On generalized Hölder spaces

D. Kreit & S. Nicolay

Fractals and Related Fields III

Porquerolles, September 19–25 2015
A function \( f \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) belongs to \( \Lambda^s(x_0) \) iff there exists a polynomial of degree at most \( s \) s.t.

\[
\sup_{|h| \leq 2^{-j}} |f(x_0 + h) - P(h)| \leq C2^{-js},
\]

for \( j \) sufficiently large.
A function \( f \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) belongs to \( \Lambda^s(x_0) \) iff there exists a polynomial of degree at most \( s \) s.t.

\[
\sup_{|h| \leq 2^{-j}} |f(x_0 + h) - P(h)| \leq C 2^{-js},
\]

for \( j \) sufficiently large.

A function \( f \in L^\infty(\mathbb{R}^d) \) belongs to \( \Lambda^s(\mathbb{R}^d) \) iff the previous inequality is satisfied for every \( x_0 \) with a uniform constant \( C \).
A function $f \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ belongs to $\Lambda^s(x_0)$ iff there exists a polynomial of degree at most $s$ s.t.

$$\sup_{|h| \leq 2^{-j}} |f(x_0 + h) - P(h)| \leq C2^{-js},$$

for $j$ sufficiently large.

A function $f \in L^\infty(\mathbb{R}^d)$ belongs to $\Lambda^s(\mathbb{R}^d)$ iff the previous inequality is satisfied for every $x_0$ with a uniform constant $C$.

One can try to be sharper by replacing the sequence $(2^{-js})_j$ with a more general sequence $\sigma = (\sigma_j)_j$:

$f \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ belongs to $\Lambda^{\sigma, M}(x_0)$ if there exists a polynomial of degree at most $M$ s.t.

$$\sup_{|h| \leq 2^{-j}} |f(x_0 + h) - P(h)| \leq C\sigma_j,$$

for $j$ sufficiently large.
Introduction

Independance of the polynomial from the scale

Alternative definitions

Wavelets

The Hölder exponent

Generalized Besov spaces

Generalization

\[ B^s_{p,q}(\mathbb{R}^d) \xrightarrow{\text{generalization}} B^{1/\sigma}_{p,q}(\mathbb{R}^d), \text{ where } \sigma \text{ is some sequence} \]
$B^s_{p,q}(\mathbb{R}^d)$ \quad \xrightarrow{\text{generalization}} \quad B^{1/\sigma}_{p,q}(\mathbb{R}^d)$, where $\sigma$ is some sequence

$\Lambda^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d)$
Generalized Besov spaces

\[ B^s_{p,q}(\mathbb{R}^d) \xrightarrow{\text{generalization}} \Lambda^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d) \xrightarrow{\text{generalization}} B^{1/\sigma}_{p,q}(\mathbb{R}^d), \text{ where } \sigma \text{ is some sequence} \]

\[ \Lambda^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d) \xrightarrow{\text{generalization}} \Lambda^\sigma(\mathbb{R}^d) = B^{1/\sigma}_{\infty,\infty} \]
Generalized Besov spaces

\[ B^{s}_{p,q}(\mathbb{R}^d) \xrightarrow{\text{generalization}} B^{1/\sigma}_{p,q}(\mathbb{R}^d), \text{ where } \sigma \text{ is some sequence} \]

\[ \Lambda^s(\mathbb{R}^d) = B^{s}_{\infty,\infty}(\mathbb{R}^d) \xrightarrow{\text{generalization}} \Lambda^\sigma(\mathbb{R}^d) = B^{1/\sigma}_{\infty,\infty} \]

\[ \Lambda^\sigma(\mathbb{R}^d) \xrightarrow{\text{pointwise version}} \Lambda^\sigma(x_0) \]
A sequence of real positive numbers is called admissible if

\[
\frac{\sigma_{j+1}}{\sigma_j}
\]

is bounded.
A sequence of real positive numbers is called admissible if

\[ \frac{\sigma_{j+1}}{\sigma_j} \]

is bounded.
For such a sequence, we set

\[ s(\sigma) = \lim_{j} \frac{\log_2(\inf_{k \in \mathbb{N}} \frac{\sigma_{j+k}}{\sigma_j})}{j} \]

and

\[ \bar{s}(\sigma) = \lim_{j} \frac{\log_2(\sup_{k \in \mathbb{N}} \frac{\sigma_{j+k}}{\sigma_j})}{j} . \]
- the open unit ball centered at the origin is denoted $B$,
- the set of polynomials of degree at most $n$ is denoted $P[n]$,
- $[s] = \sup\{n \in \mathbb{Z} : n \leq s\},$
the open unit ball centered at the origin is denoted $B$,
the set of polynomials of degree at most $n$ is denoted $P[n]$,
$[s] = \sup\{n \in \mathbb{Z} : n \leq s\}$,
if $f$ is defined on $\mathbb{R}^d$,
\[
\Delta_h^1 f(x) = f(x + h) - f(x)
\]
the open unit ball centered at the origin is denoted $B$, 
the set of polynomials of degree at most $n$ is denoted $\mathbf{P}[n]$, 
$[s] = \sup\{n \in \mathbb{Z} : n \leq s\}$, 
if $f$ is defined on $\mathbb{R}^d$, 
\[
\Delta_h^{1} f(x) = f(x + h) - f(x)
\]
and 
\[
\Delta_h^{n+1} f(x) = \Delta_h^{1} \Delta_h^n f(x),
\]
for any $x, h \in \mathbb{R}^d$
**Definition**

Let $s > 0$ and $\sigma$ be an admissible sequence; a function $f \in L^\infty(\mathbb{R}^d)$ belongs to $\Lambda^{\sigma,M}(\mathbb{R}^d)$ iff there exists $C > 0$ s.t.

$$\sup_{|h| \leq 2^{-j}} \| \Delta_h^{[M]+1} f \|_\infty \leq C \sigma j$$
Definition
Let $s > 0$ and $\sigma$ be an admissible sequence; a function $f \in L^\infty(\mathbb{R}^d)$ belongs to $\Lambda^\sigma, M(\mathbb{R}^d)$ iff there exists $C > 0$ s.t.

$$\sup_{|h| \leq 2^{-j}} \| \Delta_h^{[M] + 1} f \|_\infty \leq C \sigma j$$

Proposition
Let $s > 0$ and $\sigma$ be an admissible sequence; a function $f \in L^\infty(\mathbb{R}^d)$ belongs to $\Lambda^\sigma, M(\mathbb{R}^d)$ iff there exists $C > 0$ s.t.

$$\inf_{P \in P_{[M]}} \| f - P \|_{L^\infty(2^{-j}B + x_0)} \leq C \sigma j,$$

for any $x_0 \in \mathbb{R}^d$ and any $j \in \mathbb{N}$. 
Definition

A function $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ belongs to $\Lambda^{\sigma,M}(x_0)$ iff there exists $C > 0$ and $J \in \mathbb{N}$ s.t.

$$\inf_{P \in \mathcal{P}[M]} \|f - P\|_{L^\infty(2^{-j}B + x_0)} \leq C \sigma_j,$$

for any $j \geq J$. 


**Definition**

A function $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ belongs to $\Lambda^\sigma,M(x_0)$ iff there exists $C > 0$ and $J \in \mathbb{N}$ s.t.

$$\inf_{P \in P[M]} \left\| f - P \right\|_{L^\infty(2^{-j}B+x_0)} \leq C \sigma_j,$$

for any $j \geq J$.

**Definition**

A function $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ belongs to $\Lambda^\sigma,M(x_0)$ iff there exists $C > 0$ and $J \in \mathbb{N}$ s.t. for any $j \geq J$, there exists $P_j \in P[M]$ for which

$$\sup_{|h| < 2^{-j}} |f(x_0 + h) - P_j(x_0 + h)| \leq C \sigma_j.$$
A function \( f \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) belongs to \( \Lambda^s(x_0) \) \((s \in \mathbb{R})\) iff there exists \( C > 0 \), a polynomial \( P \) of degree less than \( s \) and \( J \in \mathbb{N} \) s.t. for any \( j \geq J \),

\[
\sup_{|h|<2^{-j}} |f(x_0 + h) - P(x_0 + h)| \leq C2^{-js}.
\]
A function $f \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ belongs to $\Lambda^s(x_0)$ ($s \in \mathbb{R}$) iff there exists $C > 0$, a polynomial $P$ of degree less than $s$ and $J \in \mathbb{N}$ s.t. for any $j \geq J$,

$$\sup_{|h|<2^{-j}} |f(x_0 + h) - P(x_0 + h)| \leq C2^{-js}.$$ 

There is one polynomial, independant from the scale.
Lemma

If $M < s(\sigma^{-1})$, the sequence of polynomials occuring in the definition of $\Lambda^{\sigma,M}(x_0)$ satisfies

$$\|D^\beta P_k - D^\beta P_j\|_{L^\infty(x_0+2^{-k}B)} \leq C2^j|\beta|\sigma_j,$$

for any multi-index $\beta$ s.t. $|\beta| \leq M$ and $k \geq j \geq J$. 
Lemma

If $M < s(\sigma^{-1})$, the sequence of polynomials occurring in the definition of $\Lambda^{\sigma,M}(x_0)$ satisfies

$$\|D^\beta P_k - D^\beta P_j\|_{L^\infty(x_0 + 2^{-k}B)} \leq C 2^j |\beta| \sigma_j,$$

for any multi-index $\beta$ s.t. $|\beta| \leq M$ and $k \geq j \geq J$.

In particular, $(D^\beta P_j(x_0))_j$ is a Cauchy sequence.
Lemma

If $M < s(\sigma^{-1})$, and $(P_j)_j$ is a sequence of polynomials in the definition of $\Lambda^{\sigma,M}(x_0)$, for any multi-index $\beta$ s.t. $|\beta| \leq M$, the limit

$$f_\beta(x_0) = \lim_{j} D^\beta P_j(x_0)$$

is independant of the chosen sequence $(P_j)_j$. 
Lemma

If $M < s(\sigma^{-1})$, and $(P_j)_j$ is a sequence of polynomials in the definition of $\Lambda^{\sigma,M}(x_0)$, for any multi-index $\beta$ s.t. $|\beta| \leq M$, the limit

$$f_{\beta}(x_0) = \lim_{j} D^\beta P_j(x_0)$$

is independent of the chosen sequence $(P_j)_j$.

$f_{\beta}(x_0)$ is the $\beta$-th Peano derivative of $f$ at $x_0$. 
Theorem

If \( M < s(\sigma^{-1}) \), then \( f \in \Lambda^{\sigma,M}(x_0) \) iff there exist \( C > 0 \) and a polynomial \( P \in \mathbb{P}[M] \) s.t.

\[
\| f - P \|_{L^\infty(x_0+2^{-j}B)} \leq C\sigma j,
\]

for \( j \) sufficiently large. The polynomial is unique.
Theorem

If $M < \mathcal{s}(\sigma^{-1})$, then $f \in \Lambda^{\sigma,M}(x_0)$ iff there exist $C > 0$ and a polynomial $P \in \mathbb{P}[M]$ s.t.

$$
\|f - P\|_{L^\infty(x_0 + 2^{-j}B)} \leq C\sigma_j,
$$

for $j$ sufficiently large. The polynomial is unique.

One has

$$
P(x) = \sum_{|\beta| \leq M} f_\beta(x_0) \frac{(x - x_0)^\beta}{|\beta|!}.
$$
For $s \in (0, \infty)$, let

- $\sigma_j = 2^{-js}$
- $M = [s(\sigma^{-1})] = [s]$ if $s \not\in \mathbb{N}$
- $M = s - 1$ if $s \in \mathbb{N}$
For $s \in (0, \infty)$, let

- $\sigma_j = 2^{-js}$
- $M = [s(\sigma^{-1})] = [s]$ if $s \notin \mathbb{N}$
- $M = s - 1$ if $s \in \mathbb{N}$

We have

$$\Lambda^s(x_0) = \Lambda^{\sigma,M}(x_0).$$
For $s \in (0, \infty)$, let

- $\sigma_j = 2^{-js}$
- $M = \lfloor s(\sigma^{-1}) \rfloor = [s]$ if $s \notin \mathbb{N}$
- $M = s - 1$ if $s \in \mathbb{N}$

We have

$$\Lambda^s(x_0) = \Lambda^{\sigma,M}(x_0).$$

**Corollary**

If $M < s(\sigma^{-1})$, one has

$$\Lambda^{\sigma,M}(x_0) \subset \Lambda^M(x_0).$$
Let

\[ B_h^M(x_0, j) = \{ x : [x, x + (M + 1)h] \subset x_0 + 2^{-j}B \}. \]
Let

\[ B_h^M(x_0,j) = \{ x : [x, x + (M + 1)h] \subseteq x_0 + 2^{-j}B \}. \]

**Proposition**

Let \( f \in L^\infty_{\text{loc}}(\mathbb{R}^d) \); one has \( f \in \Lambda^{\sigma, M}(x_0) \) iff there exist \( C, J > 0 \) s.t.

\[
\sup_{h \in B_j} \| \Delta_h^{M+1} f \|_{L^\infty(B_h^M(x_0,j))} \leq C \sigma_j,
\]

for any \( j \geq J \).
Let $\rho$ a radial function s.t. $\rho \in C^\infty_c(B)$, $\rho(B) \subset [0, 1]$ and $\|\rho\|_1 = 1$. 
Let $\rho$ a radial function s.t. $\rho \in C^\infty_c(B)$, $\rho(B) \subset [0, 1]$ and $\|\rho\|_1 = 1$.

One sets, for any $j \in \mathbb{N}_0$,

$$\rho_j = 2^{-jd} \rho(\cdot / 2^j).$$
Let $\rho$ a radial function s.t. $\rho \in C^\infty_c(B)$, $\rho(B) \subset [0, 1]$ and $\|\rho\|_1 = 1$.

One sets, for any $j \in \mathbb{N}_0$,

$$\rho_j = 2^{-jd} \rho(\cdot / 2^j).$$

**Lemma**

Let $N \in \mathbb{N}_0$; if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ satisfies

$$\sup_{k \geq j} \|f * \rho_k - f\|_{L^\infty(x_0 + 2^{-j}B)} \leq C \sigma_j,$$

for $j \geq J$, then, for any multi-index $\beta$ s.t. $|\beta| \leq N$, one has

$$\|D^\beta(f * \rho_j - f * \rho_{j-1})\|_{L^\infty(x_0 + 2^{-j}B)} \leq C 2^{jN} \sigma_j,$$

for any $j \geq J$. 
**Proposition**

If \( f \in \Lambda^{\sigma,M}(x_0) \), then there exists \( \Phi \in C^\infty_c(\mathbb{R}^d) \) s.t.

\[
\sup_{k \geq j} \| f - f \ast \Phi_k \|_{L^\infty(x_0 + 2^{-j}B)} \leq C \sigma j,
\]

for \( j \) sufficiently large.
Proposition

If \( f \in \Lambda^{\sigma,M}(x_0) \), then there exists \( \Phi \in C^\infty_c(\mathbb{R}^d) \) s.t.

\[
\sup_{k \geq j} \| f - f \ast \Phi_k \|_{L^\infty(x_0 + 2^{-j}B)} \leq C \sigma j,
\]

for \( j \) sufficiently large.

Conversely, if \( \sigma \to 0 \), \( f \in \Lambda^\epsilon(\mathbb{R}^d) \) for some \( \epsilon > 0 \) and \( f \) satisfies the previous relation for some function \( \Phi \in C^\infty_c(\mathbb{R}^d) \), then \( f \in \Lambda^{\sigma,M}(x_0) \) for any \( M \) s.t. \( M + 1 > \bar{s}(\sigma^{-1}) \).
Under some general conditions, there exist a function \( \phi \) and \( 2^d - 1 \) functions \( \psi^{(i)} \) called wavelets s.t.

\[
\{ \phi(\cdot - k) : k \in \mathbb{Z}^d \} \cup \{ \psi^{(i)}(2^j \cdot - k) : k \in \mathbb{Z}^d, j \in \mathbb{N}_0 \}
\]

forms an orthogonal basis of \( L^2(\mathbb{R}^d) \).
Under some general conditions, there exist a function $\phi$ and $2^d - 1$ functions $\psi^{(i)}$ called wavelets s.t.

$$\{\phi(\cdot - k) : k \in \mathbb{Z}^d\} \bigcup \{\psi^{(i)}(2^j \cdot - k) : k \in \mathbb{Z}^d, j \in \mathbb{N}_0\}$$

forms an orthogonal basis of $L^2(\mathbb{R}^d)$.

Any function $f \in L^2(\mathbb{R}^d)$ can be decomposed as follows,

$$f(x) = \sum_{k \in \mathbb{Z}^d} C_k \phi(x - k) + \sum_{j \geq 0, k \in \mathbb{Z}^d, 1 \leq i < 2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

with

$$C_k = \int f(x) \phi(x - k) \, dx, \quad c_{j,k}^{(i)} = 2^{dj} \int f(x) \psi^{(i)}(2^j x - k) \, dx.$$
We assume

- \( \phi, \psi^{(i)} \in C^n(\mathbb{R}^d) \) with \( n > M \),
- \( D^\beta \phi, D^\beta \psi^{(i)} (|\beta| \leq n) \) have fast decay,
- \( \text{supp}(\psi^{(i)}) \subset 2^{-j_0}B \) for some \( j_0 \).
We assume

- $\phi, \psi^{(i)} \in C^n(\mathbb{R}^d)$ with $n > M$,
- $D^\beta \phi, D^\beta \psi^{(i)} (|\beta| \leq n)$ have fast decay,
- $\text{supp}(\psi^{(i)}) \subset 2^{-j_0} B$ for some $j_0$.

We set

- $\lambda = \lambda(i, j, k) = \frac{k}{2j} + \frac{i}{2j+1} + [0, \frac{1}{2j+1})^d$
- $c_\lambda = c^{(i)}_{j, k}$
- $\psi_\lambda = \psi^{(i)}(2^j \cdot -k)$.
The wavelet leaders are defined by

\[ d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}| \]
The wavelet leaders are defined by

$$d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|$$

If $3\lambda$ denotes the $3^d$ dyadic cubes adjacent to $\lambda$ and $\lambda_j(x_0)$ the dyadic cube of length $2^{-j}$ containing $x_0$, one sets

$$d_j(x_0) = \sup_{\lambda \subset 3\lambda_j(x_0)} d_\lambda$$
Theorem

If \( f \in \Lambda^{\sigma,M}(x_0) \), then there exists \( C > 0 \) s.t.

\[
d_j(x_0) \leq C\sigma_j,
\]

for \( j \) sufficiently large.
Theorem

If \( f \in \Lambda^{\sigma,M}(x_0) \), then there exists \( C > 0 \) s.t.

\[
d_j(x_0) \leq C \sigma_j,
\]

for \( j \) sufficiently large.

Conversely, if \( \sigma \to 0 \), \( f \in \Lambda^\epsilon(\mathbb{R}^d) \) for some \( \epsilon > 0 \) and \( f \) satisfies the previous relation, then \( f \in \Lambda^{\tau,M}(x_0) \), where

- \( \tau \) is the sequence defined by \( \tau_j = \sigma_j|\log_2 \sigma_j| \),
- \( M \) is any number satisfying \( M + 1 > \bar{s}(\sigma^{-1}) \).
In the usual case, we have

$$s < t \Rightarrow \Lambda^t(x_0) \subset \Lambda^s(x_0).$$
In the usual case, we have

\[ s < t \Rightarrow \Lambda^t(x_0) \subset \Lambda^s(x_0). \]

The Hölder exponent of \( f \) at \( x_0 \) is

\[ h_f(x_0) = \sup\{s > 0 : f \in \Lambda^s(x_0)\}. \]
If, for any $s > 0$, $\sigma^{(s)}$ is an admissible sequence, the application

$$\sigma(\cdot) : s > 0 \mapsto \sigma^{(s)}$$

is called a family of admissible sequences.
If, for any $s > 0$, $\sigma^{(s)}$ is an admissible sequence, the application

$$\sigma(\cdot): s > 0 \mapsto \sigma^{(s)}$$

is called a family of admissible sequences. A family of admissible sequences is decreasing for $x_0$ if

$$s < t \Rightarrow \Lambda^{\sigma^{(t)},[t]}(x_0) \subset \Lambda^{\sigma^{(s)},[s]}(x_0).$$
If, for any $s > 0$, $\sigma^{(s)}$ is an admissible sequence, the application

$$\sigma^{(\cdot)} : s > 0 \mapsto \sigma^{(s)}$$

is called a family of admissible sequences. A family of admissible sequences is decreasing for $x_0$ if

$$s < t \Rightarrow \Lambda^{\sigma^{(t)},[t]}(x_0) \subset \Lambda^{\sigma^{(s)},[s]}(x_0).$$

Let $\sigma^{(\cdot)}$ a family of decreasing sequences for $x_0$ and $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$; the Hölder exponent of $f$ at $x_0$ for $\sigma^{(\cdot)}$ is

$$h_f^{\sigma^{(\cdot)}}(x_0) = \sup\{s > 0 : f \in \Lambda^{\sigma^{(s)},[s]}(x_0)\}.$$
How to check if a family of admissible sequences is decreasing?

Let

$$\Theta(m) = \sup_{k \in \mathbb{N}} \frac{\sigma_{k+1}(m)}{\sigma_k(m)}, \quad \Theta(m) = \inf_{k \in \mathbb{N}} \frac{\sigma_{k+1}(m)}{\sigma_k(m)},$$
How to check if a family of admissible sequences is decreasing?

Let

$$\Theta(m) = \sup_{k \in \mathbb{N}} \frac{\sigma_{k+1}(m)}{\sigma_k(m)}, \quad \underline{\Theta}(m) = \inf_{k \in \mathbb{N}} \frac{\sigma_{k+1}(m)}{\sigma_k(m)},$$

**Proposition**

A family of admissible sequences is decreasing for $x_0$ if it satisfies the following conditions:

- if $m \leq s < t < m + 1$ with $m \in \mathbb{N}_0$, $\sigma_j(t) \leq C \sigma_j(s)$ for $j$ sufficiently large
How to check if a family of admissible sequences is decreasing?

Let

$$\Theta(m) = \sup_{k \in \mathbb{N}} \frac{\sigma_{k+1}^{(m)}}{\sigma_k^{(m)}}$$
$$\Theta(m) = \inf_{k \in \mathbb{N}} \frac{\sigma_{k+1}^{(m)}}{\sigma_k^{(m)}}$$

**Proposition**

A family of admissible sequences is decreasing for \( x_0 \) if it satisfies the following conditions:

- if \( m \leq s < t < m + 1 \) with \( m \in \mathbb{N}_0 \), \( \sigma_j^{(t)} \leq C \sigma_j^{(s)} \) for \( j \) sufficiently large

- for any \( m \in \mathbb{N} \), at least one of the following conditions is satisfied: there exists \( \epsilon_0 > 0 \) s.t. for any \( \epsilon \in (0, \epsilon_0) \),
  
  \[
  \begin{align*}
  \sigma_j^{(m)} &\leq C \sigma_j^{(m-\epsilon)} \\
  \text{if } 1 < 2^m \Theta^{(m)} &\Rightarrow (\Theta^{(m)})^j \leq C \sigma_j^{(m-\epsilon)} \\
  \text{if } 1 > 2^m \Theta^{(m)} &\Rightarrow 2^{-jm} \leq C \sigma_j^{(m-\epsilon)} \\
  \text{if } 1 = 2^m \Theta^{(m)} &\Rightarrow j2^{-jm} \leq C \sigma_j^{(m-\epsilon)} \\
  \end{align*}
  \]
  
  \[
  \begin{align*}
  2^{-jm} &\leq C \sigma_j^{(m-\epsilon)} \\
  \text{if } 1 < 2^m \Theta^{(m)} &\Rightarrow \sigma_j^{(m)} \leq C \sigma_j^{(m-\epsilon)} \\
  \text{if } 1 > 2^m \Theta^{(m)} &\Rightarrow \sigma_j^{(m)} (2^m \Theta^{(m)})^{-j} \leq C \sigma_j^{(m-\epsilon)} \\
  \text{if } 1 = 2^m \Theta^{(m)} &\Rightarrow j \sigma_j^{(m)} \leq C \sigma_j^{(m-\epsilon)} \\
  \end{align*}
  \]
Thank you