau_g) + \alpha\,F_g + m_g = 0$$
$$J_1\;(\alpha\,\kappa_n)'\,B + \alpha\,\kappa_g\,(B\,\alpha\,\tau_g - C\,u_3) + \alpha\,F_n + m_n = 0$$
$$\left(J_1^2\,\psi'' + \alpha'\,\tau_g + \alpha\,\tau_g'\right)C + m_3 = 0$$

with 7 boundary conditions

$$\{\beta\,(S_a)\,,\beta'\,(S_a)\,,\psi\,(S_a)\}$$
$$\{\alpha\,(S_b)\,,\beta\,(S_b)\,,\beta'\,(S_b)\,,\psi'\,(S_b)\}$$

# Governing Equations

- Mixed order nonlinear BVP (differential-algebraic equations)

$$J_1\, F_g' + \mathcal{G}_g\left[\alpha, \beta, \boldsymbol{F}, \boldsymbol{U}\right] + f_g = 0$$

$$J_1\, F_n' + \mathcal{G}_n\left[\alpha, \beta, \boldsymbol{F}, \boldsymbol{U}\right] + f_n + p = 0$$

$$J_1\, \alpha'\, A + \mathcal{G}_3\left[\alpha, \beta, \boldsymbol{F}, \boldsymbol{U}\right] + f_3 = 0$$

$$J_1^3\, \beta''' B + \mathcal{H}_g\left[\alpha, \beta, \boldsymbol{U}\right] + \alpha\, F_g + m_g = 0$$

$$J_1^2\, \beta'' B + \mathcal{H}_n\left[\alpha, \beta, \boldsymbol{U}\right] + \alpha\, F_n + m_n = 0$$

$$J_1^2\, \psi''\, C + \mathcal{H}_3\left[\alpha, \beta, \boldsymbol{U}\right] + m_3 = 0$$

with 7 boundary conditions

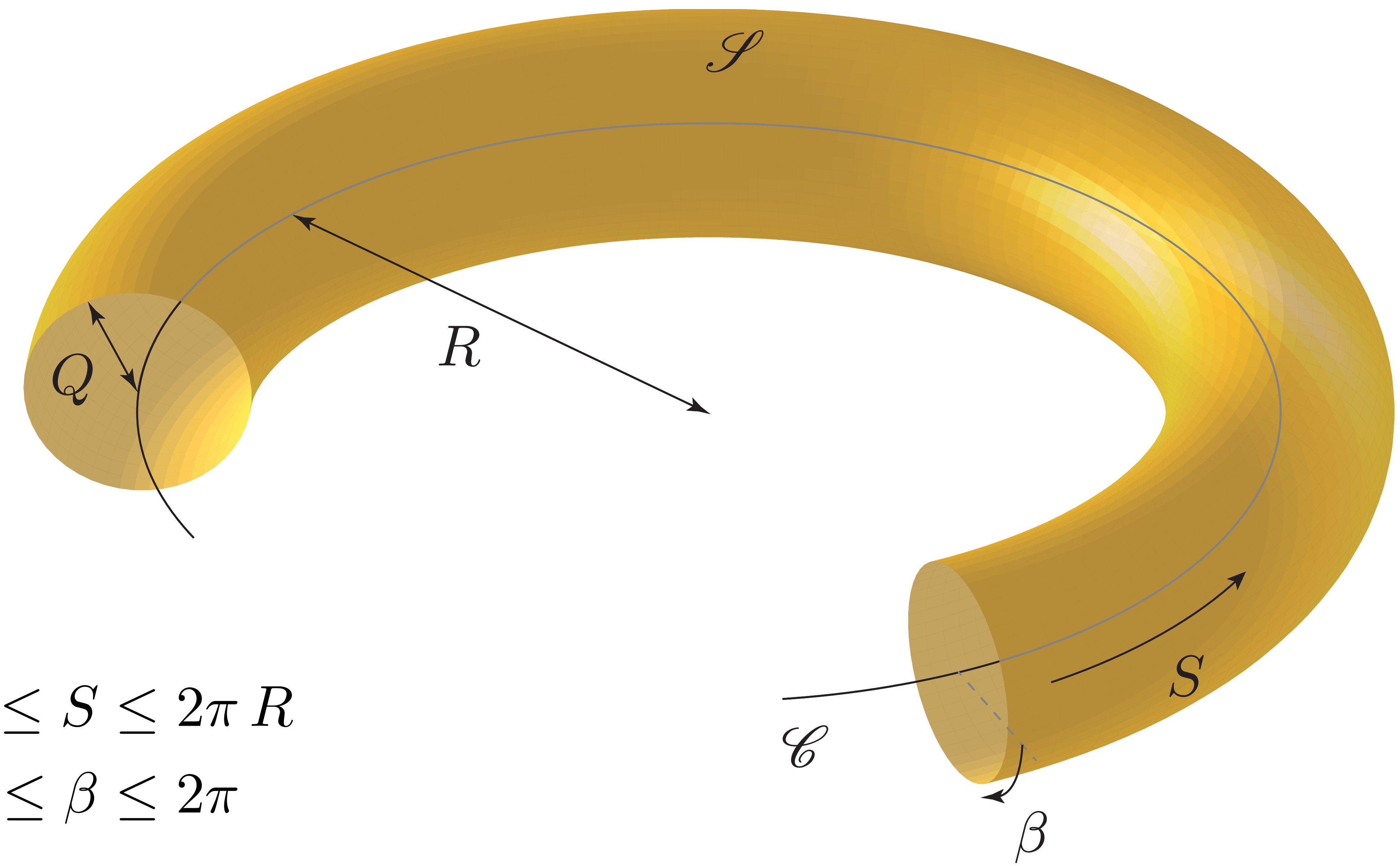
$$\left\{\beta\left(S_a\right), \beta'\left(S_a\right), \psi\left(S_a\right)\right\}$$

$$\left\{\alpha\left(S_b\right), \beta\left(S_b\right), \beta'\left(S_b\right), \psi'\left(S_b\right)\right\}$$

# Weightless Elastica on the Torus $\quad (\alpha = 1)$

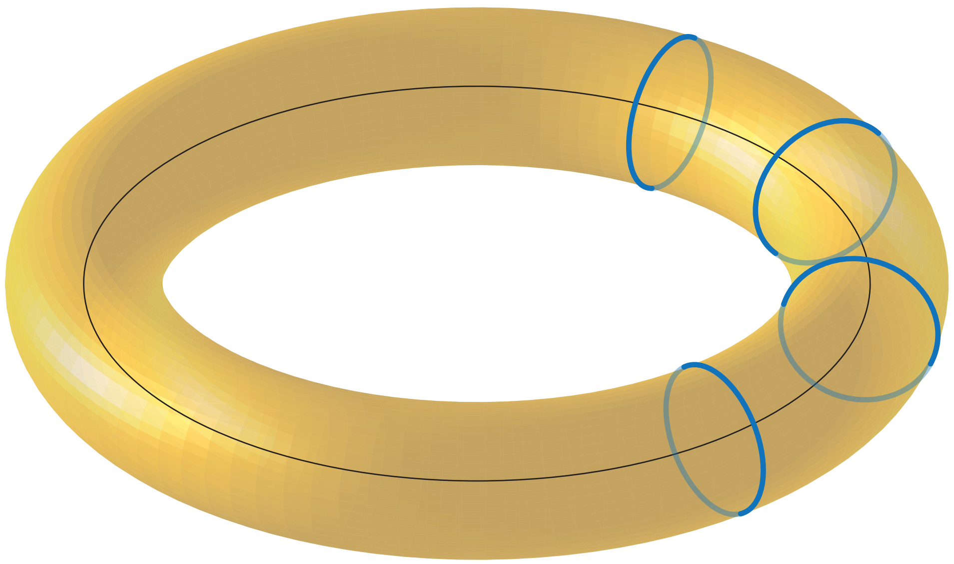
- Constant major $R$ and minor $Q$ radii $\quad (R > Q)$

$$\boldsymbol{\mathcal{S}}\,(S, \beta) = (R + Q\,\cos\beta)\cos\frac{S}{R}\,\boldsymbol{e}_1 + (R + Q\,\cos\beta)\sin\frac{S}{R}\,\boldsymbol{e}_2 + Q\,\sin\beta\,\boldsymbol{e}_3$$

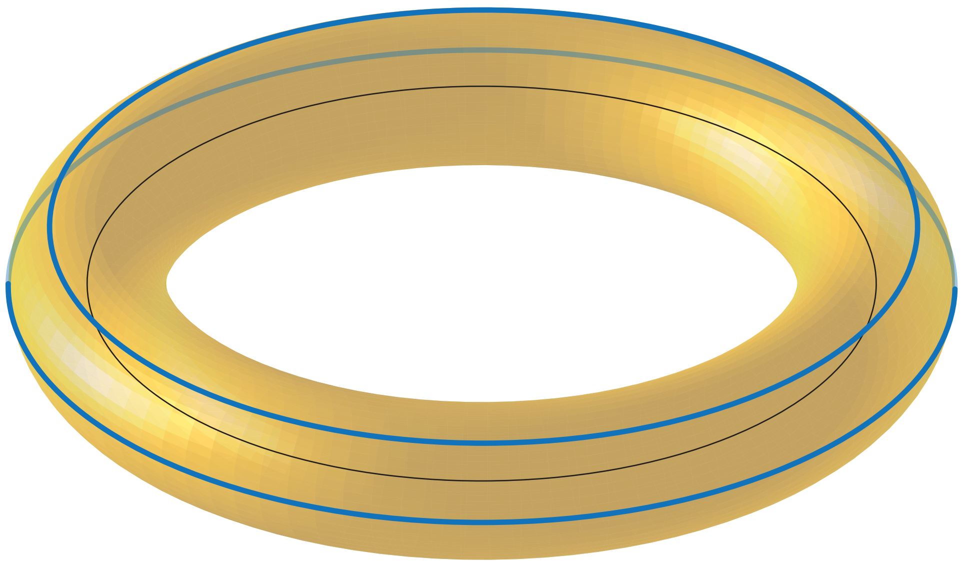



$$0 \le S \le 2\pi\,R$$
$$0 \le \beta \le 2\pi$$

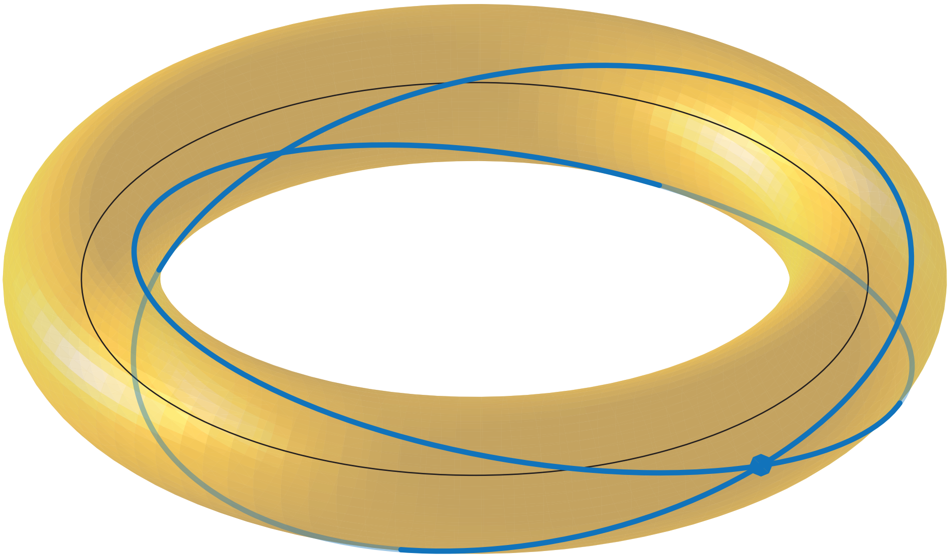
# Closed Solutions on the Torus



Meridians $S = \text{cst.}$



Parallels $\beta = \text{cst.}$



Villarceau circles

4 families of circles

# Elastic Torus Knots

- Closed solutions topologically equivalent to $(p, q)$-torus knots
  - $p$ times through the hole of the torus for $q$ revolutions
- Collocation method (Ascher et al., 1979)

$$\beta^* \in \mathcal{P}_{k+3,\Pi} \cap C^2\,[0, 2\pi\, q]$$

$$\psi^* \in \mathcal{P}_{k+2,\Pi} \cap C^1\,[0, 2\pi\, q]$$

$$F_g^*, F_3^* \in \mathcal{P}_{k+1,\Pi} \cap C^0\,[0, 2\pi\, q]$$

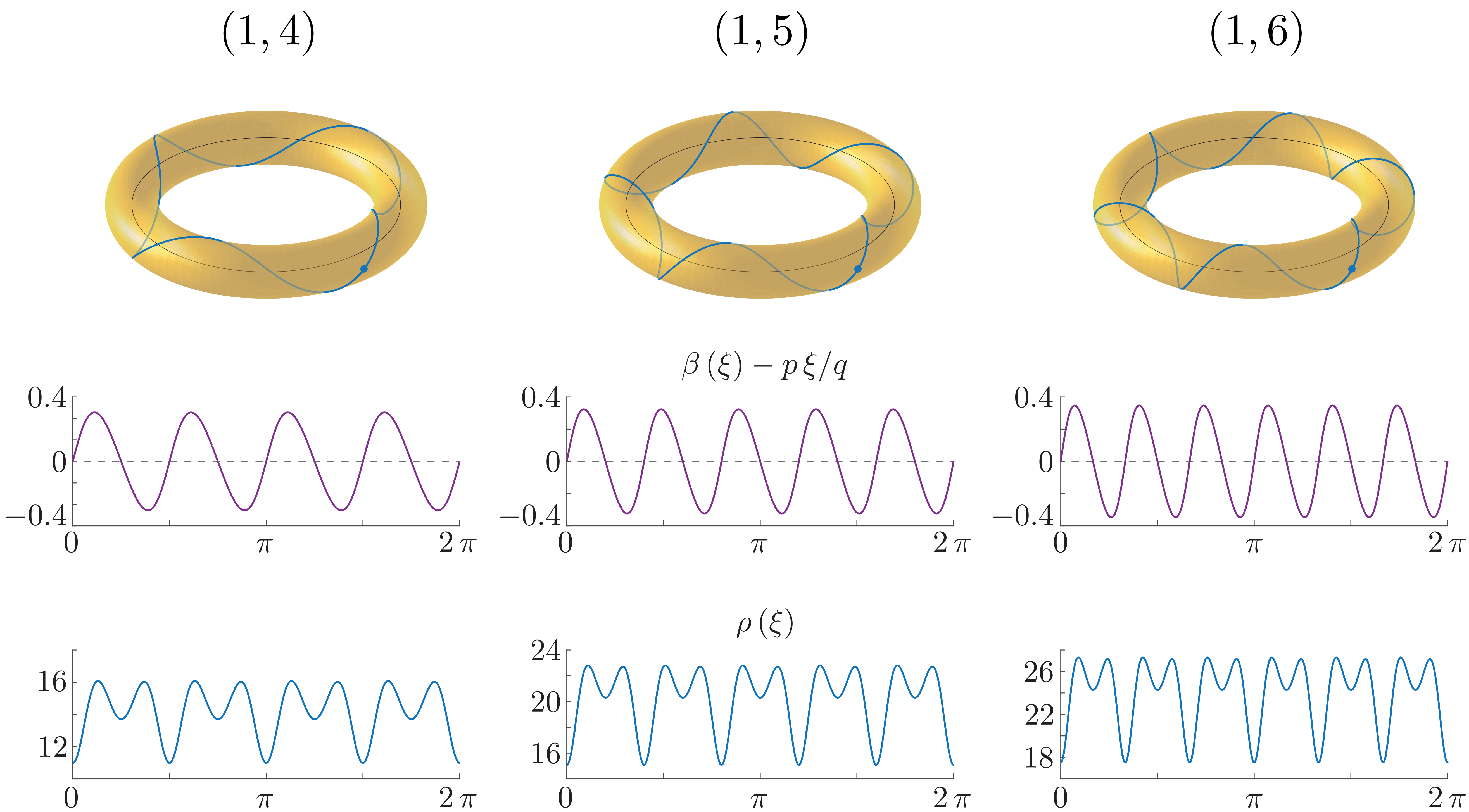
  - Periodic boundary conditions

$$\psi\,(0) = \psi'\,(2\pi\, q) = F_3\,(2\pi\, q) = 0$$

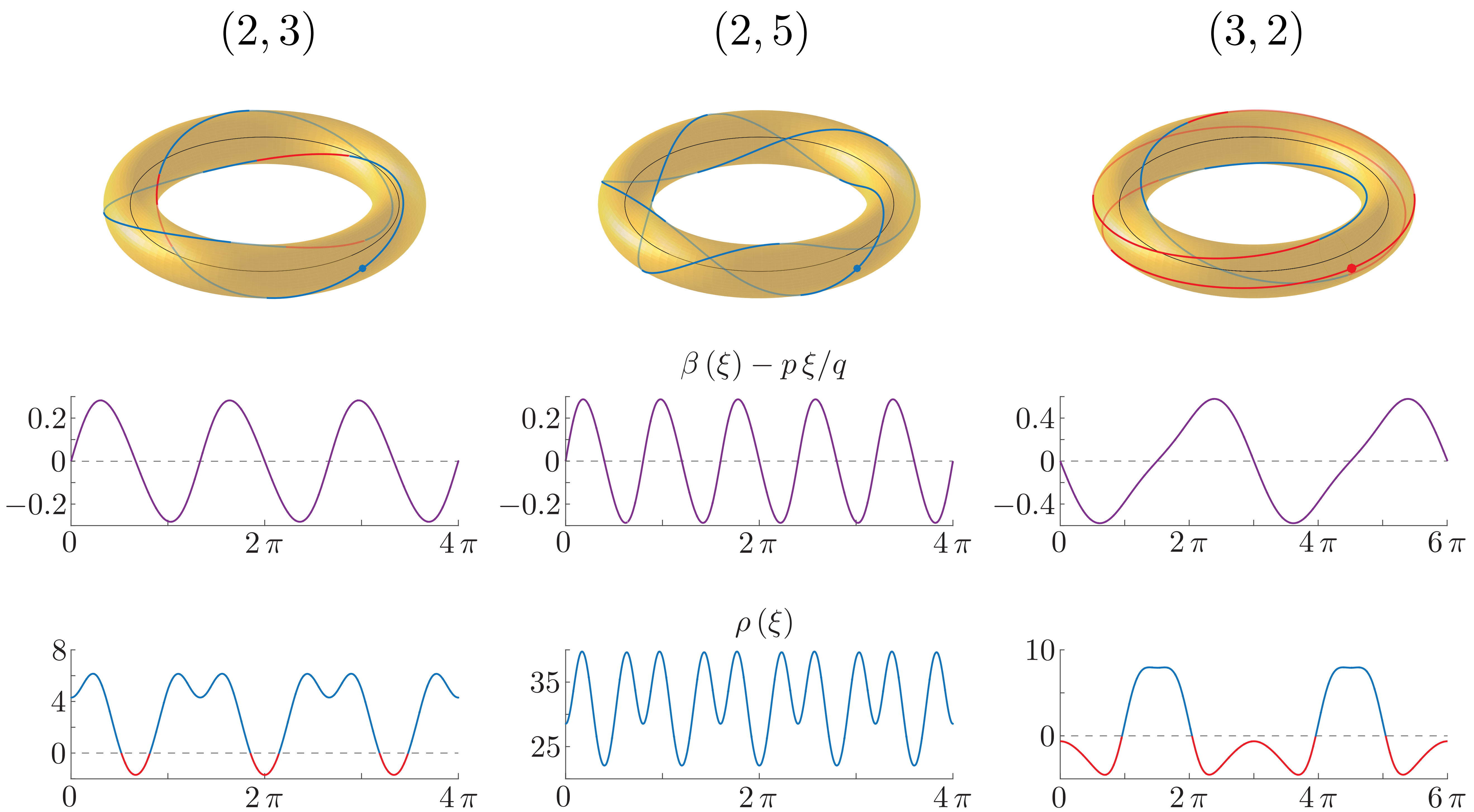
$$\beta\,(2\pi\, q) - \beta\,(0) = 2\pi\, p \qquad \beta'\,(0) = \beta'\,(2\pi\, q) = c$$

with $c$ such that $F_n\,(0) = F_n\,(2\pi\, q)$

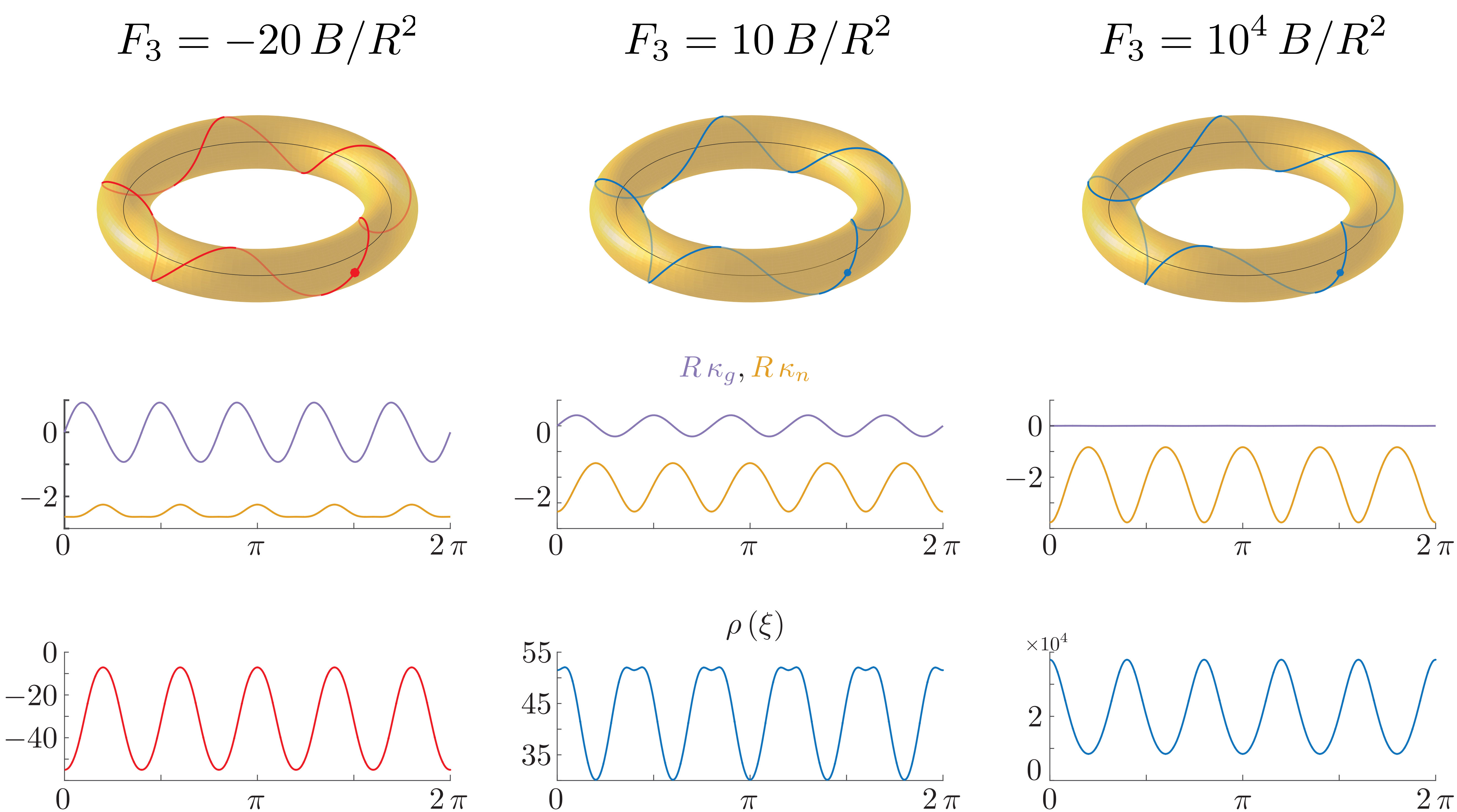
# Elastic Torus Knots (trivial)



# Elastic Torus Knots (nontrivial)



# Elastic Torus Knots



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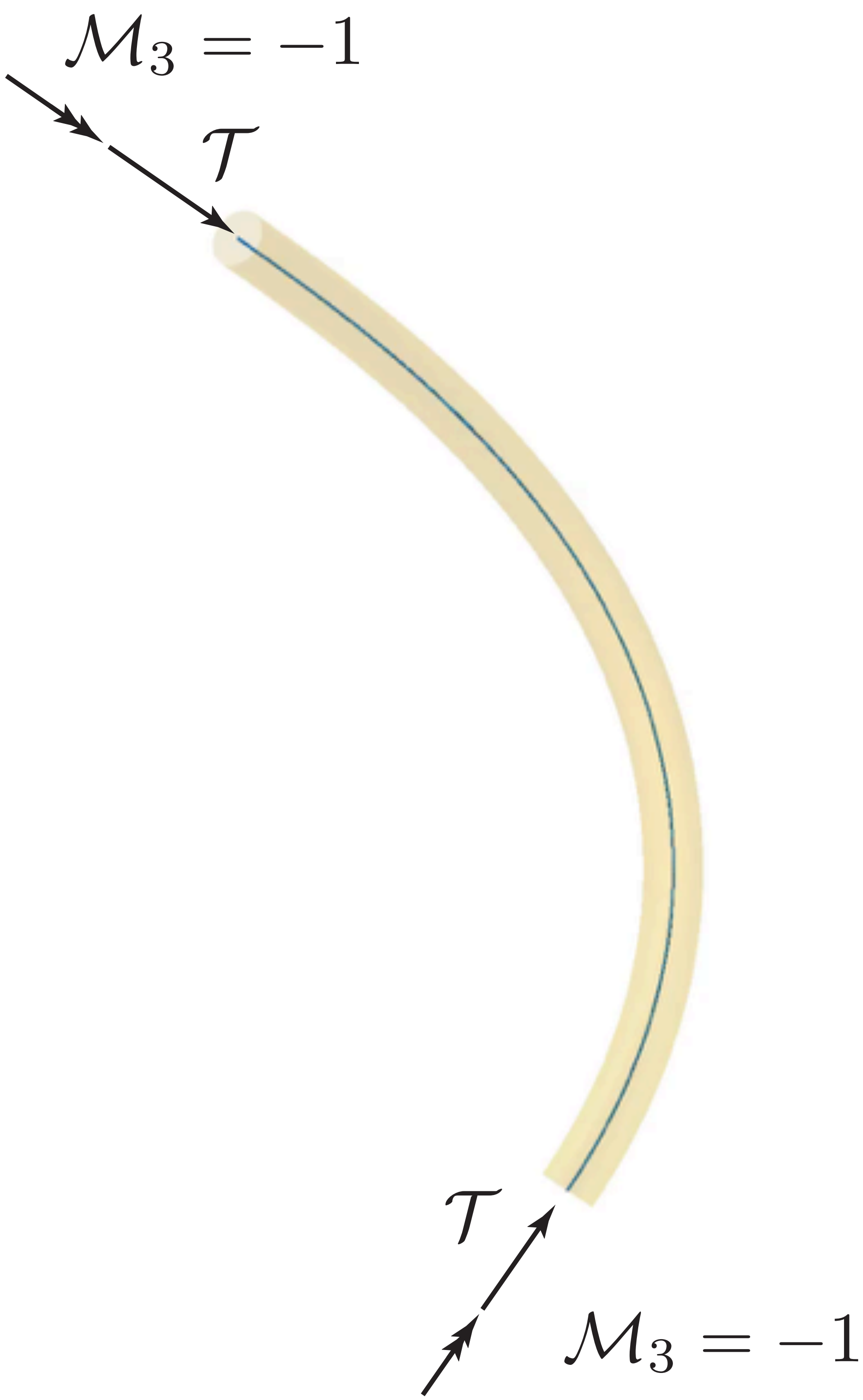
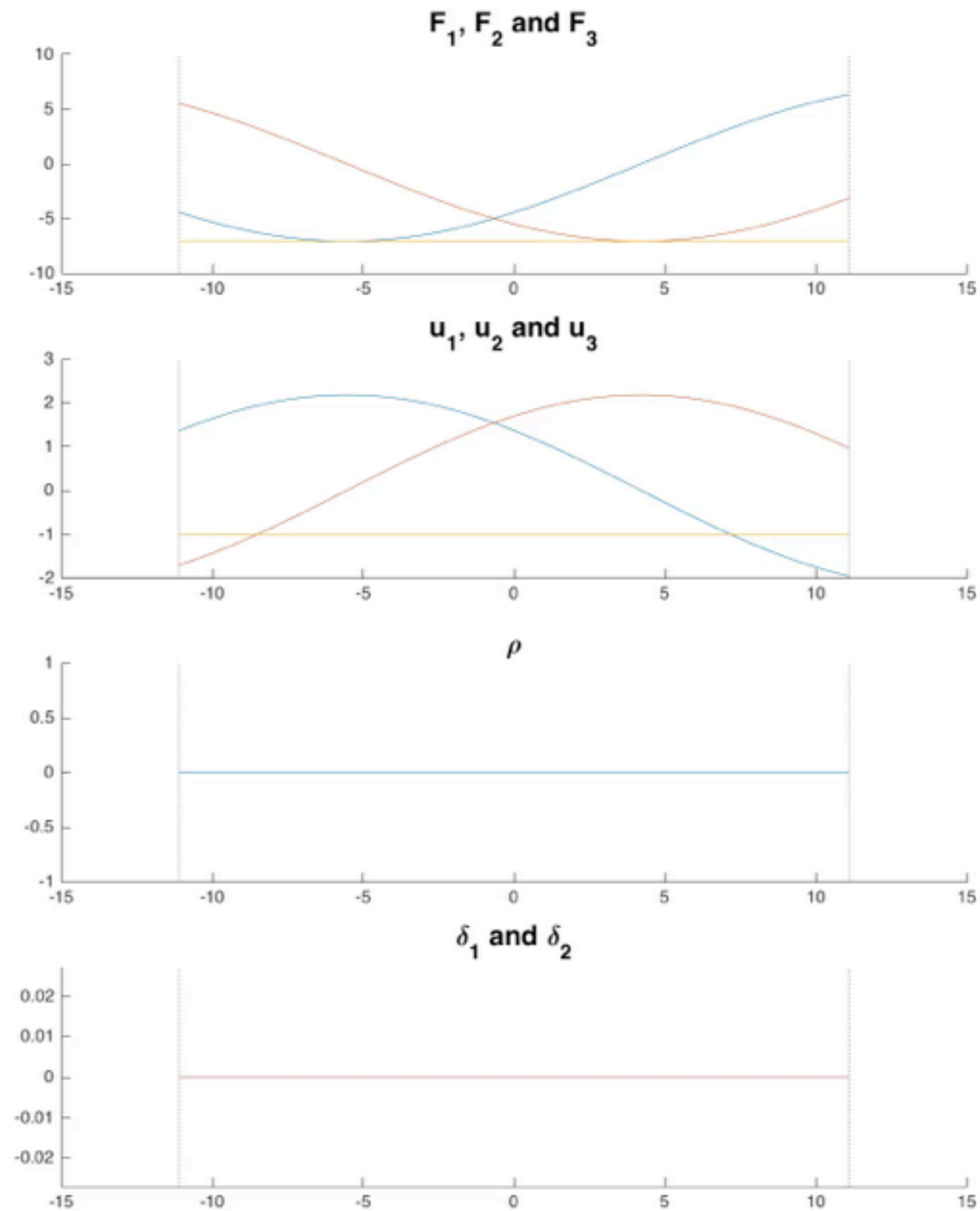
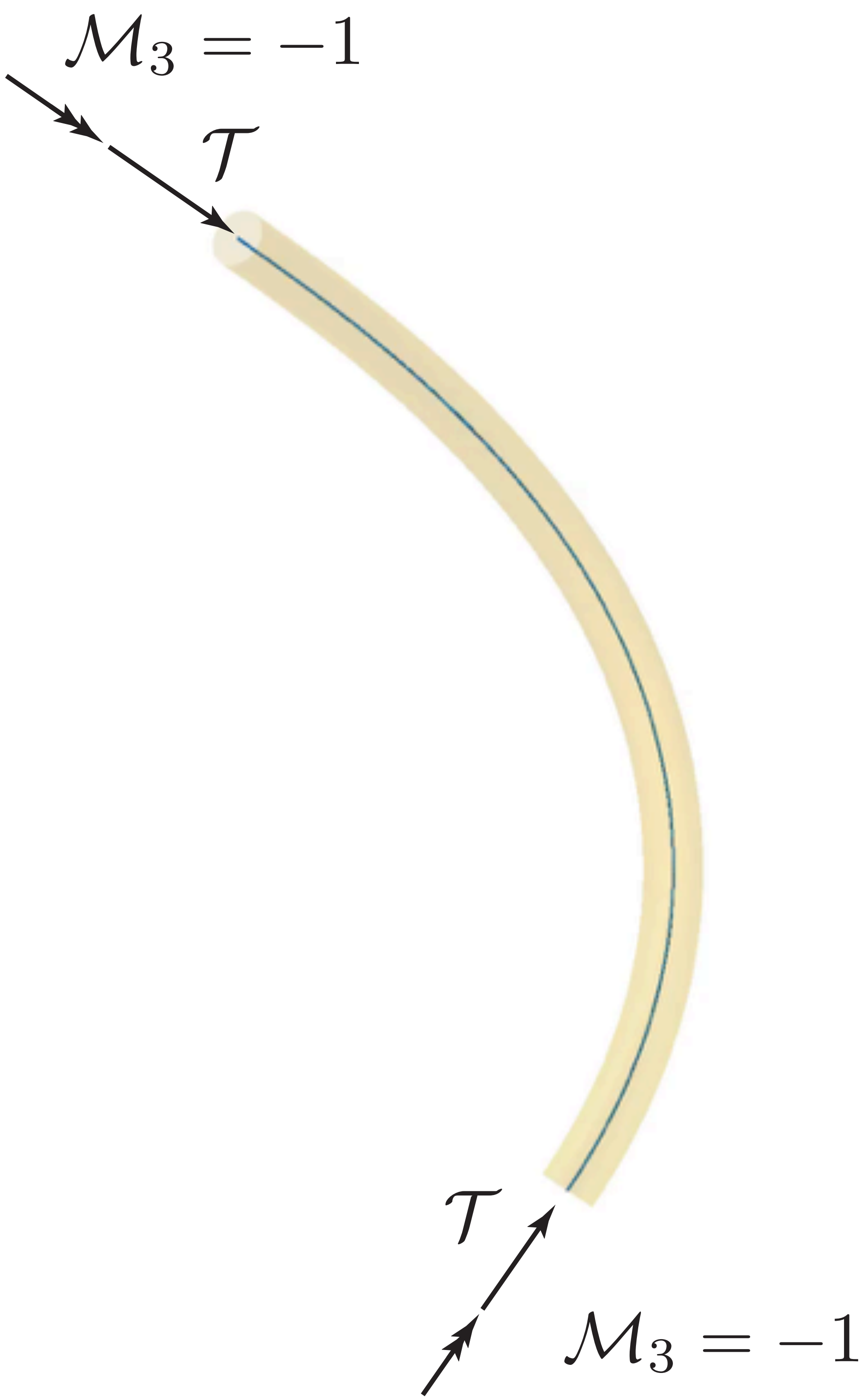
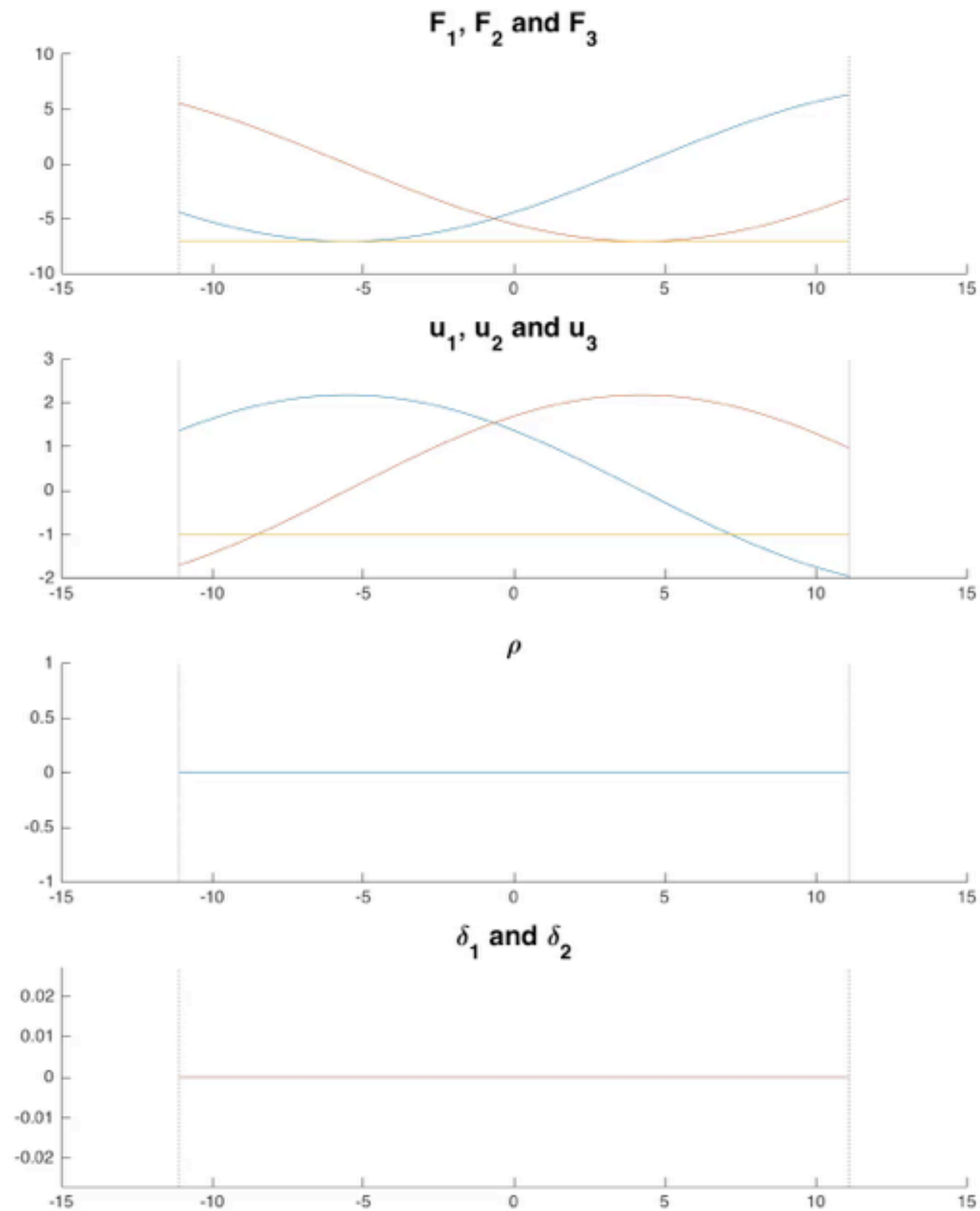
# Constrained Helical Buckling



# Constrained Helical Buckling



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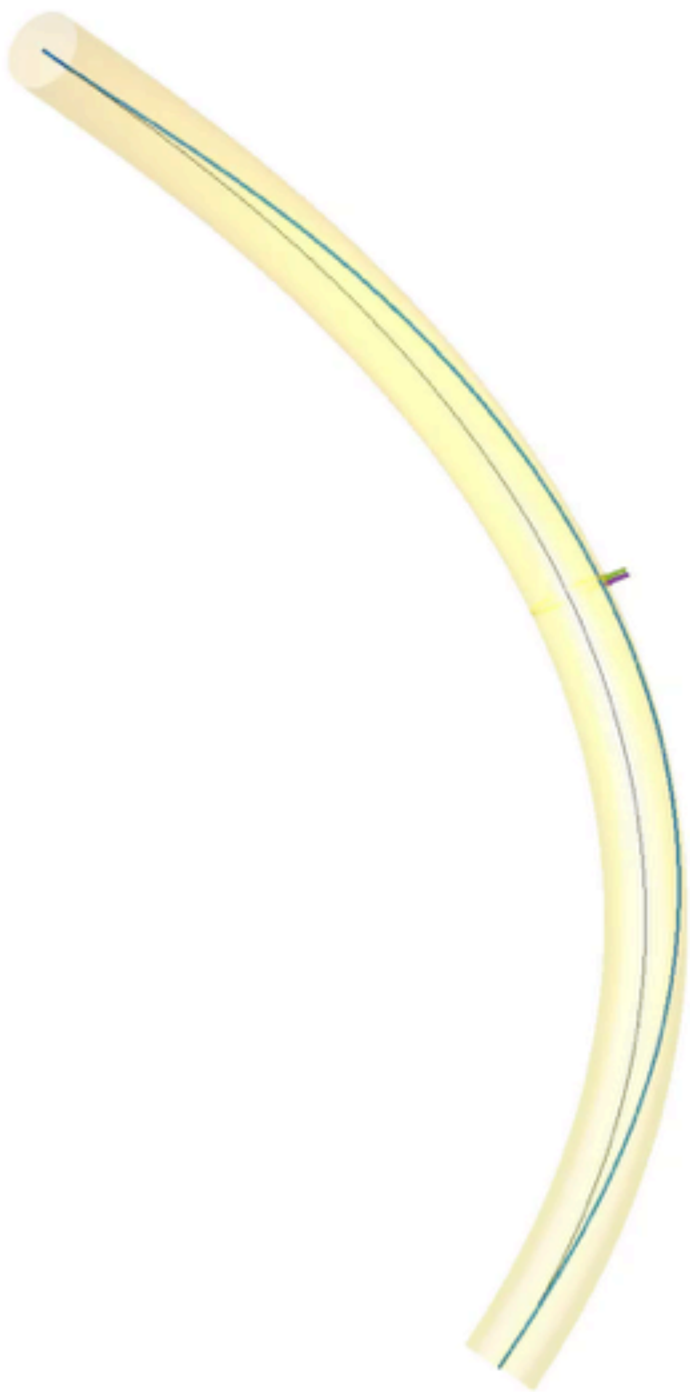

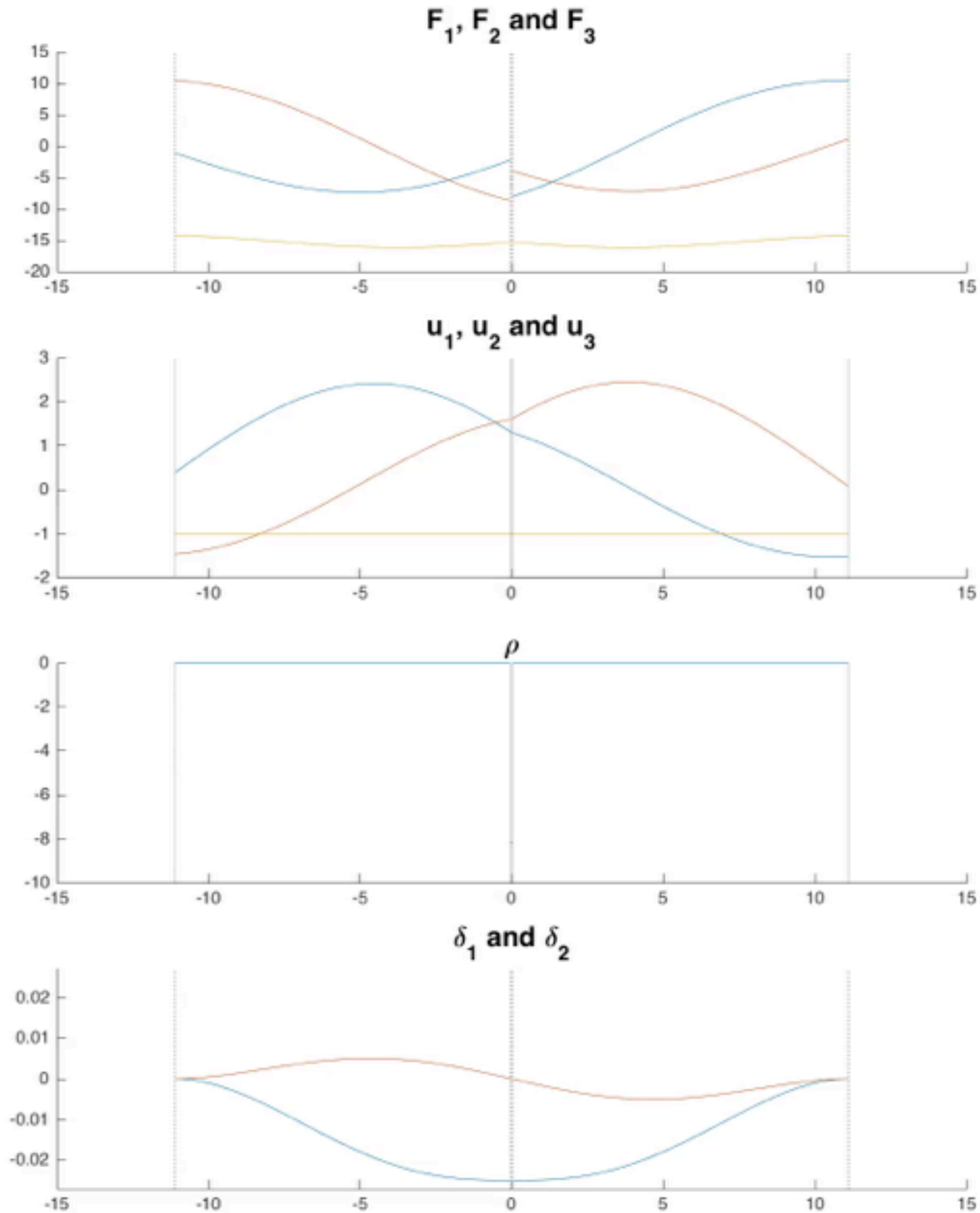

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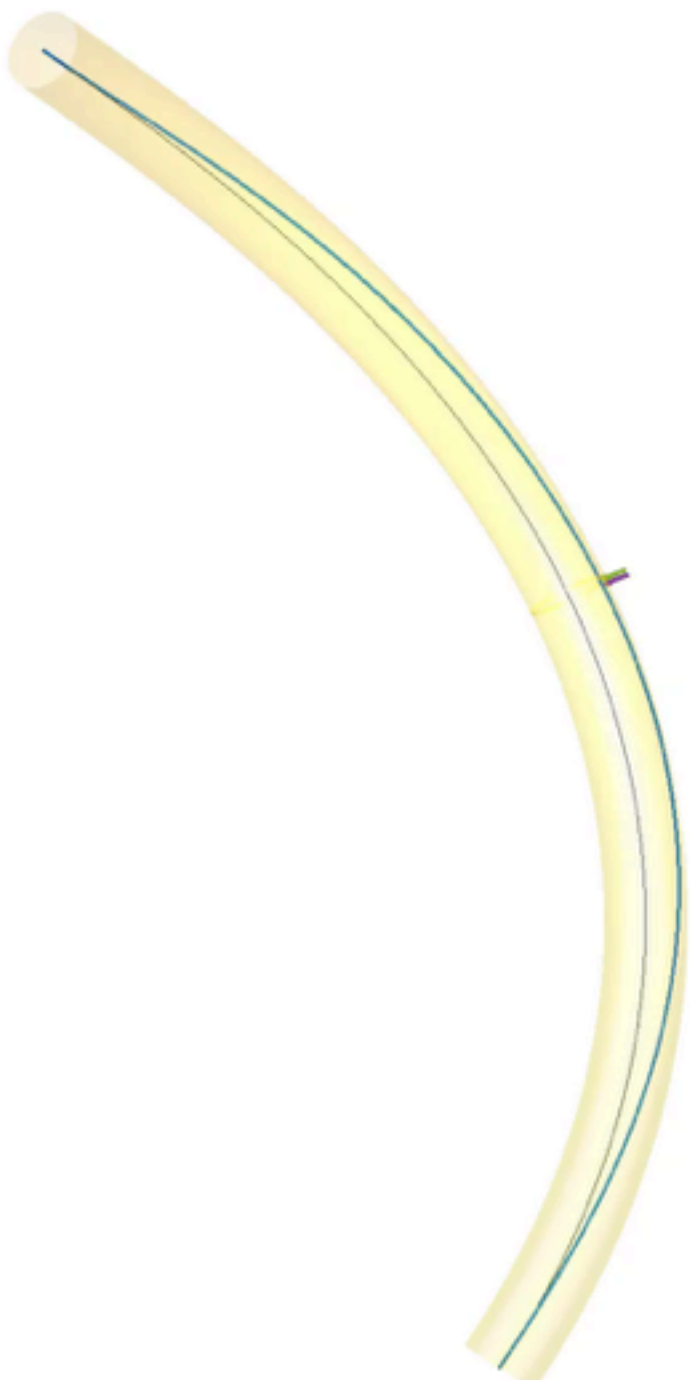
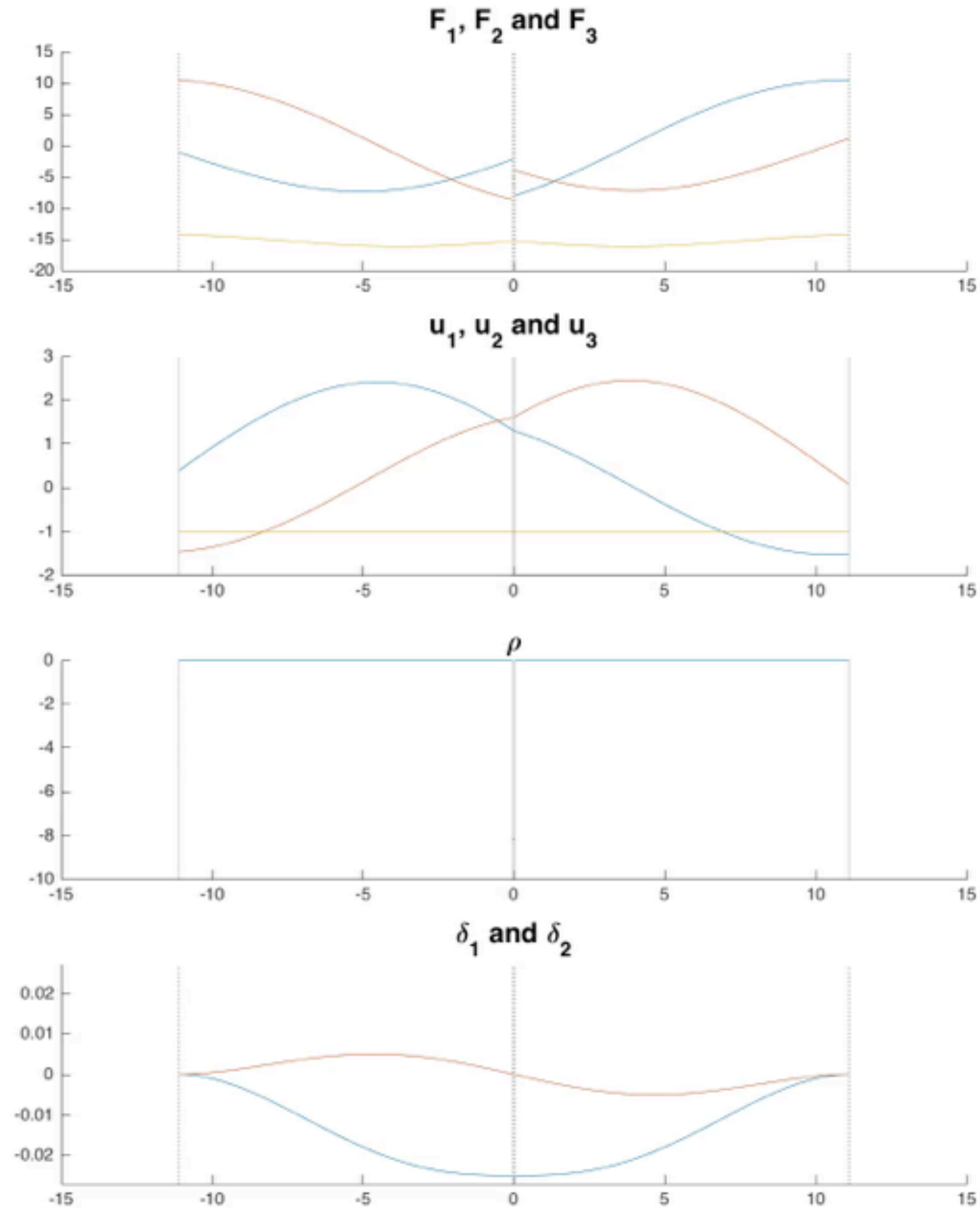
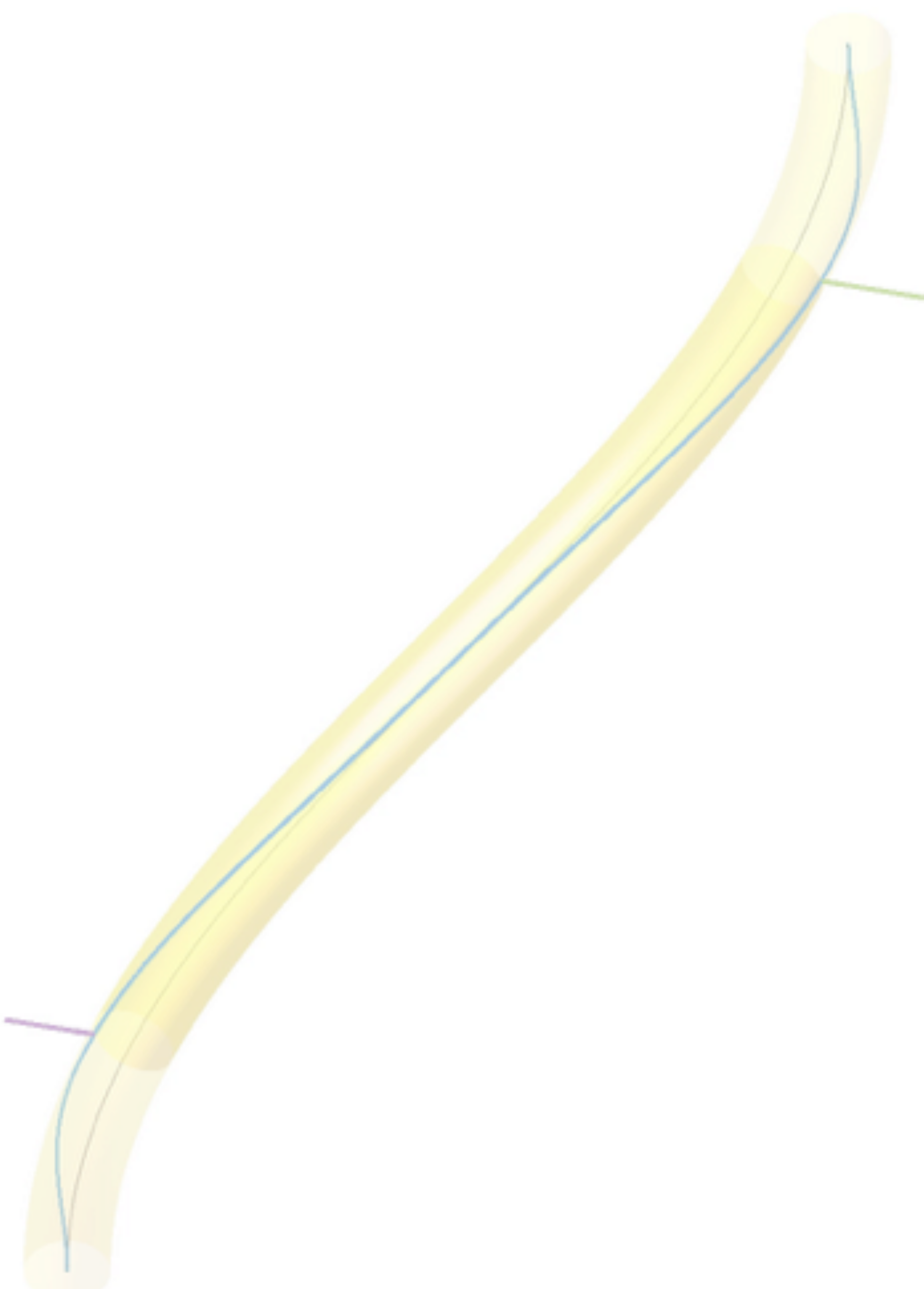
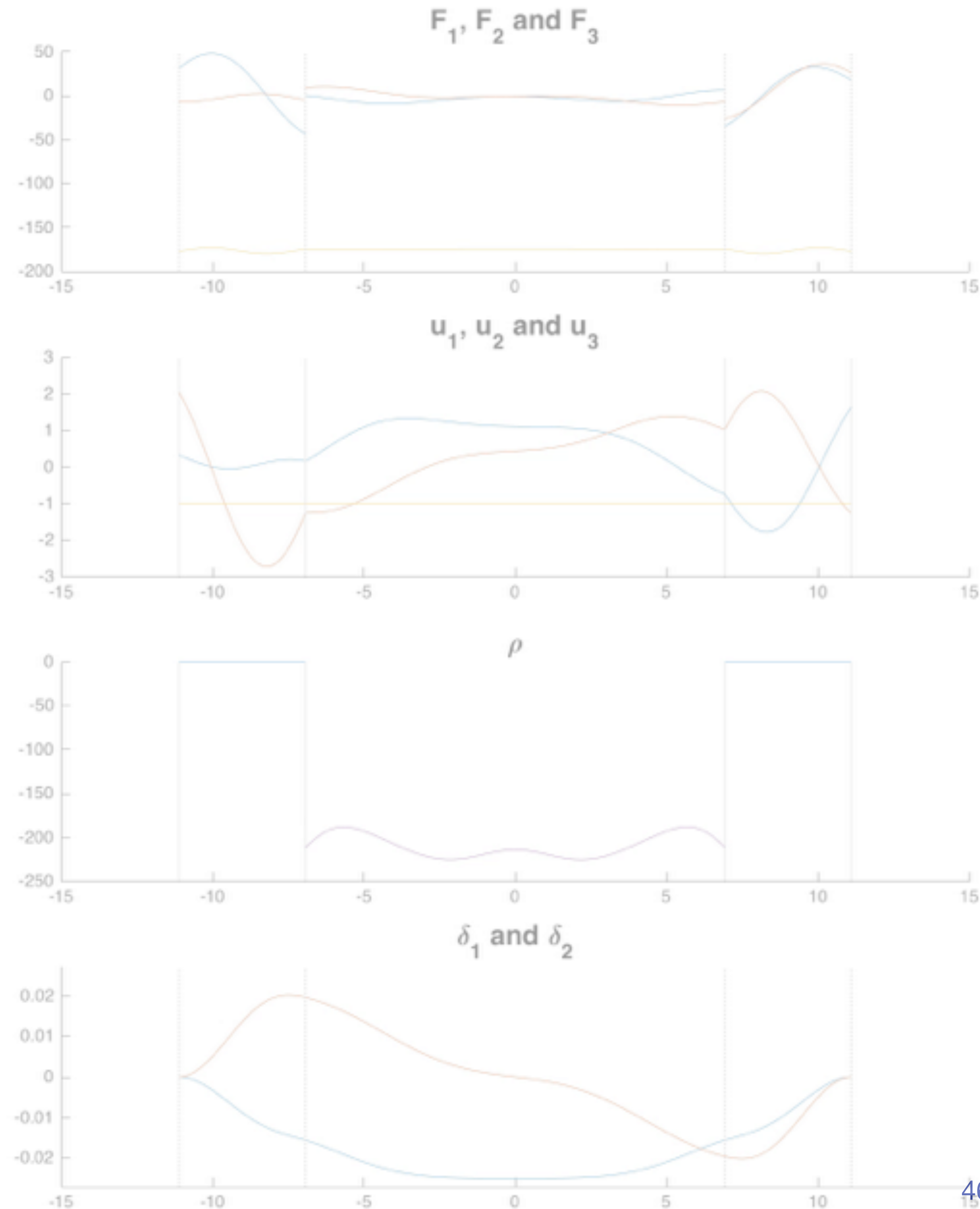




# Constrained Helical Buckling





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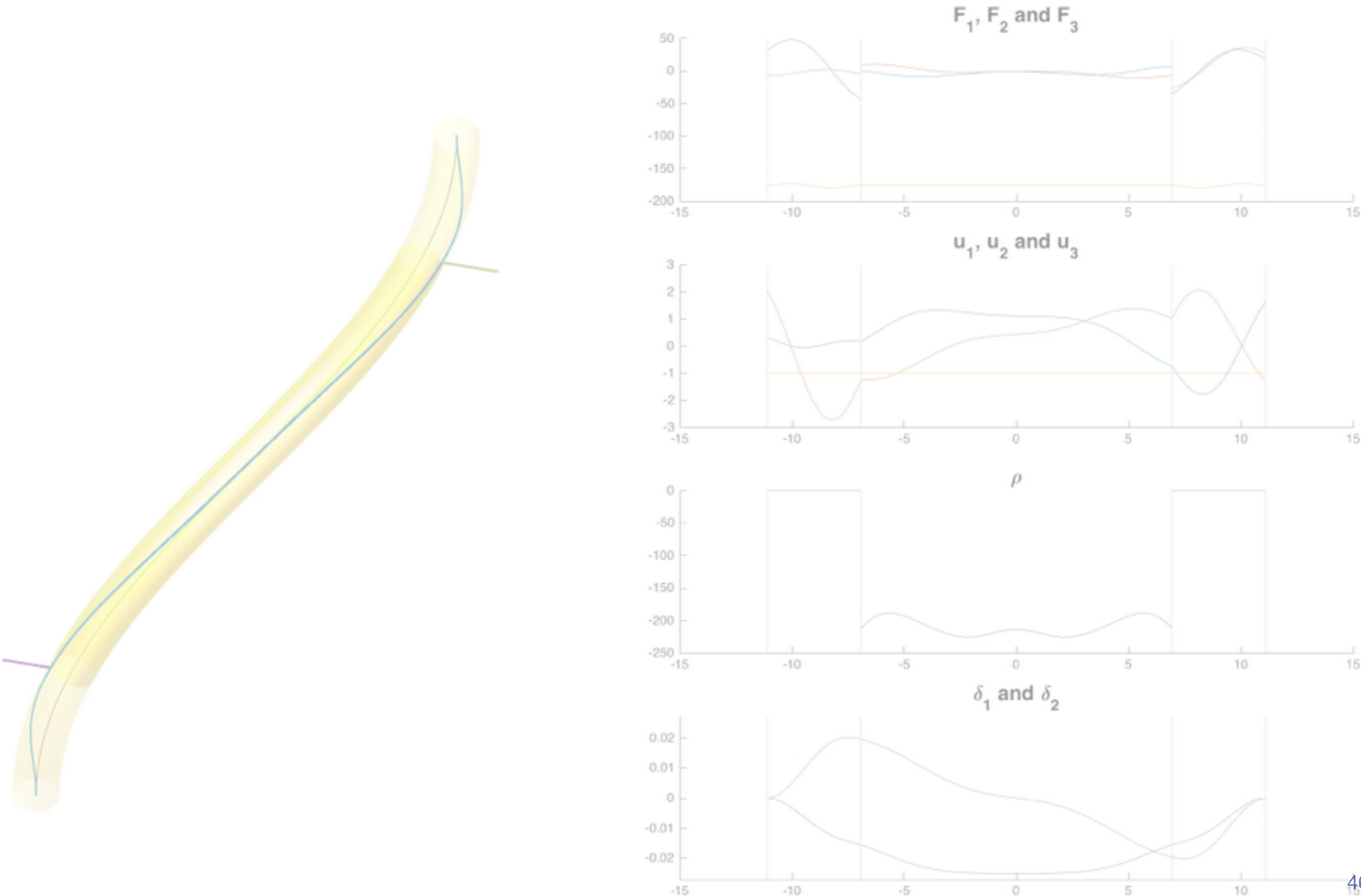
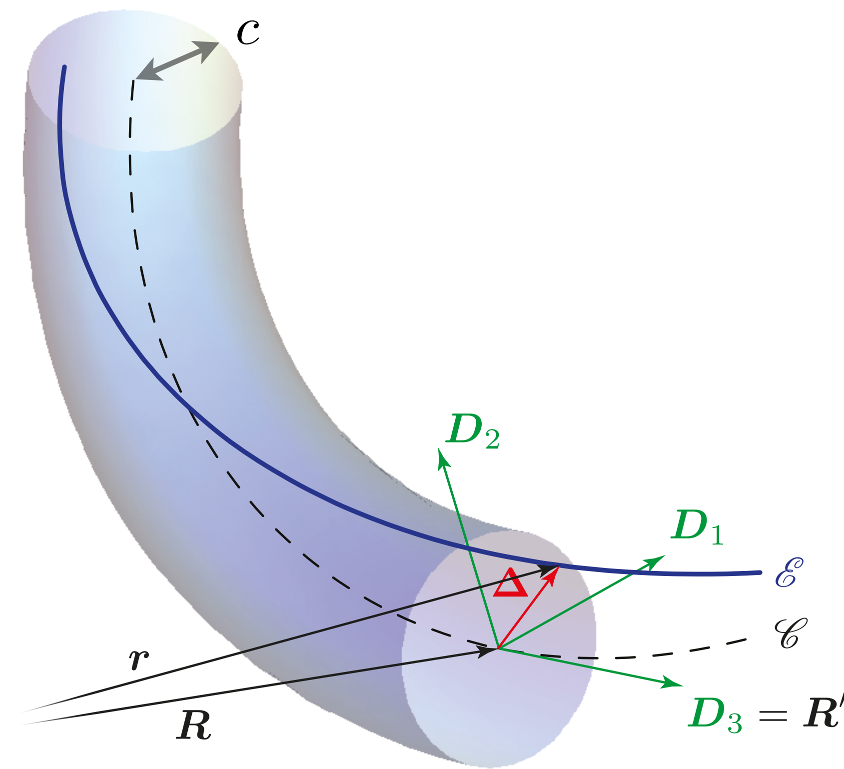


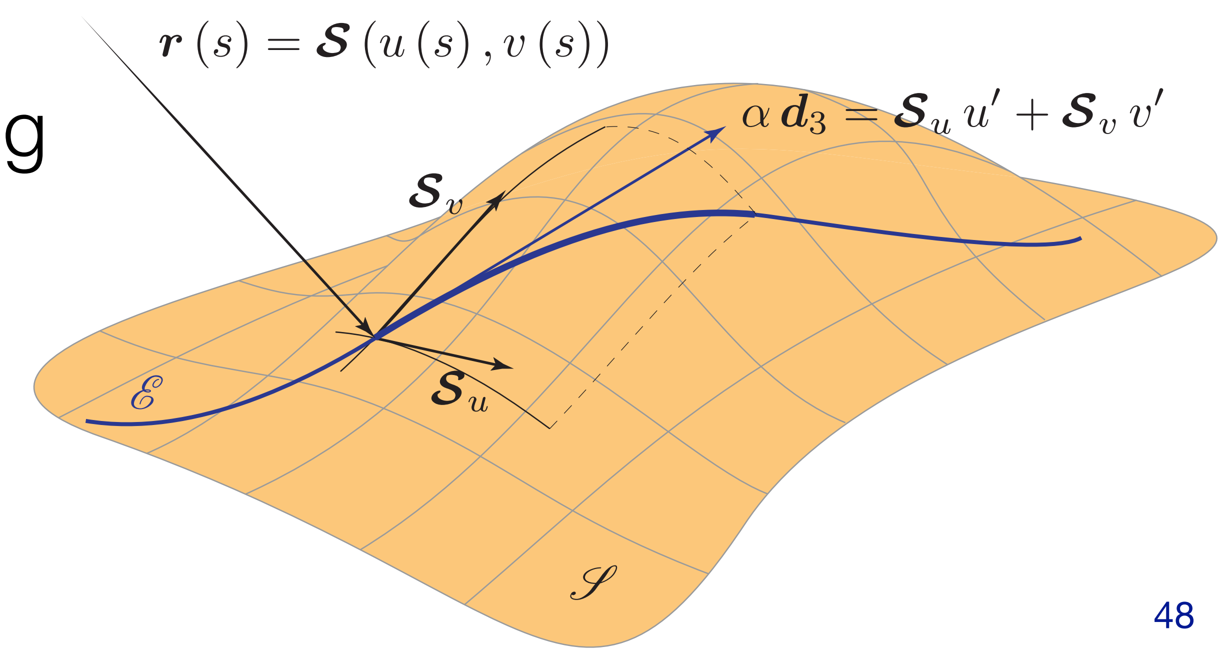
1. Introduction
2. Eulerian formulation of elastic rods deforming in space
3. Surface constrained elastic rods
4. Eulerian formulation adapted to normal ringed surface
5. Applications
6. **Conclusion**

# Conclusion

- Eulerian formulation of elastic rods deforming in space $\{\varphi(S), \Delta_1(S), \Delta_2(S)\}$
  - Suppression of the isoperimetric constraints
  - Simplification of the contact detection
  - Discard parasitic solutions (curling)

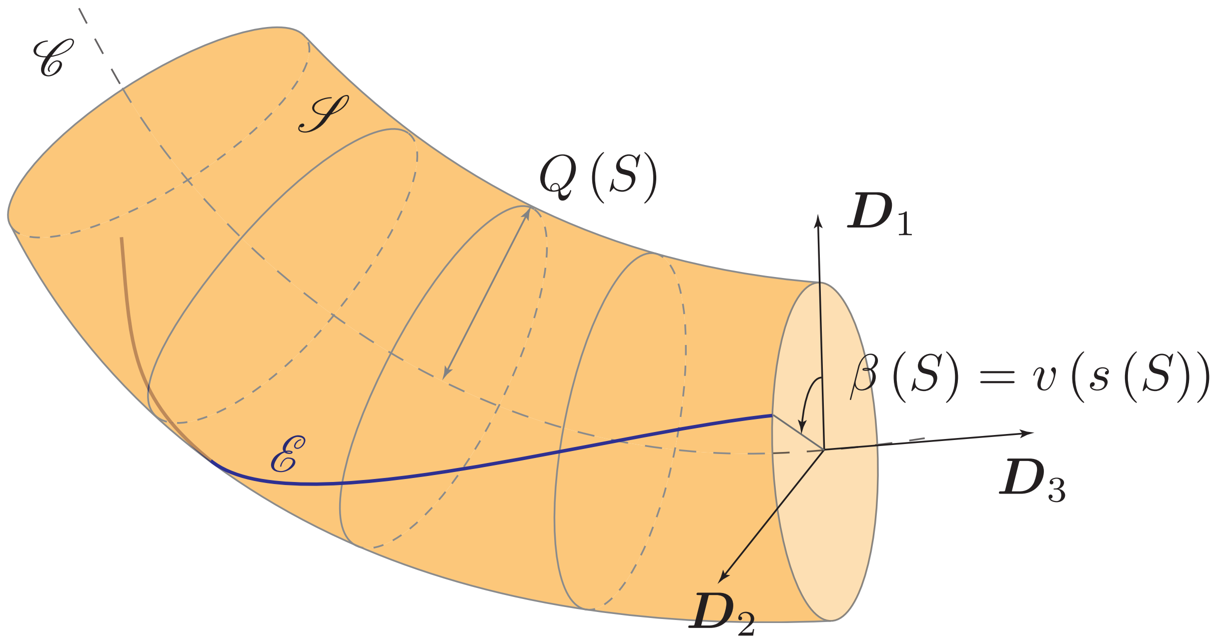



- Lagrangian formulation of surface bound elastic rods $\{\psi(s), u(s), v(s)\}$
  - Particularization of the rod governing equations (geometric invariants)
  - Surface geometry $\rightleftharpoons$ pressure

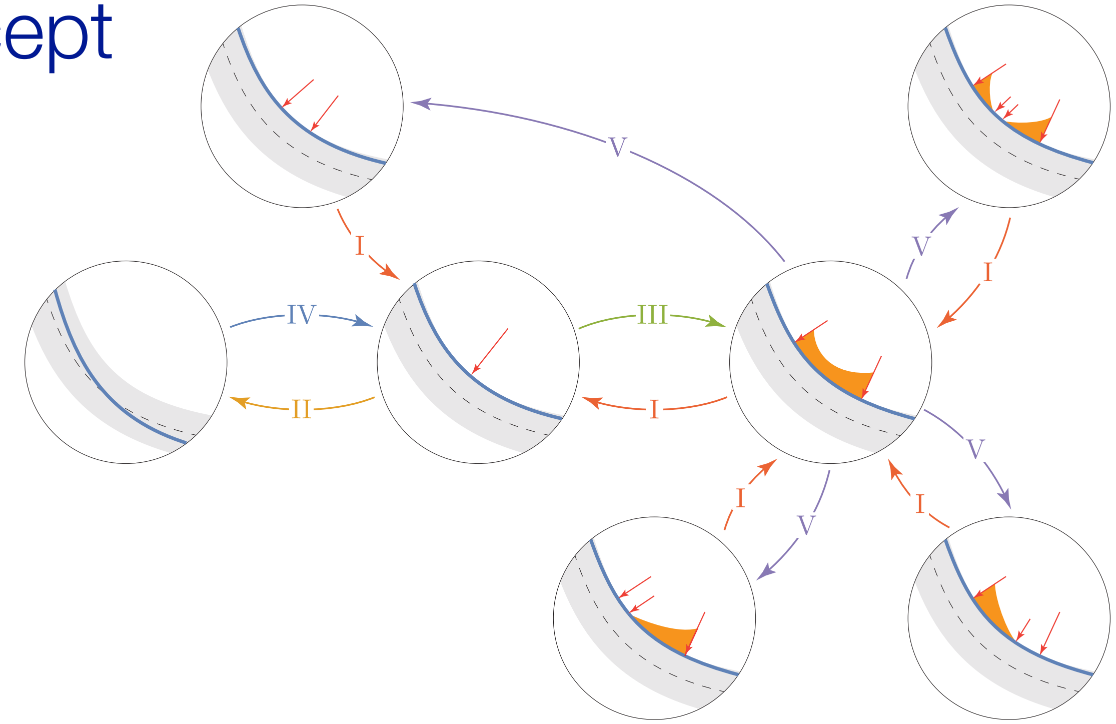


# Conclusion

- Eulerian formulation of elastic rods deforming on a normal ringed surface $\{\psi(S), \beta(S)\}$
  - Explicit representation of the rod centerline
  - Suppression of the isoperimetric constraints




- Computational model $\rightleftharpoons$ Proof of concept
  - Segmentation strategy
  - Propagation of the solution
  - Contact pattern switcher



# Perspectives

- Extension of the Eulerian formulation to account for
  - Rod dynamic (inertial terms) $\mapsto \boldsymbol{\Delta}(S,t), \beta(S,t)$
  - Effects of friction (history dependent) $\mapsto f_g(p), f_3(p)$
  - Constraint deformability $\mapsto Q(\beta, p)$
- Verification, validation and calibration of the computation model
  - Benchmarks, experiments, commercial codes, etc.
  - Improve the contact pattern switcher
- Enhancement of the numerical implementation
  - Adaptative meshing, Jacobian update, etc.
  - Alternative numerical implementation

# Thank you