Self-consistent approximations in relativistic plasmas: Quasiparticle analysis of the thermodynamic properties*

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Abstract

We generalize the concept of conserving, Φ-derivable, approximations to relativistic field theories. Treating the interaction field as a dynamical degree of freedom, we derive the thermodynamic potential in terms of fully dressed propagators, an approach which allows us to resolve the entropy of a relativistic plasma into contributions from its interacting elementary excitations. We illustrate the derivation for a hot relativistic system governed by electromagnetic interactions.

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*It is a pleasure to dedicate this paper to Leo Kadanoff on his sixtieth birthday. Immediately after Leo and I received our Ph.D.'s from Harvard in 1960, we began working together at the Institute for Theoretical Physics in Copenhagen (now the Niels Bohr Institute) on the problem of how to construct self-consistent approximations to two-particle propagators that preserved the basic conservation laws. Our solution to the problem was written up in the paper *Conservation laws and correlation functions*[1], and the concept generalized to deriving self-consistent approximations from a functional, Φ, of the single particle Green's function in Ref.[2]. I hope that Leo enjoys revisiting these early ideas, which may prove useful in understanding modern problems of the thermodynamics and transport properties of systems with long-ranged gauge fields. – Gordon Baym

I. INTRODUCTION

The motivations for studying relativistic plasmas are numerous. Nuclear matter at very high densities is expected, from the asymptotic freedom of QCD, to be in the form of a plasma of deconfined quarks and gluons [3]. Such plasmas were present in the early universe and may exist in the cores of neutron stars; current experiments aim to produce and study them in collisions of ultrarelativistic nuclei [4]. Electromagnetic plasmas are also of interest as Abelian models of quark-gluon plasmas. In addition, problems of relativistic plasmas are closely related to current issues in condensed matter theory. In gauge fields models of high T_c superconductors and of the fractional quantum Hall effect [5], the magnetic component of the interaction plays a fundamental role; this component is suppressed in normal metals by a factor $(v_F/c)^2$ (where v_F is the Fermi velocity), but is important in both strongly correlated systems and relativistic gases.

The thermodynamics and quasiparticle modes of relativistic plasmas have been the subject of much earlier work. In order to understand the equation of state of interacting relativistic plasmas, the thermodynamic potential has been calculated up to the first six orders of perturbation theory in the coupling constant [6–8]. Studies of the plasma microscopic properties have also revealed the existence of well-defined quasiparticle excitations and collective modes [9,10]. In particular, the fermionic spectrum has certain remarkable features not encountered in non-relativistic plasmas: the spectrum has a gap at zero momentum, and splits into two branches at small momenta [10].

Many physical quantities of relativistic plasmas are infrared divergent when evaluated in perturbation theory, as a consequence of the lack of static screening of magnetic interactions. Taking into account dynamical screening of such interactions at long wavelengths (the anomalous skin effect [11]), equivalent to including effects of Landau damping in the photon polarization operator, eliminates the divergences in transport coefficients [12] and in the rate of energy loss of fast particles [13]. Polarization effects also modify the interaction between low energy quasiparticle modes. The "Hard Thermal Loop" expansion scheme, proposed by Braaten and Pisarski [14], handles the effects of the medium diagrammatically by introducing vertex and self-energy corrections at small four-momentum. This description can also be cast in terms of kinetic equations [15].

Even after taking into account corrections from medium effects, difficulties still remain. Due to a lack of static screening in the magnetic component of gauge interactions, the fermion quasiparticle damping rate at finite temperatures is still divergent in perturbation theory. Blaizot and Iancu have shown that [16] in QED at finite temperatures, including multiple scattering à la Bloch-Nordsieck in the vacuum leads to well-defined, divergence-free, quasiparticle modes. This structure of the fermion propagator also appears in gauge field models of the fractional quantum Hall effect [17]. However, it is not clear whether the infrared structure in the fermion spectrum actually affects physical observables such as transport coefficients and the specific heat [18]. Even quantities that are well-controlled order by order in perturbation theory can cause difficulties. The (asymptotic) expansion of the thermodynamical potential converges very slowly, which has lead to suggestions to reorganize the perturbation expansion in terms of dressed fermion states rather than those of a free interacting gas [19].

Our aim here is to give a general framework for analyzing the effects of different characteristics of the fermion spectrum, the presence of a gap and the infrared structure, on thermodynamic quantities. Our analysis is based on Φ-derivable conserving approximations, first introduced [2,20] in the context of quantum transport theories. Among various applications, their use in the study of liquid ^{3}He has allowed one to interpret the $T^3 \log T$ term in the low temperature specific heat in terms of repeated scattering of particle-hole pairs [21,22]. The technique has been extended to Bose condensed systems in Refs. [23]. A generalization of the conserving approximation techniques to relativistic plasmas at zero temperature provided a controlled expansion of the ground state energy of electromagnetic and quark plasmas up to order g^4 [7]. In this latter work, the free energy is given as a functional of the fully dressed fermion and boson propagators as well as fully dressed vertices. In our present analysis, we keep track of both fermion and boson elementary excitations and treat both the matter and interaction fields as dynamical quantities. This allows us to decompose the entropy in the form

$$s = \sum_{\sigma,\mathbf{p}} \int \frac{d\omega_p}{2\pi} \sigma_f(\omega_p) A_s(\omega_p, p) + \sum_{\mathrm{pol},\mathbf{q}} \int \frac{d\omega_q}{2\pi} \sigma_b(\omega_q) B_s(\omega_q, q) + s', \tag{1}$$

where the entropy densities of free fermion and boson gases, $\sigma_f = -f \log f - (1-f)\log(1-f)$ and $\sigma_b = -n \log n + (1+n)\log(1+n)$ are here weighted by the spectral densities A_s and B_s of the interacting system.

In this paper, we develop Φ-derivable approximations for a hot relativistic QED plasma, a gas of electrons and positrons in a thermal bath of photons at temperature T, where $T \gg m$, although we expect the approach to be valid also for non-Abelian theories including QCD. For simplicity, we set the electron mass, m, to zero. We then illustrate the technique by computing the entropy within the one-loop approximation, and comment briefly on the general structure of the entropy spectral densities A_s and B_s.

II. THE THERMODYNAMIC POTENTIAL

We first review Φ-derivable approximations in non-relativistic field theories [2]. The thermodynamical potential of a non-relativistic Fermi system can be written as a functional of the fully dressed fermion propagator G as,

$$\beta\Omega = \Phi[G] + \mathrm{tr}\,\Sigma G - \mathrm{tr}\,\log(-G). \tag{2}$$

Here $\beta = 1/T$ and the symbol tr denotes a trace over spins and coordinates. In four-dimensional notation,

$$\mathrm{tr}\,X \equiv \sum_{\sigma,\sigma'} \int_0^{-i\beta} d1 X(1, 1^+), \tag{3}$$

where 1 is shorthand for the coordinates $(t_1, \mathbf{r_1})$, while 1^+ indicates that the time variable in the second argument of X is $t_1 + i0^+$. The functional $\Phi[G]$ is defined as the sum of all skeleton graphs contributing to the diagrammatic expansion of the potential Ω, where all lines are fully dressed propagators, G, instead of bare ones, G_0. In contrast with the familiar diagrammatic expansion of a scattering diagram in the vacuum, the expansion for $\Phi[G]$ contains a symmetry factor $1/n$ for each diagram with n fermion lines. Upon variation of G, the functional $\Phi[G]$ satisfies the relation

$$\delta\Phi[G] = \mathrm{tr}\,\Sigma\delta G, \tag{4}$$

where Σ is the electron self-energy. Pictorially, this relation means that removing one of the n fermion lines in a given Φ-diagram produces a self-energy diagram with $n-1$ fermion lines. Since there are n ways of removing a line, no symmetry factor appears on the right side of Eq. (4). The self-energy Σ is here a functional of the fully dressed propagator G, and it satisfies Dyson's equation $G^{-1} = G_0^{-1} - \Sigma$, where the bare propagator G_0 is given by $G_0^{-1} = \delta_{\sigma,\sigma'}(\omega_n - p^2/2m)$ for a non-relativistic system of fermions.

The principle behind conserving approximations is the following: one selects a particular subset of diagrams for the functional, $\Phi_a[G]$, from which one deduces an approximate self-energy functional $\Sigma_a \equiv \delta\Phi_a[G]/\delta G$. Dyson's equation provides then a self-consistent relation for the approximate propagator G_a. As shown in [2], the approximation for G leads to current densities and an energy-momentum tensor that obey the continuity equations expressing charge, particle number, energy and momentum conservation. The basic ingredient of the approximation scheme of Eq. (2) which enforces conservation laws is the stationarity property of Ω, $\delta\Omega = 0$, upon a variation of G that keeps G_0 constant, as can easily be checked from Eqs. (2) and (4).

We now generalize this approach to relativistic field theories, illustrating the method for electromagnetism, where the Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{5}$$

In QED, the free energy $\beta\,\Omega$ is given in terms of the fully dressed electron and photon propagators G and D, by

$$\beta\,\Omega = \Phi[G,D] - \mathrm{Tr}\,\Sigma G + \mathrm{Tr}\,\log(-\gamma^0 G) + \frac{1}{2}\mathrm{Tr}\,\Pi D - \frac{1}{2}\mathrm{Tr}\,\log(-D) \equiv W[G,D]. \tag{6}$$

Here, the functional $\Phi[G, D]$ is the sum of all the skeleton diagrams of the thermodynamic potential, expressed in terms of fully dressed G and D instead of the bare electron and photon propagators, G_0 and D_0. Under a simultaneous variation of the propagators G and D,

$$\delta\Phi[G, D] = \mathrm{Tr}\, \Sigma\delta G - \frac{1}{2}\mathrm{Tr}\, \Pi\delta D, \tag{7}$$

as one easily sees by removing an electron line or a photon line in a given diagram contributing to Φ. The electron self-energy $\Sigma[G, D]$ and the polarization operator $\Pi[G, D]$ are here functionals of G and D and satisfy Dyson's equations

$$G^{-1} = G_0^{-1} - \Sigma, \tag{8}$$

$$D^{-1} = D_0^{-1} - \Pi. \tag{9}$$

To prove that the functional $W[G, D]$ in Eq. (6) is identical to the thermodynamic potential $\beta\Omega$ of an electromagnetic plasma, we scale the coupling by a parameter λ in the interaction Lagrangian,

$$\mathcal{L}_{\mathrm{int}} = -\lambda e\bar{\psi}\not{A}\psi. \tag{10}$$

We work in Coulomb gauge. In the non-interacting limit, $\lambda \to 0$, both Σ and Π vanish, and we recover the result for a gas of free electrons and photons

$$\beta\Omega_0 = W[G_0, D_0] = \mathrm{Tr}\ \log(-\gamma_0 G_0) - \mathrm{Tr}\ \log(-D_{0,T}), \tag{11}$$

where only transverse polarizations (subscript T) appear in the photon contribution [24]. To prove that $\beta\,\Omega = W[G, D]$ it suffices to show that

$$\beta\frac{\partial\Omega_\lambda}{\partial\lambda} = \frac{\partial W}{\partial\lambda}[G_\lambda, D_\lambda], \tag{12}$$

for arbitrary λ.

Consider first the quantity $\beta\partial\Omega_\lambda/\partial\lambda$. From the definition of the thermodynamical potential,

$$\Omega_\lambda \equiv -T \log \mathrm{tr}\left[\exp\left(-\beta\left(H_\lambda - \mu Q\right)\right)\right] \tag{13}$$

(where Q is the charge operator, the Hamiltonian is $H_\lambda = H_0 - \int d^3x\, \mathcal{L}_{\rm int}$, and tr denotes a sum over all the quantum states of the system,) we find

$$\beta \frac{\partial \Omega_\lambda}{\partial \lambda} = \frac{\beta}{\lambda} \langle H_{\rm int} \rangle = -\frac{\beta}{\lambda} \int d^3x\, \langle \mathcal{L}_{\rm int} \rangle. \tag{14}$$

Here, $\langle X \rangle$ denotes the thermal average $\mathrm{tr}\left[\exp(-\beta(H_\lambda - \mu Q))\, X\right]/\mathrm{tr}\left[\exp(-\beta(H_\lambda - \mu Q))\right]$. We evaluate next $\partial W[G_\lambda, D_\lambda]/\partial\lambda$. A given diagram contributing to $\Phi[G_\lambda, D_\lambda]$ depends on λ implicitly through the electron and photon propagators, G_λ and D_λ, but also explicitly through the coupling λe, Eq. (10). However, the terms arising from the dependence on λe through G_λ and D_λ cancel in the partial derivative, since $W[G, D]$ is stationary under a variation of G and D. From the definition of $\Phi[G, D]$, the variation of W is

$$\begin{aligned} \delta W[G, D] = {} & \mathrm{Tr}\, \Sigma \delta G - \frac{1}{2} \mathrm{Tr}\, \Pi \delta D - \mathrm{Tr}\, \delta(\Sigma G) + \frac{1}{2} \mathrm{Tr}\, \delta(\Pi D) + \mathrm{Tr}\, \delta(\log(-\gamma_0 G)) \\ & -\frac{1}{2} \mathrm{Tr}\, \delta(\log(-D)). \end{aligned} \tag{15}$$

Using $\delta \log(-\gamma_0 G) = -G \delta G^{-1}$ and Dyson's equation, $\delta G^{-1} = -\delta\Sigma$, as well as similar relations for the photon propagators, we find that

$$\delta W[G, D] = 0, \tag{16}$$

under a variation that does not affect G_0, D_0, or the coupling constant, λe. Therefore, in the derivative of W with respect to λ only the explicit dependence in the coupling constant contributes, and we find

$$\frac{\partial W}{\partial \lambda}[G_\lambda, D_\lambda] = \left. \frac{\partial \Phi}{\partial \lambda}[G_\lambda, D_\lambda] \right|_{G,D}, \tag{17}$$

where the derivative on the right side is taken at fixed G and D.

Next, to compute the right side of Eq. (17), we use the invariance properties of Φ. As each vertex of a given Φ-diagram is connected to two electron lines and one photon line, the functional Φ remains constant under the scaling transformations

$$\Phi[G_\lambda, D_\lambda] = \Phi[s^{-f} G_\lambda, s^b D_\lambda; s\lambda], \quad f = 1 + b/2, \tag{18}$$

$$\Phi[G_\lambda, D_\lambda] = \Phi[s^{-\bar{f}} G_\lambda, s^{\bar{b}} D_\lambda; \lambda], \quad \bar{f} = \bar{b}/2, \tag{19}$$

where the third arguments of Φ on the right sides are the scaling factors of the coupling constant. Taking the derivative of Eq. (18) with respect to s, one finds

$$\lambda \left. \frac{\partial \Phi}{\partial (s\lambda)} \right|_{G,D} - f s^{-f-1} \mathrm{Tr} \, \frac{\delta \Phi}{\delta G} \, G_\lambda + b s^{b-1} \mathrm{Tr} \, \frac{\delta \Phi}{\delta D} \, D_\lambda = 0, \tag{20}$$

which, for $s = 1$, gives

$$\lambda \left. \frac{\partial \Phi}{\partial \lambda} \right|_{G,D} = f \, \mathrm{Tr} \, \Sigma_\lambda G_\lambda + \frac{b}{2} \, \mathrm{Tr} \, \Pi_\lambda D_\lambda = f \, \mathrm{Tr} \, \Sigma_\lambda G_\lambda + (f-1) \, \mathrm{Tr} \, \Pi_\lambda D_\lambda. \tag{21}$$

Similarly, the derivative of Eq. (19) with respect to s gives

$$\mathrm{Tr} \, \Sigma_\lambda G_\lambda + \mathrm{Tr} \, \Pi_\lambda D_\lambda = 0. \tag{22}$$

Therefore, Eqs. (22), (21) and (17) give

$$\frac{\partial W}{\partial \lambda}[G_\lambda, D_\lambda] = \frac{1}{\lambda} \mathrm{Tr} \, \Sigma_\lambda G_\lambda. \tag{23}$$

To complete the proof, we relate $\partial \mathrm{Tr} \, \Sigma_\lambda \, \Sigma_\lambda / \partial \lambda$ to $\langle H_{\mathrm{int}} \rangle$, starting with the equation of motion for the time-ordered fermion propagator $G_\lambda(1, 1') \equiv -i \langle T \psi(1) \bar{\psi}(1') \rangle$, which, from the Lagrangian, Eq. (5), is

$$i \partial\!\!\!/_1 G_\lambda(1, 1') + i \lambda e \langle T A\!\!\!/(1) \psi(1) \bar{\psi}(1') \rangle = \delta(1 - 1'). \tag{24}$$

Comparing to Dyson's equation in real space,

$$\int d\bar{1} \, G_0^{-1}(1, \bar{1}) G_\lambda(\bar{1}, 1') - \int d\bar{1} \, \Sigma_\lambda(1, \bar{1}) G_\lambda(\bar{1}, 1') = \delta(1 - 1'), \tag{25}$$

we identify

$$\begin{aligned} \mathrm{Tr} \, \Sigma_\lambda G_\lambda \equiv \int d1 \int d\bar{1} \, \Sigma_\lambda(1, \bar{1}) \, G_\lambda(\bar{1}, 1^+) &= i \lambda e \int_0^{-i\beta} d1 \langle \bar{\psi}(1) A\!\!\!/(1) \psi(1) \rangle \\ &= \beta \lambda e \int d\mathbf{r_1} \langle \bar{\psi}(\mathbf{r}_1, 0) A\!\!\!/(\mathbf{r}_1, 0) \psi(\mathbf{r}_1, 0) \rangle = \beta \langle H_{\mathrm{int}} \rangle. \end{aligned} \tag{26}$$

Therefore,

$$\beta \frac{\partial W}{\partial \lambda} = \frac{1}{\lambda} \mathrm{Tr} \, \Sigma_\lambda \, G_\lambda = \frac{\beta}{\lambda} \langle H_{\mathrm{int}} \rangle = \beta \frac{\partial \Omega}{\partial \lambda}, \tag{27}$$

which completes the proof.

III. THE ENTROPY

We now turn to the derivation of the entropy of the plasma in terms of the fully dressed propagators G and D, which is given by the derivative

$$S = -\left.\frac{\partial \Omega}{\partial T}\right|_{V,\mu} = -\left.\frac{\partial}{\partial T}(TW[G,D])\right|_{V,\mu}, \tag{28}$$

at constant volume V and chemical potential μ. We work in frequency-momentum space, where the electron and boson propagators have the spectral representations

$$G(\omega_n, \mathbf{p}) = \int \frac{d\omega_p}{2\pi} \frac{A(\omega_p, \mathbf{p})}{\omega_n - \omega_p}, \tag{29}$$

$$D_T(\omega_n, \mathbf{q}) = \int \frac{d\omega_q}{2\pi} \frac{B_T(\omega_q, \mathbf{q})}{\omega_n - \omega_q}, \tag{30}$$

$$D_L(\omega_n, \mathbf{q}) = \frac{1}{q^2} + \int \frac{d\omega_q}{2\pi} \frac{B_L(\omega_q, \mathbf{q})}{\omega_n - \omega_q}, \tag{31}$$

where the subscripts L and T denote longitudinal and transverse polarizations. The Matsubara frequencies are $\omega_n = i\pi(2n+1)T + \mu$ for electrons and $\omega_n = 2i\pi nT$ for photons. The functional $W[G,D]$ depends therefore on the temperature through the spectral functions A and B, through the complex frequencies ω_n, as well as through factors T at the vertices [25]. Because $W[G,D]$ is stationary under a simultaneous variation of the propagators G and D, Eq. (16), it is also stationary under variations of A and B that ignore the other temperature dependences. Thus, the contribution to the entropy coming from the temperature derivatives of the spectral densities alone vanishes. In the following, we therefore evaluate every derivative with respect to the temperature at constant A and B.

We start from the expression of the thermodynamical potential $\beta\,\Omega = W[G,D]$, Eq. (6). Evaluating the frequency sum by standard contour integration techniques [25], we find

$$\Omega = T\Phi[G,D] + \sum_{\mathbf{p}} \int \frac{d\omega_p}{\pi} f(\omega_p)\, \mathrm{tr}\, \mathrm{Im}\, \left[\Sigma G + \log(-\gamma_0 G^{-1})\right]$$
$$+ \sum_{\mathbf{q}} \fint \frac{d\omega_q}{2\pi}\, n(\omega_q) \sum_{l=L,T} g_l\, \mathrm{Im}\left[\Pi_l\, D_l + \log(-D_l^{-1})\right], \tag{32}$$

where $f(\omega_p) = (\exp(\beta(\omega_p - \mu)) + 1)^{-1}$, and $n(\omega_q) = (\exp(\beta\omega_q) - 1)^{-1}$ are the Fermi and Bose occupation factors, tr is a trace over spinor indices, the degeneracy factors g_l are $g_L = 1$ for longitudinal modes and $g_T = 2$ for transverse ones. Here, $\mathrm{Im}\,[F]$, means $\mathrm{Im}\,[F(\omega + i0^+)]$. The integration over ω_q is a principal value integral, since in deforming the contour integral onto the real axis, one needs to go around the (Bose) frequency $\omega_{n=0} = 0$. Differentiating Eq. (32) with respect to T at constant A and B, we decompose the entropy into a sum of three terms,

$$S = -\left.\frac{\partial\Omega}{\partial T}\right|_{\mu,V,A,B} = S_f + S_b + S', \tag{33}$$

where

$$S_f \equiv -\sum_{\mathbf{p}} \int \frac{d\omega_p}{\pi}\frac{\partial f}{\partial T}\mathrm{tr}\,\left(\mathrm{Im}\Sigma\,\mathrm{Re}G + \mathrm{Im}\,\log(-\gamma_0 G^{-1})\right), \tag{34}$$

$$S_b \equiv -\sum_{\mathbf{q},l} g_l \fint \frac{d\omega_q}{2\pi}\frac{\partial n}{\partial T}\left(\mathrm{Im}\Pi_l\,\mathrm{Re}D_l + \mathrm{Im}\log(-D_l^{-1})\right), \tag{35}$$

$$\begin{aligned} S' \equiv & -\left.\frac{\partial(T\Phi)}{\partial T}\right|_{A,B} - \sum_{\mathbf{p}} \int \frac{d\omega_p}{\pi}\frac{\partial f}{\partial T}\mathrm{tr}\,(\mathrm{Im}G\,\mathrm{Re}\Sigma) \\ & -\sum_{\mathbf{q},l} g_l \fint \frac{d\omega_q}{2\pi}\frac{\partial n}{\partial T}\,\mathrm{Im}D_l\,\mathrm{Re}\Pi_l. \end{aligned} \tag{36}$$

In this decomposition, the terms S_f and S_b are the contributions from the electron and photon elementary modes, while S' is a correction term arising from the interactions.

We illustrate the derivation in the one–loop approximation, which corresponds to taking the diagram Φ depicted in Fig. 1, and the self-energy diagrams of Fig. 2. Applying the usual Feynman rules, and carrying the sum over frequencies, we find

$$T\Phi = \frac{e^2}{16\pi^3 V}\sum_{\mathbf{p},\mathbf{q},l} g_l \int d\omega_p d\omega_{p'} d\omega_q P_l B_l(\omega_q, q)\frac{nf\,(1-f') - (1+n)\,(1-f)\,f'}{\omega_p + \omega_q - \omega_{p'}} + T\Phi_{\mathrm{HF}}, \tag{37}$$

$$\Sigma(z,p) = \frac{e^2}{4\pi^2 V}\sum_{\mathbf{q},l} g_l \int d\omega_q d\omega_{p'} S_l(\omega_{p'}, p') B_l(\omega_q, q)\,\frac{(1-f')\,n + f'\,(1+n)}{z + \omega_q - \omega_{p'}} + \Sigma_{\mathrm{HF}}, \tag{38}$$

$$\Pi_l(z,q) = \frac{e^2}{4\pi^2 V}\sum_{\mathbf{p}} \int d\omega_p d\omega_{p'} P_l\,\frac{f\,(1-f') - f\,(1-f')}{z + \omega_p - \omega_{p'}}, \tag{39}$$

where $f \equiv f(\omega_p)$, $f' \equiv f(\omega_{p'})$ and $n \equiv n(\omega_q)$, z is complex, $\mathbf{p}' = \mathbf{p} + \mathbf{q}$, and the matrix elements P_l and S_l are given by

$$P_L(\omega_p,p;\omega_{p'},p') \equiv \mathrm{tr}\,\left[\gamma^0 A(\omega_p,p)\gamma^0 A(\omega_{p'},p')\right], \tag{40}$$

$$P_T(\omega_p,p;\omega_p,p') \equiv \frac{1}{2}(\delta_{ij}-\hat{\mathbf{q}}_i\hat{\mathbf{q}}_j)\,\mathrm{tr}\,\left[\gamma^i A(\omega_p,p)\gamma^j A(\omega_{p'},p')\right], \tag{41}$$

$$S_L(\omega_{p'},p') \equiv \gamma^0 A(\omega_{p'},p')\gamma^0, \tag{42}$$

$$S_T(\omega_{p'},p') \equiv \frac{1}{2}(\delta_{ij}-\hat{\mathbf{q}}_i\hat{\mathbf{q}}_j)\,\gamma^i A(\omega_{p'},p')\gamma^j. \tag{43}$$

The terms $\Phi_{\rm HF}$ and $\Sigma_{\rm HF}$ in Eqs. (37) and (38) arise from the static term $1/q^2$ in the photon propagator of Eq. (31),

$$T\Phi_{\rm HF} = -\frac{e^2}{8\pi^2 V}\sum_{\mathbf{p},\mathbf{p}'}\int d\omega_p d\omega_{p'}\,\mathrm{tr}\,\left[\gamma^0 A(\omega_p,p)\gamma^0 A(\omega_{p'},p')\right]\frac{ff'}{q^2}, \tag{44}$$

$$\Sigma_{\rm HF} = -\frac{e^2}{2\pi V}\sum_{\mathbf{p}'}\int d\omega_{p'}\,\gamma^0 A(\omega_{p'},p')\gamma^0\,\frac{f'}{q^2}. \tag{45}$$

In the one-loop approximation S' in fact vanishes. (A similar result was observed in the SPA approximation in liquid ^{3}He by Riedel [22].) To see how the various terms in Eq. (36) cancel, we first note that the same combination of matrix elements appears in each term of S'. Since $\mathrm{Im}[G(\omega_p+i0^+)] = -A(\omega_p,p)/2$ and $\mathrm{Im}[D_l(\omega_p+i0^+)] = -B_l(\omega_q,q)/2$,

$$\begin{aligned} S' = &-\frac{e^2}{8\pi^2 V}\sum_{\mathbf{p},\mathbf{p}'}\int d\omega_p d\omega_{p'}\,\mathrm{tr}\,\left[\gamma^0 A(\omega_p,p)\gamma^0 A(\omega_{p'},p')\right]\frac{\mathcal{S}_{\rm HF}}{q^2} \\ &+\frac{e^2}{16\pi^3 V}\sum_{\mathbf{p},\mathbf{q},l} g_l \,⨍\, d\omega_p d\omega_{p'} d\omega_q\, P_l(\omega_p,p;\omega_{p'},p')B_l(\omega_q,q)\,\frac{\mathcal{S}}{\omega_p+\omega_q-\omega_{p'}}, \end{aligned} \tag{46}$$

where the $\mathcal{S}$ and $\mathcal{S}_{\rm HF}$ denote combinations of statistical factors and their T derivatives. Using Eqs. (37) through (45), symmetrizing the contributions of $\mathrm{Re}\,\Sigma_{\rm HF}$ and $\mathrm{Re}\,\Sigma$ by the transformation $(\omega_p,\mathbf{p}) \leftrightarrow (\omega_{p'},\mathbf{p}')$, $(\omega_q,\mathbf{q}) \leftrightarrow (-\omega_q,-\mathbf{q})$, and using the fact that the spectral functions B_l are odd functions of their arguments, we have

$$\mathcal{S}_{\rm HF} = -\partial\left\{ff'\right\}/\partial T + f'\partial f/\partial T + f\partial f'/\partial T = 0, \tag{47}$$

$$\begin{aligned} \mathcal{S} = &-\partial\left\{nf(1-f') - (1+n)(1-f)f'\right\}/\partial T + \left(f(1-f') - f'(1-f)\right)\partial n/\partial T \\ &+ \left(n(1-f') + (1+n)f'\right)\partial f/\partial T + \left(nf + (1+n)(1-f)\right)\partial f'/\partial T = 0. \end{aligned} \tag{48}$$

Thus the correction term S' vanishes in the one-loop approximation.

This cancelation takes place only in the lowest order diagram for Φ. A general analysis of Φ diagrams in the context of non-relativistic normal Fermi liquids [21] shows that the contributions to S' come only from those graphs that have at least two vanishing energy denominators. These terms correspond graphically to the set of diagrams that can be cut in three and only three different pieces when one removes a set of fermion lines. Generalizing this analysis to the diagrams representing the functional $\Phi[G, D]$, we see that there is only one possible cut in the diagram of Fig. 1 and it generates the vanishing denominator $\omega_q + \omega_p - \omega_{p'}$. The simplest diagram that contributes to S' is shown in Fig. 3, where we have drawn a set of cuts that give two vanishing energy denominators.

IV. INTERPRETATION OF THE ENTROPY FORMULA

A. Exchange and correlation entropy

When the two electron lines in Fig. 1 are replaced by bare propagators and the photon line is replaced by a dressed propagator in the RPA approximation, i.e., with the lowest order bubble diagram insertions, we recover the set of diagrams that contribute to the exchange and correlation terms considered by Akhiezer and Peletminskiĭ [6]. It is instructive to see how the expression of the entropy that we have derived, Eqs. (34), (35) and (36), generates these exchange and correlation terms correctly when expanded in the first few powers of the coupling constant e. We start with the exchange entropy [6],

$$S_s \equiv \frac{1}{2}\frac{\partial}{\partial T}\left(T\,\mathrm{Tr}\,\Pi^{(2)} D_0\right) = \sum_{\mathbf{q}} \fint \frac{d\omega_q}{2\pi}\frac{\partial}{\partial T}\left(n\,\mathrm{Im}\,\Pi^{(2)}\,D_0\right), \tag{49}$$

where $\Pi^{(2)}$ is the photon self–energy of Fig. 2 with two bare electron propagators. [In this section, we do not write the sum over polarization states explicitely.] Expanding Eqs. (34) and (35) to first order in e^2, we find

$$S^{(2)} = \sum_{\mathbf{p}} \int \frac{d\omega_p}{\pi}\frac{\partial f}{\partial T}\mathrm{tr}\,\mathrm{Im}G_0\,\mathrm{Re}\,\Sigma[D_0] + \sum_{\mathbf{q}} \fint \frac{d\omega_q}{2\pi}\frac{\partial n}{\partial T}\mathrm{Im}\,D_0\,\mathrm{Re}\,\Pi^{(2)}. \tag{50}$$

Then expanding Eq. (36) for $S' = 0$ to order e^2, we recognize that the right side of Eq. (50) is $S^{(2)} = -\partial(T\Phi^{(2)})/\partial T$, which from the identity $\Phi^{(2)} = (-1/2)\mathrm{Tr}\,\Pi^{(2)}\,D_0$ becomes $S^{(2)} = (1/2)\partial(T\,\mathrm{Tr}\,\Pi^{(2)}\,D_0)/\partial T = S_s$.

The sum of the exchange and the correlation terms that contribute to the entropy is [6]

$$\begin{aligned} S_c + S_s &\equiv -\frac{1}{2}\frac{\partial}{\partial T}\left(T\,\mathrm{Tr}\,\log(-1+D_0\Pi^{(2)})\right) \\ &= -\sum_{\mathbf{q}} \fint \frac{d\omega_q}{2\pi}\left(\frac{\partial n}{\partial T}\,\mathrm{Im}\left[\log(-1+D_0\Pi^{(2)})\right] - n\,\mathrm{Im}\left[D\,\frac{\partial \Pi}{\partial T}^{(2)}\right]\right), \end{aligned} \tag{51}$$

where D is now the propagator in the RPA approximation, $D^{-1} = D_0^{-1} - \Pi^{(2)}$. Using arguments similar to those we used to derive the identity $S' = 0$, we can decompose the last term on the right side into the following terms:

$$\begin{aligned} \sum_{\mathbf{q}} \fint \frac{d\omega_q}{2\pi}\, n\,\mathrm{Im}\left[D\,\frac{\partial \Pi}{\partial T}^{(2)}\right] &= \sum_{\mathbf{p}} \int \frac{d\omega_p}{\pi}\,\frac{\partial f}{\partial T}\mathrm{Im}\,G_0\,\mathrm{Re}\,\Sigma[D] \\ &\quad - \sum_{\mathbf{q}} \fint \frac{d\omega_q}{2\pi}\,\frac{\partial n}{\partial T}\,\mathrm{Im}\,\Pi^{(2)}\,\mathrm{Re}\,D. \end{aligned} \tag{52}$$

The two terms on the right side correspond respectively to S_f expanded to linear order in $\Sigma[D]$, and to the first term in S_b. Adding the first term on the right side of Eq. (51), we have therefore shown that our expression for $S = S_f + S_b$ gives the correct correlation term in the entropy when one uses bare electrons and the RPA photon propagator in Eqs. (34)-(36).

B. Decomposition in elementary excitation modes

The formulae we obtained for the entropy terms S_f, Eq. (34), and S_b, Eq. (35), allow us to perform a spectral analysis by decomposing S_f and S_b into integrals over the elementary excitations of the matter field and the electromagnetic field. We closely follow the derivation that was carried out by Carneiro and Pethick for liquid ^{3}He [21]. The key is to transform the temperature derivatives of the statistical factors f and n into the following expressions

$$\frac{\partial f}{\partial T} = -\frac{\partial \sigma_f}{\partial \omega_p}(\omega_p), \tag{53}$$

$$\frac{\partial n}{\partial T} = -\frac{\partial \sigma_n}{\partial \omega_q}(\omega_q), \tag{54}$$

where again $\sigma_f \equiv -f\log f - (1-f)\log(1-f)$ and $\sigma_n \equiv -n\log n + (1+n)\log(1+n)$ are the entropy contributions from an electron mode of energy ω_p and from a photon mode of frequency ω_q. Integrating Eqs. (34) and (35) by parts, we see that S_f and S_b obey the spectral representations

$$S_f = \sum_{\mathbf{p}} \int \frac{d\omega_p}{2\pi}\, \sigma_f(\omega_p)\, A_s(\omega_p, p), \tag{55}$$

$$S_b = \sum_{\mathbf{q}} \int_0^\infty \frac{d\omega_q}{2\pi}\, \sigma_b(\omega_q) B_s(\omega_q, q), \tag{56}$$

where A_s and B_s are defined by

$$A_s(\omega_p, p) = \mathrm{tr}\, \frac{\partial}{\partial \omega_p} \left\{ \mathrm{Re} G\, \Gamma + 2 \mathrm{Im}\, \log(-\gamma_0 G) \right\}, \tag{57}$$

$$B_s(\omega_q, q) = \sum_l g_l\, \frac{\partial}{\partial \omega_q} \left\{ \mathrm{Re} D_l\, L_l + 2 \mathrm{Im}\, \log(-D_l) \right\}, \tag{58}$$

while $\Gamma(\omega_p, p) \equiv -2\, \mathrm{Im}\, \Sigma(\omega_p + i0^+, p)$ and $L_l(\omega_q, q) \equiv -2\, \mathrm{Im}\, \Pi_l(\omega_q + i0^+, q)$ are the imaginary parts of the electron and photon self–energies. In deriving Eq. (56), we have used the fact that the boson propagator D is an even function of its arguments to reduce the domain of integration in Eq. (56) to positive frequencies ω_q.

The sum over spinor indices (the trace tr) in Eq. (57) can easily be decomposed into two contributions, one from electron states with a chirality equal to their helicity (subscript +) and one from states with opposite helicities and chiralities (subscript −), by writing

$$\begin{aligned} G &= \frac{\gamma_0 - \gamma \cdot \hat{\mathbf{p}}}{2}\, G_+ + \frac{\gamma_0 + \gamma \cdot \hat{\mathbf{p}}}{2}\, G_- \\ \Sigma &= \frac{\gamma_0 - \gamma \cdot \hat{\mathbf{p}}}{2}\, \Sigma_- + \frac{\gamma_0 + \gamma \cdot \hat{\mathbf{p}}}{2}\, \Sigma_+, \end{aligned} \tag{59}$$

which gives

$$\mathrm{tr}\, \mathrm{Re}\, G\, \Gamma = 2 \left(\mathrm{Re}\, G_+\, \Sigma_+ + \mathrm{Re}\, G_-\, \Sigma_- \right), \tag{60}$$

where the factor 2 is from the spin sum. Hence, $A_s = 2 \sum_\pm A_{s\pm}$.

We now turn to the entropy spectral functions A_s and B_s. In the one-loop approximation of Fig. 1, the self-energies Σ and Π of Eqs. (38) and (39) depend on the fully dressed spectral functions A and B. These functions depend in turn on the self-energies Σ and Π through Dyson's equations, Eq. (9). The problem of determining A and B, and the functions A_s

and B_s, is therefore a self-consistent one. However, we can make general statements about their structure on the basis of the following arguments. If the system develops well-defined excitation modes, we expect the functions A and B to consist of narrow peaks at the locations of the quasiparticles and collective modes, as well as wide bands of continuum states. The continuum states that contribute to the spectral function A are composed of particle-photon states, while those contributing to B are particle-hole and particle-antiparticle states. In lowest order of perturbation theory, the continua extend over frequency ranges $-p < \omega_p < p$ and $-q < \omega_q < q$, respectively [9,10]. We expect that in the present self-consistent problem, interactions only slightly modify these frequency ranges. The quasiparticle modes are at frequencies well outside the spectrum of continuum states, $|\omega_p| > p$ and $\omega_q > q$.

The structure of the entropy spectral functions A_s and B_s is qualitatively similar to that of A and B. The photon spectral function B_s, as we see from its definition, Eq. (58), has a support over the frequency range of the continuum states contributing to B. It also contains a contribution from the oscillation modes of the electromagnetic field, i.e., the longitudinal and transverse plasmon modes, with frequencies $\omega_L(q)$ and $\omega_T(q)$ and momentum $\mathbf{q}$, where $\omega_{L,T}(q) > q$. As we show, the spectral function B_s takes a simple form in the vicinity of these frequencies

$$B_{s,l}(\omega_q, q) \simeq \frac{(Z_l L_l)^3/2}{((\omega_q - \omega_l(q))^2 + (Z_l L_l/2)^2)^2}, \tag{61}$$

where $Z_l \equiv \partial \mathrm{Re} D_l^{-1}/\partial\omega_q$, $Z_L \simeq \omega_L/(2q^2)$ and $Z_T \simeq 1/(2\omega_T)$. Evaluating the logarithmic term in Eq. (58), we have for each polarization state l,

$$2\mathrm{Im}\log(-D_l) = 2\pi\Theta(-\mathrm{Re}D_l^{-1}) - 2\arctan\frac{L_l}{2\mathrm{Re}D_l^{-1}}, \tag{62}$$

where the values of the arc tangent are in the range $[-\pi/2, \pi/2]$. The function $L_l/(2\omega_q)$ is the interaction rate of an excitation of frequency ω_q. In the vicinity of the modes, $\omega_q \simeq \omega_{L,T}(q)$, $L_l/2\omega_q$ is perturbatively smaller than the modes frequencies $\omega_{L,T}(q)$ [26], and varies slowly around $\omega_{L,T}(q)$. Thus, with $\mathrm{Re}D_l = \mathrm{Re}D_l^{-1}/((\mathrm{Re}D_l^{-1})^2 + (L_l/2)^2)$, and $\mathrm{Re}D_l^{-1} \simeq (\omega_q - \omega_l(q))/Z_l$, we can evaluate the right side of Eq. (58) keeping L_l constant, and derive Eq. (61).

The spectral functions $B_{s,l}$ are therefore sharply peaked around the mode frequency ω_l. Comparing to the photon spectral densities B_l, which have a Lorentzian form close to the poles, $B_l \simeq (Z_l L_l)/((\omega_q - \omega_l)^2 + (Z_l L_l/2)^2)$, we see that $B_{s,l}$ have a stronger peak and smaller wings. The electron spectral function, $A_s = 2\sum_{\pm} A_{s\pm}$, has a structure similar to that of B_s: continuum states contribute at small energies $|\omega_p| \lesssim p$, while $A_{s\pm}$ exhibit a sharp variation close to the quasiparticle energies $\omega_{0\pm}$,

$$A_{s\pm}(\omega_p, p) \simeq \frac{(Z_\pm \Gamma_\pm)^3/2}{((\omega_p - \omega_{0\pm})^2 + (Z_\pm \Gamma_\pm/2)^2)^2}, \tag{63}$$

where $Z_\pm$ is the quasiparticle mode residue. In deriving Eq. (63), we have neglected the variation of the (one-loop order) interaction rate $\Gamma_\pm$, which for energies ω_p in the vicinity of $\omega_{0\pm}$ is $\sim \mathcal{O}(e^2 T \log(q_D/|\omega_p - \omega_{0\pm}|))$, where $q_D \sim eT$ is the Debye momentum [16].

Both the electron and the photon excitation spectra exhibit collective modes at small momenta. However, the low energy modes contribute to the total entropy in very different orders in the coupling constant. The contribution from long wavelength modes of the electromagnetic field has been calculated by Akhiezer and Peletmniskiĭ [6]; the result is that the sum of plasmon oscillations and continuum modes contributes to the correlation term in the entropy, of order e^3T^3. We can easily understand this order of magnitude by noting that the contribution to the entropy of all states of energies of order of the plasmon frequency $\omega_{L,T}(0) \sim eT$ is $\sim \sum_{\mathbf{q}} \propto q_c^3$, where the momentum cutoff is $q_c \sim eT$. The small energy modes in the electron spectrum give a smaller contribution, of order e^5T^3. To see this, we expand to lowest order the difference of S_f from its expression for a non-interacting gas, and concentrate on the phase space of small energy modes, $\omega_p \sim eT$,

$$\Delta S_f \simeq 2 \sum_{\pm} \sum_{\mathbf{p}} \int \frac{d\omega_p}{\pi} \frac{\partial f}{\partial T} A_{0\pm}(\omega_p, p) \,\mathrm{Re}\,\Sigma_{\pm}^{(2)}(\omega_p, p). \tag{64}$$

For small energies, $\omega_p \sim eT$, $\partial f/\partial T$ is $\sim \omega_p/(4T^2)$, and $A_{0\pm}(\omega_p, p)\,\mathrm{Re}\,\Sigma_{\pm}^{(2)}(\omega_p, p)$ is $\simeq \pm 2\pi m_f^2\, \delta(\omega_p \mp p)/p$, where m_f is the gap at zero momentum. Thus, $\Delta S_f \sim m_f^2/T^2 \sum_{\mathbf{p}} \sim e^2 p_c^3 \sim e^5 T^3$, for a cutoff momentum $p_c \sim eT$. The reason why ΔS_f is higher by two orders in the coupling constant than the contribution to the total entropy from long wavelength modes of the electromagnetic field is the following. An expansion in power series of e of the logarithm term in B_s, Eq. (58), leads to infrared divergences term by term [6]; hence, to evaluate the correlation term in the entropy, one must include the logarithm as it stands. However, an expansion of the logarithm term in A_s, Eq. (57), is divergence-free and results in a prefactor $m_f^2/T^2 \sim e^2$.

Although the one-loop approximation is mathematically tractable, it has important limitations. To take into account correctly the low energy part of the electron spectrum, which enters only in order e^5T^3, one needs to consider Φ-diagrams of higher orders, such as the one depicted in Fig. 3, contributing at least in order e^4. Also, the diagram of Fig. 1 does not always provide the correct width for the entropy spectral functions evaluated close to the quasiparticle modes. For instance, the lifetime of a photon mode of energy $\sim T$ is limited by Compton scattering and inverse pair annihilation. The self-consistent approximation of Fig. 1 includes only the direct terms of these processes, as can be seen by considering an electron self-energy insertion in one of the fermion lines of the bubble diagram shown in Fig. 2 and then taking the imaginary part of the corresponding photon self-energy. The cross terms arise from the imaginary part of the photon self-energy obtained by removing one photon line in Fig. 3. Finally, in calculating the widths of low energy modes, one also needs to include vertex corrections, as described by the "Hard Thermal Loop" perturbation scheme, see Ref. [14].

We conclude by stressing the fact that our analysis of the entropy provides a framework for studying effects of the infrared structure in the fermion propagator on thermodynamical quantities. Kim et al. have proposed, in the context of the fractional quantum Hall effect [18], that one should be careful in calculating thermodynamical quantities by summing over fermion degrees of freedom, since by doing so, one may encounter infrared divergent terms. Kim et al. attribute this problem to the fact that the composite fermion propagator is not a gauge invariant quantity and suggest that the thermodynamical quantities should be calculated instead by summing over boson degrees of freedom only. In the present analysis, the dangerous term is the first component of S_f, $\sim \sum_{\mathbf{p}} \int (d\omega_p/2\pi)\partial f/\partial T \,\mathrm{Re}\, G\Gamma$, with $\Gamma \propto e^2 T \log(q_D/|\omega_p - p|)$, and it does not lead to a divergence, as the principal part $\fint dx \log|x|/x$ vanishes. This term is actually finite and is part of the exchange entropy term, of order e^2. On the other hand, for energies ω_p in the vicinity of p, both the spectral densities $A \simeq \Gamma/((\omega_p - p)^2 + (\Gamma/2)^2)$ and $A_s \simeq \Gamma^3/2/((\omega_p - p)^2 + (\Gamma/2)^2)^2$ vanish as $\sim 1/\Gamma \sim \log^{-1}|\omega_p - p|$, as $\omega_p \sim p$. This logarithmic behavior is symptomatic of a breakdown of perturbation theory. In a future publication we will examine corrections in the free energy similar to those considered in the calculation of the fermion lifetime in a hot plasma [16].

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FIGURE CAPTIONS

Figure 1: Φ in the one-loop approximation.

Figure 2: a) One-loop electron self-energy. b) One-loop photon self-energy.

Figure 3: Φ-diagram to explicit order e^4. The dashed lines illustrate the cuts that give rise to a non-vanishing term in S'.

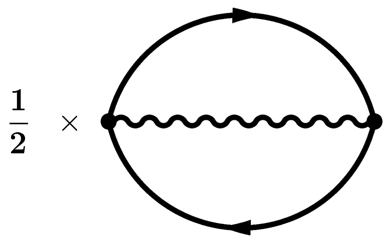

Fig. 1

a) b)

Figs. 2

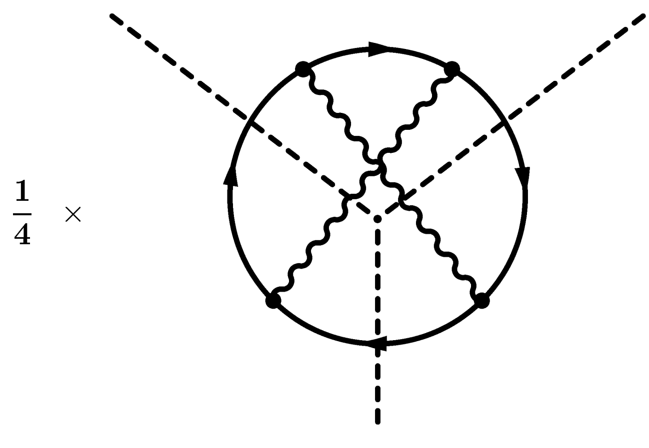

Fig. 3