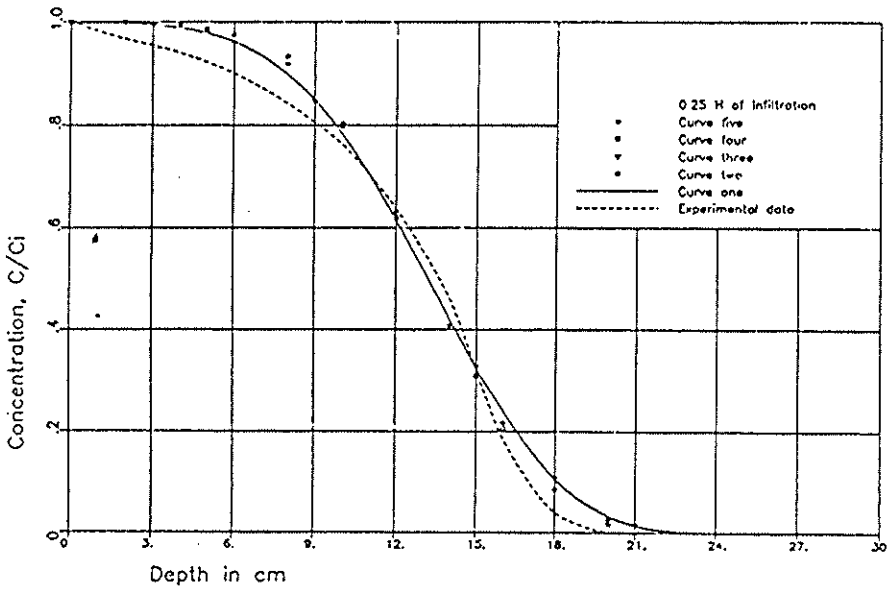

FIGURE 5.- Solute profiles. Experimental and simulated data.

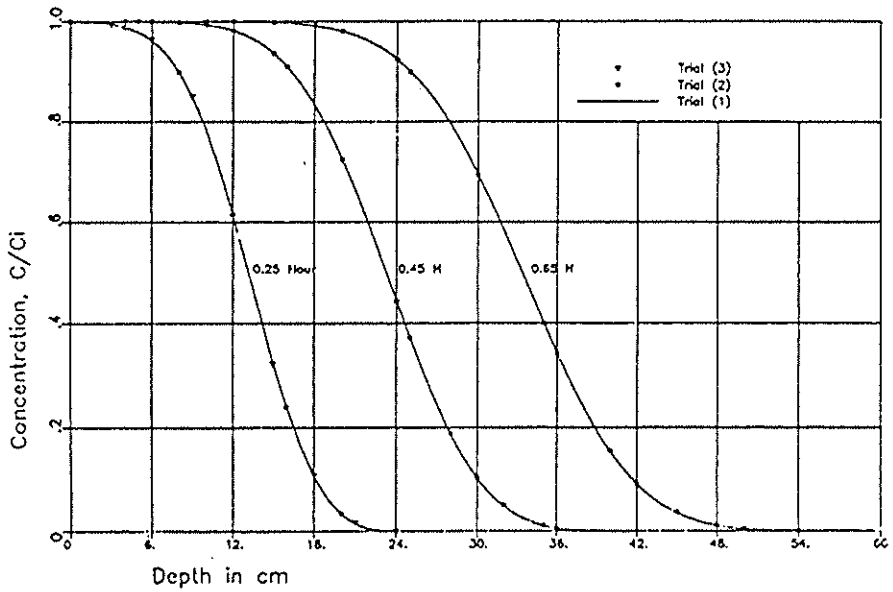

FIGURE 6.- Solute profiles. Different parameter sets.

Numerical Simulation of Transient Unconfined Seepage Problems

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ABSTRACT

The paper deals with finite elements modeling of transient unconfined aquifers. First, one presents the main characteristics of the large water tables collected in order to supply the drinking water of great towns. In the second section, the transient flow constitutive laws are recalled. In a confined aquifer, the Darcy law is valid. In an unconfined aquifer, the storage varies with the pore saturation. A new law is proposed to model this phenomenon. Following, the local flow equilibrium equation is set up. The virtual power principle expresses the same idea at a global scale. It is the basis of the finite elements scheme. The time integration and the iteration technique are discussed. Finally, two applications are presented.

INTRODUCTION

The City of Liège, as many other big towns, gets a stock of drinking water by pumping out of an unconfined aquifer. The present paper deals with a new numerical code developed and used to simulate the evolution of such an aquifer. The most important features of the numerical code are the following :

- The fluctuations of a water table are transient, this phenomenon is linked to the storage capacity in soil pores. The rain fall supply of the aquifer, the water outlet at rivers and the water collecting are also time dependent.

- The geometry of the geological layers where the aquifer is lying is really complex. The thickness varies from point to point. In some areas, layers disappear. Due to the water table fluctuations, the upper layers are sometimes fully or partially saturated, sometimes unsaturated.

- The permeabilities and the storage coefficient are varying in each layer. The permeabilities can be orthotropic.

- Imposed potential, imposed pressure and impermeable limits exist. They are sinuous.

- Faults and dry valleys modify significantly the water flow.

- Water collecting is done through wells or collecting galeries.

The finite elements method seems to be the best one for the modeling of such problems. Especially, isoparametric finite elements can follow easily geometrically complex layers or boundaries. Soil flow elements, gallery elements and rainfall elements can be combined. Therefore, the finite elements method is a very flexible one. All developments presented hereafter have been implemented in the finite element code LAGAMINE which has been developed since 6 years in the M.S.M. department, University of Liège (Charlier[1]).

The computation of the water table position is the most difficult problem to be solved. A large number of authors have developed finite element codes searching the free surface and adapting the mesh by geometrical analysis. See for example Taylor and Brown[11], Desai[7,8], Cividini and Gioda[12] ... Bruch[3] uses the boundary elements method and a variable mesh. Piette and Cescotto[5,6] use an invariable mesh, but eliminate the integration points lying above the water table. All these analyses are based on a particular free surface shape. The free surface motion is simple and generally vertical or subvertical. Therefore, an important part of the finite element code is dedicated to geometrical computations. The hereafter presented method does not depend on the free surface shape. It is based on a modification of the constitutive relation, taking into account the degree of pore saturation.

Transient flow and three-dimensional unconfined problems have been studied only by a small number of authors.(e.g. Bruch[3], Neuman[7]). The proposed method is easily extended to 3D problems.

THE FLOW CONSTITUTIVE LAW

The constitutive relation of the fluid flow in porous media is the well known Darcy law. It links the fluid velocity $\underline{f}$ to the pore pressure gradient. We note here :

p the pressure
$\underline{\underline{k}}$ the permeability tensor (anisotropic media)
γ the fluid specific weight
$\underline{e}_z$ the gravity direction vector (normed).

In transient state, the fluid storage H in the soil pores varies with respect to the pore pressure and to the saturation degree. Therefore, a second constitutive relation, named here the storage law, must be used.

Confined aquifer

In a confined aquifer, the Darcy flow law has the following form :

$$\underline{f} = \frac{\underline{\underline{k}}}{\gamma} \underline{\text{grad}}\ p - \underline{k}\ \underline{e}_z \qquad (1)$$

The storage law expresses the modification of the pore volume when the pore pressure varies. It is linked to the elastic or elastoplastic soil skeleton strains.

The fluid volume variation is $\dot{H}$:

$$\dot{H} = C\ \dot{p} \qquad (2)$$

C is the confined storage coefficient. It is a function of the porosity and the skeleton mechanical characteristics. $\underline{\underline{k}}$ and C are generally supposed to be constant. Then the problem is linear and computations are not too expensive. But variation of the flow coefficient with respect to the pore pressure can easily be taken in account (Charlier[1]).

Unconfined aquifer

An unconfined aquifer has a water table which is moving in transient state. This motion is governed by a special storage law. Most authors are linking it to the fluid velocity $\underline{f}$ and the porosity n. If $\underline{v}$ is the normal to the table, one obtains the following surface equilibrium relation :

$$\underline{v}^T \underline{v} = n\ \underline{f}^T \underline{v} \qquad (3)$$

This is a geometrical evolution law. It does not describe completely the table displacement vector, but only one of its components. The other components are generally determined on the hypothesis of an only vertical table displacement. The implementation of this hypothesis is a main limitation of this technique.

We have developed a new method which is not based on boundary geometrical considerations but on a volumic storage equation. When the soil is saturated, the pore pressure is positive (if the atmospheric pressure is taken as zero of the pressure scale), and the storage constitutive law is governed by equation (2) (figure 1). At the opposite, when the soil is unsaturated, the pore pressure is null, and the fluid contents H is comprised between zero and the effective porosity n. Negative

$$p > 0 \rightarrow H = n + C\,p$$

$$p = 0 \rightarrow 0 \leq H \leq n \qquad (4)$$

$$p < 0 \rightarrow H = 0 \text{ (unphysical)}$$

This relation (4) is equivalent to the equation (3): for each one, the free surface motion is governed by the saturation-unsaturation of the soil pores.

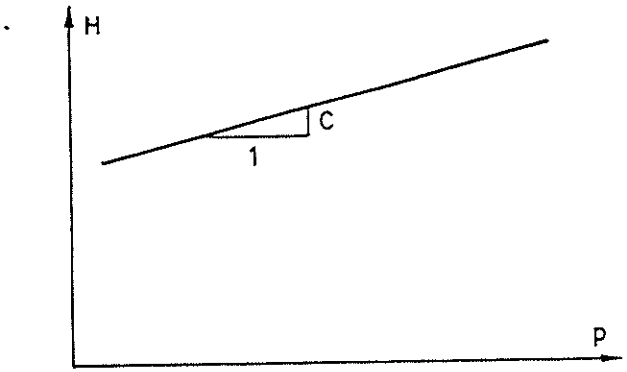

Figure 1. Storage law for a confined aquifer.

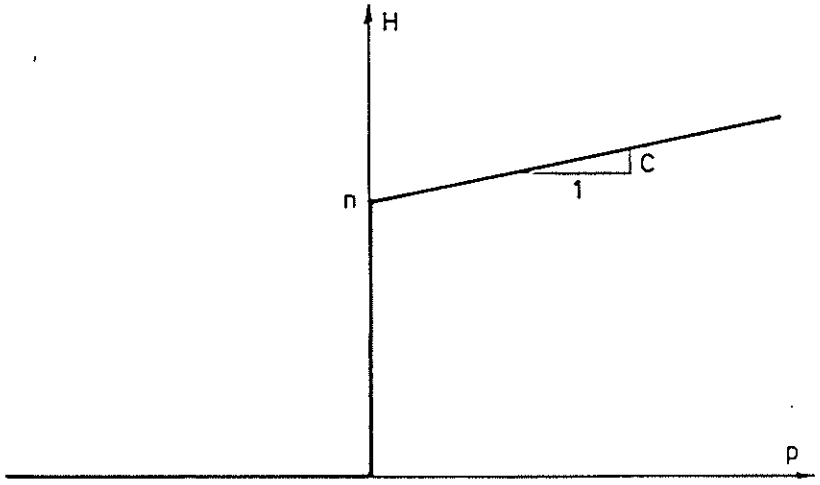

Figure 2. Storage law for an unconfined aquifer.

Unfortunately, the transient seepage modeling is not based on the relation $H = f(p)$ but on the derivative of this relation :

$$\dot{H} = f'(p)\,\dot{p}$$

As it clearly appears on figure 2, there is a discontinuity of f' at $p = 0$, and this is difficult to implement in a finite element code. Therefore, one can smooth the equation (4) by a polynomial function or an arc-tangent function. We have adopted the second solution in our code LAGAMINE (figure 3) :

$$H = n \left(\frac{1}{\pi} \operatorname{arctg} \left(\frac{p}{\alpha} \right) + \frac{1}{2} \right) + C\langle p \rangle \qquad (5)$$

with $\langle p \rangle = p$ if $p > 0$

$\langle p \rangle = 0$ if $p \leq 0$

The smaller the α coefficient, the better the approximation of equation (4) by equation (5). This method is therefore a kind of penalty method.

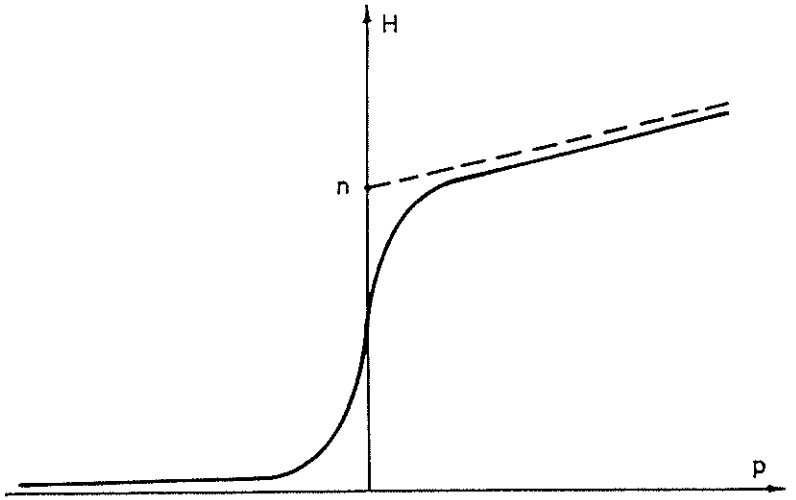

Figure 3. Relaxed storage law for an unconfined aquifer.

It is to be noted that the relaxed storage law implies the disappearance of the completely unsaturated domain. The negative pore pressure domain contains a small amount of water (as small as the penalty is good). How does it flow ? The Darcy law (equation (1)) is valid only for saturated soils. Therefore, we use a slightly modified Darcy law, which takes in account the amount of saturation :

$$\underline{f} = \frac{H}{n} \left(\frac{k}{\gamma} \underline{\text{grad}}\, p - \underline{k}\, \underline{e}_z \right) \qquad (6)$$

This new law (6) is quite equivalent to the Darcy law (3) in the saturated domain. It assumes a slight flow in the unsaturated region due to the gravity term: the isobars have the same interval in the saturated and unsaturated region, and large negative pore pressures can appear. But this is only a numerical effect, because the unsaturated flows are really small with regard to the saturated flow.

It is to be noted that the same kind of transformation from a discontinuous incremental law $\dot{H} = f(\dot{p})$ to a total law $H = F(p)$ has been proposed by Comini, de Guidice, Lewis and Zienkiewicz [2] for heat conduction problem with reference to phase change. In that paper, H is the enthalpy, p is the temperature and the sharp rise of the figure 2 expresses the phase change.

THE FLOW EQUILIBRIUM AND THE VIRTUAL POWER PRINCIPLE.

The development of a numerical code modeling seepage problems is based on two kinds of assumptions :
- constitutive relations, i.e. the Darcy law and the storage law ;
- equilibrium equations.

In this section, we recall the well known seepage equilibrium equations, which state the fluid volume conservation.

When the fluid crosses a boundary of the studied domain (with the exterior normal $\underline{n}$) with a velocity v, the surface equilibrium equation is :

$$\underline{n}^T \underline{f} + \underline{n}^T \underline{v} = 0 \qquad (7)$$

If one studies the volume conservation in a small soil volume, one finds the following contributions :

- the fluid volume crossing through the boundaries is

$$div\, \underline{f} \quad \underline{\text{grad}}\ \underline{f}\ dv$$

- the change of the pore volume is $\dot{H}\ dv$

- the captation or supply of fluid is $\bar{Q}\ dv$

Adding these terms, one obtains the volumic equilibrium equation :

$$div\, \underline{f} \quad \underline{\text{grad}}\ \underline{f} + \dot{H} - \bar{Q} = 0 \qquad (8)$$

The exterior virtual power is the power developed by the imposed fluid velocities ($\bar{Q}$, $\bar{v}$) when one applies a virtual allowable perturbation of the pore pressures δp, i.e. a pressure perturbation with respect to the imposed pore pressure :

The internal virtual power is developed by the internal fluid velocities $\underline{f}$ and the storage $\dot{H}$ under the same pressure perturbation :

$$\delta W_I = \int_v [\dot{H}\, \delta p - \underline{f}^T\, \underline{\text{grad}}\, (\delta p)]\ dv \qquad (10)$$

The global equilibrium of the fluid flow in the whole studied domain is obtained when the internal and external virtual powers are equal for any virtual pressure perturbation :

$$\delta W_E = \delta W_I \qquad (11)$$

THE FINITE ELEMENTS.

Using the virtual power equations (9), (10), (11), one can easily build isoparametric finite elements and implement them in a classical finite elements code.

If L is the node number and N_l are the shape functions, one obtains the coordinates $\underline{x}$ and the pore pressure p inside one element as functions of the nodal coordinates $\underline{X}_l$ and pressure P_l :

$$\underline{x} = \sum_{l=1}^{L} N_l\, \underline{X}_l \qquad (12)$$

$$p = \sum_{l=1}^{L} N_l\, P_l \qquad (13)$$

The internal virtual power in one element is then

$$\delta W_I = \sum_{l=1}^{L} \int_v [\dot{H}\, N_l - \underline{f}^T\, \underline{\text{grad}}\, (N_l)]\ dv\ \delta P_l \qquad (14)$$

In each element, the volumic integration is performed numerically using the Gaussian scheme. One obtains thus "equivalent nodal flows" F_l, similar to nodal forces in a structural finite elements code :

$$F_l = \sum_{IP} [\dot{H}\, N_l - \underline{f}^T\, \underline{\text{grad}}\, (N_l)]\ w_{IP} \qquad (15)$$

$$\delta W_I = \sum_{l=1}^{L} F_l\, \delta P_l \qquad (16)$$

w_{IP} are the Gaussian weights at the integration points (IP).

In our code LAGAMINE, we have developed the following elements (figure 4) :

- plane axisymmetric parabolic elements : 8-nodes squares and 6-nodes triangles ;
- 3D elements with variable node numbers on each edge, from 1 to 3 : linear, parabolic and cubic polynomials can be used.
- 3D lines, i.e. lines with a little cross section and a large permeability (to model collecting galeries for example).
- alimentation 2D and 3D elements (compatible with the other elements).

We will develop in a near future a 3D thin fault element.

TIME STEPPING AND TIME INTEGRATION.

For each numerical technique, one tries to perform as large time step as possible. During each time step, one must compute the nodal flows F_l with accuracy. In a large number of finite elements codes, the nodal flows F_l are only computed at a one moment θ of the time step ($0 \leq \theta \leq 1$). If $\theta \geq 0,5$, the time integration scheme is unconditionally stable. When large variations of the fluid storage are arising, this computation scheme does not achieve enough accuracy. The storage A of the local volumic equilibrium equation (8) becomes very large at some points, during some time steps. Therefore the computation of the nodal flows F_l at one preference moment does not represent the real behaviour. A better solution is the computation of average or weighted nodal flows F_l. If ξ is an undimensional time measure and $W(\xi)$ a weighting function, the average nodal flows are

$$\bar{F}_l = \left(\int_0^1 W(\xi)\, F_l\, d\xi\right) / \left(\int_0^1 W(\xi)\, d\xi\right) \qquad (17)$$

with $0 \leq \xi \leq 1$ during the time step.

If $W(\xi) = 1$, the numerical scheme looks like the mean point scheme ($\theta = 1/2$). If $W(\xi) = \xi$, it looks like the Galerkin scheme ($\theta = 2/3$).

THE ITERATIONS.

Nodal pore pressures p_l at the end of a time step are accurate when the internal weighted nodal flows F_l and the weighted nodal flows Q_l imposed on the boundaries are equal or quite equal (with respect to some imposed precision). Otherwise, one must iterate in order to improve the pore pressures. The best iteration technique is the well-known Newton-Raphson one. It is based on the tangent iteration matrix $\underline{K}$ (the stiffness

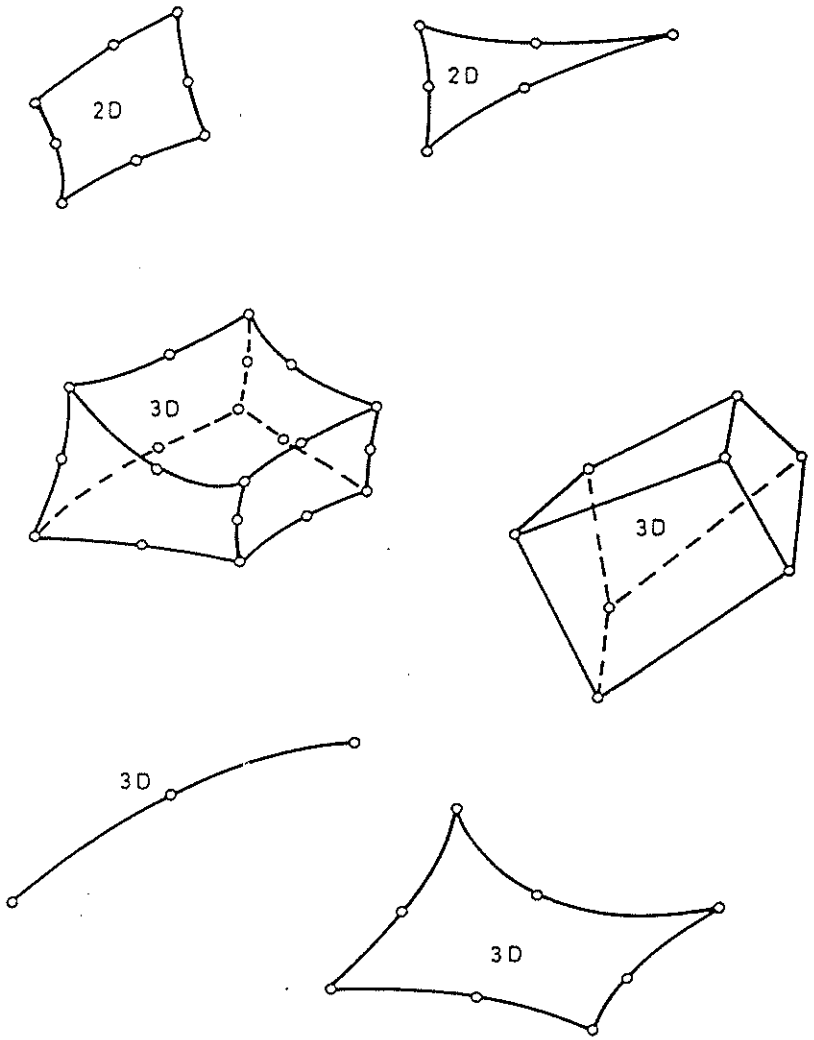

Figure 4 - Finite elements library.

matrix in structural mechanics), obtained by derivation of the weighted nodal flows which respect to the nodal pore pressures

$$K_{ij} = \frac{\partial \bar{F}_i}{\partial p_j} \qquad (18)$$

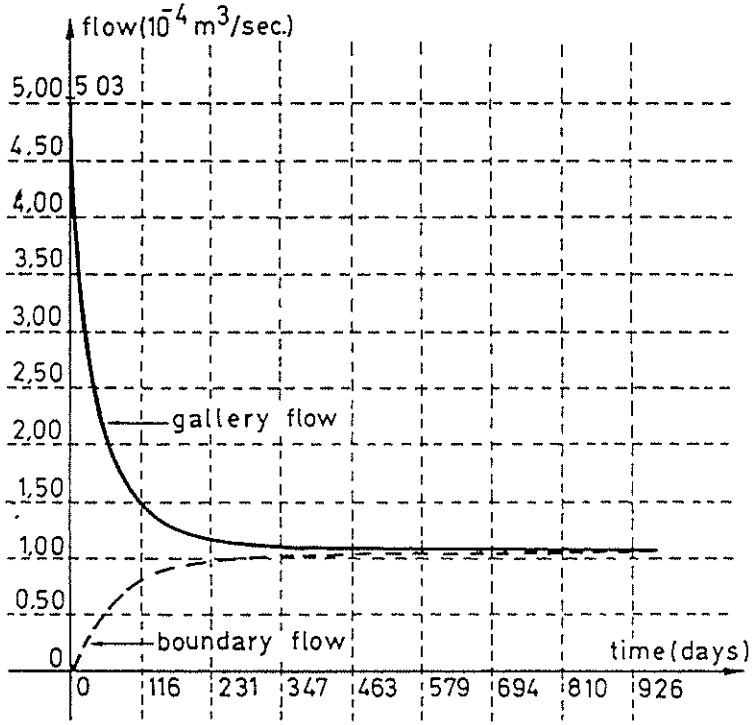

Figure 8. Input and collected flows.

2. Comini G., de Guidice S., Lewis R.W. and Zienkiewicz O.C. (1974). Finite element solution of non-linear heat conduction problems with special reference to phase change. Int. J. Num. Meth. in Eng., Vol. 8, pp. 613 - 624.

3. Bruch E. (1985). Résolution par éléments frontières des écoulements permanents en milieu poreux à surface libre éventuellement indéterminée. Travail de fin d'études. F.S.Appl., Université de Liège.

4. Ghaboussi J. and Wilson E.L. (1973). Flow of compressible fluid in porous elastic media. Int. J. Num. Meth. Eng., Vol. 5, pp. 419 - 442.

5. Piette Ch. (1976). Application des éléments finis à la détermination de la surface piézométrique d'une nappe d'eau souterraine. Travail de fin d'études, F.S.Appl., Université de Liège.

6. Piette Ch. and Cescotto S. (1983). Application des éléments finis à la détermination de la surface piézométrique d'une nappe d'eau souterraine. Actes de la journée d'étu-

7. Neuman S.P. (1973). Saturated - unsaturated seepage by finite elements. ASCE, Hydraulics Div., Vol. 99, HY12, pp. 2233 - 2250.

8. Desai C.S. (1976). Finite element residual scheme for unconfined flow. Int. J; Num. Meth. Eng., Vol. 10, pp. 1415 - 1418.

9. Desai C.S. and Li G.C. (1983). A residual flow procedure and application for free surface flow in porous media. Int. J. Advances in Water Resources. Vol. 6, pp. 27 - 35.

10. Bathe K.J. and Khoshgoftaar M.R. (1979). Finite element free surface seepage analysis without mesh iteration. Int. J. Num. Anal. Meth. Geomech., Vol. 3, pp. 13 - 22.

11. Taylor R.L. and Brown C.B. (1967). Darcy flow solution with a free surface. ASCE, Hydraulics Div., Vol. 93, HY2, pp. 25 - 33.

12. Gioda G. and Cividini A. (1986). An analysis of unconfined steady state seepage. Eur. Conf. ECONMIG'86, Stuttgart.

13. Radu J.P., Charlier R. and Cescotto S. (1987). Simulation numérique des écoulements transitoires à surface libre en milieu poreux. Actes du 1er Congrès National Belge de Mécanique Théorique et Appliquée, Bruxelles.

14. Bruch E. and Grilli S. (1987). Computation of the transient seepage problem in anisotropic porous media by the boundary element method. 9th Int. Conf. on BEM, Stuttgart.

15. Charlier R., Radu J.P., Dassargues A. and Monjoie A.(1987) Modélisation de la nappe aquifère de Hesbaye. Rapport MSM-LGIH n° HESB/871, Université de Liège.

16. Dassargues A., Radu J.P. and Charlier R. (1988). Finite elements modelling of a large water table aquifer in transient conditions. Proc. of the 1st Int. Conf. in Africa on Comp. Meth. and Water Resources, Rabat. Morocco.