

Optimization of laminated composite structures: problems, solution procedures and applications

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Abstract

In this chapter the optimal design of laminated composite structures is considered. A review of the literature is proposed. It aims at giving a general overview of the problems that a designer must face when he works with laminated composite structures and the specific solutions that have been derived. Based on it and on the industrial needs an optimization method specially devoted to composite structures is developed and presented. The related solution procedure is general and reliable. It is based on fibers orientations and ply thicknesses as design variables. It is daily used in an (European) industrial context for the design of composite aircraft box structures located in the wings, the center wing box, and the vertical and horizontal tail plane. This approach is based on sequential convex programming and consists in replacing the original optimization problem by a sequence of approximated sub-problems. A very general and self adaptive approximation scheme is used. It can consider the particular structure of the mechanical responses of composites, which can be of different nature when both fibers orientations and plies thickness are design variables. Several numerical applications illustrate the efficiency of the proposed approach.

1. Introduction

According to their high stiffness and strength to weight ratios, composite materials are well suited for high-tech aeronautics applications. A large amount of parameters is needed to qualify a composite construction, e.g. the stacking sequence, the plies thickness and the fibers orientations. It results that the use of optimization techniques is necessary, especially to tailor the material to specific structural needs. The chapter will cover this subject and is divided in three main parts.

After recalling the goal of optimization, the different laminates parameterizations will be presented with their limitations (the pros and the cons) in the frame of the optimal design of composite structures. The issues linked to the modeling of structures made of such materials and the problems solved in the literature will be reviewed. The key role of fibers orientations in the resulting laminate properties will be discussed. Finally the outlines of a pragmatic solution procedure for industrial applications will be drawn. Throughout this section, a profuse and state-of-the-art review of the literature will be provided.

Secondly, a general solution procedure daily used in industrial problems including fibers reinforced composite materials will be described. The related optimization algorithm is based on sequential convex programming and has proven to be very reliable. This algorithm is presented in details and validated by comparing its performances to other optimization methods of the literature.

Finally, it will be shown how this optimization algorithm can efficiently solve several kinds of composite structures designs problems: amongst others, solutions for topology optimization with orthotropic materials will be presented, important considerations about the optimal design of composites including buckling criteria will be discussed, optimization with respect to damage tolerance will be considered (crack delamination in a laminated structure). On top of that, some key points of the solution procedure based on this optimization algorithm applied to the pre-sizing of (European) industrial composite aircraft box structures will be presented.

2. The optimal design problem and available optimization methods

The goal of optimization is to reach the best solution of a problem under some restrictions. Its mathematical formulation is given in (2.1), where $g_0(\mathbf{x})$ is the objective function to be minimized, $g_j(\mathbf{x})$ are the constraints to be satisfied at the solution, and $\mathbf{x}=\{x_i, i=1, \dots, n\}$ is the set of design variables. The value of those design variables change during the optimization process but are limited by an upper and a lower bound when they are continuous, what will be the case in the sequel.

$$\begin{aligned} \min g_0(\mathbf{x}) \\ g_j(\mathbf{x}) \leq g_j^{\max} \quad j=1, \dots, m \\ \underline{x}_i \leq x_i \leq \bar{x}_i \quad i=1, \dots, n \end{aligned} \quad (2.1)$$

The problem (2.1) is illustrated in Figure 2.1, where 2 design variables x_1 and x_2 are considered. The isovalues of the objective function are drawn, as well as the limiting values of the constraints. The solution is found via an iterative process. \mathbf{x}^k is the vector of design variables at the current iteration k , and \mathbf{x}^{k+1} is the estimation of the solution at the iteration $k+1$. Typically a local solution \mathbf{x}_{local}^* will be reached when a gradient based optimization method is used. The best solution \mathbf{x}_{global}^* can only be found when all the design space is looked over: this last can be accessed with specific optimization methods that include a non deterministic procedure, as the genetic algorithms.

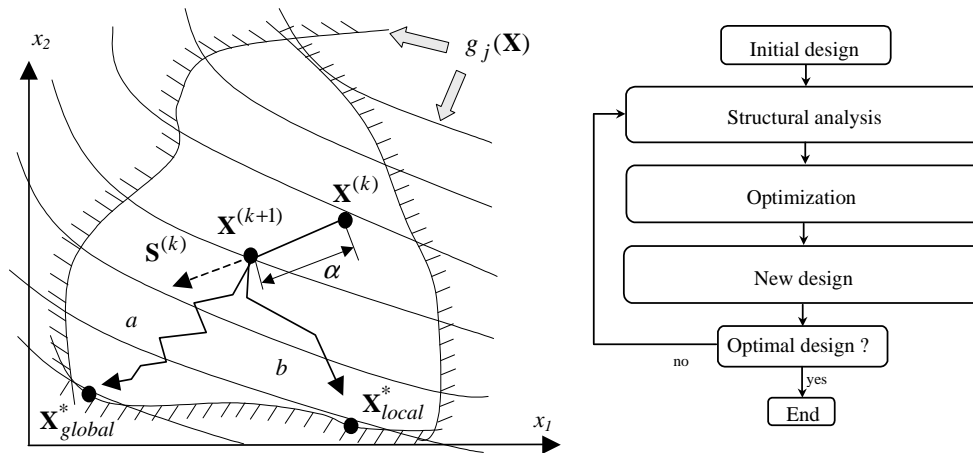


Figure 2.1. Illustration of an optimization problem and its solution

In structural optimization the design functions can be global as the weight, the stiffness, the vibration frequencies, the buckling loads, or local as strength constraints, strains and failure criteria. When the design variables are linked to the transverse properties of the structural members (e.g. the cross section area of a bar in a truss), the related optimization problem is called optimal sizing (Figure 2.2a). The value of some geometric items (e.g. a radius of an ellipse) can also be variable: in this case, we are talking about shape optimization (Figure 2.2b). Topology optimization aims at spreading a given amount of material in the structure for a maximum stiffness. Here, holes can be automatically created during the optimization process (Figure 2.2c). Finally the optimization of the material can be addressed, e.g. the local design of laminated composite structure with respect to fibers orientations, ply thickness and stacking sequence (Figure 2.2d).

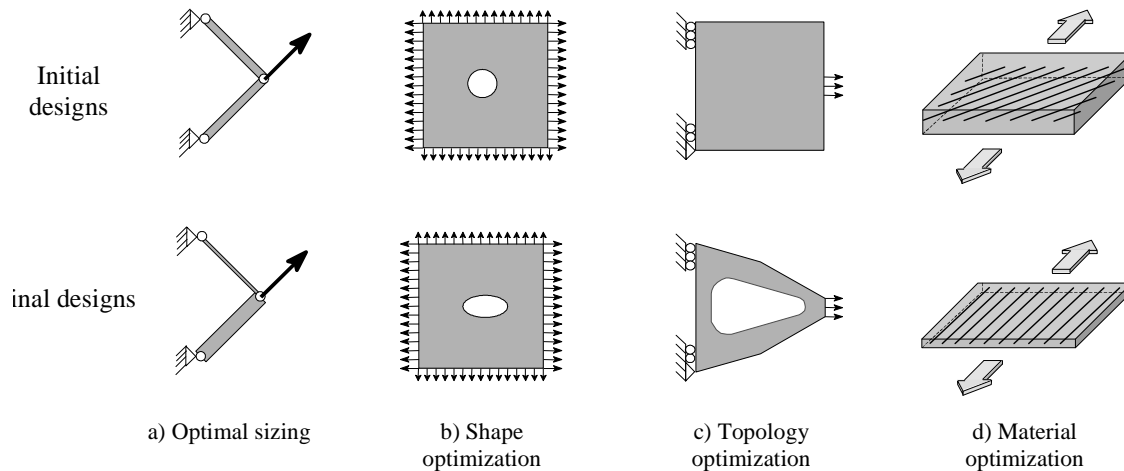


Figure 2.2. The structural optimization problems

The structural optimization problems are non linear and non convex, and several local minimum exist. It is usually accepted that a local solution $\mathbf{x}_{\text{local}}$ gives satisfaction. The global solution $\mathbf{x}_{\text{global}}$ can only be determined with very large computational resources. In some cases when the problem includes a very large amount of constraints, a feasible solution is acceptable.

A lot of methods exist to solve the problem (2.1). Morris (1982), Vanderplaats (1984), and Haftka and Gurdal (1992) present techniques based on the mathematical programming approach used in structural optimization. Most of them are compared by Barthelemy and Haftka (1993), and Schittkowski et al. (1994). Non deterministic methods, such as the genetic algorithm (Goldberg, 1989), are studied by Potgieter and Stander (1998), and Arora et al. (1995). Those authors also present a review of the methods used in global optimization. Optimality criteria for the specific solution of fibers optimal orientations in membrane (Pedersen 1989) and in plates (Krog 1996) must be mentioned as well. Finally the response surfaces methods are also used for optimizing laminated structures (Harrison et al. 1995, Liu et al. 2000, Rikards et al. 2006, Lanzi and Giavotto 2006).

The approximation concepts approach, also called Sequential Convex Programming, developed in the seventies by Fleury (1973), Schmit and Farschi (1974), and Schmit and Fleury (1980) has allowed to efficiently solve several structural optimization problems: the optimal sizing of trusses, shape optimization (Braibant and Fleury, 1985), topology optimization (Duysinx, 1996, 1997, and Duysinx and Bendsøe, 1998), composite structures optimization (Bruyneel and Fleury 2002, Bruyneel 2006), as well as multidisciplinary optimization problems (Zhang et al., 1995 and Sigmund, 2001). In sizing and shape optimization the solution is usually reached within 10 iterations. For topology optimization, since a very large number of design variables are included in the problem, a larger number of design cycles is needed for converging with respect to stabilized design variables values over 2 iterations.

Those approximation methods consist in replacing the solution of the initial optimization problem (2.1) by the solution of a sequence of approximated optimization problems, as illustrated in Figure 2.3.

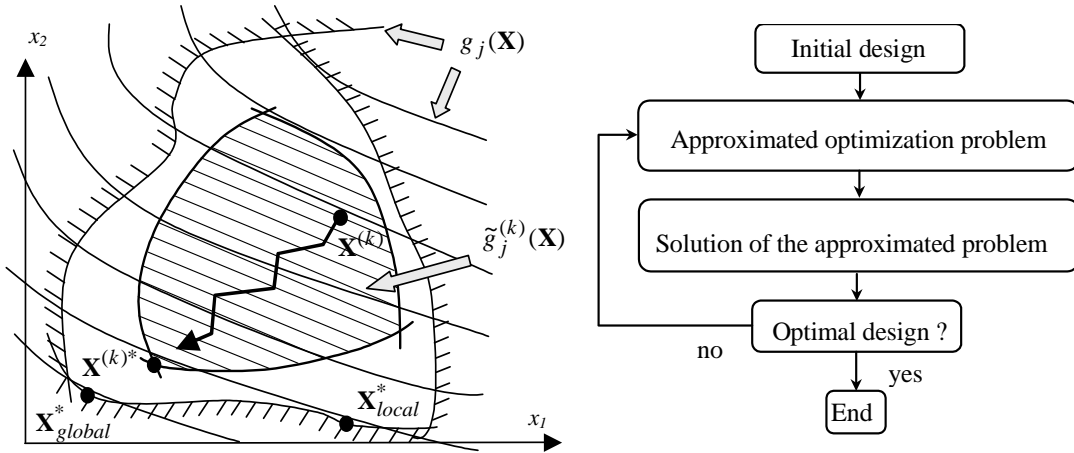


Figure 2.3. Definition of an approximated optimization problem based on the information at the current design point $\mathbf{x}^{(k)}$. The corresponding feasible domain is defined by the constraints of (2.2)

Each function entering the problem (2.1) is replaced by a convex approximation $\tilde{g}_j^{(k)}(\mathbf{X})$ based on a Taylor series expansion in terms of the direct design variables x_i or intermediate ones as for example the inverse design variables $1/x_i$. For a current design $\mathbf{x}^{(k)}$ at iteration k , the approximated optimization problem writes:

$$\begin{aligned} \min \quad & \tilde{g}_0^{(k)}(\mathbf{x}) \\ \tilde{g}_j^{(k)}(\mathbf{x}) \leq & g_j^{\max} \quad j = 1, \dots, m \\ \underline{x}_i^{(k)} \leq x_i \leq & \bar{x}_i^{(k)} \quad i = 1, \dots, n \end{aligned} \quad (2.2)$$

where the symbol \sim is related to an approximated function. The explicit and convex optimization problem (2.2) is itself solved by dedicated methods of mathematical programming (see Section 7). Building an approximated problem requires to carry out a structural and a sensitivity analyses (via the finite elements method). Solving the related explicit problem does no longer necessitate a finite element analysis (expensive in CPU for large scale problems).

The solution obtained with this approach doesn't correspond to the global optimum, but to a local one, since gradients and deterministic information are used. Nevertheless this local solution is found very quickly and several initial designs could be used to try to find a better solution, as proposed by Cheng (1986). Finally it must be noted that when a very large number of constraints is considered in the optimal design problem (say more than 10^5) the user is often satisfied with a feasible solution.

3. Parameterizations of laminated composite structures

Before presenting the several possible parameterizations of laminates, with their advantages and their disadvantages, the classical lamination theory is briefly recalled in order to introduce the notation that will be used throughout the chapter. See Tsai and Hahn (1980), Gay (1991) and Berthelot (1992) for details.

3.1 The classical lamination theory

3.1.1 Constitutive relations for a ply

Fibers reinforced composite materials are orthotropic along the fibers direction, that is in the local material axes (x, y, z) illustrated in Figure 3.1. Homogeneous macroscopic properties are assumed at the ply and at the laminate levels.

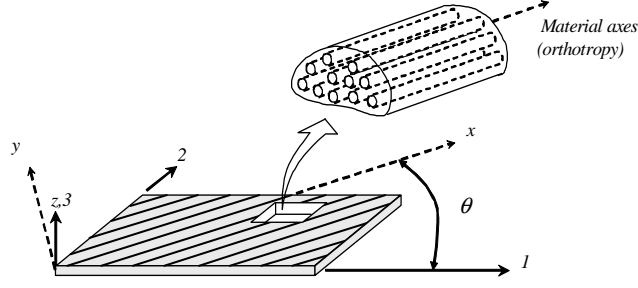


Figure 3.1. The unidirectional ply with its material and structural axes

For a linear elastic behaviour, the stress-strain relations in the material axes are given by the Hook's law $\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are the strain and stress tensors, respectively, while \mathbf{Q} is the matrix collecting the stiffness coefficients in the orthotropic axes. For a plane stress assumption, it comes that

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} mE_x & mv_{yx}E_x & 0 \\ mv_{xy}E_y & mE_y & 0 \\ 0 & 0 & G_{xy} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & 0 \\ 0 & 0 & Q_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad m = \frac{1}{1 - \nu_{xy}\nu_{yx}} \quad (3.1)$$

The stresses and strains can be written in the structural coordinates (1,2,3) as in (3.2) and (3.3) where θ is the angle between the local and structural axes, defined in Figure 3.1.

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & -\cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & \cos \theta \sin \theta \\ 2 \cos \theta \sin \theta & -2 \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (3.2)$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & -2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & 2 \cos \theta \sin \theta \\ \cos \theta \sin \theta & -\cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} \quad (3.3)$$

For a ply with an orientation θ with respect to the structural axes, the constitutive relations write:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \quad (3.4)$$

where the matrix of the stiffness coefficients in the structural axes takes the form:

$$\mathbf{Q}_{(1,2,3)} = \begin{Bmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \\ Q_{16} \\ Q_{26} \end{Bmatrix}_{(1,2,3)} = \begin{bmatrix} c^4 & s^4 & 2c^2s^2 & 4c^2s^2 \\ s^4 & c^4 & 2c^2s^2 & 4c^2s^2 \\ c^2s^2 & c^2s^2 & c^4 + s^4 & -4c^2s^2 \\ c^2s^2 & c^2s^2 & -2c^2s^2 & (c^2 - s^2)^2 \\ c^3s & -cs^3 & cs^3 - c^3s & 2(cs^3 - c^3s) \\ cs^3 & -c^3s & (c^3s - cs^3) & 2(c^3s - cs^3) \end{bmatrix} \begin{Bmatrix} Q_{xx} \\ Q_{yy} \\ Q_{xy} \\ Q_{ss} \end{Bmatrix}_{(x,y,z)} \quad (3.5)$$

with

$$c = \cos \theta \quad s = \sin \theta$$

The variation of the Q 's with respect to the angle θ is plotted in Figure 3.2. It is observed that the stiffness coefficients are highly non linear in terms of the fibers orientation.

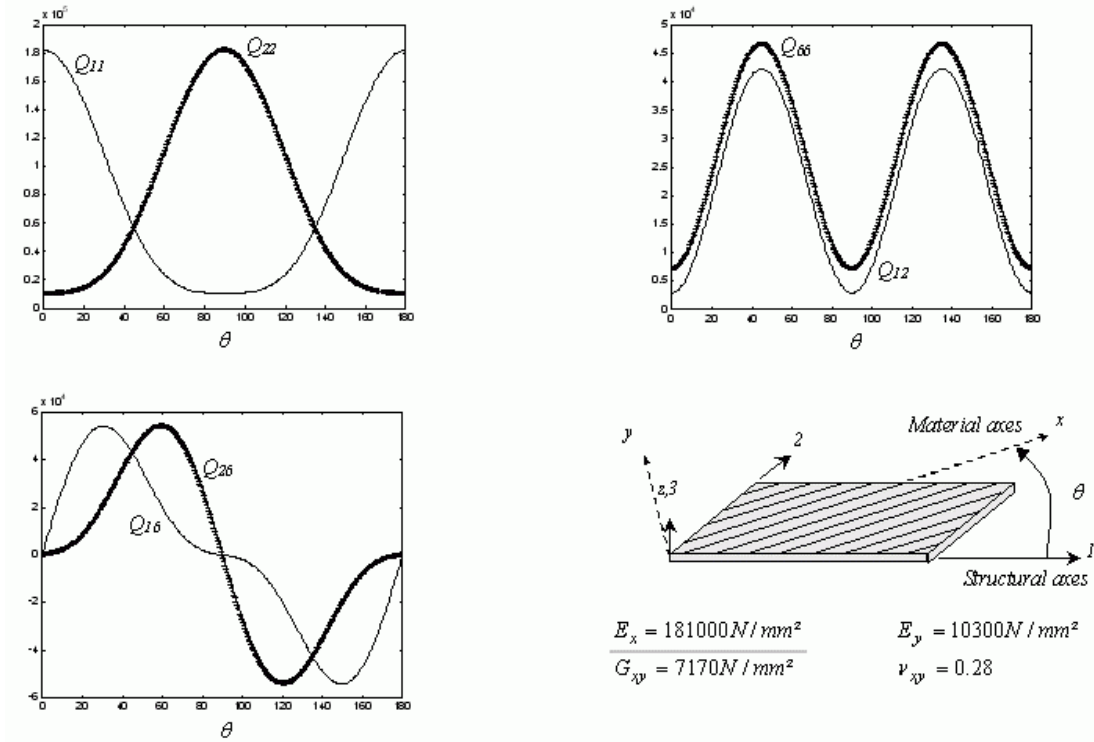


Figure 3.2. Stiffness coefficients in N/mm^2 in the structural axes for several values of the fibers orientation in a carbon/epoxy material T300/5208 (after Tsai and Hahn, 1980)

Based on the fact that the trigonometric functions entering the matrix in (3.5) can be written in the following way:

$$\begin{aligned}
 \cos^4 \theta &= \frac{1}{8} (3 + 4 \cos 2\theta + \cos 4\theta) & \cos \theta \sin^3 \theta &= \frac{1}{8} (2 \sin 2\theta - \sin 4\theta) \\
 \cos^3 \theta \sin \theta &= \frac{1}{8} (2 \sin 2\theta + \sin 4\theta) & \sin^4 \theta &= \frac{1}{8} (3 - 4 \cos 2\theta + \cos 4\theta) \\
 \cos^2 \theta \sin^2 \theta &= \frac{1}{8} (1 - \cos 4\theta)
 \end{aligned} \tag{3.6}$$

Tsai and Pagano (1968) derived an alternative expression for the Q 's coefficients in the structural axes given in (3.7):

$$\mathbf{Q}_{(1,2,3)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ & Q_{22} & Q_{26} \\ sym & & Q_{66} \end{bmatrix} = \gamma_0 + \gamma_1 \cos 2\theta + \gamma_2 \cos 4\theta + \gamma_3 \sin 2\theta + \gamma_4 \sin 4\theta \tag{3.7}$$

where the parameters γ are functions of the *lamina invariants* U_1 - U_5 :

$$\gamma_0 = \begin{bmatrix} U_1 & U_4 & 0 \\ & U_1 & 0 \\ sym & & U_5 \end{bmatrix} \quad \gamma_1 = \begin{bmatrix} U_2 & 0 & 0 \\ & -U_2 & 0 \\ sym & & 0 \end{bmatrix} \tag{3.8}$$

$$\gamma_2 = \begin{bmatrix} U_3 & -U_3 & 0 \\ & U_3 & 0 \\ sym & & -U_3 \end{bmatrix} \quad \gamma_3 = \begin{bmatrix} 0 & 0 & \frac{U_2}{2} \\ & 0 & \frac{U_2}{2} \\ sym & & 0 \end{bmatrix} \quad \gamma_4 = \begin{bmatrix} 0 & 0 & U_3 \\ & 0 & -U_3 \\ sym & & 0 \end{bmatrix}$$

and

$$\begin{aligned} U_1 &= \frac{1}{8}(3Q_{xx} + 3Q_{yy} + 2Q_{xy} + 4Q_{ss}) & U_4 &= \frac{1}{8}(Q_{xx} + Q_{yy} + 6Q_{xy} - 4Q_{ss}) \\ U_2 &= \frac{1}{2}(Q_{xx} - Q_{yy}) & U_5 &= \frac{1}{8}(Q_{xx} + Q_{yy} - 2Q_{xy} + 4Q_{ss}) \\ U_3 &= \frac{1}{8}(Q_{xx} + Q_{yy} - 2Q_{xy} - 4Q_{ss}) \end{aligned}$$

3.1.2 Constitutive relations for a laminate

Composite structures are thin membranes, plates or shells made of n unidirectional orthotropic plies stacked on the top of each other. Such structures can support in and out-of plane loadings. In the following the constitutive relations for a laminate made of several individual plies are derived. The notations are defined in Figure 3.3. In the case of plane stress, i.e. the effects of transverse shear is neglected, in-plane normal and shear loads \mathbf{N} , as well as the flexural and torsional moments \mathbf{M} are applied to the laminate. Those loadings are computed by considering the stress state in each ply with the relations (3.9):

$$\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} dz \quad \mathbf{M} = \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} z dz \quad (3.9)$$

For a first order cinematic theory, where the displacement through the laminate's thickness is linear in the z coordinate measured with respect to the mid-plane of the plate/shell (Figure 3.3), the vector of laminate's strains $\boldsymbol{\varepsilon}_l$ is linked to the in-plane strains and the curvatures via the relation $\boldsymbol{\varepsilon}_l = \boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}$. With this definition it turns that the constitutive relations for a laminate are given by (3.10) where \mathbf{A} , \mathbf{B} and \mathbf{D} are the in-plane, coupling and bending stiffness matrices of the laminate.

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{Bmatrix} \quad (3.10)$$

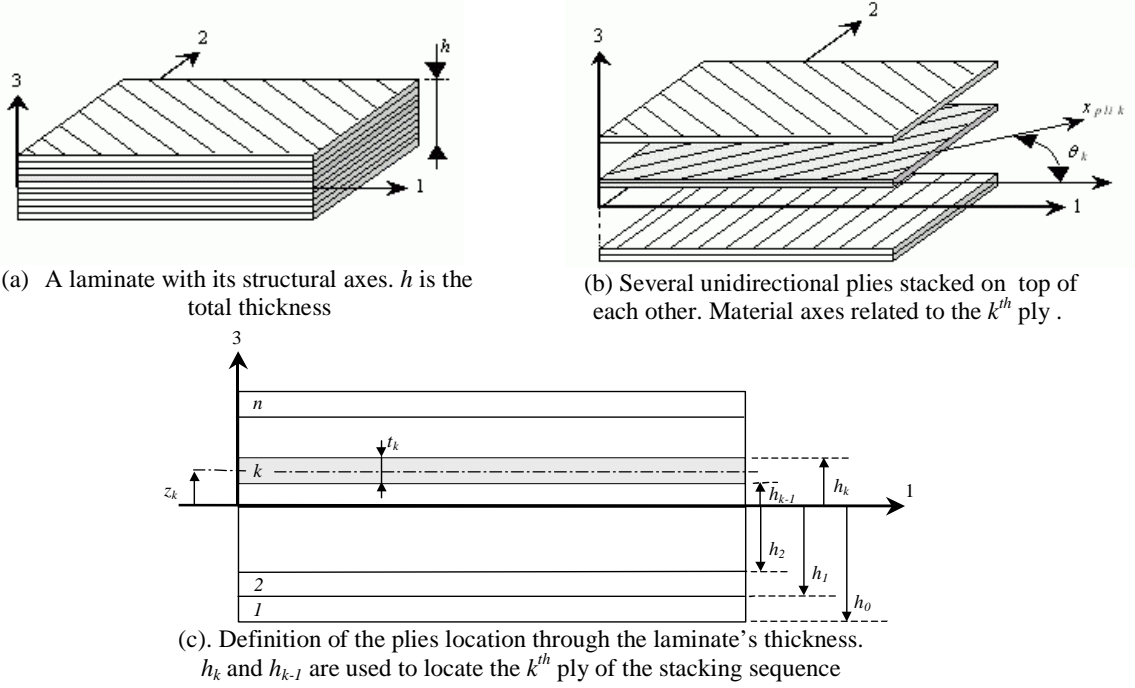


Figure 3.3. A laminate with n layers
 (a) Structural axes (b) Material axes of ply k
 (c) Position of each ply in the stacking sequence

3.2 The possible parameterizations of laminates

There exist several parameterizations for the laminates depending on the way the coefficients of the stiffness matrices in (3.10) are computed and depending on the definition of the design variables. The advantages and disadvantages of those different parameterizations are compared in the perspective of the optimal design of the laminated composite structures.

3.2.1 Parameterization with respect to thickness and orientation

When the *ply thickness* and the related *fibers orientation* are chosen to describe the laminate, the coefficients of the stiffness matrices can be written as follows:

$$\begin{aligned}
 A_{ij} &= \sum_{k=1}^n [Q_{ij}(\theta_k)](h_k - h_{k-1}) & \Leftrightarrow & \quad A_{ij} = \sum_{k=1}^n [Q_{ij}(\theta_k)]t_k \\
 B_{ij} &= \frac{1}{2} \sum_{k=1}^n [Q_{ij}(\theta_k)](h_k^2 - h_{k-1}^2) & \Leftrightarrow & \quad B_{ij} = \sum_{k=1}^n [Q_{ij}(\theta_k)]t_k z_k \\
 D_{ij} &= \frac{1}{3} \sum_{k=1}^n [Q_{ij}(\theta_k)](h_k^3 - h_{k-1}^3) & \Leftrightarrow & \quad D_{ij} = \sum_{k=1}^n [Q_{ij}(\theta_k)]\left(t_k z_k^2 + \frac{t_k^3}{12}\right) \quad i, j = 1, 2, 6
 \end{aligned} \tag{3.11}$$

where z_k and h_k define the position of the k^{th} ply in the stacking sequence. t_k and θ_k are the ply thickness and the fibers orientation, respectively (Figure 3.3).

With such a parameterization the local values (e.g. the stresses in each ply of the laminate) are available via the relations (3.1) and (3.4). On top of that the design problem is written in terms of the physical parameters used for the manufacturing of the laminated structures. Finally several different materials can be considered in the laminate when the parameterization (3.11) is used.

However when fibers orientations are allowed to change during the structural design process the resulting mechanical properties are generally strongly non linear (see Figure 3.2) and non convex, and local minima appear in the optimization problem. This is also illustrated in Figure 3.4 that draws the variation of the strain energy density in a laminate over 2 fibers orientations. In Figure 3.5 it is shown that the structural responses entirely differ when either ply thickness or ply orientation is considered in the design, resulting in mixed monotonous-non monotonous structural behaviors. It turns that the optimal design task is more complicated since the optimization method should be able to efficiently take into account simultaneously both different behaviors.

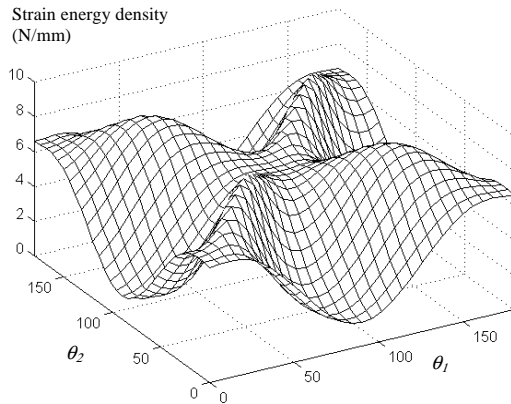


Figure 3.4. Variation of the strain energy density in a $[\theta_1/\theta_2]_S$ laminate with respect to the fibers orientations θ_1 and θ_2

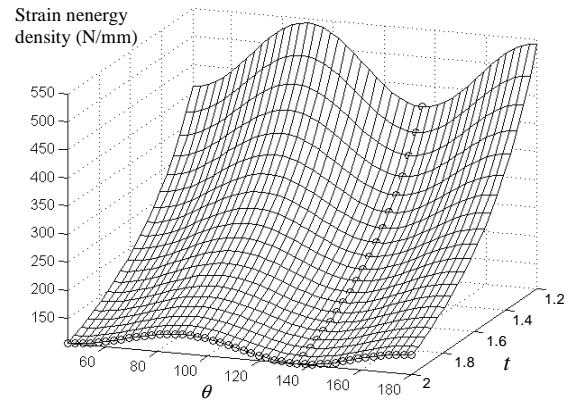


Figure 3.5. Variation of the strain energy density in an unidirectional ply with respect to its thickness t and its fibers orientation θ

Additionally using such a parameterization increases the number of design variables that may appear in the optimal design problem since the thickness and fibers orientation of each ply are possible variables. Finally optimizing with respect to the fibers orientations is known to be very difficult and few publications are available on the subject. For a sake of completion, the sensitivity analysis of the structural responses of composites with respect to those variables can be found in Mateus et al. (1991), Geier and Zimmerman (1994), and Dems (1996).

3.2.2 Parameterization with sub-laminates

The design parameters are no longer defined based on single unidirectional plies but instead on predefined sub-laminates. Each sub-laminate is itself made of several single unidirectional plies. The design parameters are assigned to the sub-laminates and no longer to each individual ply. Examples of sub-laminates may be $[0/45/-45/90]$, $[0/60/-60]$ or $[0/90]$. This parameterization allows to decrease the number of design variables. However the control at the ply level is lost. The previously presented parameterization in terms of ply thickness and orientation is a limiting case.

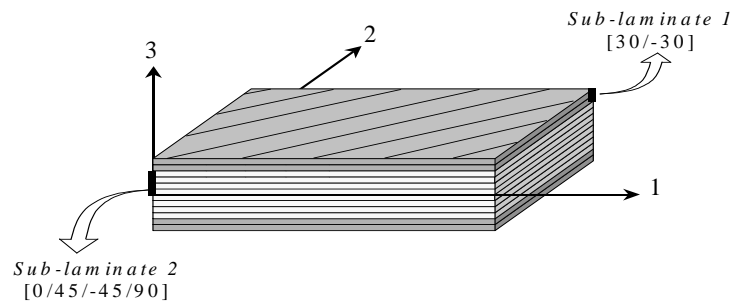


Figure 3.6. Parameterization with sub-laminates. Here the symmetric laminate is made of 2 sub-laminates

3.2.3 The lamination parameters

The stiffness matrix in (3.10) can be expressed with the *lamina invariants* defined in (3.8) together with the *lamination parameters*. For a given base material identical for each ply of the laminate the lamination parameters are given by (3.12) in the structural axes:

$$\xi_{[1,2,3,4]}^{\mathbf{A},\mathbf{B},\mathbf{D}} = \int_{-h/2}^{h/2} z^{0,1,2} [\cos 2\theta(z), \cos 4\theta(z), \sin 2\theta(z), \sin 4\theta(z)] dz \quad (3.12)$$

The lamination parameters are the zero, first and second order moments relative to the plate mid-plane of the trigonometric functions (3.6) entering the rotation formulae for the ply stiffness coefficients (3.5). With this definition the stiffness matrices \mathbf{A} , \mathbf{B} and \mathbf{D} in (3.10) write:

$$\begin{aligned} \mathbf{A} &= h\gamma_0 + \gamma_1\xi_1^{\mathbf{A}} + \gamma_2\xi_2^{\mathbf{A}} + \gamma_3\xi_3^{\mathbf{A}} + \gamma_4\xi_4^{\mathbf{A}} \\ \mathbf{B} &= \gamma_1\xi_1^{\mathbf{B}} + \gamma_2\xi_2^{\mathbf{B}} + \gamma_3\xi_3^{\mathbf{B}} + \gamma_4\xi_4^{\mathbf{B}} \\ \mathbf{D} &= \frac{h^3}{12}\gamma_0 + \gamma_1\xi_1^{\mathbf{D}} + \gamma_2\xi_2^{\mathbf{D}} + \gamma_3\xi_3^{\mathbf{D}} + \gamma_4\xi_4^{\mathbf{D}} \end{aligned} \quad (3.13)$$

Twelve lamination parameters exist in total and characterize the global stiffness of the laminate. This number is independent of the number of plies that contains the laminate. In most applications the lamination parameters are normalized with respect to the total thickness of the laminate (Grenestedt, 1992, and Hammer, 1997). In the case of symmetric laminates the 4 lamination parameters $\xi^{\mathbf{B}}$ defining the coupling stiffness \mathbf{B} vanish. Moreover when the structure is either subjected to in-plane loads or to out-of-plane loads only the 4 lamination parameters related to the in-plane stiffness $\xi^{\mathbf{A}}$ or the out-of-plane stiffness $\xi^{\mathbf{D}}$ must be considered, respectively. In the case of composite membrane or plates presenting orthotropic material properties 2 lamination parameters are sufficient to characterize the problem. Lamination parameters are not independent variables. Feasible regions of the lamination parameters exist which provide realizable laminates. Grenestedt and Gudmundson (1993) demonstrated that the set of the 12 lamination parameters is convex. It is also observed from (3.13) that the constitutive matrices \mathbf{A} , \mathbf{B} and \mathbf{D} are linear with respect to the lamination parameters. This means that the optimization problem is convex if it includes functions related to the global stiffness of the laminate, as for example the structural stiffness, vibration frequencies and buckling loads (Foldager, 1999).

Feasible regions were determined for specific laminate configurations (e.g. Miki, 1982 and Grenestedt, 1992), but the region for the 12 lamination parameters has not yet been determined. Recently the relations between the lamination parameters were derived for ply angles restricted to 0, 90, 45 and -45 degrees by Liu et al. (2004) for membrane and bending effects, and by Diaconu and Sekine (2004) for membrane, coupling and bending effects.

One of the feasible regions of lamination parameters is illustrated in Figure 3.7 in the case of a symmetric and orthotropic laminated plate subjected to bending. As the plate is assumed orthotropic in bending $\xi_1^{\mathbf{D}}$ and $\xi_2^{\mathbf{D}}$ are enough to identify the stiffness of such a problem. Those two lamination parameters take their values on the outline delimited by the points A , B , C , and in the dashed zone. Any combination of the lamination parameters that is outside of this region will produce a laminate which is not realizable. When this plate is simply supported and subjected to a uniform pressure, the vertical displacement is a function of $\xi_1^{\mathbf{D}}$ and $\xi_2^{\mathbf{D}}$. The iso-values of this structural response are the parallel lines illustrated in Figure 3.7. According to Grenestedt (1990), the plate stiffness increases in the direction of the arrow. The stiffest plate is then characterized by the point D in Figure 3.7, which corresponds to a $[(\pm\theta)_s]$ laminate, defined by a single parameter θ .

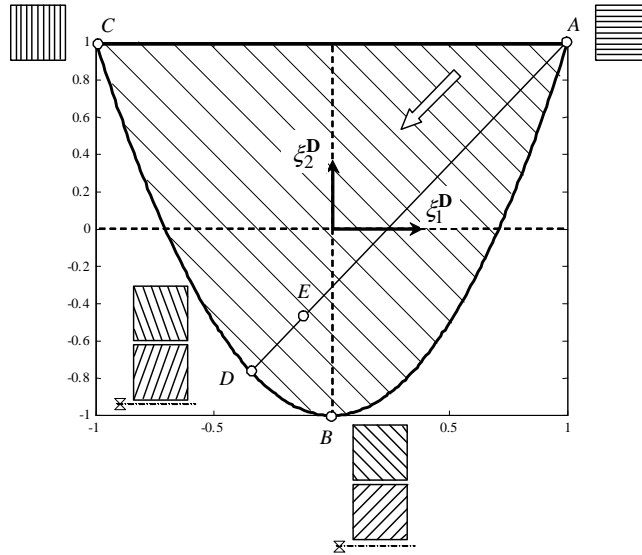


Figure 3.7. Feasible domain (outline plus dashed zone) of the lamination parameters for a symmetric and orthotropic laminated plate subjected to a uniform pressure (after Grenestedt, 1990). The points A , B , C correspond to $[0]$, $[(\pm 45)_n]_S$ and $[90]$ laminates, respectively. The point D defines a $[(\pm \theta)_n]_S$ laminate. The point E is a combination of laminates defined on the outline. The laminate of maximum stiffness is located on the outline (point D)

This kind of parameterization has allowed to show that optimal solutions – in terms of the stiffness – are often related to simple laminates with few different ply orientations. For example only one orientation is necessary for characterizing the optimal laminate in a flexural problem (Figure 3.7), and at most 3 different ply orientations are sufficient to define the optimal stacking sequence in the case of a membrane of maximum stiffness (Lipton, 1994). Table 3.1 summarizes some of those important results.

When using such a parameterization the number of design variables is very small (12 in the most general case) irrespective to the number of plies that contains the laminate. As seen in Figure 3.7 the design space is convex, and only one set of lamination parameters characterizes the optimal solution. However according to the relations (3.8) and (3.13) only one kind of material can be used in the laminate: defining a different material for the core of a sandwich panel is for example not allowed (Tsai and Hahn, 1980). Additionally the local structural responses (e.g. the stresses in each ply) can not be expressed in terms of the lamination parameters since those last are defined at the global (laminate) level and are linked to the structural stiffness. However the global strains of the laminate (but not in each ply) can be computed with relation (3.10) and used in the optimization, as is done by Herencia et al. (2006). The feasible regions of the 12 lamination parameters is not yet determined. As said before those regions are only known for specific laminate configurations. This strongly limit their use in the frame of the optimal design of composite structures. Finally when the optimal values of the lamination parameters are known, coming back to corresponding thicknesses and orientations is a difficult problem and the solution is non unique (Hammer, 1997). Foldager et al. (1998) proposed a technique based on a mathematical programming approach while Autio (2000) used a genetic algorithm to find this solution when the number of layers is limited or for prescribed standardized ply angles.

Kind of structure	Laminate configuration	Criteria	Optimal sequence	Reference
Plate	Symmetric/orthotropic	Stiffness Vibration Buckling	$[(\pm\theta)_n]_S$ $[(\pm\theta)_n]_S$ $[(\pm\theta)_n]_S$	Greestedt (1990), Miki and Sugiyama (1993)
	Symmetric	Buckling	$[\theta]_S$	Greestedt (1991)
Membrane	Symmetric	Stiffness	$[(\alpha/90+\alpha)_n]_S$	Fukunaga and Sekine (1993)
	General	Stiffness	$[\theta], [\alpha/90+\alpha]$	Hammer (1997)
Cylindrical shell	Symmetric/orthotropic	Buckling	$[(\pm\theta)_n]_S, [0/90]_S,$ quasi-isotropic	Fukunaga and Vanderplaats (1991b)

Table 3.1. Summary of some important results obtained with the lamination parameters

3.2.4 Combined parameterization

As shown by Foldager et al. (1998) and Foldager (1999), composite structures can be designed by combining two parameterizations: the lamination parameters on one hand, and the plies thickness and fibers orientations on the other hand. The benefit of the approach relies on using a convex design space with respect to the lamination parameters, while keeping in the problem's definition the physical variables in terms of thickness and orientation. This iterative procedure – between both design spaces – consists in determining a first (local) solution in terms of thicknesses and orientations. A new search direction towards the global optimum is then computed by evaluating the first order derivative of the objective function at the local solution with respect to the lamination parameters. The global optimum is reached when this sensitivity is close to zero. Otherwise a new design point is calculated in the space of the fibers orientations, and the process continues, usually by adding new plies in the laminate. As seen in Figure 3.8, the structural response is not convex with respect to θ while it is convex in terms of the lamination parameter ξ . With this technique the knowledge of the feasible regions of the lamination parameters is not mandatory.

Although efficient, this solution procedure can only be used for global structural responses like the stiffness, the vibration frequencies and the buckling load.

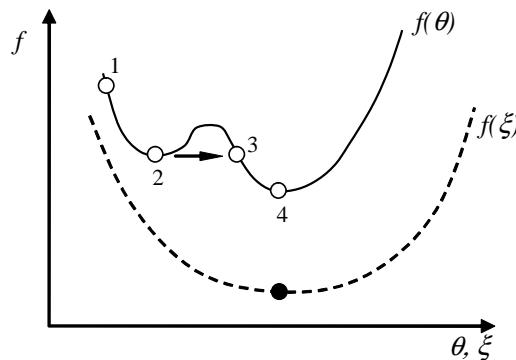


Figure 3.8. Illustration of the optimization process after Foldager et al. (1998) in both spaces of the lamination parameters ξ and the fibers orientation θ

3.2.5 Alternative parameterization

In order to decrease the non linearities introduced by the fibers orientation variables, Fukunaga and Vanderplaats (1991a) proposed to parameterize the laminated composite membranes with the following intermediate variables:

$$x_i = \sin 2\theta_i \quad \text{or} \quad x_i = \cos 2\theta_i$$

based on the relation (3.12) and (3.13). This formulation was tested by Vermaut et al. (1998) for the optimal design of laminates with respect to strength and weight restrictions. As in the previous section, the main difficulty is to compute the orientations corresponding to the optimal intermediate variables values x_i .

4. Specific problems in the optimal design of composite structures

For designing laminated composite structures a very large number of data must be considered (material properties, plies thickness and fibers orientation, stacking sequence) and complex geometries must be modelled (aircraft wings, car bodies). Therefore the finite element method is used for the computation of the structural mechanical responses. Usually mass, structural stiffness, ply strength and strain, as well as buckling loads are the functions used in the optimization problem. The design variables are classically the parameters defining the laminate: fibers orientations, plies thickness, and indirectly the number of plies and the stacking sequence. Some specific problems appear in the formulation of the optimization problem for laminated structures. They are reported hereafter.

- *Large number of design variables.* Even for a parameterization in terms of the lamination parameters, the number of design variables can easily reach a large value when the plies thickness and fibers orientations are allowed to change over the structure, leading to non homogeneous plies (Figure 4.1) and curvilinear fibers formats (Hyer and Charette 1991, Hyer and Lee 1991, Duvaut et al. 2000). In industrial applications (Krog et al. 2007), thicknesses related to specific orientations (0° , $\pm 45^\circ$, 90°) are used and several independent regions are defined throughout the composite structure, what increases the number of design variables.
- *Large number of design functions.* Not only global structural responses related to the stiffness are relevant in a composite structure optimization, but also the local strength of each ply. Damage tolerance and local buckling restrictions are important as well. For an aircraft wing, it is usual to include about 300000 constraints in the optimization problem (Krog et al. 2007).

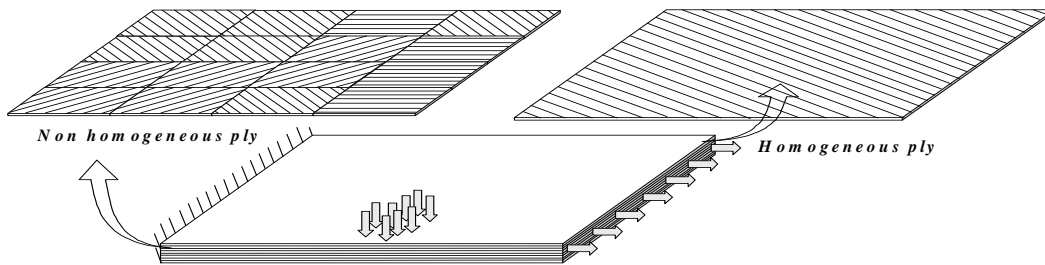


Figure 4.1. Homogeneous and non homogeneous ply in a laminate

- *Problems related to the topology optimization of composite structures.* In topology optimization one is looking for the optimal distribution of a given amount of material in a predefined design space that maximizes the structural stiffness (Figure 4.2).

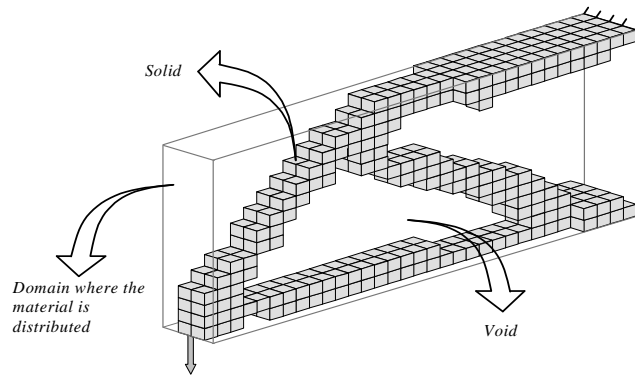


Figure 4.2. Illustration of a topology optimization problem (after Bruyneel, 2002)

For composite structures, and due to the stratification of the material, it results that 2 topology optimization problems must be defined and solved simultaneously: the optimal distribution of plies at a given altitude in the laminate (Figure 4.3) and the transverse topology optimization where the optimal local stacking sequence is looked for (Figure 4.4). Continuity conditions between adjacent laminates should also be imposed.

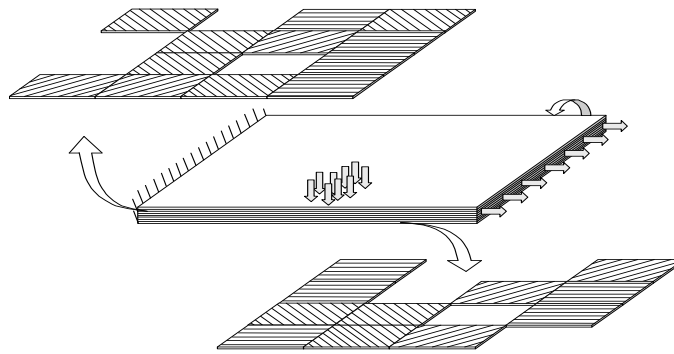


Figure 4.3 Topology optimization at a given altitude in the non homogeneous laminate

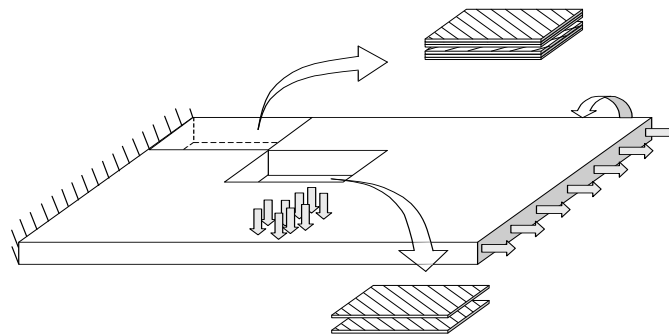


Figure 4.4. Transverse topology optimization in a composite structure

- *Specific non linear behaviors of laminated structures.* In order to improve the accuracy in the model, non linear effects, and especially the design with respect to the limit load, should be considered in the formulation of the optimization of composite structures. This dramatically increases the computational time of the finite element analysis, and can only be used for studying small structural parts such as super-stringers, i.e. some stiffeners and the panel (Colson et al., 2007). Although simple fracture mechanics criteria have been considered (Papila et al. 2001), damage tolerance and propagation of the cracks (delamination) should be taken into account in the same way.

- *Uncertainties on the mechanical properties of composites.* There is a larger dispersion in the mechanical properties of the fibers reinforced composite materials than for metals. Moreover, some uncertainties concerning the orientations and the plies thickness exist. Robust optimization should be used in these cases (Mahadevan and Liu, 1998, Chao et al., 1993, Chao, 1996, and Kristindottir et al., 1996).
- *Strong link with the manufacturing process.* Contrary to the design with metals, there is a strong link between the material design, the structural design and the manufacturing process when dealing with composite materials. The constraints linked to manufacturing can strongly influence the design and the structural performances (Henderson et al., 1999, Fine and Springer, 1997, Manne and Tsai, 1998) and should be taken into account to formulate in a rational way the design problem (Karandikar and Mistree, 1992).
- *Singular optima in laminates design problems.* When strength constraints are considered in the design problem, and if the lower bounds on the plies thickness is set close to 0 (i.e. some plies can disappear at the solution from the initial stacking sequence), it can be seen (Schmit and Farschi, 1973, Bruyneel and Fleury, 2001) that the design space can become degenerated. In this case the optimal design can not be reached with gradient based optimization methods. Such a degenerated design space is illustrated in Figure 4.5. It is divided into a feasible and an infeasible region according to the limiting value of the Tsai-Wu criteria. In this example a $[0/90]_s$ laminate's weight is to be minimized under an in-plane load N_1 . The optimal solution is a $[0]$ laminate. Unfortunately this optimal laminate configuration can not be reached with a gradient based method since the 90 degree plies are still present in the problem even if their thickness is close to zero, and the related Tsai-Wu criterion penalizes the optimization process. A first solution consists in using the ϵ -relaxed approach (Cheng and Guo 1997), which slightly modifies the design space in the neighborhood of the solution and allows the optimization method to reach the true optimum $[0]^*$. Alternatively (Bruyneel and Fleury, 2001, and Bruyneel and Duysinx, 2006) when fibers orientations are design variables the shape of the design space changes, the gap between the true optimal solution and the one constrained by plies with a vanishing thickness $[0/x]^*$ decreases and the real optimal solution becomes attainable (Figure 4.5). Optimizing over the fibers orientations allows to circumvent the singularity of the design space.

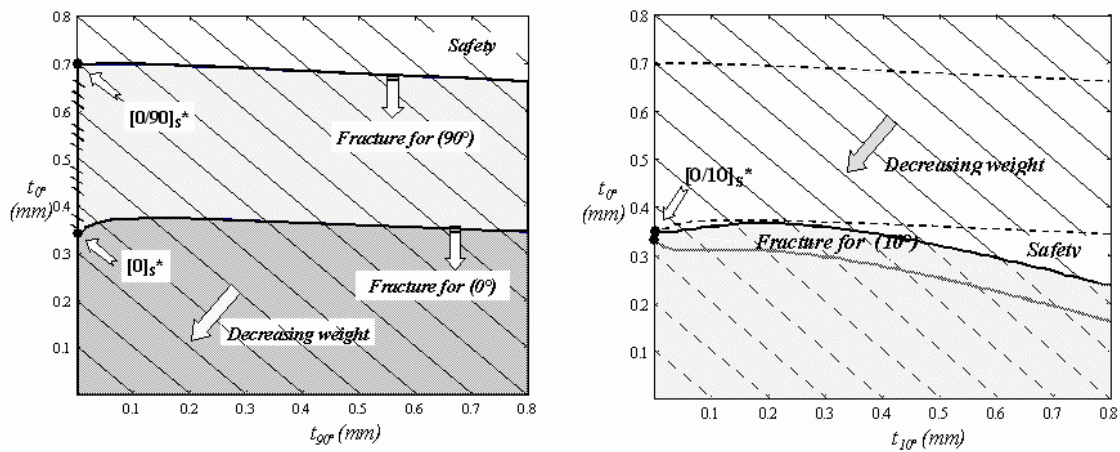


Figure 4.5. Design space for $[0/90]_s$ and $[0/10]_s$ laminates.

* represents the obtained solutions, optimum or not

- *Importance of the fibers orientations in the laminate design.* Besides their efficiency in avoiding the singularity in the optimization process as just explained before fibers orientations play a key role in the design of composite structures. Modifying their value allows for great weight savings, as illustrated in Figure 4.6. Let's consider that the initial laminate design corresponds to fibers

orientation and ply thickness at point A. A first way to obtain a feasible design with respect to strength restrictions is to increase the ply thickness and go to B, which penalizes the structural weight. Another solution consists in modifying the fibers orientation, here at constant thickness (point C). A better solution is to simultaneously optimize with respect to both kinds of design variables (point D). However taking into account such variables in the optimization problem is a real issue, and providing a reliable solution procedure is a challenge.

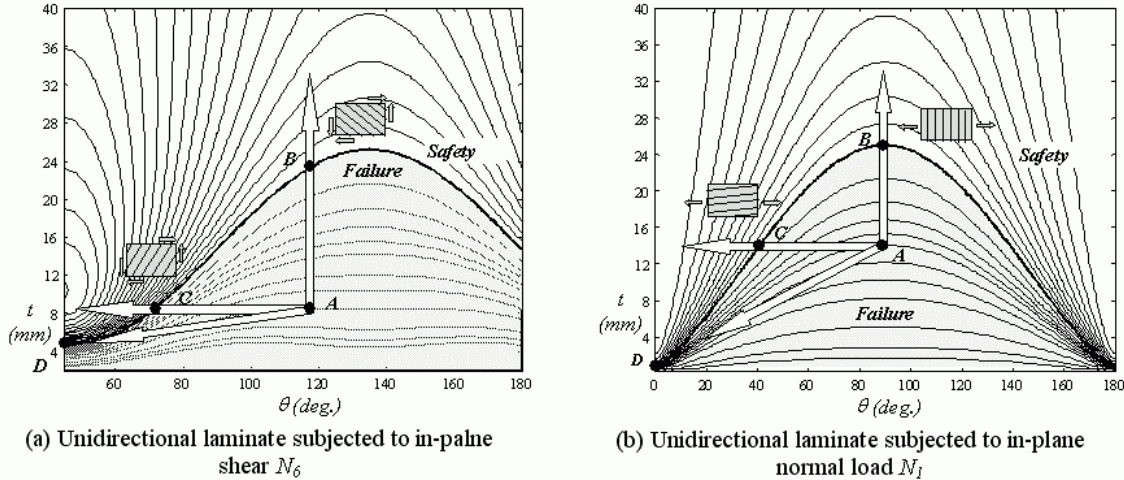


Figure 4.6. Design space for an unidirectional laminate subjected to either N_1 or N_6 . Iso-values of the Tsai-Wu criterion. The ply thickness and fibers orientations are the design variables

- The optimal stacking sequence.* A large part of the research effort on composites has been dedicated to the solution of the optimal stacking sequence problem. As it is a combinatorial problem including integer variables, genetic algorithms have been used (Haftka and Gurdal, 1992, Le Riche and Haftka, 1993). The topology optimization formulation of Figure 4.4 was used by Beckers (1999) and (Stegmann and Lund, 2005) to solve this problem with discrete and continuous design variables, respectively. Another approach, still based on the discrete character of the problem, is proposed by Carpentier et al. (2006). It consists in using a lay-up table defined based on buckling, geometric and industrial rules considerations. This table, which satisfies the ply drop-off continuity restrictions is determined numerically. Once it is obtained a given laminate total thickness corresponds to a stacking sequence (via a column of the table). The optimization process then consists in optimizing the local thickness of a set of contiguous laminates defining the structure. Each laminate has equivalent homogenized properties with $0, \pm 45$ and 90° plies. Based on the lay-up table, the stacking sequence is therefore known everywhere in the structure for different local optimal thicknesses and the composite material can be draped.

Local thickness									
	28	26	24	22	20	18	16	14	12
90									
135		90							
0		135	90						
45		0	135	90					
90		45	0	135	90				
135		90	45	0	135	90			
135		135	90	45	0	135	90		
0		0	135	90	45	0	135	90	
45		45	0	135	135	45	0	135	90
0		0	45	0	0	0	45	0	135
45		45	0	45	45	45	0	45	0
45		45	45	45	45	45	45	45	45
90		90	90	90	90	90	90	90	90
135		135	135	135	135	135	135	135	135

Figure 4.7. Illustration of a lay-up table for $0, \pm 45$ and 90° plies

5. Problems solved in the literature

5.1 Structural responses

When designing laminated composite structures the functions entering the optimization problem (2.1) are classically the stiffness, the vibration frequencies, the structural stability and the plies' strength. (see Abrate, 1994, for a detailed review of the literature). It is interesting to note that for orthotropic laminates maximizing the stiffness, the frequency or the first buckling load will provide the same solution (Pedersen, 1987 and Grenestedt, 1990). On top of that, it should be noted that optimizing a laminated structure against plies strength or stiffness will result in different designs. It results that the local (stress) effects are very important in the optimal design of composite structures (Tauchert and Adibhatla, 1985, Fukunaga and Sekine, 1993, and Hammer, 1997).

5.2 Optimal design with respect to fibers orientations

Determining the optimal fibers orientation is a very difficult problem since the structural responses in terms of such variables are highly non linear, non monotonous and non convex. However it has just been show in the previous section that the design of laminated composite structures is very sensitive with respect to those variables. As explained by the editors of commercial optimization software (Thomas et al., 2000) there is a need for an efficient treatment of such parameters.

A small amount of work has been dedicated to the optimal design of laminated structures with respect to the fibers orientations. Several kinds of approaches have been investigated and are reported in the literature:

- **Approach by optimality criteria**

Optimal orientations of orthotropic materials that maximize the stiffness in membrane structures were obtained by Pedersen (1989, 1990 and 1991), and by Diaz and Bendsøe (1992) for multiple load cases. When the unidirectional ply is only subjected to in-plane loads, Pedersen (1989) proposed to place the fibers in the direction of the principal stresses. The resulting optimality criterion was used in topology optimization including rank-2 materials (Bendsøe, 1995). This technique was used by Thomsen (1991) in the optimal design of non homogeneous composite disks. This criterion was extended by Krog (1996) to Mindlin plates and shells.

- **Approach based on the mathematical programming**

As soon as 1971, Kicher and Chao solved the problem with a gradients based method. Hirano (1979a and 1979b) used the zero order method of Powell (conjugate directions) for buckling optimization of laminated structures. Tauchert and Adibhatla (1984 and 1985) used a quasi-Newton technique (DFP) able to take into account linear constraints for minimizing the strain energy of a laminate for a given weight. Cheng (1986) minimized the compliance of plates in bending and determined the optimal orientations with an approach based on the steepest descent method.

Martin (1987) found the minimum weight of a sandwich panel subjected to stiffness and strength restrictions with a method based on the Sequential Convex Programming (Vanderplaats, 1984). Watkins and Morris (1987) used a similar procedure with a robust move-limits strategy (see also Hammer 1997).

In Foldager (1999), the method used for determining the optimal fibers orientations is not cited but belongs according to the author to the family of mathematical programming methods.

SQP, the feasible directions method and the quasi-Newton BFGS were used by Mahadevan and Liu (1998), Fukunaga and Vanderplaats (1991a), and Mota Soares et al. (1993, 1995 and 1997),

respectively. Those mathematical programming methods are reported and explained in Bonnans et al. (2003).

- **Approach with non deterministic methods**

Genetic algorithms have been employed by several authors for determining the optimal stacking sequence of laminated structures (Le Riche and Haftka, 1993, Kogiso et al., 1994 and Potgieter and Stander, 1998) or in the treatment of fibers orientations (Upadhyay and Kalyanarama, 2000).

5.3 Formulations of the optimization problem

Thickness and orientation variables were treated in several ways in the literature. They have been considered either simultaneously as in Pedersen (1991), and Fukunaga and Vanderplaats (1991a), or separately (Mota Soares et al. 1993, 1995 and 1997, and Franco Correia et al. 1997).

Weight, stiffness and strength criteria have been separately introduced in the design problem and taken into account in a bi-level approach by (Mota Soares et al., 1993, 1995, 1997 and Franco Correia et al., 1997): at the first level the weight is kept constant and the stiffness is optimized over fibers orientations ; at the second level the ply thicknesses are the only variables in an optimization problem that aims at minimizing the weight with respect to strength and/or displacements restrictions. A similar approach can be found in Kam and Lai (1989), and Soeiro et al. (1994). Fukunaga and Sekine (1993) also used a bi-level approach for determining laminates with maximal stiffness and strength in non homogeneous composite structures (Figure 4.2) subjected to in-plane loads. In Hammer (1997), both problems are separately solved and the initial configuration for optimizing with respect to strength is the laminate previously obtained with a maximal stiffness consideration.

6. Optimal design of composites for industrial applications

Based on the several possible laminate parameterizations and on the previous discussion it was concluded in Bruyneel (2002, 2006) that an industrial solution procedure for the design of laminated composite structures should preferably be based on fibers orientations and ply thicknesses, instead of intermediate non physical design variables such as the lamination parameters. Using those variables allows optimizing very general structures (membranes, shells, volumes, subjected to in- and out-of-plane loads, symmetric or not) and provides a solution that is directly interpretable by the user.

On the other hand, an optimization procedure used for industrial applications should be able to consider a large number of design variables and constraints, and find the solution (or at least a feasible design) in a small number of design cycles. Additionally, the optimization formulation should be as much general as possible, and not only limited to specific cases (e.g. not only thicknesses, not only membrane structures, not only orthotropic configurations,...). For those reasons, a solution procedure based on the approximation concepts approach seems to be inevitable. Interesting local solutions can be found by resorting to other optimization methods (e.g. response surfaces coupled with a genetic algorithm) but on structures of limited size. For the pre-design of large composite structures like a full wing or a fuselage, or when non linear responses are defined in the analysis (post-buckling, non linear material behavior), the approximation concepts approach proved to be a fast method not expensive in CPU time for solving industrial problems (Krog and al, 2007, Colson et al., 2007).

It results that robust approximation schemes must be available to efficiently optimize laminated structures. The characteristics of such a reliable approximation are explained in the following, and tests are carried out to show the efficiency and the applicability of the method.

7. Optimization algorithm for industrial applications

7.1 The Approximation Concepts Approach

In the approximation concepts approach, the solution of the primary optimization problem (2.1) is replaced with a sequence of explicit approximated problems generated through first order Taylor series expansion of the structural functions in terms of specific intermediate variables (e.g. direct x_i or inverse $1/x_i$ variables). The generated structural approximations built from the information known at least at the current design point (via a finite element analysis), are convex and separable. As will be explained latter a dual formulation can then be used in a very efficient way for solving each explicit approximated problem.

According to section 2, it is apparent that the approximation concepts approach is well adapted to structural optimization including sizing, shape and topology optimization problems. However, the use of the existing schemes (section 7.2) can sometimes lead to bad approximations of the structural responses and slow convergence (or no convergence at all) can occur (Figure 7.1).

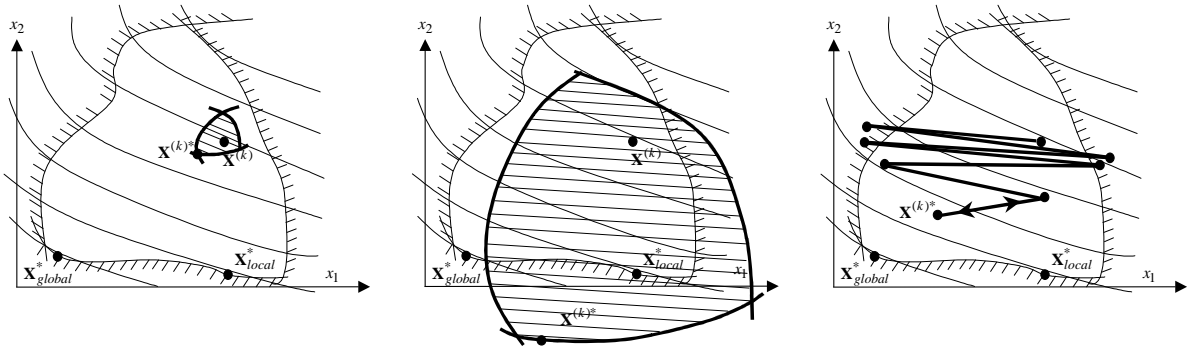


Figure 7.1. Difficulties appearing in the approximation of highly non linear structural responses.
a. A too conservative approximation b. A too few conservative approximation and unfeasible intermediate solutions c. An approximation not adapted to the problem, leading to zigzagging

Such difficulties are met for laminates optimization: their structural responses are mixed, i.e. monotonous with regard to plies thickness and non monotonous when fibers orientations are considered (Figure 3.5). Additionally, the non monotonous structural behaviors in terms of orientations are difficult to manage (Figure 3.4). It results that the selection of a right approximation scheme is a real challenge. In the next section a generalized approximation scheme is presented that is able to effectively treat those kinds of problems. This optimization algorithm will identify the structural behavior (monotonous or not) according to the involved design variable (orientation or thickness), and will automatically generate the most reliable approximation for each structural function included in the optimization problem. In section 8 numerical tests will compare the efficiency of the proposed approximation scheme and the existing ones for laminates optimization including both thickness and orientation variables.

7.2 Selection of an accurate approximation scheme

7.2.1 Monotonous approximations

Based on the first order derivatives of the structural responses included in the optimization problem, *linear approximations* can be built at the current design point \mathbf{x}^k . It is a first order Taylor series expansion in terms of the direct design variables x_i (7.1).

$$\tilde{g}_j^{(k)}(\mathbf{x}) = g_j(\mathbf{x}^{(k)}) + \sum_i \frac{\partial g_j(\mathbf{x}^{(k)})}{\partial x_i} (x_i - x_i^{(k)}) \quad (7.1)$$

As it is very simple this approximation is most of the time not efficient for structural optimization but can anyway be used with some specific move-limits rules (Watkins and Morris, 1987) that prevent the intermediate design point to go too far from the current one and to generate large oscillations during the optimization process (Figures 7.1b and 7.1c).

Since the stresses vary as $1/x_i$ in isostatic trusses where x_i is the cross section area of the bars, a linear approximation in terms of the inverse design variables is more reliable for the optimal sizing of thin structures. The resulting *reciprocal approximation* is given in (7.2).

$$\tilde{g}_j^{(k)}(\mathbf{x}) = g_j(\mathbf{x}^{(k)}) - \sum_i (\mathbf{x}_i^{(k)})^2 \frac{\partial g_j(\mathbf{x}^{(k)})}{\partial x_i} \left(\frac{1}{x_i} - \frac{1}{x_i^{(k)}} \right) \quad (7.2)$$

The Conlin scheme developed by Fleury and Braibant (1986) is a convex approximation based on (7.1) and (7.2). It is reported in (7.3) and illustrated in Figure 7.2.

$$\tilde{g}_j^{(k)}(\mathbf{x}) = g_j(\mathbf{x}^{(k)}) + \sum_+ \frac{\partial g_j(\mathbf{x}^{(k)})}{\partial x_i} (x_i - x_i^{(k)}) - \sum_- (x_i^{(k)})^2 \frac{\partial g_j(\mathbf{x}^{(k)})}{\partial x_i} \left(\frac{1}{x_i} - \frac{1}{x_i^{(k)}} \right) \quad (7.3)$$

The symbols \sum_+ and \sum_- in (7.3) denote the summations over terms having positive and negative first order derivatives. When the first order derivative of the considered structural response is positive a linear approximation in terms of the direct variables is built, while a reciprocal approximation is used on the contrary.

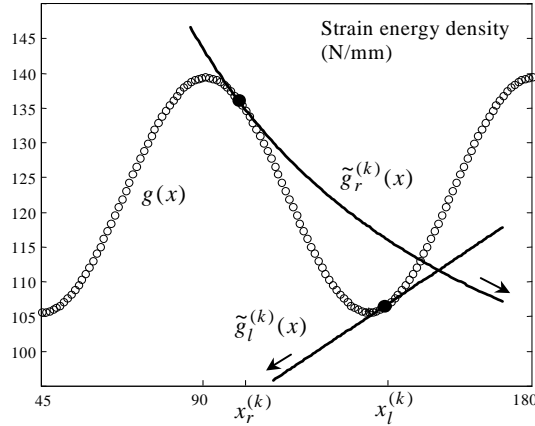


Figure 7.2. The Conlin approximation

Conlin can only work with positive design variables since an asymptote is imposed at $x_i=0$. On top of that, the curvature of this approximation is imposed by the derivative at the current design point and can not be adapted to better fit the problem.

The Method of Moving Asymptotes or MMA (Svanberg 1987) generalizes Conlin by introducing two sets of new parameters, the lower and upper asymptotes, L_i and U_i , that can take positive or negative values, in order to adjust the convexity of the approximation in accordance with the problem under consideration. The asymptotes are updated following some rules provided by Svanberg (1987). The parameters p_{ij} and q_{ij} are built with the first order derivatives.

$$\tilde{g}_j^{(k)}(\mathbf{x}) = g_j(\mathbf{x}^{(k)}) + \sum_+ p_{ij}^{(k)} \left(\frac{1}{U_i^{(k)} - x_i} - \frac{1}{U_i^{(k)} - x_i^{(k)}} \right) + \sum_- q_{ij}^{(k)} \left(\frac{1}{x_i - L_i^{(k)}} - \frac{1}{x_i^{(k)} - L_i^{(k)}} \right) \quad (7.4)$$

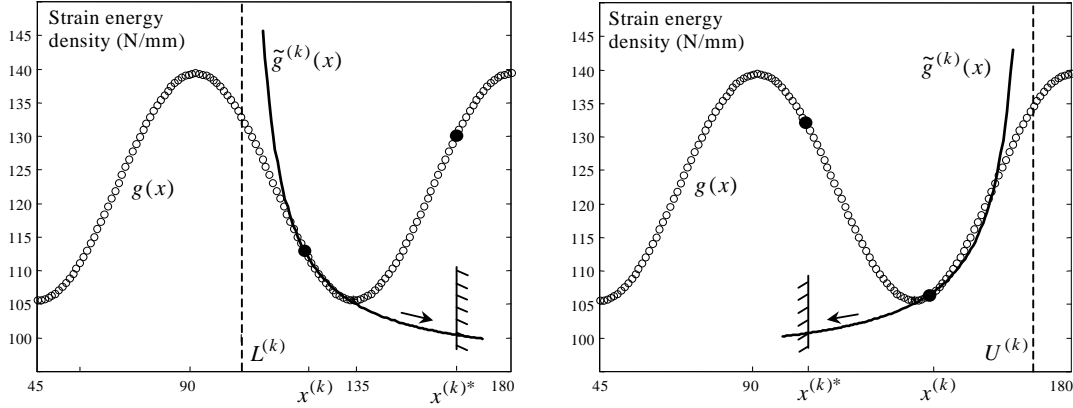


Figure 7.3 The MMA approximation

As it will be seen later those monotonous schemes are not efficient for optimizing structural functions presenting non monotonous behaviors, as in Figure 3.4.

7.2.2 Non monotonous approximations

Based on MMA, Svanberg (1995) developed the Globally Convergent MMA approximation (GCMMA). As illustrated in Figure 7.4 it is non monotonous and still only based on the information at the current design point (functions values, first order derivatives, asymptotes values). Here both U_i and L_i are used simultaneously. It was not the case in (7.4).

$$\tilde{g}_j^{(k)}(\mathbf{x}) = g_j(\mathbf{x}^{(k)}) + \sum_i p_{ij}^{(k)} \left(\frac{1}{U_i^{(k)} - x_i} - \frac{1}{U_i^{(k)} - x_i^{(k)}} \right) + \sum_i q_{ij}^{(k)} \left(\frac{1}{x_i - L_i^{(k)}} - \frac{1}{x_i^{(k)} - L_i^{(k)}} \right) \quad (7.5)$$

Using this method can lead to slow convergence given that it can generated too conservative approximations of the design functions (Figure 7.1a).

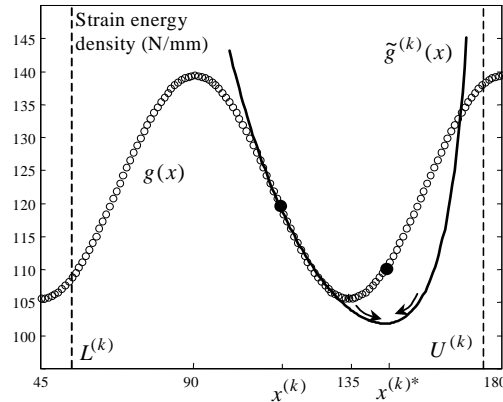


Figure 7.4. The GCMMA approximation

In order to improve the quality of this approximation it was proposed in Bruyneel and Fleury (2002) and Bruyneel et al. (2002) to use the gradients at the previous iteration to improve the quality of the approximation, leading to the definition of the Gradient Based MMA approximations (GBMMA). In those methods the p_{ij} and q_{ij} parameters of (7.5) are computed based on the function value and gradient at the current design point and on the gradient at the previous iteration. The rules defined by Svanberg (1995) for updating the asymptotes are used.

7.2.3 Mixed approximation of the MMA family

When dealing with structural optimization problems including design variables of two different natures, for example in problems mixing ply thickness and orientation variables, one is faced to a difficult task because of the simultaneous presence of monotonous and non-monotonous behaviors with respect to the set of design variables. In these conditions, most of the usual approximation schemes presented before have poor convergence properties or even fail to solve these kinds of problems. Knowing that the MMA approximation is very reliable for approximating monotonous design functions and based on the GBMMA approximations, a mixed monotonous – non monotonous scheme is presented in Bruyneel and Fleury (2002) and Bruyneel et al. (2002), which will automatically adapt itself to the problem to be approximated (7.6).

$$\begin{aligned} \tilde{g}_j^{(k)}(\mathbf{x}) = & g_j(\mathbf{x}^{(k)}) + \sum_{i \in A} p_{ij}^{(k)} \left(\frac{1}{U_i^{(k)} - x_i} - \frac{1}{U_i^{(k)} - x_i^{(k)}} \right) + \sum_{i \in A} q_{ij}^{(k)} \left(\frac{1}{x_i - L_i^{(k)}} - \frac{1}{x_i^{(k)} - L_i^{(k)}} \right) \\ & + \sum_{+, i \in B} p_{ij}^{(k)} \left(\frac{1}{U_i^{(k)} - x_i} - \frac{1}{U_i^{(k)} - x_i^{(k)}} \right) + \sum_{-, i \in B} q_{ij}^{(k)} \left(\frac{1}{x_i - L_i^{(k)}} - \frac{1}{x_i^{(k)} - L_i^{(k)}} \right) \end{aligned} \quad (7.6)$$

In (7.6) the symbols $\sum(+,i)$ and $\sum(-,i)$ designate the summations over terms having positive and negative first order derivatives, respectively. A and B are the sets of design variables leading to a non monotonous and a monotonous behavior respectively, in the considered structural response. At a given stage k of the iterative optimization process, a monotonous, non monotonous or linear approximation is automatically selected, based on the tests (7.7), (7.8) and (7.9) computed for given structural response $g_j(\mathbf{X})$ and design variable x_i .

$$\frac{\partial g_j(\mathbf{x}^k)}{\partial x_i} \times \frac{\partial g_j(\mathbf{x}^{k-1})}{\partial x_i} > 0 \Rightarrow \quad \text{MMA (monotonous)} \quad (7.7)$$

$$\frac{\partial g_j(\mathbf{x}^k)}{\partial x_i} \times \frac{\partial g_j(\mathbf{x}^{k-1})}{\partial x_i} < 0 \Rightarrow \quad \text{GBMMA (non monotonous)} \quad (7.8)$$

$$\frac{\partial g_j(\mathbf{x}^k)}{\partial x_i} - \frac{\partial g_j(\mathbf{x}^{k-1})}{\partial x_i} = 0 \Rightarrow \quad \text{linear expansion} \quad (7.9)$$

The selection of a right approximation is illustrated in Figure 7.5: when a monotonous approximation is used for approximating a non monotonous function, oscillations can appear, while a non monotonous approximation is too conservative when the function is monotonous. The best approximation is therefore selected based on tests (7.7) to (7.9). This strategy proved to be reliable for simple laminates design (Bruyneel and Fleury 2002) and for general laminated composite structures design problems (Bruyneel 2006, Bruyneel et al. 2007, Krog et al. 2007), for truss sizing and configuration (Bruyneel et al. 2002), for topology optimization which includes a large amount of design variables (Bruyneel and Duysinx 2005). It has been made available in the BOSS Quattro optimization toolbox (Radovic and Remouchamps, 2002). In the following this solution procedure based on a mixed approximation scheme is called Self Adaptive Method (SAM). Based on this approximation scheme, it is possible to resort to the other ones (GBMMA, MMA, Conlin and the linear approximation) by setting specific values to the asymptotes and by limiting the approximations to the sets A or B in (7.6).

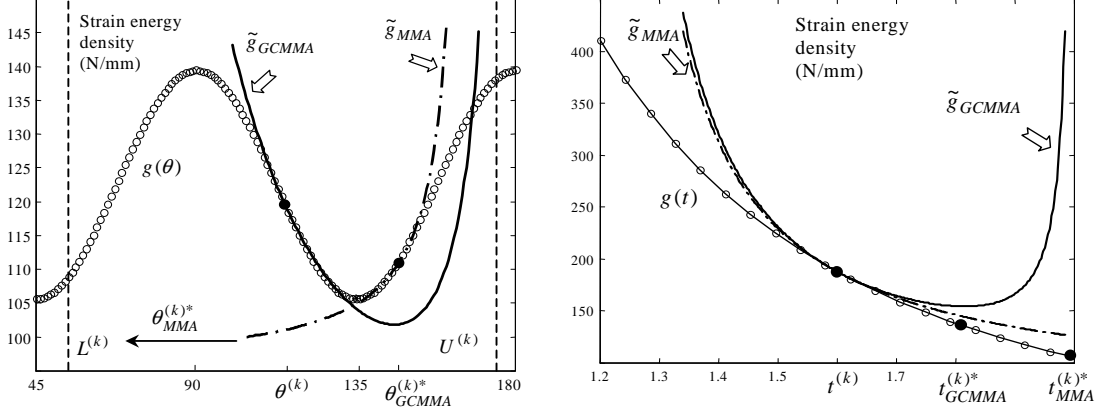


Figure 7.5. The mixed SAM approximation

A summary of the approximations that will be compared in the following is presented in Table 7.1.

Approximation	Author	Behavior
MMA	Svanberg (1987)	Monotonous
GCMMA	Svanberg (1995)	Non monotonous
SAM	Bruyneel (2006)	Mixed monotonous/non monotonous

Table 7.1. Summary of the approximations that will be compared in the numerical tests

7.3 Solution procedure for mono and multi-objective optimizations

Since the approximations are convex and separable the solution of each optimization sub-problem (Figure 2.3) is achieved by using a dual approach. Based on the theory of the duality, solving the problem (2.2) in the space of the primal variables x_i is equivalent to maximize a function (7.10) that depends on the Lagrangian multipliers λ_j , also called dual variables:

$$\begin{aligned} & \max_{\lambda} \min_{\mathbf{x}} L(\mathbf{x}, \lambda) \\ & \lambda_j \geq 0 \quad j = 0, \dots, m (\lambda_0 = 1) \end{aligned} \quad (7.10)$$

Solving the primal problem (2.2) requires the manipulation of one design function, m structural restrictions and $2 \times n$ side constraints (for mono-objective problems). When the dual formulation is used, the resulting quasi-unconstrained problem (7.10) includes one design function and m side constraints, if the side constraints in the primal problem are treated separately. In relation (7.10), $L(\mathbf{x}, \lambda)$ is the Lagrangian function of the optimization problem, which can be written

$$L(\mathbf{x}, \lambda) = \sum_j \lambda_j (c_j + \sum_i \frac{p_{ij}^k}{U_i^k - x_i} + \sum_i \frac{q_{ij}^k}{x_i - L_i^k}) \quad (7.11)$$

according to the general definition of the involved approximations $\tilde{g}_j(X)$ of the functions. The parameter λ_j is the dual variable associated to each approximated function $\tilde{g}_j(X)$. Given that the approximations are separable, the Lagrangian function is separable too. It turns that:

$$L(\mathbf{x}, \boldsymbol{\lambda}) = \sum_i L_i(x_i, \boldsymbol{\lambda})$$

and the Lagrangian problem of (7.10)

$$\min_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda})$$

can be split in n one dimensional problems

$$\min_{x_i} L_i(x_i, \boldsymbol{\lambda}) \quad (7.12)$$

The primal-dual relations are obtained by solving (7.12) for each primal variable x_i :

$$\frac{\partial L_i(x_i, \boldsymbol{\lambda})}{\partial x_i} = 0 \Rightarrow x_i = x_i(\boldsymbol{\lambda}) \quad (7.13)$$

Relation (7.13) asserts the stationnarity conditions of the Lagrangian function over the primal variables x_i . Once the primal-dual relations (7.13) are known, (7.10) can be replaced by

$$\begin{aligned} \max_{\boldsymbol{\lambda}} l(\boldsymbol{\lambda}) &\Leftrightarrow \max_{\boldsymbol{\lambda}} L(\mathbf{x}(\boldsymbol{\lambda}), \boldsymbol{\lambda}) \\ \lambda_j &\geq 0 \quad j = 1, \dots, m \end{aligned} \quad (7.14)$$

Solving problem (2.2) is then equivalent to maximize the dual function $l(\boldsymbol{\lambda})$ with non negativity constraints on the dual variables (7.14). As it is explained by Fleury (1993), the maximization (7.14) is replaced by a sequence of quadratic sub-problems. Each sub-problem is itself partially solved by a first order maximization algorithm in the dual space.

In the case of a multi-objective formulation the optimization problem writes :

$$\begin{aligned} \min_{\mathbf{x}} \max_{l=1, \dots, nc} g_{l0}(\mathbf{x}) \\ g_j(\mathbf{x}) \leq \bar{g}_j \quad j = 1, \dots, m \end{aligned} \quad (7.15)$$

where nc is the number of load cases. Using the bound formulation (Olhoff, 1989) the problem (7.15) can be written as:

$$\begin{aligned} \min \frac{1}{2} \beta^2 \\ g_{l0}(\mathbf{x}) \leq \beta \quad l = 1, \dots, nc \\ g_j(\mathbf{x}) \leq \bar{g}_j \quad j = 1, \dots, m \end{aligned} \quad (7.16)$$

where β is the multiobjective factor, that is an additional design variable in the optimization problem. Instead of solving (7.16) problem (7.17) is considered where a new variable δ is introduced for the possible relaxation of the set of constraints.

$$\begin{aligned} \min \frac{1}{2} \beta^2 + \frac{1}{2} (\delta + p)^2 + \frac{C}{2} \sum_i (x_i - x_i^{(k-1)})^2 \\ g_{j0}(\mathbf{x}) \leq \beta_j \bar{g}_{j0} \quad j = 1, \dots, nobj \\ g_j(\mathbf{x}) \leq \bar{g}_j (1 + \delta) \quad j = 1, \dots, m \end{aligned} \quad (7.17)$$

\bar{g}_{j_0} are target values on the objective functions. The dual approach described for mono-objective optimisation problems is then applied to (7.17).

8. Applications of the optimization solution procedure

In the following examples (except the simple laminate designs and the topology optimization problem), the structural and semi-analytical sensitivity analyses are carried out with SAMCEF (<http://www.samcef.com>). The Boss Quattro optimisation tool box (<http://www.samcef.com>) is used for defining and solving the optimisation problem (Radovic and Remouchamps 2002).

8.1 Laminate subjected to in- and out-of-plane loadings

A symmetric 4 plies laminate made of carbon/epoxy is considered. The load case and the initial configuration are provided in Table 8.1. The fibers orientations of each ply are the design variables, while plies thicknesses are kept constant. The optimization consists in minimizing the laminate's strain energy density, i.e. maximizing its stiffness. The evolution of this objective function with respect to the 2 angles θ_1 and θ_2 is reported in Figure 8.1, with the initial and optimal design points. A restriction is imposed on the relative variation of the 2 design variables. The optimization problem writes :

$$\begin{aligned} \min_{\boldsymbol{\theta}} \quad & \frac{1}{2} \boldsymbol{\varepsilon}_0^T \mathbf{A} \boldsymbol{\varepsilon}_0 + \frac{1}{2} \boldsymbol{\kappa}^T \mathbf{D} \boldsymbol{\kappa} \\ & |\theta_2 - \theta_1| \leq 45 \\ & 0.001 \leq \theta_i \leq 180 \quad i = 1,2 \end{aligned} \quad (8.1)$$

where the stiffness matrices \mathbf{A} , \mathbf{B} and \mathbf{D} , and the laminate's strain and curvature were previously defined in Section 3.

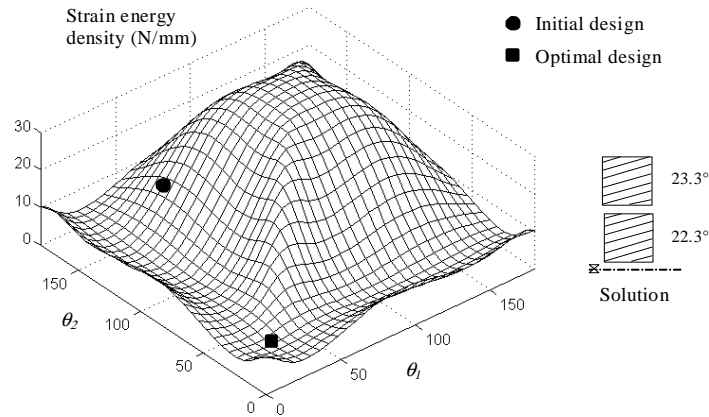


Figure 8.1. Variation of the strain energy density in the symmetric laminate subjected to the load case of Table 8.1

In-plane load case (N_1, N_2, N_6) in N/mm	Out-of-plane load case (M_1, M_2, M_6) in N	Initial orientations $\boldsymbol{\theta} = (\theta_1, \theta_2)$ in degrees	Initial thicknesses $\mathbf{t} = (t_1, t_2)$ in mm
(2000,0,1000)	(0,500,0)	(45,135)	(1,2)

Table 8.1. Problem's definition: load case and initial design

In this application the laminate is subjected not only to in-plane but also to out-of-plane loadings. Since the plies thicknesses are not identical (Table 8.1) the objective function is not symmetric with regards to the axis $\theta_1 = \theta_2$ (Figure 8.2).

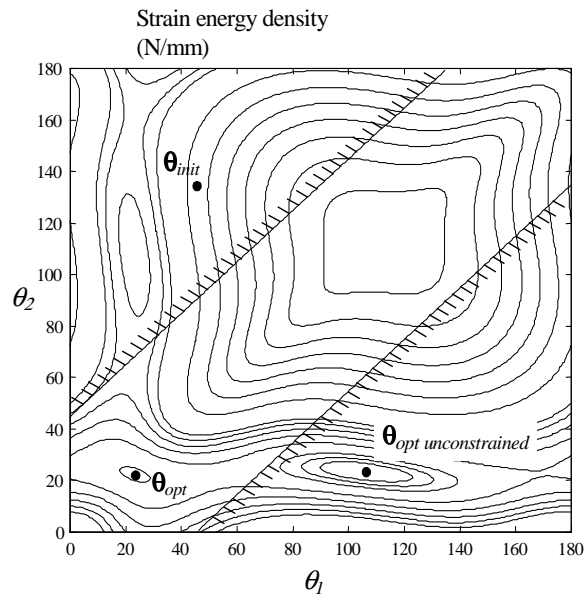


Figure 8.2. Illustration of the design space. Starting point, unconstrained and constrained optimum

The iteration histories for the 3 approximation schemes are illustrated in the Figure 8.3. The convergence of the optimization process is controlled by the relative variation of the design variables at 2 successive iterations. The MMA approximation converges in 41 iterations. 29 iterations are enough for GCMMA. When the SAM approximation is used the solution is reached in a very small number of iterations.

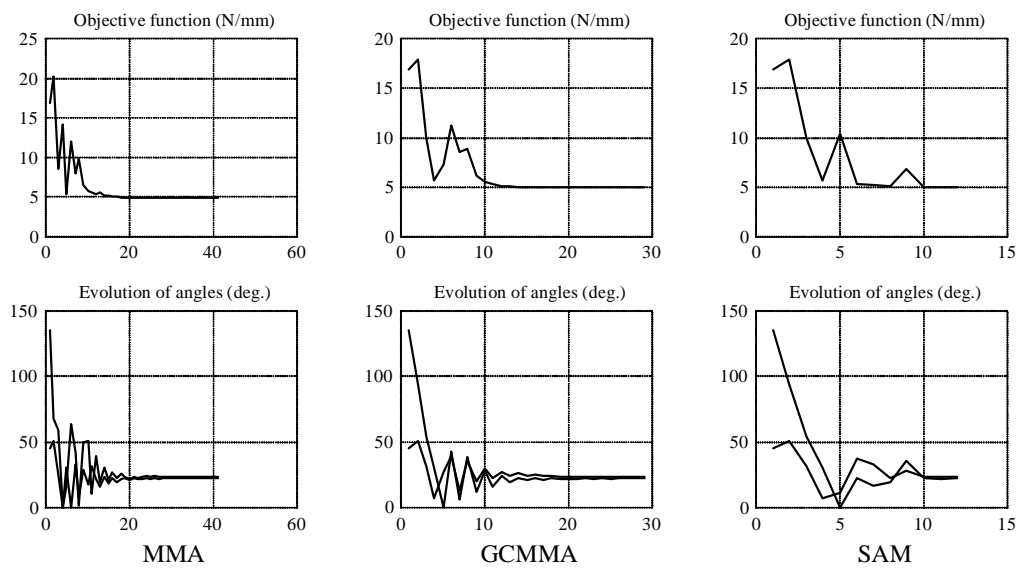


Figure 8.3. Iteration history for the 3 approximation methods

8.2 Non homogeneous laminate

In this application a non homogeneous composite membrane divided in regions of constant thickness and fibre orientations is studied. Each region is defined with an unidirectional laminate made of a glass/epoxy material. The design over stiffness is only considered here. The solution with respect to strength and stiffness is provided in Bruyneel (2006).

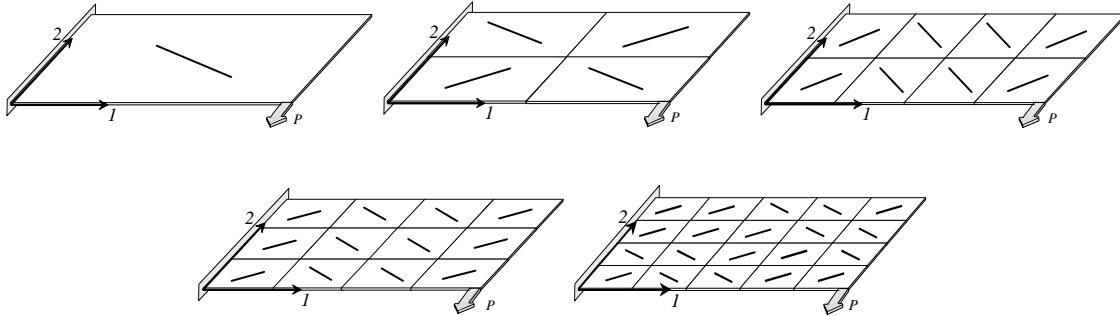


Figure 8.4. Initial configurations with 45 and -45 degrees plies orientations

The quasi-unconstrained optimization problem (8.2) consists in finding the optimal values of the plies thickness and fibers orientations in each region of the laminated composite structure that maximize the overall stiffness (i.e. that minimize the compliance – the potential energy of the applied loads). The vectors of the design variables are given by $\theta = \{\theta_i, i = 1, \dots, n\}$ and $\mathbf{t} = \{t_i, i = 1, \dots, n\}$ where n is the number of regions according to Figure 8.4. The initial thicknesses are of 1 mm.

$$\begin{aligned} & \min_{\theta, \mathbf{t}} \text{Compliance} \\ & 0^\circ \leq \theta_i \leq 180^\circ \quad i = 1, \dots, n \\ & 0.01 \text{mm} \leq t_i \leq 5 \text{mm} \end{aligned} \quad (8.2)$$

In this problem the optimal values of the thickness is 5 mm, that is their upper bound. Anyway this application illustrates the difficulties encountered when both kinds of design variables appear in the design problem. The optimal values of the compliances are reported in Figure 8.5 as a function of the number of regions. As already noticed by Foldager (1999) an increase of the number of regions of different orientations improves the overall optimal structural stiffness (i.e. it decreases the compliance).

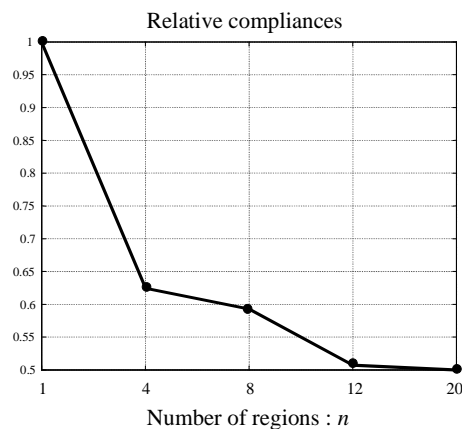


Figure 8.5. Evolution of the compliances in the problem (8.2) for the structures illustrated in Figure 8.4. The compliance of the one region structure is the reference ($n = 1$)

The optimal fibers orientations are illustrated in Figure 8.6, for the several membrane configurations of Figure 8.4. The iteration histories are reported in Figure 8.7. When the SAM method is used, about 10 iterations are enough for reaching a stationary solution with respect to a small relative variation of the objective at 2 successive iterations. The GCMMA approximation finds this solution in a larger number of design cycles. It is observed that when the SAM method is used, the structural responses in terms of both the fibers orientations and the thicknesses are well approximated, while using GCMMA, the approximation in terms of the thicknesses is too conservative, what slows down the overall convergence speed of the optimization process.

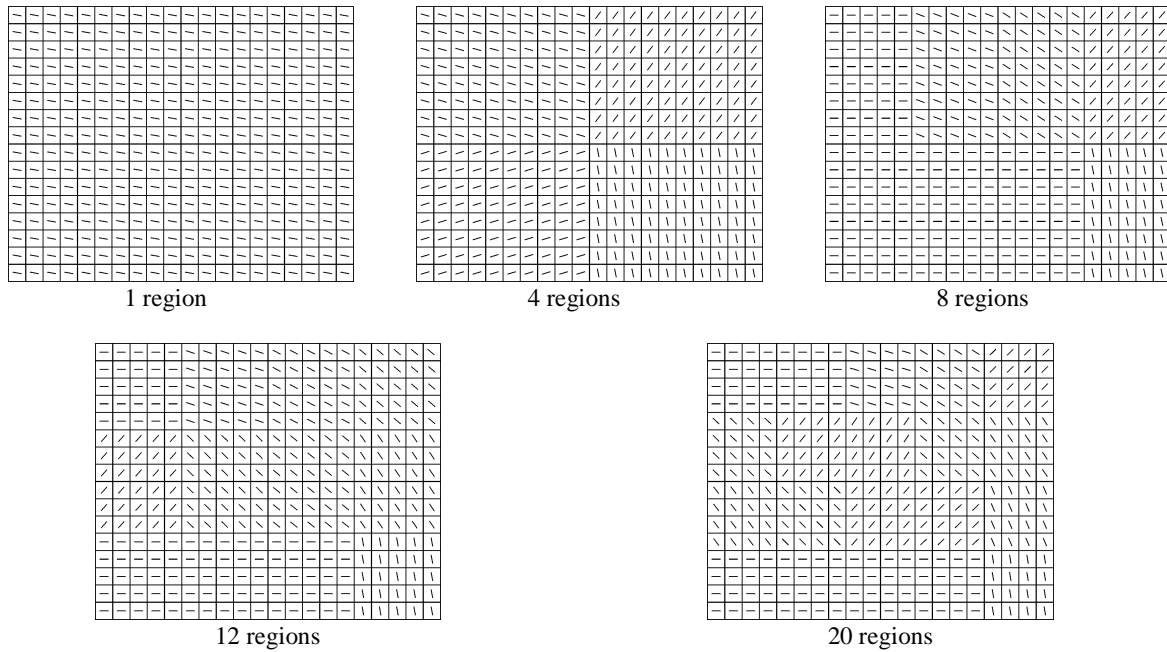
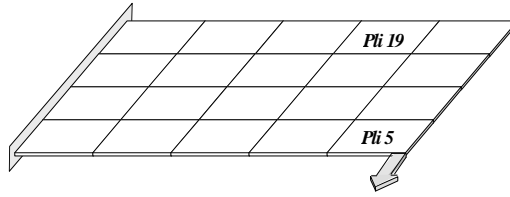
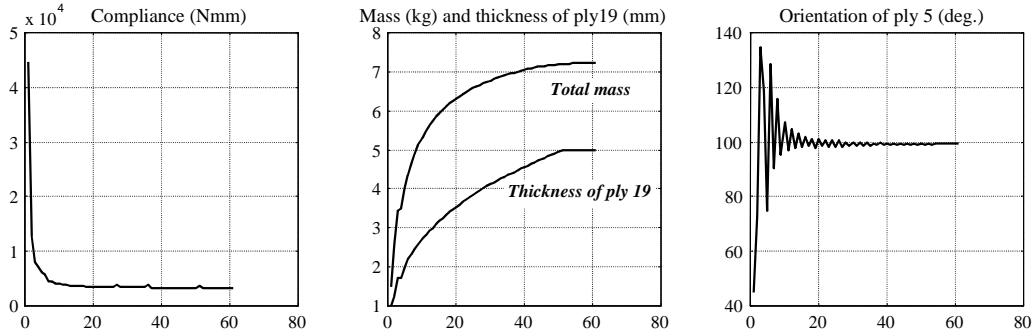


Figure 8.6. Illustration of the optimal fibers orientations for the different composite membranes illustrated in Figure 7.9

In Figure 8.8 the evolution of the vertical displacement under the load is drawn with respect to the fibers orientation in the case of the homogeneous membrane (Figure 8.4, $n=1$). The global minimum displacement is obtained for a value of 170° . When the starting point of the optimization process of the problem (8.2) is close to 45° , 0° fibers orientation is found as a local optimum. As -45° is chosen here for the initial design (i.e. 135°), the global optimum can be reached. This illustrates the fact that a gradient based method is not able to reach the global optimum, unless the starting point is in its vicinity. In Figure 8.8, the influence of the mesh refinement on the solution is presented, as well.



GCMMA



SAM

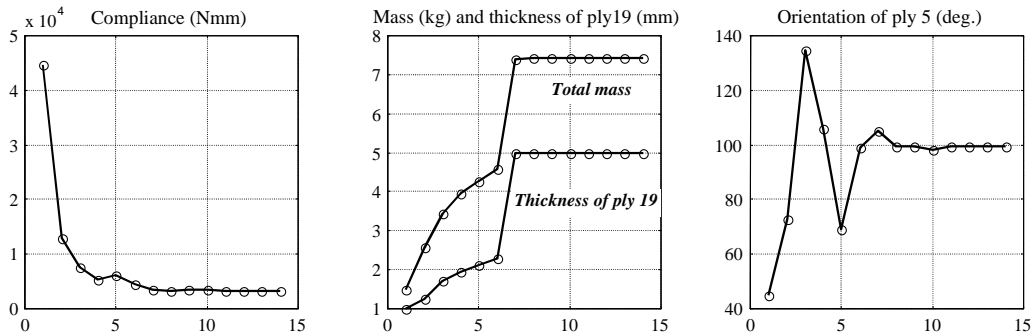


Figure 8.7. Convergence history for GCMMA and SAM for the membrane divided in 20 regions. Evolution of the thickness and the orientations of the plies number 5 and 19

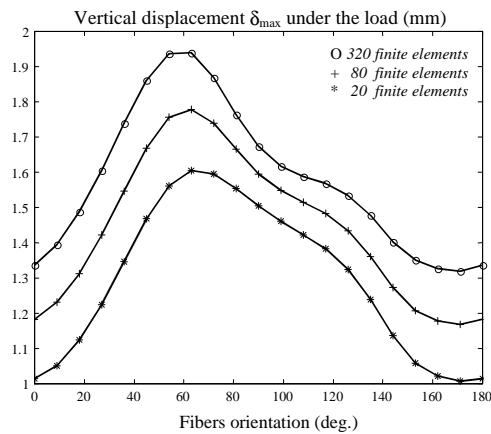


Figure 8.8. Evolution of the vertical displacement under the applied load for several discretizations of the homogeneous composite membrane (Figure 8.4, $n=1$)

8.3 Multi-objective optimization

A symmetric laminate made of 4 plies and subjected to 2 in-plane load cases is considered.

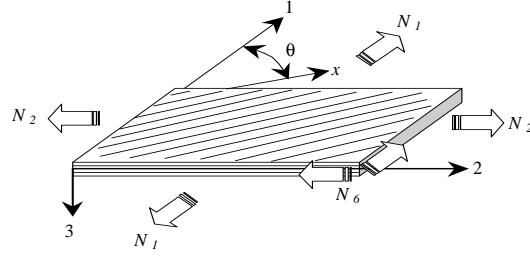


Figure 8.9. Laminate subjected to in-plane loads

The applied loads and the initial configuration are reported in Table 8.2. The load case (2) is variable : the factor k takes the values $0,1,2,\dots,8$. The extreme load cases are, on one hand $(1000,0,0)$ and on the other hand the combination of $(1000,0,0)$ and $(0,2000,0)$ N/mm.

Load case (1) (N_1, N_2, N_6) in N/mm	Load case (2) (N_1, N_2, N_6) in N/mm	Initial orientations $\theta = (\theta_1, \theta_2)$ in degrés	Initial thickness $\mathbf{t} = (t_1, t_2)$ en mm
(1000,0,0)	(0, $k \times 250$, 0)	(30,120)	(1,2)

Tableau 8.2. Definition of the problem: load case and starting point

The performance of three approximation schemes are compared : GCMMA, MMA and SAM. The optimization problem writes :

$$\begin{aligned}
 \min_{\theta, \mathbf{t}} \max_{j=1,2} \frac{1}{2} \boldsymbol{\varepsilon}_{(j)}^T \mathbf{A} \boldsymbol{\varepsilon}_{(j)} \\
 \text{TW}_{(j)}(\theta_i, t_i) \leq 1 \quad i, j = 1,2 \\
 \sum_{i=1}^4 t_i \leq 4 \\
 0.001 \leq \theta_i \leq 180 \quad i = 1,2 \\
 0.001 \leq t_i \leq 10 \quad i = 1,2
 \end{aligned} \tag{8.3}$$

where j is the number of the load case. This problem is solved by resorting the its bound formulation (Olhoff, 1989) including here 5 design variables (2 orientations, 2 thicknesses and the multi-objective factor β) and 7 constraints:

$$\begin{aligned}
 \min \frac{1}{2} \beta^2 \\
 \frac{1}{2} \boldsymbol{\varepsilon}_{(j)}^T \mathbf{A} \boldsymbol{\varepsilon}_{(j)} \leq \beta \quad j = 1,2 \\
 \text{TW}_{(j)}(\theta_i, t_i) \leq 1 \quad i, j = 1,2 \\
 \sum_{i=1}^4 t_i \leq 4 \\
 0.001 \leq \theta_i \leq 180 \quad i = 1,2 \\
 0.001 \leq t_i \leq 10 \quad i = 1,2
 \end{aligned} \tag{8.4}$$

The results are reported in Figure 8.10 for the different values of k . The solution is obtained when the relative variation of the design variables at 2 successive iterations is lower than 0.01. It is seen that a large number of iterations is needed to reach the optimum when MMA is used. GCMMA converges in a lower number of iterations. As for mono-objective problems, SAM is the most effective optimization method.

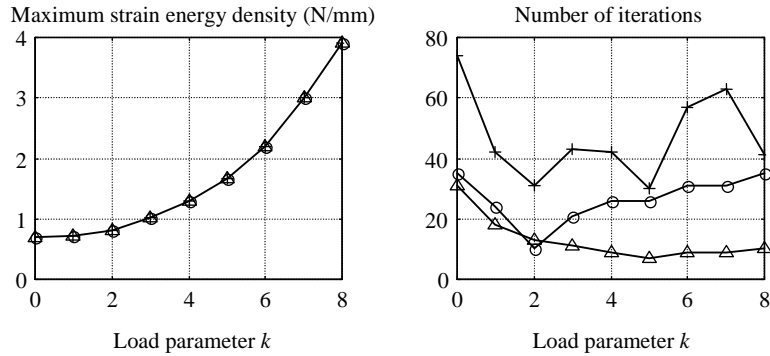


Figure 8.10. Variation of the strain energy density and number of iterations needed to reach the solution as a function of the parameter k .

+ MMA o GCMMA Δ SAM

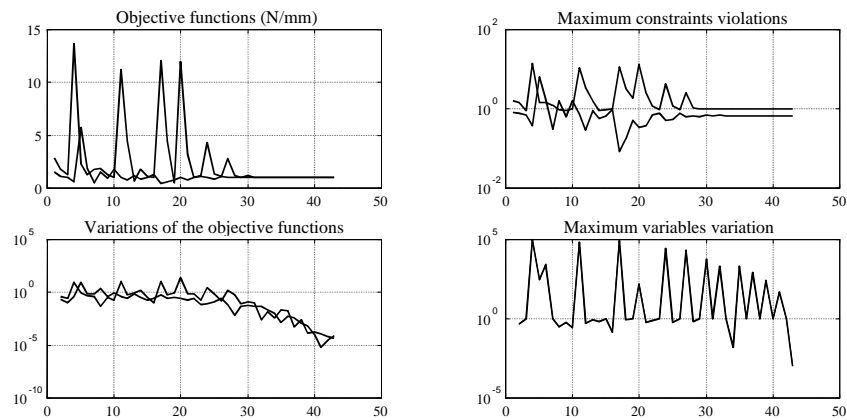


Figure 8.11. Convergence history for MMA. k is equal to 3

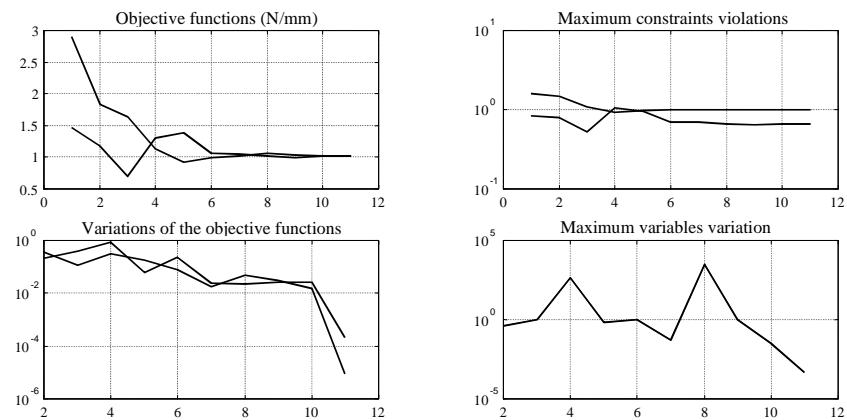


Figure 8.12. Convergence history for SAM. k is equal to 3

Figure 8.13 illustrates the optimum stacking sequence for the different values of the load parameter k . The solution corresponds to a $[0/90]_S$ with a variable proportion of 90° plies (depending on k).

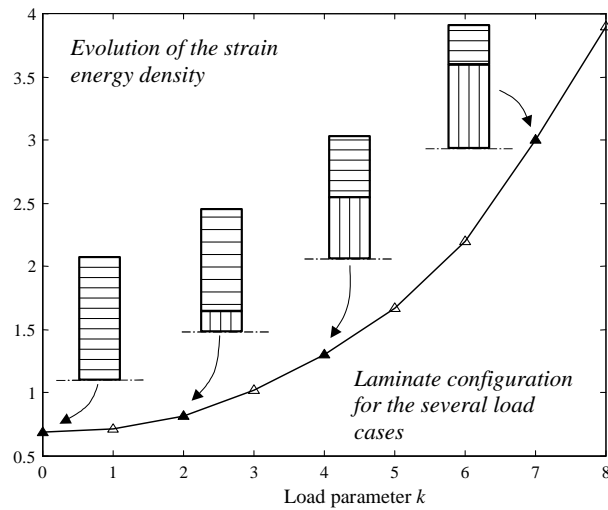


Figure 8.13. Variation of the strain energy density and configuration of the corresponding optimal laminate

Figure 8.14 describes the design space for $k = 4$. The iso-values of both objective functions are drawn. The arrow indicates the direction for an increase of the stiffness. The optimal solution is characterized here by identical values of both objective functions.

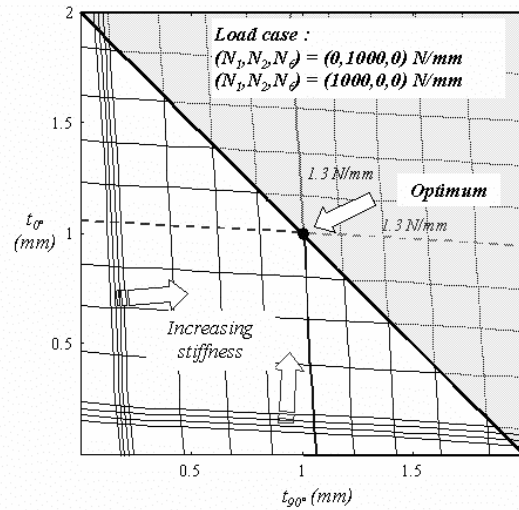


Figure 8.14. Evolution of the strain energy densities for the $[0/90]_S$ laminate Subjected to $(N_1, N_2, N_6) = (0, 1000, 0) N/mm$ and $(N_1, N_2, N_6) = (1000, 0, 0) N/mm$. t_{0° and t_{90° are the plies thickness.

The variation of the strain energy density for each single load case is illustrated in Figures 8.15 and 8.16. In those particular cases, the optimal solutions are given by only 90° or 0° orientations. This illustrates the need for a multi-objective formulation when several functions are considered as objective.

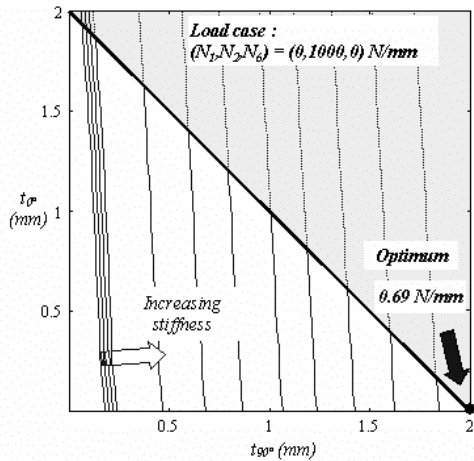


Figure 8.15. Evolution of the strain energy density in the $[0/90]_s$ laminate subjected to $N_2 = 1000\text{ N/mm}$

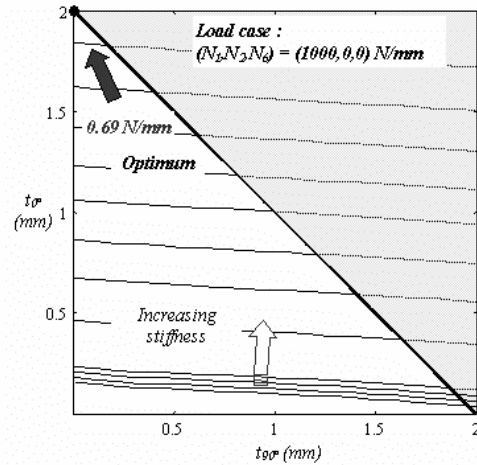


Figure 8.16. Evolution of the strain energy density in the $[0/90]_s$ laminate subjected to $N_1 = 1000\text{ N/mm}$

8.4 Optimal design with respect to stiffness and strength restrictions

In this application a stiffened laminated composite panel subjected to a uniform pressure is considered. The geometry, the boundary conditions and the stacking sequence of the different parts of the panel are illustrated in Figure 8.17. The plies thickness is equal to 0.125 mm and the base material is carbon/epoxy.

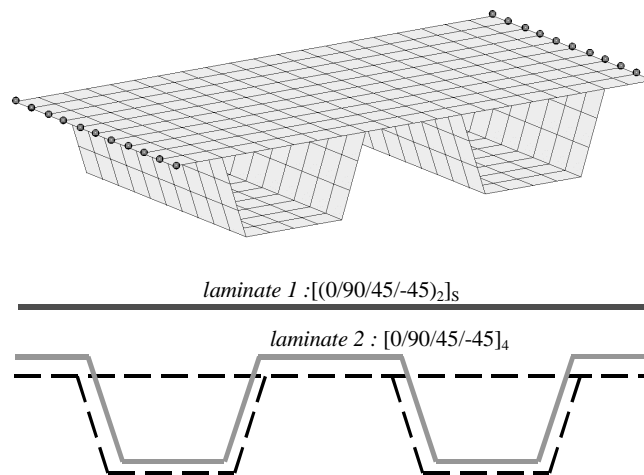


Figure 8.17. Geometry and initial stacking sequence of the stiffened panel

The optimization problem consists in maximizing the structural stiffness for a given maximum weight, knowing that a safety margin of 0.15 on the Tsai-Hill criterion on the top and the bottom of each ply must be obtained at the solution. 64 strength restrictions are defined at the plies level. The design variables are the orientations of the plies initially oriented at 0, -45, +45 and 90 degrees and the related thicknesses. The problem includes 16 design variables. The convergence histories of GCMMA and SAM are compared in Figure 8.18.

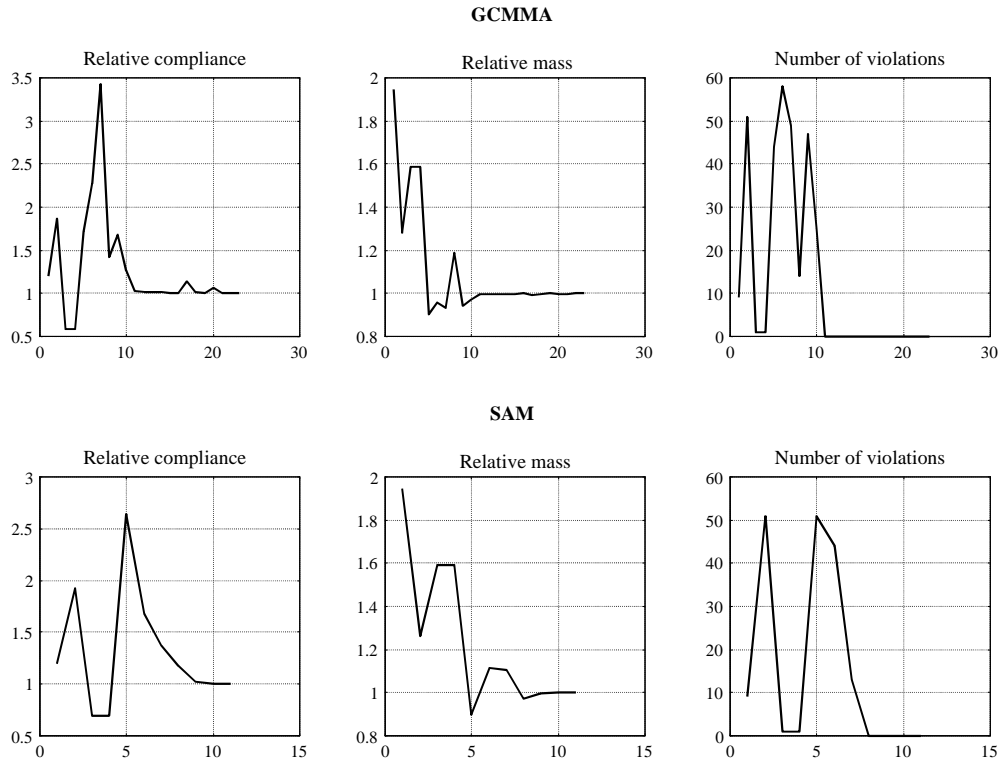


Figure 8.18. Convergence history for GCMMA and SAM

The SAM approximation succeeds in finding a solution in a very small number of iterations, with comparison to GCMMA. The optimal stacking sequence is illustrated in Figure 8.19. As already observed by Grenestedt (1990) and Foldager (1999), the optimal laminates include very few different orientations.

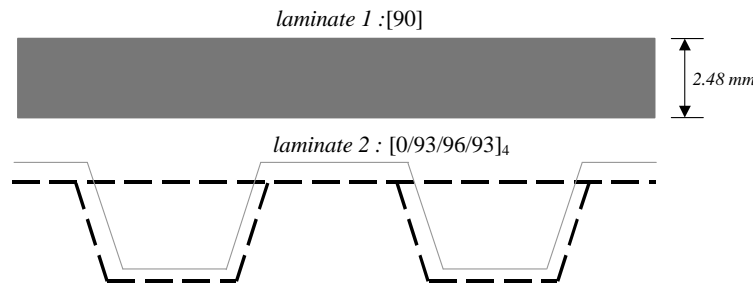


Figure 8.19. Optimal design of the stiffened panel

8.5 Optimal design under buckling considerations

Anyone who has carried out optimal sizing with a buckling criterion has experienced an undesirable effect of very slow convergence speed and possibly large variations of the design functions during the iteration history. The reasons for the bad convergence of the buckling optimisation problem are multiple, and make it difficult to solve: discontinuous character of the problem due to the localized nature of local buckling, non differentiability of the eigen-values and related problems in the sensitivity computation, modes crossing, selection of a right optimisation method, etc.

A curved composite panel including 7 hat stiffeners is considered. The load case consists of a compression along the long curved sides, and in shear on the whole outline. The structure is simply

supported on its edges. Bushing elements are used to fasten the stiffeners to the panel. In each super-stiffener (made of one stringer and the corresponding part of the whole panel), 3 design variables are used for defining the thickness of the 0° , 90° and $\pm 45^\circ$ plies in the panel and in the stiffener. 42 design variables are then defined. The goal is to find the structure of minimum weight with a minimum buckling load larger than 1.2. The results obtained in Bruyneel et al. (2007) are reported in Figures 8.20 for Conlin (Fleury and Braibant 1986) and SAM (Bruyneel 2006). The 12 first buckling loads are the design restrictions of the optimisation problem. In Figure 8.20, the evolutions of the weight and the first buckling load λ_1 over the iterations are plotted, as well as some characteristic buckling modes.

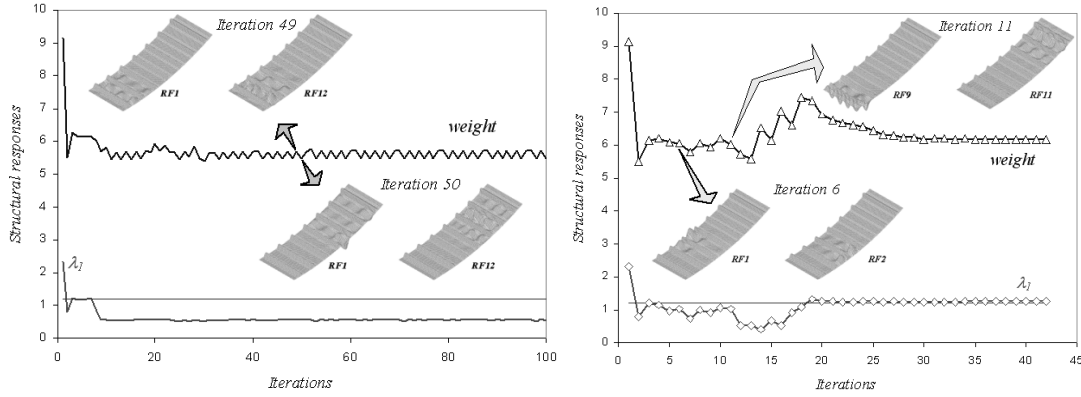


Figure 8.20. Convergence history for the buckling optimisation with Conlin (left) and SAM (right) Bruyneel et al. (2007)

It is seen that when Conlin is used (Figure 8.20, left) a solution can not be reached. With SAM (Figure 8.20, right), the solution is obtained after an erratic convergence history. Those oscillations come from the fact that local buckling modes appear during the optimisation process, and some parts of the structures are no longer sensitive to this criterion. A small thickness is therefore assigned to those parts to decrease the weight, what makes them very sensitive to buckling at the next iteration, leading to oscillations of the design variables and functions values. It was observed in Bruyneel et al. (2007) that when a large number of buckling loads are used in the optimization problem (say 100 for the problem of Figure 8.20), a solution with SAM is reached in 6 iterations, while Conlin is still no longer able to converge.

8.6 Topology optimization of laminated composite structures

The topology optimization problem of Figure 4.3 is here considered. In topology optimization of isotropic material (Bendsoe 1995), the design variable is a pseudo-density μ_i that varies between 0 and 1 in each finite element i (Figure 4.2). The so-called SIMP material law (Simply isotropic Material with Penalization) takes the following form:

$$E_i = \mu_i^p E^0 \quad \rho_i = \mu_i \rho^0 \quad (8.5)$$

where E^0 and ρ^0 are the Young modulus and the density of the base material (e.g. steel), E and ρ are the effective material properties, and p is the exponent of the SIMP law, chosen by the user ($1 < p < 4$). In linear elasticity the stresses are linked to the strains via the relation $\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\varepsilon}$. With the SIMP parameterization, it comes for a plane stress state that:

$$\mathbf{Q} = \mu_i^p \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \mu_i^p \begin{bmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & 0 \\ 0 & 0 & Q_{ss} \end{bmatrix} = \frac{\mu_i^p E^0}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

and the material stiffness matrix \mathbf{Q} depends on the density design variable μ :

$$\boldsymbol{\sigma} = \mathbf{Q}(\mu)\boldsymbol{\varepsilon} \quad (8.6)$$

For orthotropic materials in a plane stress state, the stiffness in the material axes is given by the expression (3.1) where 4 material properties E_x , E_y , ν_{xy} and G_{xy} must be provided. For a material with orthotropic axes oriented at an angle θ with respect to the reference axes the material stiffness is given by (3.5). The SIMP parameterization (8.22 and 8.23) can be extended to a Simply Anisotropic Material with Penalization (Rion and Bruyneel 2006) , and the material law for topology optimization is now written as:

$$\mathbf{Q} = \mu_i^p \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}$$

$$\boldsymbol{\sigma} = \mathbf{Q}(\mu, \theta)\boldsymbol{\varepsilon} \quad (8.7)$$

The material stiffness now depends on both kinds of design variables, i.e. the material density and the fibers orientation.

The problem in Figure 8.21 is solved with this parameterization. It includes 3750 design variables. The optimal topology and orientations obtained for an half of the structure are given in Figure 8.22. A comparison of the convergence speed for several approximations is provided in Figure 8.23.

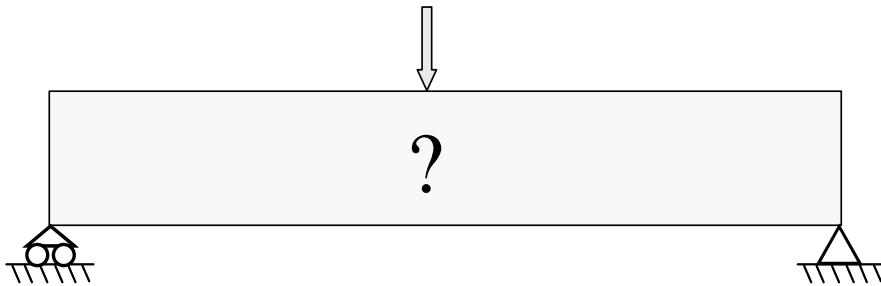


Figure 8.21. Definition of topology optimization problem. The initial structure is full of material

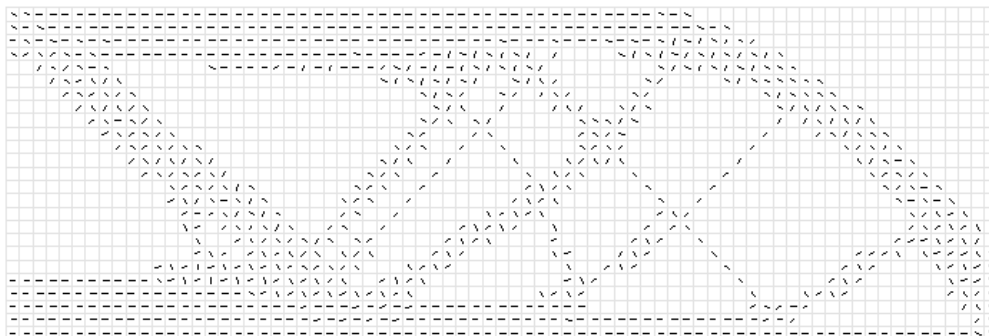


Figure 8.22. Optimal topology with orthotropic material. Only one half of the structure is drawn. The fibers orientation is plotted in the few elements that contain full material at the solution

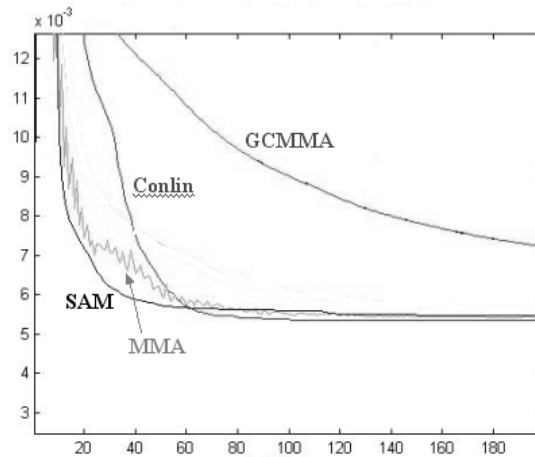


Figure 8.23. Convergence history for several approximation schemes for the topology optimization problem including orthotropic material

8.7 An industrial solution for the pre-design of composite aircraft boxes

As reported in Krog et al. (2007), the pre-design of an aircraft wing is a large scale optimization problem including (up to now) about 1000 design variables and about 300000 constraints. Those variables are linked to the total thickness of the laminate made of $0, \pm 45$ and 90° plies in the panel and to the dimensions of the cross section for the composite stiffener of each super-stringer defining the box structure (Figure 8.24). The constraints expressed as reserve factors (RF) are amongst others related to buckling and damage tolerance.

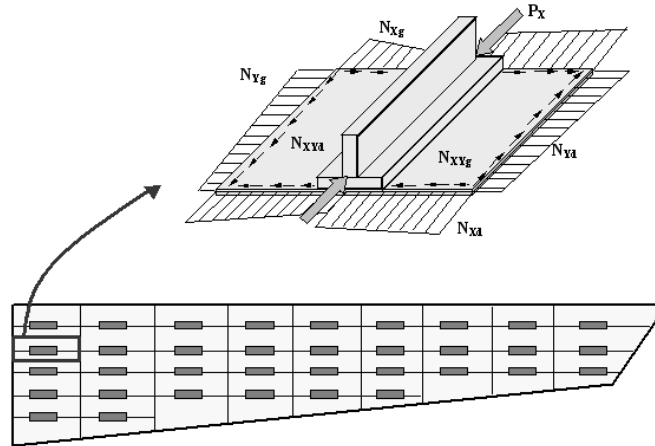


Figure 8.24. The principle of a composite wing made of super-stringers (from Krog et al., 2007)

Taking into account a so large number of design functions in the optimization problem will dramatically increase the CPU time spent in the optimizer. In order to decrease the size of the optimization problem, a technique for scanning the constraints (Figure 8.25) has been implemented in Boss Quattro (www.samcef.com). It consists in feeding the optimizer with the most critical constraints, based on their value at a given iteration. This leads to the definition of 2 sets of active and inactive constraints. The optimizer can only see the active restrictions. Those sets are not updated at each iteration but only when some inactive constraints tend to become violated after a given number of iterations (FREQ in Figure 8.25). When the SAM method (Bruyneel 2006) is used, the information at the previous design point is lost when the sets are updated, and the approximation is therefore only

built based on the information at the current design point, that is with GCMMA (Svanberg 1995), for that specific iteration.

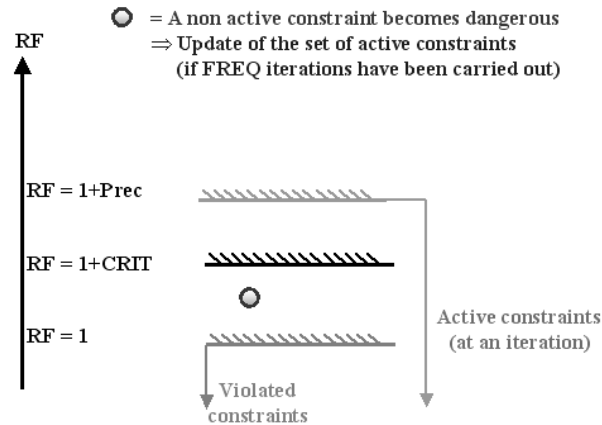


Figure 8.25. Strategy for scanning the constraints in large scale optimization problems (Krog et al. 2007)

The SAM approximation was found to be reliable in solving pre-design optimization problems of composite aircraft box structures in wings, center wing box, vertical and horizontal tail planes. Typically 30 iterations were enough to reach a stationary value of the weight and a nearly feasible design where very few constraints (less than 10) were still violated but of an amount of no more than 3 percents (RF larger than 0.97). Details of the results and of the implementation can be found in Krog et al. (2007).

8.8 Optimal design with respect to damage tolerance

A simple DCB beam is considered (Figure 8.26). The energy release rates of modes I, II and III are computed at the straight crack front with a specific virtual crack extension method described by Bruyneel et al. (2006). The stacking sequence composed of 32 plies is given by:

$$[\theta/-\theta/0/-\theta/0/\theta/\theta/0_4/\theta/0/-\theta/0/-\theta/\theta/d/-\theta/\theta/0/\theta/0/-\theta/0_4/-\theta/0/\theta/0/\theta/-\theta]$$

where d is the location of the interface where delamination will take place and θ is a variable. The goal is to find the optimal value of the orientation that will decrease the maximum value of G_I along the crack front.

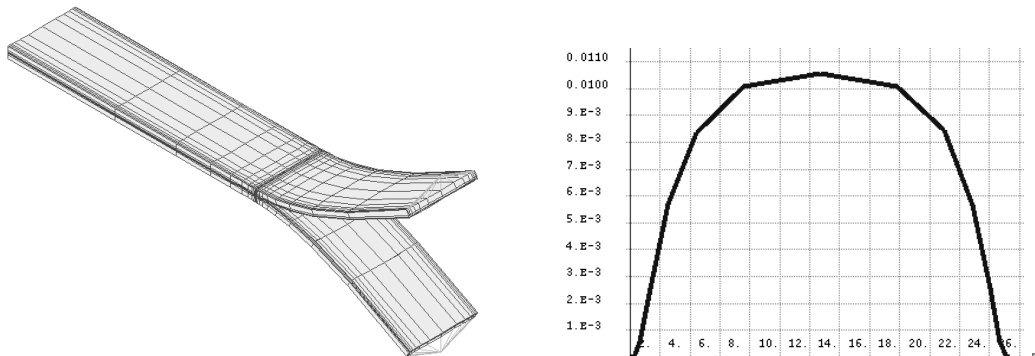


Figure 8.26. DCB beam and variation of G_I along the crack front for the initial design. On the left the displacements of the lips are multiplied by 50

The solution is provided in Figure 8.27. The optimal value for the angle θ is zero. The convergence is achieved in 5 iterations with the SAM approximation and in 15 for MMA (Figure 8.28). Although the solution of this problem is trivial, the procedure could be used for more realistic structures subjected to several complex load cases.

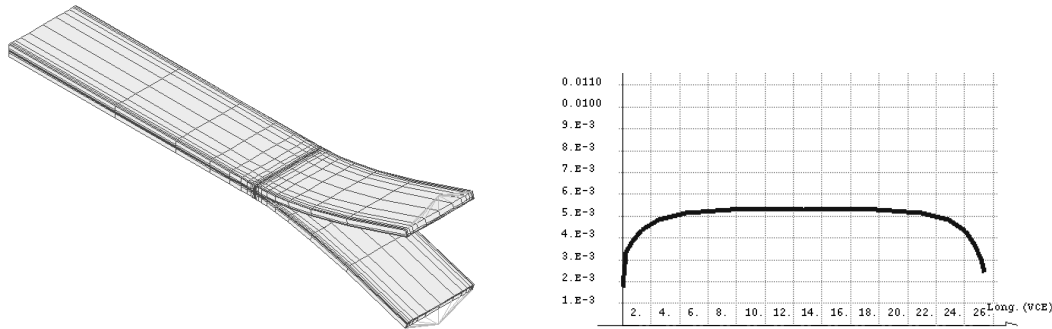


Figure 8.27. DCB beam and variation of G_I along the crack front for the optimal design
On the left the displacements of the lips are multiplied by 50

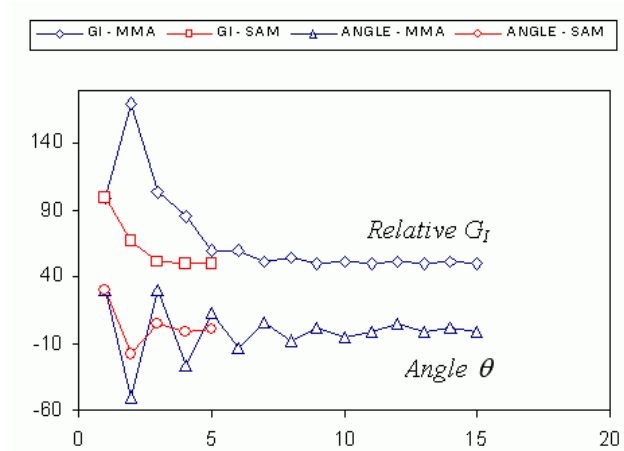


Figure 8.28. Convergence history for the optimization with respect to damage tolerance.
SAM converges in 5 iterations while MMA needs 15 iterations to reach the solution

9. Conclusion

In this chapter the optimal design of laminated composite structures was considered. After a review of the literature an optimization method specially devoted to composite structures was presented. This review helped us in selecting a formulation of the optimization problem that satisfies the industrial needs. In this context the fibers orientations and the ply thicknesses were selected as design variables. It was shown on the proposed applications that the developed solution procedure is general and reliable. It can be used for solving laminated composite problems including membrane, shells, solids, single and multiple load cases, in stiffness, buckling and strength based designs. It is routinely used in an (European) industrial context for the design of composite aircraft box structures located in the wings, the center wing box, and the vertical and horizontal tail plane. This approach is based on sequential convex programming and consists in replacing the original optimization problem by a sequence of approximated sub-problems. A very general and self adaptive approximation scheme is used. It can consider the particular structure of the mechanical responses of composites, which can be of different nature when both fibers orientations and plies thickness are design variables.

10. References

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