University of Liège
Faculty of Applied Sciences
Aerospace and Mechanical Engineering Department

9th European Solid Mechanics Conference:

Simplified fatigue resistance in mechanical engineering using topology optimization

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9th July 2015

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Outline

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Introduction

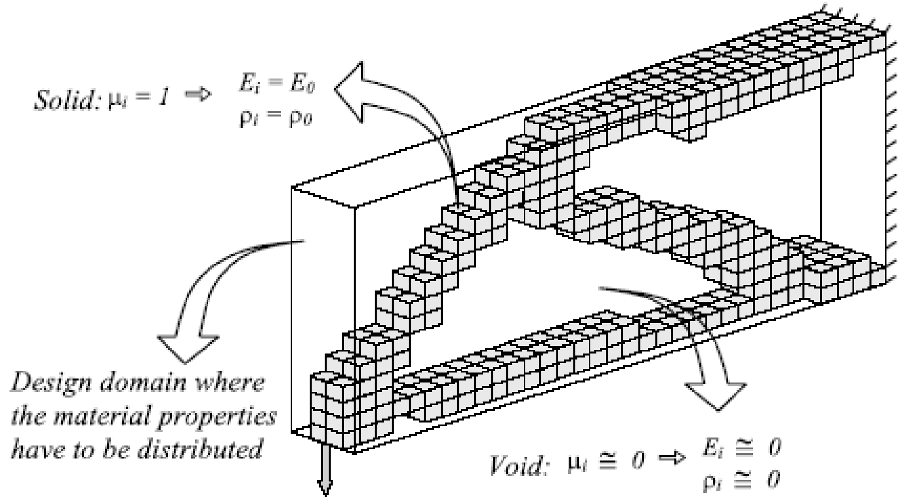
Introduction

- Principle: **Optimal distribution of material** within a given space subject to given load(s) and boundary conditions

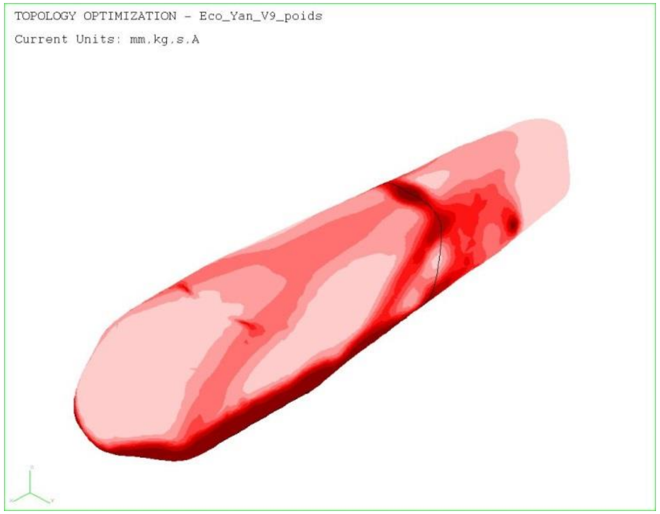
- Variables: absence/presence of material = **density** (ranging from 0 to 1)

$$E_j(x_j) = E_{min} + x_j^p(E_0 - Emin)$$

- Tool **for creativity** ➔ new very efficient concepts



min density	**Objective function**
s.t.	**constraints**



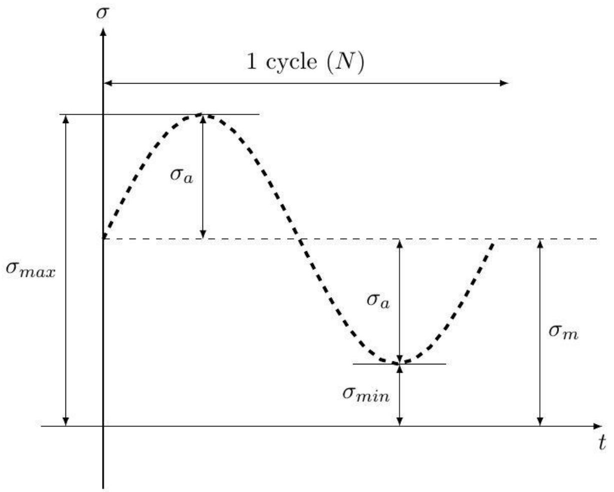
- In this work:

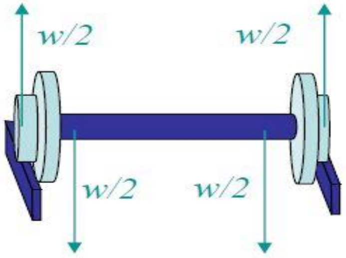
- **Objective function**: volume
- **Constraints**: Limit of the **stresses** (under fatigue considerations)

Introduction (2)

Fatigue is a critical issue when considering mechanical functioning parts in various fields of application

→ Failure of the component with a stress level below the ultimate tensile strength of the material → Cyclic loading





$$\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2}$$
$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$$

[Images: from « Fracture mechanics, damage and fatigue » L.Noels]

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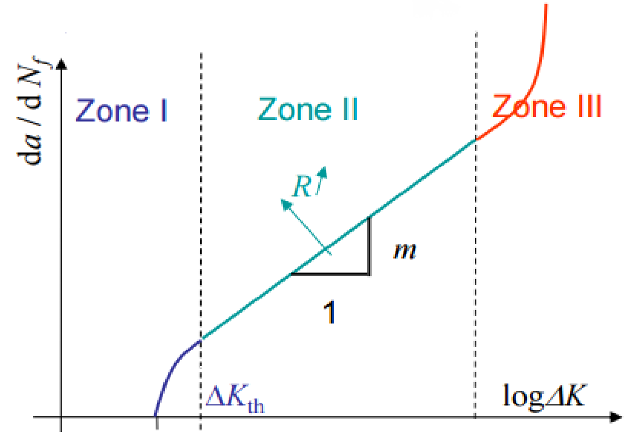
Introduction (3)

- Fatigue is responsible for almost 80% of failure in mechanical system → Crack initiation and propagation:
  - Local plastification (Zone I)
  - Cracks initiation (Zone I)
  - Crack propagation (Zone II)
  - Failure (Zone III)



[Images: from « Fracture mechanics, damage and fatigue » L.Noels]

- It is necessary to design components accounting for fatigue failure to prevent oversizing of the structure:
  - Design rules based on fatigue criteria: Sines, Crossland, Dang Van, etc.
  - Design rules based on several diagrams : Whöler, Goodman,Soderberg,etc.

→ Searching for the best "performance/weight" ratio !

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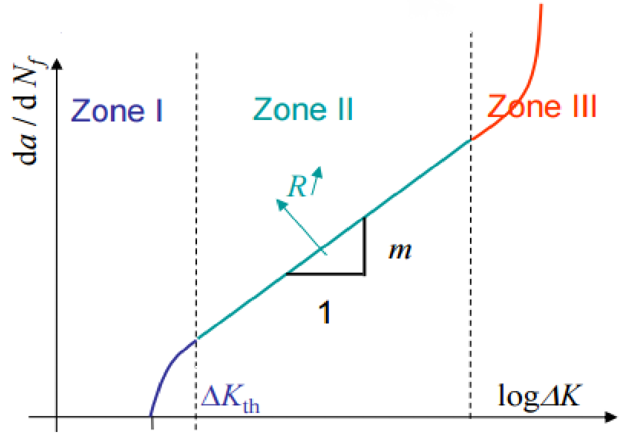

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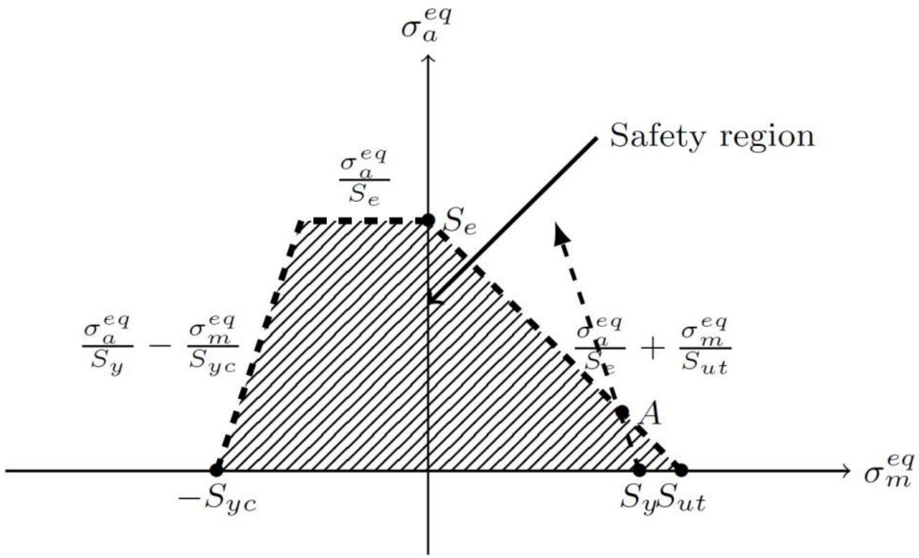
  → Searching for the best "performance/weight" ratio !

! TOPOLOGY OPTIMIZATION INCLUDING FATIGUE CONSTRAINTS !

Fatigue considerations

Simplified Haigh (Goodman) Diagram

- In this Work: Simplified Haigh (Goodman) diagram (Norton(2000)) :



$$\begin{cases} \dfrac{\sigma_a^{eq}}{S_e} + \dfrac{\sigma_m^{eq}}{S_{ut}} \leq 1 \\ \dfrac{\sigma_a^{eq}}{S_y} - \dfrac{\sigma_m^{eq}}{S_{yc}} \leq 1 \\ \dfrac{\sigma_a^{eq}}{S_e} \leq 1 \end{cases}$$

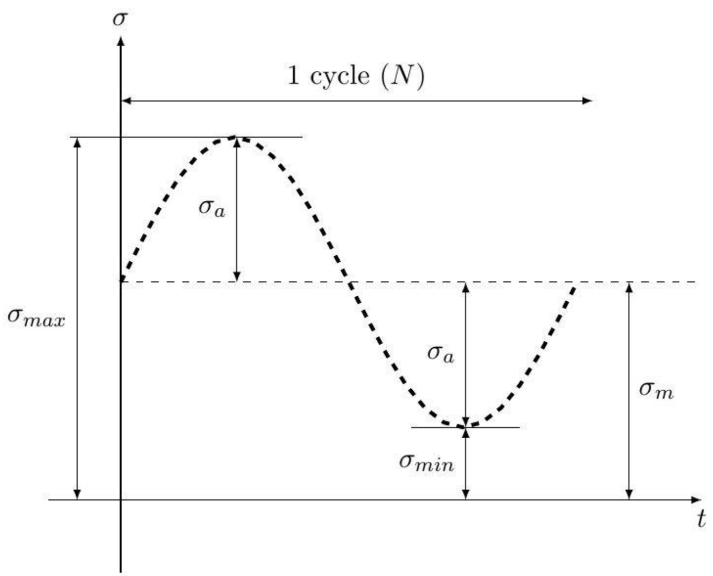
- Linear Piece-Wise criteria → Easy to evaluate with a shape well suited for stress-based topology optimization

- Infinite life is supposed !

- Fatigue design following the rule of « machine design element » (Norton(2000))

Loading consideration

- Let assume that the total stress is given by a amount of alternate component $c_a \sigma_a^{eq}$ and mean component $c_m \sigma_m^{eq}$ → superposition principle!



$$\sigma_{eq} = c_a \cdot \sigma_a^{eq} + c_m \cdot \sigma_m^{eq} \leq \sigma_{eq}^L$$

$$0 \leq c_a, c_m \leq 1$$

$$c_a + c_m = 1$$

- It means that the alternate and mean component come from the same load case.

Topology Optimization

Topology optimization formulation

- In this work: Sines (multiaxial stress) method to compute equivalent alternate and mean stresses:

$$\begin{cases} \sigma_m^{eq} = J_1(\sigma_{m,ij}) \\ \sigma_a^{eq} = \sqrt{3J_{2D}(\sigma_{a,ij})}, \end{cases} \rightarrow \underbrace{\begin{cases} J_{1,e}(\sigma_{ij}) = x_e^p \mathbf{H}_e^0 \mathbf{U}_e \\ 3J_{2D,e}(\sigma_{ij}) = x_e^{2p} \mathbf{U}_e^T \mathbf{M}_e^0 \mathbf{U}_e \end{cases}}_{\text{Element level+ SIMP law}} \rightarrow \begin{cases} \sigma_m^{eq} = x_e^p \left(c_m \mathbf{H}_e^0 \mathbf{U}_e \right) = x_e^p\, \overline{\sigma}_{m,e}^{eq} \\ \sigma_a^{eq} = x_e^p \left(c_a \sqrt{\mathbf{U}_e^T \mathbf{M}_e^0 \mathbf{U}_e} \right) = x_e^p\, \overline{\sigma}_{a,e}^{eq}, \end{cases}$$

- Introducing the local apparent stress (Duysinx et Bendsøe (1998)) $\sigma_{ij} = \langle \sigma_{ij} \rangle / x_e^q$ and recalling the fatigue criteria at the element level, using the *qp-relaxation* (Bruggi(2008)) for stresses:

$$\begin{cases} \dfrac{\sigma_{a,e}^{eq}}{S_e} + \dfrac{\sigma_{m,e}^{eq}}{S_{ut}} \leq 1 \\ \dfrac{\sigma_{a,e}^{eq}}{S_y} - \dfrac{\sigma_{m,e}^{eq}}{S_{yc}} \leq 1 \\ \dfrac{\sigma_{a,e}^{eq}}{S_e} \leq 1 \end{cases} \Rightarrow \begin{cases} \dfrac{\sigma_{a,e}^{eq}}{S_e} + \dfrac{\sigma_{m,e}^{eq}}{S_{ut}} = \dfrac{\langle \sigma_{a,e}^{eq} \rangle}{x_e^q\, S_e} + \dfrac{\langle \sigma_{m,e}^{eq} \rangle}{x_e^q\, S_{ut}} = \boxed{x_e^{(p-q)} \left(\dfrac{\overline{\sigma}_{a,e}^{eq}}{S_e} + \dfrac{\overline{\sigma}_{m,e}^{eq}}{S_{ut}} \right) \leq 1} \\ \dfrac{\sigma_{a,e}^{eq}}{S_y} - \dfrac{\sigma_{m,e}^{eq}}{S_{yc}} = \dfrac{\langle \sigma_{a,e}^{eq} \rangle}{x_e^q\, S_y} - \dfrac{\langle \sigma_{m,e}^{eq} \rangle}{x_e^q\, S_{yc}} = \boxed{x_e^{(p-q)} \left(\dfrac{\overline{\sigma}_{a,e}^{eq}}{S_y} - \dfrac{\overline{\sigma}_{m,e}^{eq}}{S_{yc}} \right) \leq 1} \\ \dfrac{\sigma_{a,e}^{eq}}{S_e} = \dfrac{\langle \sigma_{a,e}^{eq} \rangle}{x_e^q\, S_e} = \boxed{x_e^{(p-q)} \dfrac{\overline{\sigma}_{a,e}^{eq}}{S_e} \leq 1} \end{cases}$$

Topology optimization formulation (2)

- The topology optimization problem to solve reads:

$$
\begin{cases}
\min\limits_{x_0 \le x_e \le 1} & \mathcal{W} = \sum\limits_{N} x_e V_e \\
\text{s.t.} & \mathbf{K}(\mathbf{x})\ \mathbf{U} \ = \ \mathbf{F}, \\
& \mathcal{C} \ / \ \mathcal{C}_L \ \le \ 1, \quad \mathcal{C}_L = \alpha \mathcal{C}_0 \\
& x_e^{(p-q)} \left(\dfrac{\overline{\sigma}_{a,e}^{eq}}{S_e} + \dfrac{\overline{\sigma}_{m,e}^{eq}}{S_{ut}} \right) \le 1, \qquad \text{for } e = 1, ..., N \\
& x_e^{(p-q)} \left(\dfrac{\overline{\sigma}_{a,e}^{eq}}{S_y} - \dfrac{\overline{\sigma}_{m,e}^{eq}}{S_{yc}} \right) \le 1, \qquad \text{for } e = 1, ..., N \\
& x_e^{(p-q)} \dfrac{\overline{\sigma}_{a,e}^{eq}}{S_e} \le 1, \qquad \text{for } e = 1, ..., N
\end{cases}
$$

Enforcing a global compliance constraint (Bruggi and Duysinx(2012))

- With the density filter (Bruggi and Duysinx(2012)):

$$
\tilde{x}_e = \frac{1}{\sum_N H_{ej}} \sum_N H_{ej} x_j,
$$

$$
H_{ej} = \sum_N \max(0, r_{min} - \text{dist}(e, j)),
$$

Sensitivity Analysis

Sensitivity Analysis

- Sensitivity of the global compliance constraint

$$\frac{\partial \mathcal{C}}{\partial x_k} = -p x_k^{p-1} \mathbf{U}_k^T \mathbf{K}_k^0 \mathbf{U}_k,$$

- Sensitivities of the local stress constraints for the mean an alternated part

$$\frac{\partial \langle \sigma_{a,e}^{eq} \rangle}{\partial x_k} = \delta_{ek}(p-q) x_e^{p-q-1}\ \overline{\sigma}_{a,e}^{eq} + \frac{\partial \overline{\sigma}_{a,e}^{eq}}{\partial x_k} x_e^{p-q}$$

$$\frac{\partial \langle \sigma_{m,e}^{eq} \rangle}{\partial x_k} = \delta_{ek}(p-q) x_e^{p-q-1}\ \overline{\sigma}_{m,e}^{eq} + \frac{\partial \overline{\sigma}_{m,e}^{eq}}{\partial x_k} x_e^{p-q}.$$

- With $\frac{\partial \overline{\sigma}_{a,e}^{eq}}{\partial x_k}$ and $\frac{\partial \overline{\sigma}_{m,e}^{eq}}{\partial x_k}$ respectively computed as (Duysinx et Bendsøe (1998)) and Duysinx and Sigmund (1998)):

$$\frac{\partial \overline{\sigma}_{a,e}^{eq}}{\partial x_k} = -\widetilde{\mathbf{U}}^T \frac{\partial \mathbf{K}}{\partial x_k} \mathbf{U}, \quad \text{where}$$

$$\mathbf{K}\widetilde{\mathbf{U}} = \left[c_a (\mathbf{U}^T \mathbf{M}_e^0 \mathbf{U})^{-\frac{1}{2}} \mathbf{M}_e^0 \mathbf{U} \right],$$

and

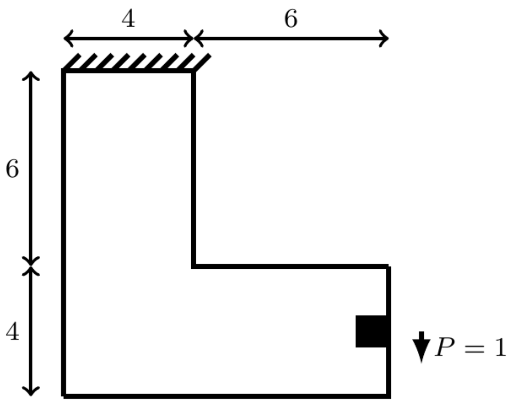
$$\frac{\partial \overline{\sigma}_{m,e}^{eq}}{\partial x_k} = -\widetilde{\mathbf{U}}^T \frac{\partial \mathbf{K}}{\partial x_k} \mathbf{U}, \quad \text{where}$$

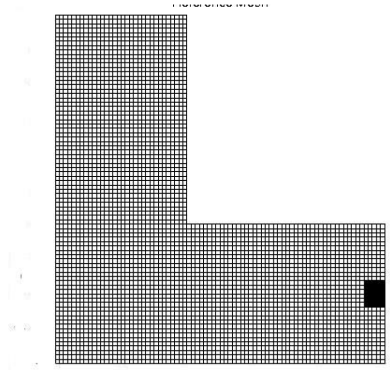
$$\mathbf{K}\widetilde{\mathbf{U}} = \left[c_m \mathbf{H}_e^0 \right],$$

- Adjoint Sensitivity Method is used because the number of active constraints is likely smaller than the number of design variables.

Examples

Example 1: L-shape lamina





- SIMP model

- Penalization p = 3
- *qp* relaxation: q = 2.6 → 2.75

- Material: Steel (normalized values !)

- E = 1MPa (normalized), ν = 0.3

- Compliance regularization constraint: $\alpha_c = 2$

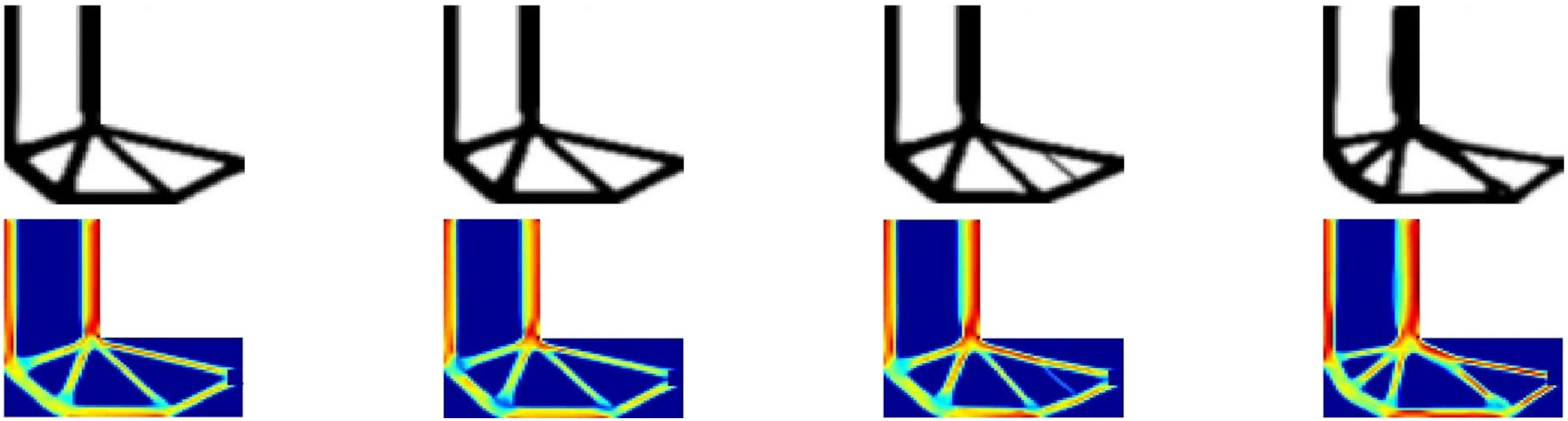
- Optimizer: MMA (Svanberg(1987))

Problem	NE	$\mathcal{W}/\mathcal{W}_0$	$\mathcal{C}/\mathcal{C}_0$	N_a^f	CPU	it.max	r_{min}
MWCS	4096	39.65	2	60	486.3	397	0.25
MWCF ($c_a = 0.3; c_m = 0.7$)	4096	39.74	2	44	316.7	222	0.25
MWCF ($c_a = 0.5; c_m = 0.5$)	4096	41.17	2	118	881	324	0.25
MWCF ($c_a = 0.7; c_m = 0.3$)	4096	43.7	2	263	1261	247	0.25

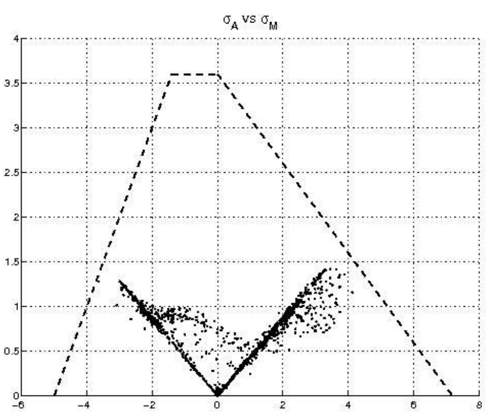
Example 1: L-shape lamina (3)

- Optimized designs + Stress maps

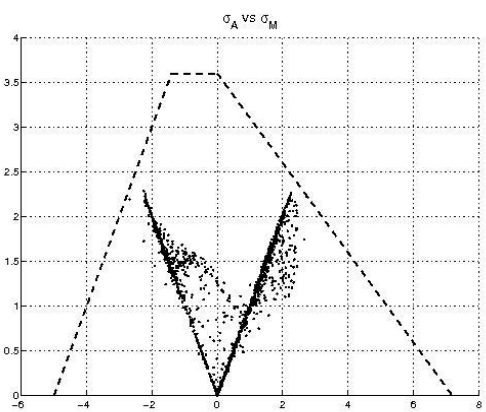
MWCS	MWCF (ca = 0.3;cm = 0.7)	MWCF (ca = 0.5;cm = 0.5)	MWCF (ca = 0.7;cm = 0.3)



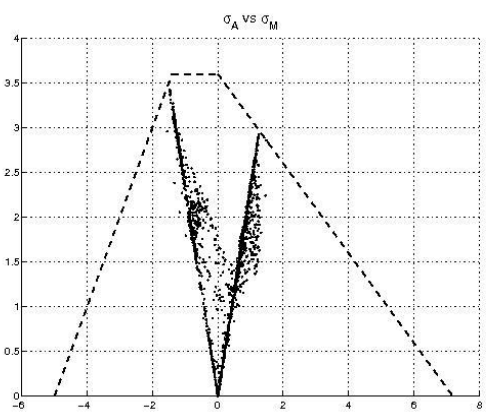
- Goodman diagrams:



MWCF (ca = 0.3;cm = 0.7)



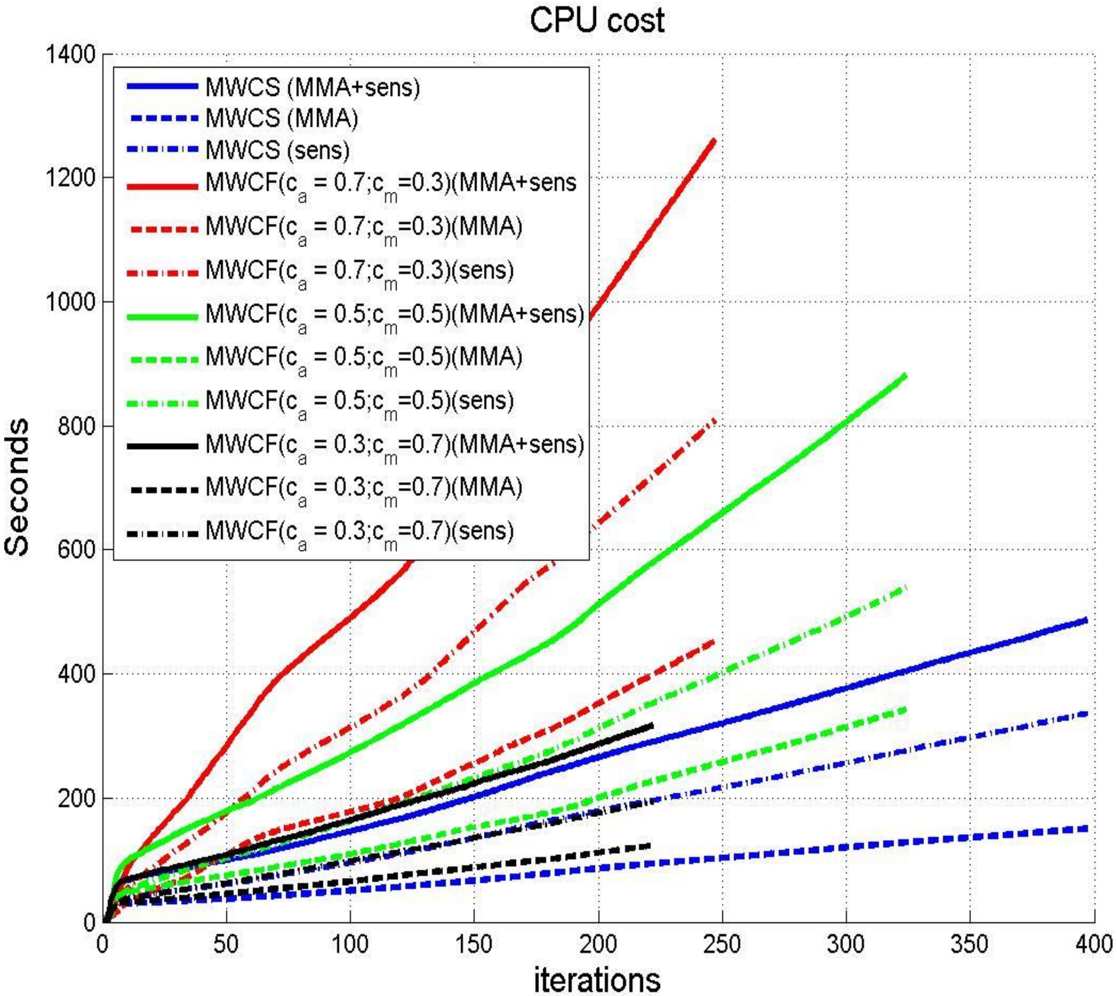
MWCF (ca = 0.5;cm = 0.5)

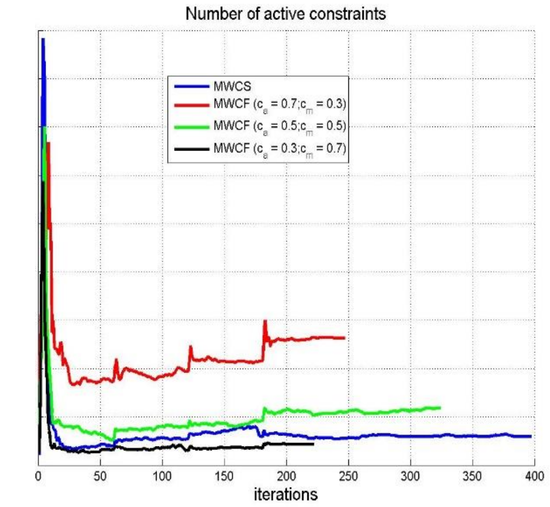


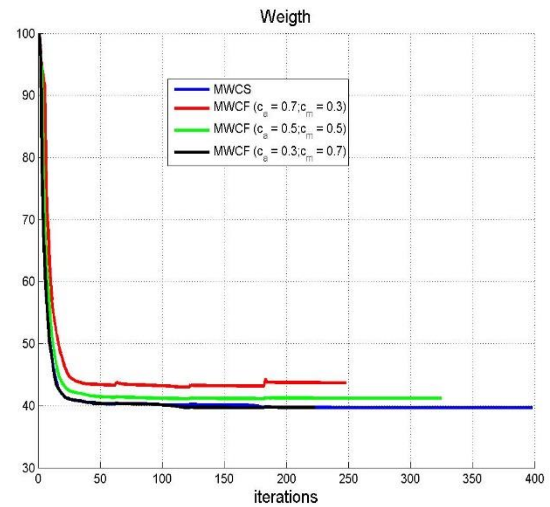
MWCF (ca = 0.7;cm = 0.3)

Example 1: L-shape lamina (3)

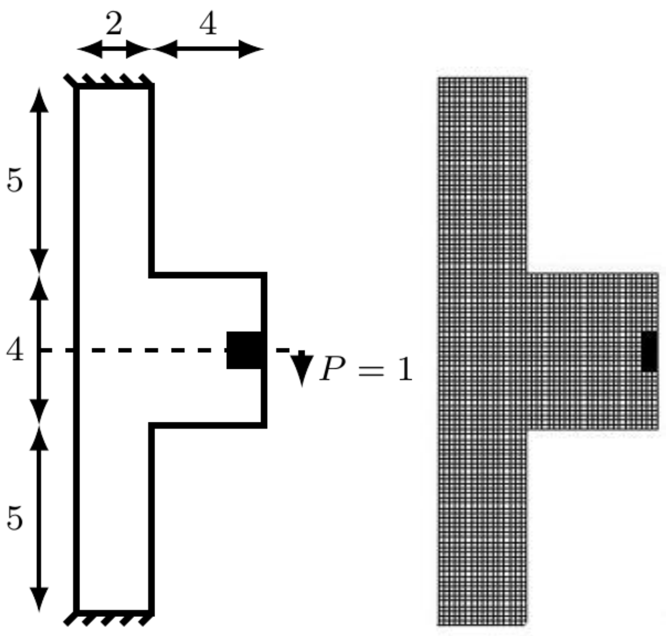
- CPU COST







Example 2: Half-H lamina



- SIMP model
  - Penalization p = 3
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  - E = 1MPa (normalized), ν = 0.3
- Compliance regularization constraint: $\alpha_c = 2$
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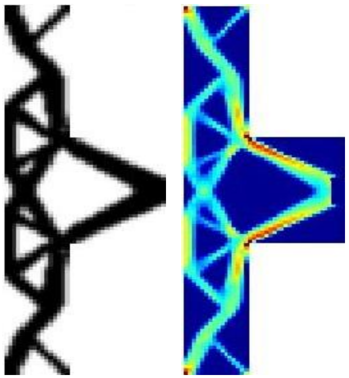
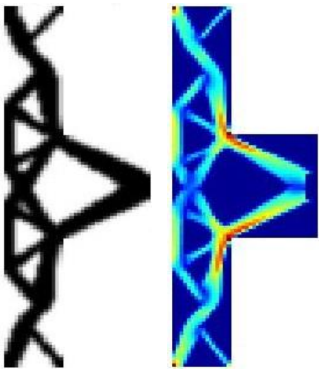
Problem	NE	$\mathcal{W}/\mathcal{W}_0$	$\mathcal{C}/\mathcal{C}_0$	N_a^f	CPU	it.max	r_{min}
MWCS	2560	49.12	2	33	45.65	134	0.1875
MWCF ($c_a = 0.3; c_m = 0.7$)	2560	50.15	2	43	111.8	154	0.1875
MWCF ($c_a = 0.5; c_m = 0.5$)	2560	50.67	2	56	231.5	272	0.1875
MWCF ($c_a = 0.7; c_m = 0.3$)	2560	51.53	2	98	192.1	231	0.1875

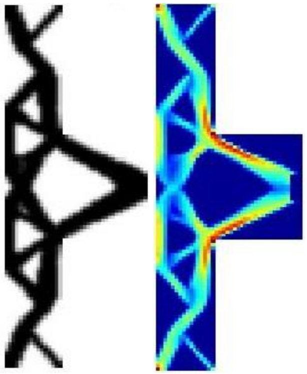
Example 2: Half-H lamina (2)

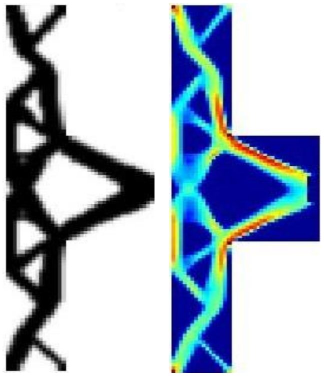
- Optimized designs + Stress maps

MWCS	MWCF (ca = 0.3;cm = 0.7)	MWCF (ca = 0.5;cm = 0.5)	MWCF (ca = 0.7;cm = 0.3)

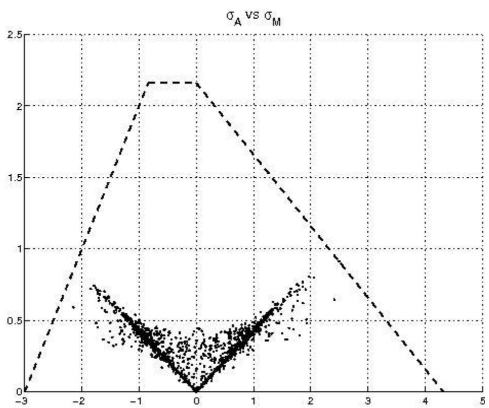




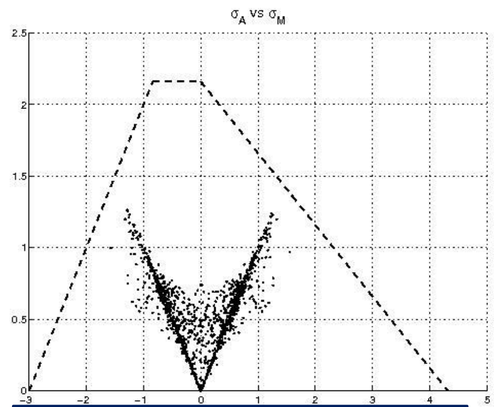




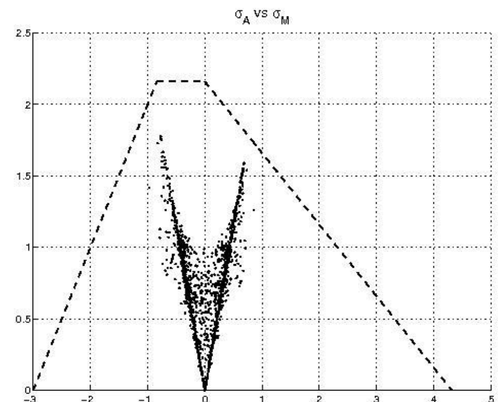
- Goodman diagrams:



MWCF (ca = 0.3;cm = 0.7)



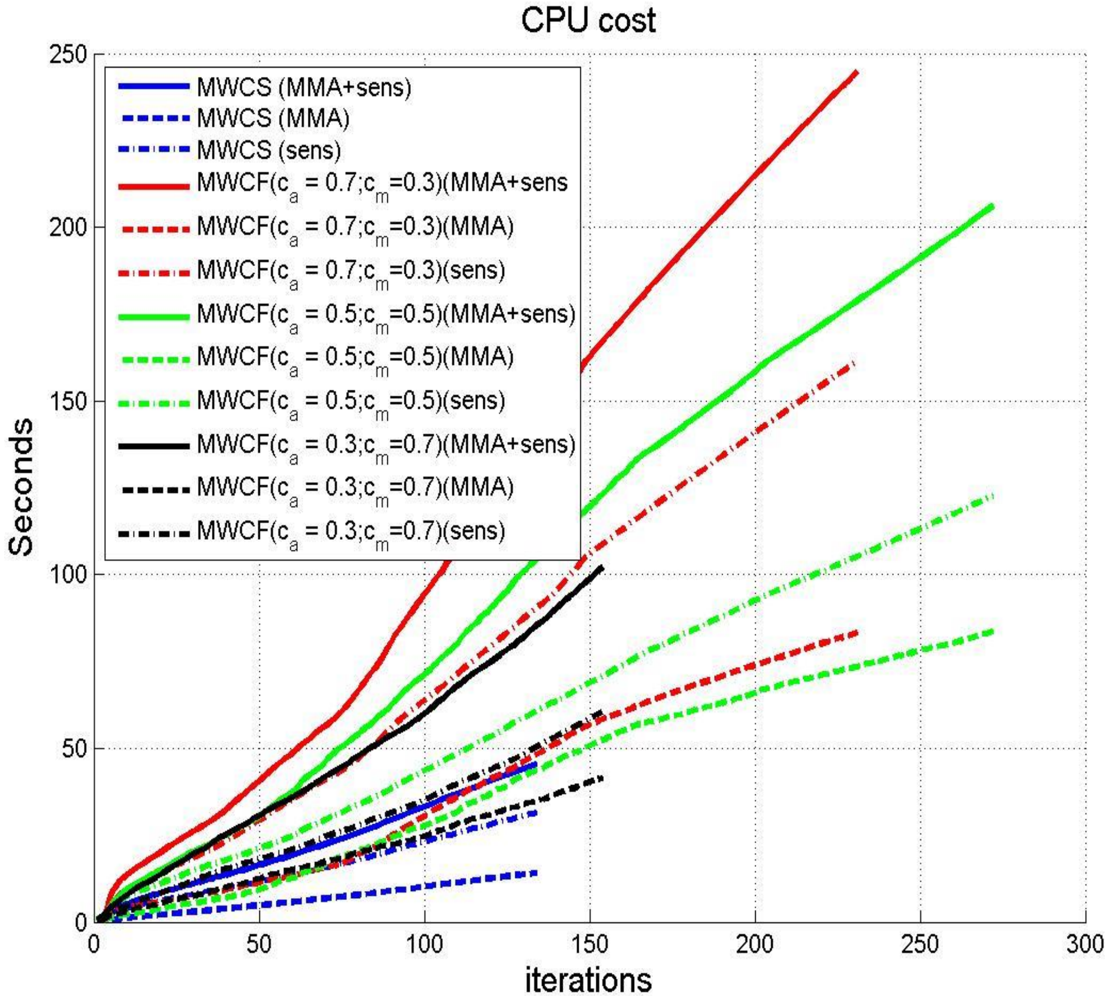
MWCF (ca = 0.5;cm = 0.5)

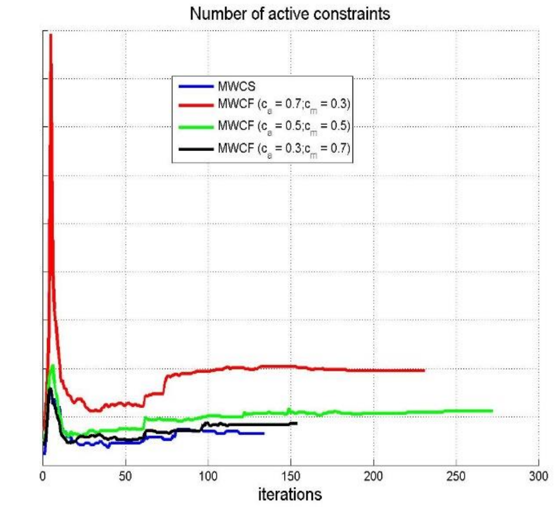


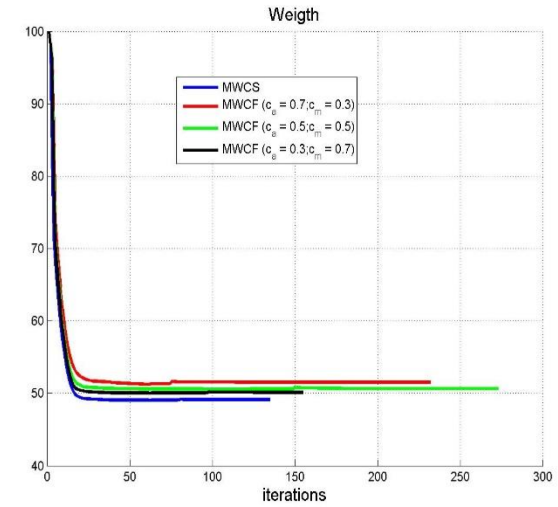
MWCF (ca = 0.7;cm = 0.3)

Example 2: Half-H lamina (3)

- CPU COST







Conclusion

Conclusion

In this work:

- Easy implementation of fatigue considerations
- More heavier structures are obtained with fatigue constraints BUT with optimized weight to sustain the fatigue allowable stress
- More rounded shapes can be obtained when large singularity occurs (typically sharp edges)
- More CPU time needed → more active constraints sent to the optimizer → Sensitivity Analysis is heavy

Future works:

- Extension to several load cases + time history of stresses → consideration of the Dang Van criterion
- Improve the numerical resolution of the optimization problem
- Implement of projection filter (e.g. Heaviside) → Additive Manufacturing !!!

References

- Bruggi M (2008) On an alternative approach to stress constraints relaxation in topology optimization. Struct Multidiscip Optim 36:125—141
- Bruggi M, Dusyinx P (2012) Topology optimization for minimum weight with compliance and stress constraints. Struct Multidisc Optim 46(3):369-384
- Duysinx P, Bendsøe MP (1998) Topology optimization of continuum structures with local stress constraints. Int J Numer Methods Eng 43: 1453—1478
- Duysinx P, Sigmund O (1998) New developments in handling stress constraints in optimal material distribution. 7th Symposium on Multidisciplinary Analysis and Optimization AIAA--98--4906: 1501—1509
- Norton R. L (2000). Machine design: An integrated approach. Upper Saddle River, N.J: Prentice Hall
- Svanberg K (1987) Method of moving asymptotes - A new method for structural optimization. Int J Numer Methods Eng 24:359--373

Thank you for your attention

The author, Maxime Collet, would like to acknowledge the Belgian National Fund for Scientific Research (FRIA) for its financial support.

Questions ?