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# Damage process sensitivity analysis using an XFEM-Level Set framework.

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## 1. Abstract
Designing efficient and lightweight structures is a key objective for many industrial applications such as in aerospace or the automotive industry. To this end, composite materials are appealing as they combine high stiffness and light weight. The main challenge slowing down the integration of such materials in real structures is their damage behavior. The latter should be considered in the design process of the structures. This work focuses on developing a systematic approach to designing structures that can sustain an acceptable amount of degradation or exhibit a low sensitivity to damage. An optimization approach is chosen to achieve this goal. To deal with complex geometries and to allow for large shape modifications in the optimization process, the extended finite element method (XFEM) is advantageously combined with a level set description of geometry. The degradation of materials is modeled by using a non-local damage model, motivated by the work of James and Waisman [1] on a density approach to topology optimization. To solve design problems with damage constraints by gradient-based optimization method, a sensitivity analysis of the damage process is developed. Damage propagation and growth is an irreversible process. Therefore, the path dependence of the structural response needs to be accounted for in the sensitivity analysis. In this paper, we present an analytical approach for efficiently and accurately evaluating the design sensitivities, considering both direct and adjoint formulations. Finally, the sensitivity analysis approach is studied with simple benchmark problems and compared with the results obtained by finite differences.

**2. Keywords:** damage, sensitivity analysis, XFEM, level set.

## 3. Introduction
This research work focuses on introducing the effect of degradation in the design process of structures. Taking into account the potential degradation of materials at the design stage allows creating structures that can sustain an acceptable amount of degradation and exhibit a low sensitivity to damage. To achieve this goal, a shape optimization approach is chosen. The work aims at developing systematic tools to evaluate the influence of damage on the optimal design of structures. First, a sensitivity analysis of the non-linear damage process is developed and is the key objective of this study. The analysis of the structures is performed exploiting both a level set description of the geometry, which enables dealing with moving boundaries and complex structures, and the extended finite element method (XFEM), which allows working on fixed non-conforming meshes. The degradation of materials is represented using a non-local damage model, which reduces the material stiffness by a scalar damage value $D$. To perform the sensitivity analysis, an analytical approach is proposed. As damage is an irreversible process, the history of the structural response has to be accounted for in the sensitivity computation. Finally, the proposed analytical approach is validated and compared against finite differences for a simple benchmark example: a bimaterial bar in tension. Ongoing work is devoted to apply the approach to academic examples.

## 4. Framework
This research work aims at accounting for the damage behavior of materials in the design process. To this end, damage constraints are introduced in the optimization problem. These constraints include either bounds on the maximum damage value or a global restriction on the percentage of damage tolerated in the structures for a given load level.

To predict the structural response, the XFEM is combined with a level set description of the geometry. The XFEM allows working with non-conforming mesh and avoids costly remeshing operations required in classical shape optimization. The level set description enables an easy handling of moving boundaries and certain topological modifications as geometric entities can merge or disappear. The basics of these methods, necessary for further developments, are outlined in the following sections.

4.1. Level set description of the geometry

The level set method was introduced in 1988 by Osher and Sethian [5] to describe propagating fronts. In a $n$ dimensional space, the method represents the boundaries implicitly resorting to a level set function $\phi$ of dimension $n+1$. An iso-level of this function $\phi$, generally the iso-zero level, is selected to represent the boundaries and so the distribution of the materials on a given domain. The level set function $\phi$ can be expressed as a function of the spatial coordinates $\mathbf{x}$ and some design parameters $\mathbf{s}$ as:

$$\begin{cases} \phi(\mathbf{x},\mathbf{s}) > 0, & \forall \mathbf{x} \in \Omega_A \\ \phi(\mathbf{x},\mathbf{s}) = 0, & \forall \mathbf{x} \in \Gamma_{AB} \\ \phi(\mathbf{x},\mathbf{s}) < 0, & \forall \mathbf{x} \in \Omega_B \end{cases}, \tag{1}$$

where $\Omega_A$ is the domain filled with material A, $\Omega_B$ the domain filled with material B, and $\Gamma_{AB}$ the interface between materials A and B.

Working on a discrete mesh, the level set function $\phi$ is typically represented through its nodal values $\phi_i$ that are interpolated using classical finite element shape functions $N_i(\mathbf{x})$ to capture the singular behavior:

$$\phi^h(\mathbf{x},\mathbf{s}) = \sum_i N_i(\mathbf{x})\ \phi_i. \tag{2}$$

4.2. The extended finite element method

The XFEM offers a convenient way to represent discontinuities and singularities within the elements by adding particular shape functions to the approximation field:

$$u^h = \sum_{i \in I} N_i(\mathbf{x})\ u_i + \sum_{i \in I^\star} N_i^\star(\mathbf{x})\ a_i, \tag{3}$$

where $I$ is the set of all the mesh nodes, $N_i(\mathbf{x})$ the classical finite element shape functions, $u_i$ the degrees of freedom related to $N_i(\mathbf{x})$, $I^\star$ the set of enriched nodes, $N_i^\star(\mathbf{x})$ the enriched shape functions, $a_i$ the additional degrees of freedom related to $N_i^\star(\mathbf{x})$.

There exists different types of enrichment functions $\psi$ depending on the type of discontinuity across material interfaces. This work focuses on materials exhibiting different properties. This kind of interface are characterized by a continuous displacement field, but a discontinuous strain field. A commonly used enrichment function for material interface is the ridge function proposed by Moës [3]:

$$\psi(\mathbf{x}) = \sum_i N_i(\mathbf{x})\,|\phi_i| - \left|\sum_i N_i(\mathbf{x})\phi_i\right|. \tag{4}$$

The enriched shape functions are obtained multiplying the enrichment function $\psi$ and the classical finite element shape functions $N_i(\mathbf{x})$:

$$N_i^\star(\mathbf{x}) = N_i(\mathbf{x})\ \psi(\mathbf{x}). \tag{5}$$

## 5. Non-linear damage analysis

The degradation of materials is accounted for using a non-local damage model, motivated by the work of James and Waisman [1], who implemented a non-local approach to perform topology optimization. The non-local damage model and the computational scheme for predicting the structural response are summarized in the following subsections.

5.1. Non-local damage model

The damage is represented by a scalar value $D$. The degradation of the structure is described by the evolution of this scalar value from $D=0$, corresponding to the undamaged state of the material to $D=1$, where the material is fully degraded and unable to sustain any higher load. In a finite element model, the damage parameter $D$ is evaluated at each Gauss point. The evolution law governing the growth of material degradation is given as a function of the displacements:

$$\mathbf{D} = g(\mathbf{u}). \tag{6}$$

Typically, damage laws provide the evolution of damage as a function of the stresses or the strains. Degradation is generally initiated at a prescribed stress or strain threshold leading to an abrupt change in the damage values and a non-smooth dependence in stresses or strains. To avoid these problems degrading the convergence, the damage law is smoothed using some regularization function $f^S$:

$$\mathbf{D}^S = f^S(\mathbf{D}). \tag{7}$$

The damage is considered non-local to avoid the localisation of damage in a thin strip of mesh elements. The damage value at each Gauss point is then influenced by the damage values at neighboring Gauss points, i.e. Gauss point located within a given distance $l_c$ to the considered Gauss point. The value of the damage at a given Gauss point is then computed as:

$$D_i^{NL} = \frac{\sum_{j \in N} W(d_{ij}) D_j^S}{\sum_{i \in N} W(d_{ij})}, \tag{8}$$

where the subscript $_i$ accounts for the treated Gauss point, $N$ is the set of Gauss points in the neighborhood of the treated one, $W$ a given weighting function depending on the distance $d_{ij}$ between the treated Gauss point $i$ and its neighbor $j$.

5.2. Analysis of the damage process

The damage process is non-linear and exhibits a limit point in its force-displacement curve, as shown in Figure 1. The evaluation of the structural response of the system is conducted using a path-following procedure. This form of structural response can not be analyzed by classical solvers as Newton-Raphson, unless working with low damage values. As a first step, a Newton-Raphson solver is used. Later, a displacement control or a Riks-Crisfield solver will be used to follow the force-displacement curve further than the limit point.

The procedure followed by the iterative solver to evaluate the structural response of the system is illustrated in Figure 1. The problem is solved simultaneously for the displacement variables $\mathbf{u}$ and the damage variables $\mathbf{D}$, collected in the vector $\mathbf{y}$. The residuals are also collected in a vector $\mathbf{R}$ given as:

$$\mathbf{R}^{(k)} = \begin{bmatrix} \mathbf{R}_u^{(k)} \\ \mathbf{R}_D^{(k)} \end{bmatrix} = \begin{bmatrix} \mathbf{K}^{(k)}\mathbf{u}^{(k)} - \mathbf{f}^{(k)} \\ \mathbf{D}^{(k)} - g\left(\mathbf{u}^{(\star)}\right) \end{bmatrix}, \tag{9}$$

where $\mathbf{K}^{(k)}$ is the stiffness matrix, $\mathbf{u}^{(k)}$ the displacements variables, $\mathbf{f}^{(k)}$ the external forces, $\mathbf{D}^{(k)}$ the damage variables at iteration k and $\mathbf{u}^{(\star)}$ the displacement variables at an iteration $\star$ where the maximum damage was reached so far.

As showed in (9), the residuals can depend on several or all the prior iterations and can be expressed in the most general way as:

$$\mathbf{R}^{(k)} = \mathbf{R}^{(k)}\left(\mathbf{s}, \mathbf{y}^{(k)}, \mathbf{y}^{(k-1)}, \ldots, \mathbf{y}^{(1)}\right) \tag{10}$$

**6. Sensitivity analysis**

To solve optimization problems with damage constraints, a sensitivity analysis has to be developed. An important feature of the degradation process is that the damage propagation is irreversible. Therefore, the path dependence of the structural response has to be taken in account in the sensitivity analysis. An analytical approach to the sensitivity analysis, based on the work by Michaleris et al. [2], is proposed and explained in the following sections.

6.1. Derivative of an objective/constraint function

Let us consider an objective or constraint function $\mathscr{F}$ given as follow:

$$\mathscr{F} = \mathscr{F}\left(\mathbf{s},\ \mathbf{y}^{(k)}\right) \tag{11}$$

where $\mathbf{s}$ are the design parameters and $\mathbf{y}^{(k)}$ are the discrete state variables at iteration k, i.e. the displacements $\mathbf{u}$ and the damage $\mathbf{D}$.

The derivative of this function with respect to a particular design parameter $s_i$ can be expressed as:

$$\frac{d\mathscr{F}}{ds_i} = \frac{\partial \mathscr{F}}{\partial s_i} + \frac{\partial \mathscr{F}}{\partial \mathbf{y}^{(k)}} \frac{d\mathbf{y}^{(k)}}{ds_i} \tag{12}$$

where $\frac{d}{ds_i}$ and $\frac{\partial}{\partial s_i}$ are the total and the partial derivative with respect to the design parameter $s_i$.

To evaluate this expression, the total derivative of the state variables $\mathbf{y}^{(k)}$ with respect to the design variable $s_i$ is computed through the derivatives of the residuals $\mathbf{R}^{(k)}$. Taking the derivative of (10) with respect to a particular design parameter $s_i$, one gets:

$$\frac{d\mathbf{R}^{(k)}}{ds_i} = \frac{\partial \mathbf{R}^{(k)}}{\partial s_i} + \frac{\partial \mathbf{R}^{(k)}}{\partial \mathbf{y}^{(k)}} \frac{d\mathbf{y}^{(k)}}{ds_i} + \frac{\partial \mathbf{R}^{(k)}}{\partial \mathbf{y}^{(k-1)}} \frac{d\mathbf{y}^{(k-1)}}{ds_i} + \ldots + \frac{\partial \mathbf{R}^{(k)}}{\partial \mathbf{y}^{(1)}} \frac{d\mathbf{y}^{(1)}}{ds_i} \tag{13}$$

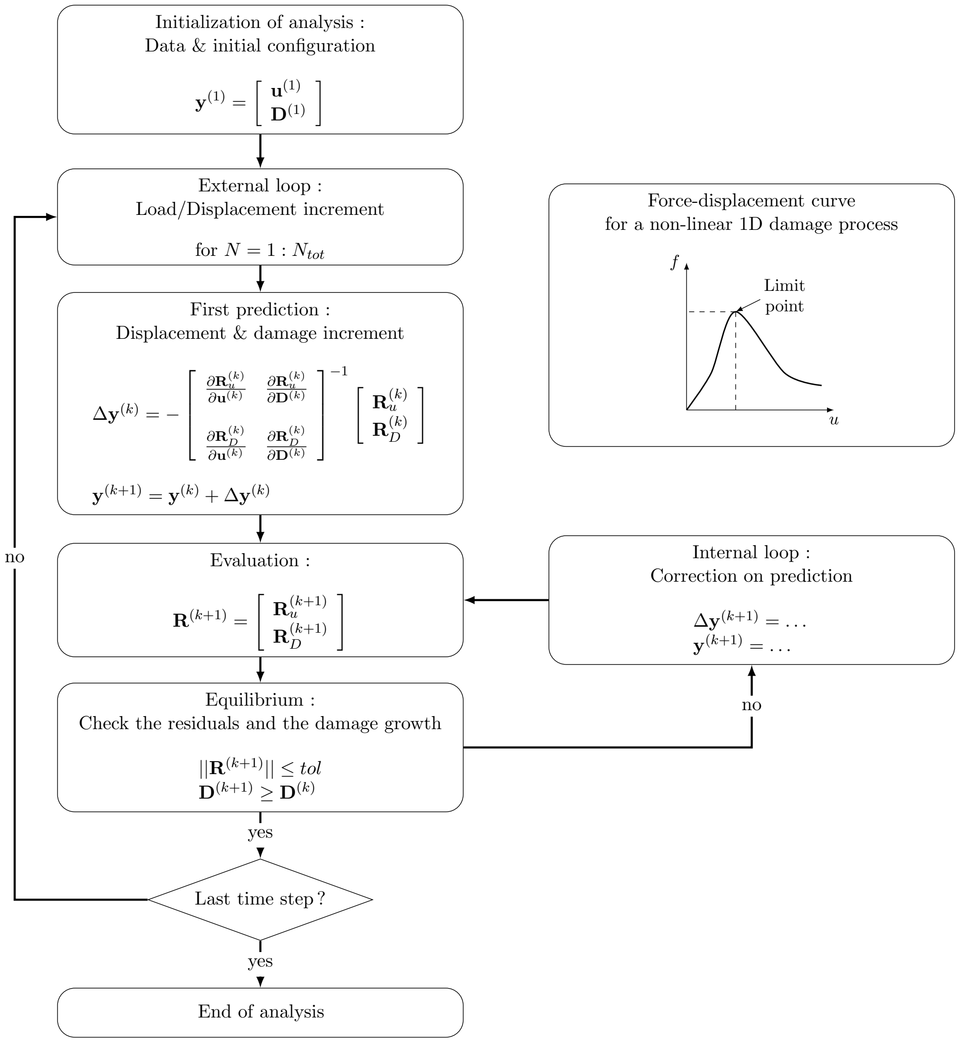



Figure 1: Procedure of the non-linear damage analysis.

Starting from (13) evaluated at the first iteration and proceeding to the last iteration, the derivatives of the problem variables $\mathbf{y}$ can be successively evaluated at each iteration provided that the derivatives of the residuals $\frac{\partial \mathbf{R}^{(k)}}{\partial s_i}$ and $\frac{\partial \mathbf{R}^{(k)}}{\partial \mathbf{y}^{(l)}}$, $l = 1, \dots, k$ are known.

The derivative of the residuals $\mathbf{R}^{(k)}$ at iteration k with respect to a particular design parameter $s_i$ can be expressed as:

$$\frac{\partial \mathbf{R}^{(k)}}{\partial s_i} = \begin{bmatrix} \frac{\partial \mathbf{R}_u^{(k)}}{\partial s_i} \\ \\ \frac{\partial \mathbf{R}_D^{(k)}}{\partial s_i} \end{bmatrix}, \text{ where } \quad \begin{aligned} \frac{\partial \mathbf{R}_u^{(k)}}{\partial s_i} &= \frac{\partial \mathbf{K}^{(k)}}{\partial s_i}\mathbf{u}^{(k)} - \frac{\partial \mathbf{f}^{(k)}}{\partial s_i}, \\ \frac{\partial \mathbf{R}_D^{(k)}}{\partial s_i} &= -\frac{\partial g\left(\mathbf{u}^{(\star)}\right)}{\partial s_i}. \end{aligned} \tag{14}$$

The derivative of the residuals $\mathbf{R}^{(k)}$ at iteration $k$ with respect to the state variable $\mathbf{y}^{(l)}$ at any iteration $l = 1, \dots, k$ can be expressed as:

$$\frac{\partial \mathbf{R}^{(k)}}{\partial \mathbf{y}^{(l)}} = \begin{bmatrix} \frac{\partial \mathbf{R}_u^{(k)}}{\partial \mathbf{u}^{(l)}} & \frac{\partial \mathbf{R}_u^{(k)}}{\partial \mathbf{D}^{(l)}} \\ \\ \frac{\partial \mathbf{R}_D^{(k)}}{\partial \mathbf{u}^{(l)}} & \frac{\partial \mathbf{R}_D^{(k)}}{\partial \mathbf{D}^{(l)}} \end{bmatrix}, \text{ where } \quad \begin{aligned} &\frac{\partial \mathbf{R}_u^{(k)}}{\partial \mathbf{u}^{(l)}} = \begin{cases} \mathbf{K}^{(k)} & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}, && \frac{\partial \mathbf{R}_u^{(k)}}{\partial \mathbf{D}^{(l)}} = \frac{\partial \mathbf{K}^{(k)}}{\partial \mathbf{D}^{(l)}}\mathbf{u}^{(k)}, \\ &\frac{\partial \mathbf{R}_D^{(k)}}{\partial \mathbf{u}^{(l)}} = -\frac{\partial g\left(\mathbf{u}^{(\star)}\right)}{\partial \mathbf{u}^{(l)}}, && \frac{\partial \mathbf{R}_D^{(k)}}{\partial \mathbf{D}^{(l)}} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}. \end{aligned} \tag{15}$$

All the derivatives are evaluated analytically starting from the discretized governing equations and taking their derivatives with respect to the design parameters **s**. The procedure to compute the analytical derivatives, within an XFEM-level set framework for shape optimization of bimaterial structures, is detailed in [4].

## 7. Application

The sensitivity analysis method described above is illustrated and validated with a simple benchmark example: a bimaterial bar in tension. The bar is loaded with a force $F$ that increases monotonously at each iteration of the path-following procedure. The setting of the problem is illustrated in Figure 2, where a single mesh element is used to model the bar. The location of the interface, given by $s$, is used as design variable . All the parameters of the problem are summarized in Table 1. The damage law used to evaluate the propagation of the degradation is given as:

$$D_{gp} = 1 - \exp\left(1 - \frac{\varepsilon_{gp}}{\varepsilon_{th}}\right), \tag{16}$$

where $\varepsilon_{gp}$ is the strain at the considered Gauss point and $\varepsilon_{th}$ the strain threshold from which the material degradation is initiated. The damage law is then smoothed using a Kreisselmeier-Steinhauser function:

$$D^S_{gp} = \frac{1}{\eta_S} \ln\left(1 + \exp\left(\eta_S\, D_{gp}\right)\right), \tag{17}$$

where $\eta_S$ is a smoothing parameter.

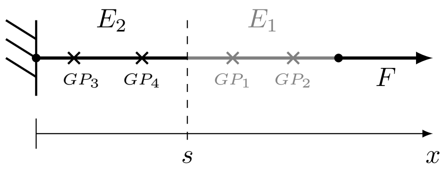


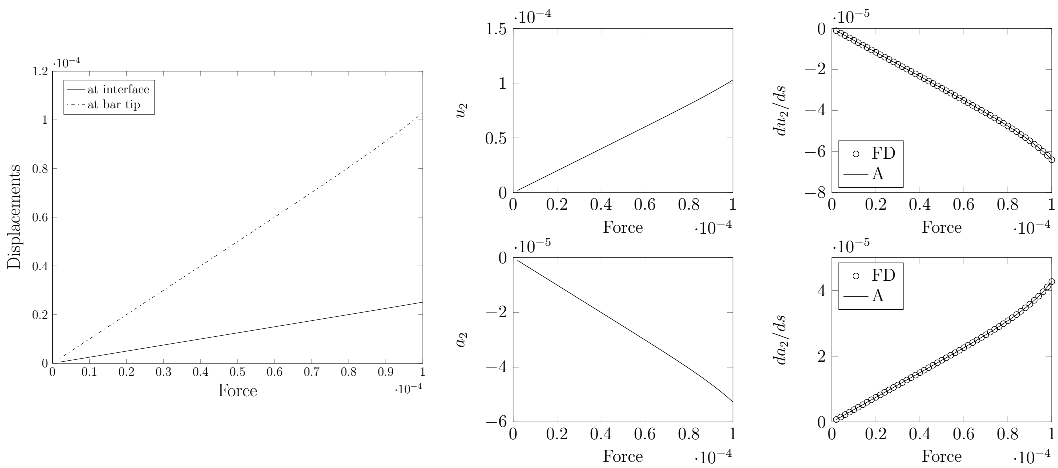

Figure 2: First benchmark - bar in tension.

Table 1: Bimaterial bar in tension - parameters

| | | |
|---|---|---|
| Dimensions | $[m]$ | $L = 2$ |
| Elastic moduli | $[N/m^2]$ | $E_1 = 1,\ E_2 = 5$ |
| Load | $[N]$ | $F = 10^{-4}$ |
| Level set function | | $\phi(x,s) = x - s$ |
| Gauss points per subelement | | $n_{gp} = 2$ |
| Strain threshold | | $\varepsilon_{th} = 10^{-4}$ |
| Smoothing parameter | | $\eta_S = 7$ |

The structural response of the bar is computed by a Newton-Raphson solver, as damage is kept small. The design sensitivities are computed by the analytical approach (A) and validated against finite differences (FD).

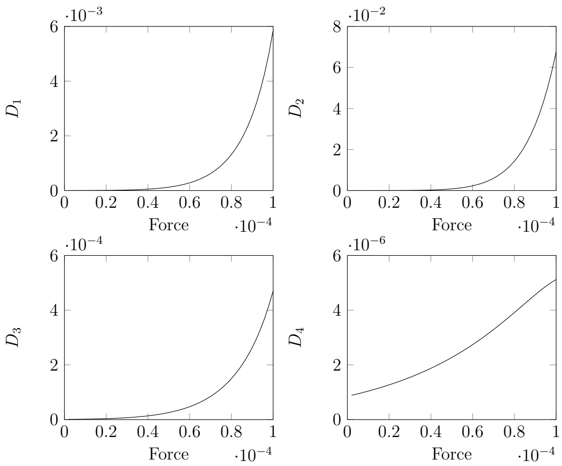
Figure 3 depicts the evolution of the displacements and its derivatives. Figure 3(a) shows the displacements at the interface and at the bar tip. Figures 3(b) gives the evolution of the free degrees of freedom of the structure $u_2$ and $a_2$. Figure 3(c) shows the evolution of the derivatives of these degrees of freedom with respect to the design parameter $s$. As can be seen, both sensitivity analysis approaches are in good agreement.



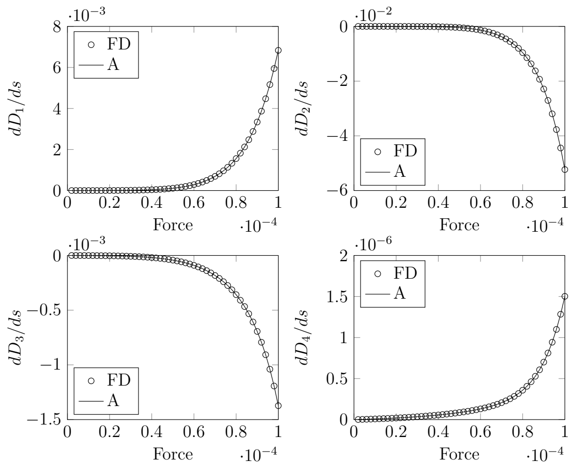

(a) Displacements of the bar (b) Degrees of freedom $u_2$ and $a_2$ (c) Displacements derivatives

Figure 3: Structural response and sensitivity analysis of the displacement variables **u**.

Figure 4 presents the results of the damage variables. Figures 4(a) shows the evolution of the damage parameter at each Gauss point of the structure; Figure 4(b) shows the evolution of the derivatives of these damage parameters with respect to the design parameter $s$. Once again, the sensitivity results are in excellent agreement.




(a) Damage




(b) Damage derivatives

Figure 4: Structural response an sensitivity analysis of the damage variables $\mathbf{D}$.

## 8. Conclusion

An analytical approach to the sensitivity analysis of damaged structures for shape optimization has been developed. The approach combines the extended finite element method and a level set description of the geometry. The level set function is given as a function of some design parameters. The degradation of materials is described by a non-local damage model, which reduces the material stiffness by scalar damage value $D$. The sensitivity analysis is performed analytically, starting from the discretized governing equations and taking their derivatives with respect to the design parameters. The proposed analytical approach is validated and compared against finite differences for a simple benchmark: a bimaterial bar in tension. The sensitivity results obtained with the proposed analytical approach are in good agreement with finite differences results. In further work, the sensitivity approach is extended to two dimensional structures and optimization problems including damage constraints are solved.

## 9. Acknowledgements

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