Lie antialgebras

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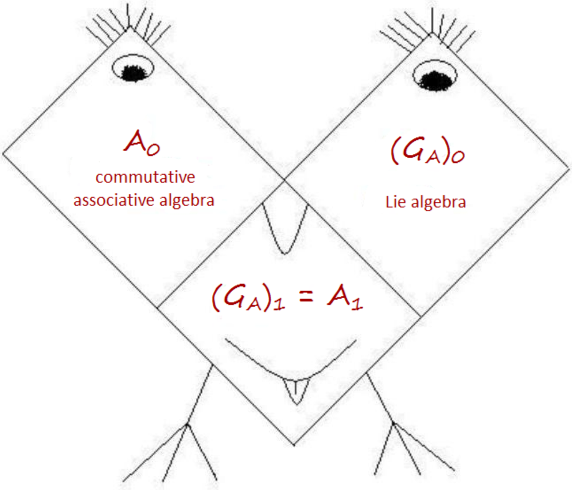
A_0
commutative
associative algebra
$(G_A)_0$
Lie algebra
$(G_A)_1 = A_1$

1 Superspaces

2 Lie Antialgebras

3 Elements of theory

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Lie algebra : $[x, y] = -[y, x]$ (skew-symmetric)

$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity)

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Three interesting examples of Lie superalgebras.

1. Symplectic orthogonal $osp(1|2)$:

$$\begin{pmatrix} 0 & u & v \\ v & p & q \\ -u & r & -p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & q \\ 0 & r & -p \end{pmatrix} \oplus \begin{pmatrix} 0 & u & v \\ v & 0 & 0 \\ -u & 0 & 0 \end{pmatrix}$$

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2. Conformal Neveu-Schwarz $\mathscr{K}(1)$: basis $\{\alpha_n; b_i \mid n \in \mathbb{Z}, i \in \mathbb{Z} + \frac{1}{2}\}$ with relations

$$[\alpha_n, \alpha_m] = (m-n)\alpha_{n+m}, \qquad [\alpha_n, b_i] = (i - \tfrac{n}{2})b_{i+n}, \qquad [b_i, b_j] = -\alpha_{i+j}.$$

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3. Space of derivations of $\mathscr{G}$ (a $\mathbb{Z}_2$-graded algebra) Der($\mathscr{G}$)

$$D : \mathscr{G} \longrightarrow \mathscr{G}, \qquad D(x.y) = D(x).y + (-1)^{\bar{D}\bar{x}} x.D(y).$$

$$[D, D'] = D \circ D' - (-1)^{\bar{D}\bar{D}'} D' \circ D$$

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Three interresting examples of Lie Antialgebras.

1. Tiny Kaplansky K_3 : basis $\{\varepsilon; a, b\}$ with relations

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Remark : Lie Antialgebras are a particular cases of Jordan superalgebras.

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Construction of the Lie superalgebra related to a Lie antialgebra.

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$(\mathscr{G}_{\mathscr{A}})_1 := \mathscr{A}_1$

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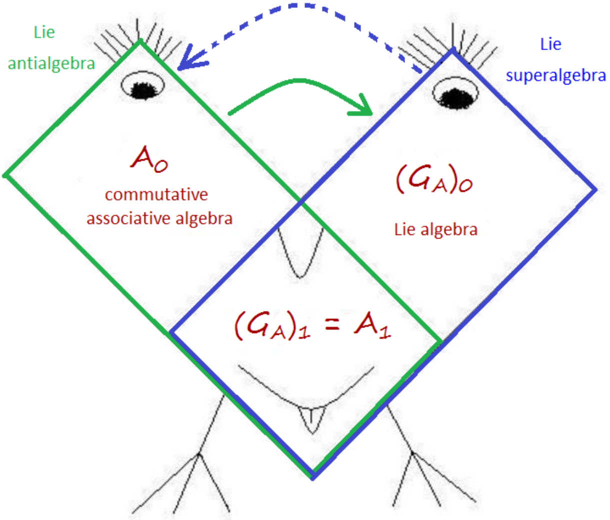
$$[a,b] = a \odot b$$

$$[a \odot b, c] = a.(b.c) + b.(a.c)$$

$$[a \odot b, c \odot d] = 2a.(b.c) \odot d + 2b.(a.d) \odot c$$

With this bracket, the space $(\mathscr{G}_{\mathscr{A}}, [\cdot,\cdot])$ is a Lie superalgebra.



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$$Der(K_3) \cong osp(1|2) \cong \mathscr{G}_{K_3},$$

$$Der(\mathscr{A}\mathscr{K}(1)) \cong \mathscr{K}(1) \cong \mathscr{G}_{\mathscr{A}\mathscr{K}(1)},$$

$$Der(\mathscr{J}_{KN}) \cong \mathscr{L}_{KN} \cong \mathscr{G}_{\mathscr{J}_{KN}}.$$

Classification of non trivial central extension of $\mathscr{A}$.

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- $c : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{B}$ bilinear map preserving parity
- $\rho : \mathscr{A} \longrightarrow End(\mathscr{B})$ even and linear map

  s.t. $(x,v).(y,w) = (x.y, \rho_x(w) + (-1)^{\bar{v}\bar{y}}\rho_y(v))$

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An extension is **trivial** if $\mathscr{A} \oplus \mathscr{B} \cong \mathscr{A} \ltimes \mathscr{B}$ (if we have $c = 0$).

Further work...

paper : "Alternated Hochschild Cohomology" (submitted by P. Lecomte & V. Ovsienko)

$\rightarrow$ help to classify extensions of Lie antialgebras.

Case of $\mathscr{B} = \mathbb{K}$:

Identify a (unique) non trivial central extension on $\mathscr{A}\mathscr{K}(1)$

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Goal :

Identify a (unique) non trivial central extension on $\mathscr{J}_{KN} \leftrightarrow \mathscr{L}_{KN}$

(Joint work with - S. Leidwanger & S. Morier-Genoud)

Thank you for your attention