Algebras generalizing the Octonions

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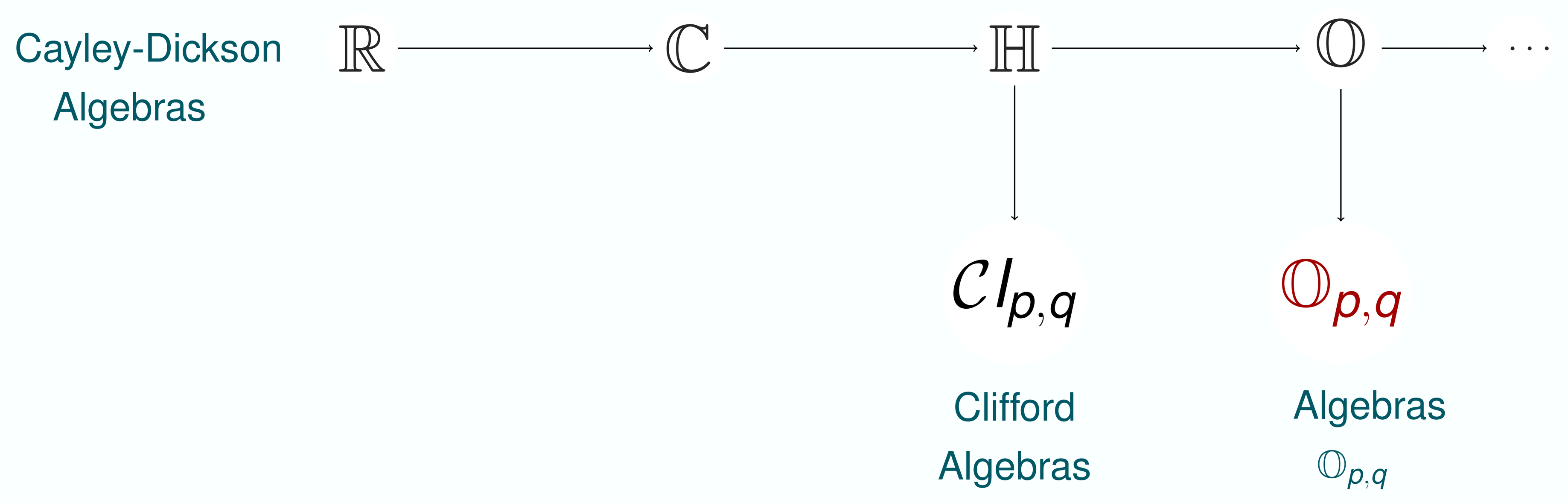
Poster presented at the Topical Workshop "Integrability and Cluster Algebras: Geometry and Combinatorics" at ICERM, US, 25 – 29 August, 2014

Overview



Applications to the classical Hurwitz problem of "Sums of Squares".

Main Definition

The *algebra* $\mathbb{O}_{p,q}$ ($p+q=n\geq 3$) is the 2^n-dimensional vector space on $\mathbb{R}$ with the basis $\{u_x : x\in\mathbb{Z}_2^n\}$, and equipped with the product

$$u_x\cdot u_y=(-1)^{f_{\mathbb{O}_{p,q}}(x,y)}u_{x+y}$$

for all $x,y\in\mathbb{Z}_2^n$, where

$$f_{\mathbb{O}_{p,q}}(x,y)=\sum_{1\leq i<j<k\leq n}(x_ix_jy_k+x_iy_jx_k+y_ix_jx_k)+\sum_{1\leq i\leq j\leq n}x_iy_j+\sum_{1\leq i\leq p}x_i.$$

In particular, the algebra $\mathbb{O}_{p,q}$ is a twisted group algebra, noted $(\mathbb{R}[\mathbb{Z}_2^n],f_{\mathbb{O}_{p,q}})$.

Generating Cubic Form

The defect of commutativity and associativity of a twisted algebra $(\mathbb{R}[\mathbb{Z}_2^n],f)$ is measured by a symmetric function $\beta:\mathbb{Z}_2^n\times\mathbb{Z}_2^n\to\mathbb{Z}_2$, and a function $\phi:\mathbb{Z}_2^n\times\mathbb{Z}_2^n\times\mathbb{Z}_2^n\to\mathbb{Z}_2$, respectively

$$u_x\cdot u_y=(-1)^{\beta(x,y)}\,u_y\cdot u_x,$$

$$u_x\cdot(u_y\cdot u_z)=(-1)^{\phi(x,y,z)}\,(u_x\cdot u_y)\cdot u_z.$$

A function $\alpha:\mathbb{Z}_2^n\longrightarrow\mathbb{Z}_2$ is called a *generating function* for $(\mathbb{R}[\mathbb{Z}_2^n],f)$, if for x, y and z in $\mathbb{Z}_2^n$,

(*i*) $f(x,x)=\alpha(x)$,

(*ii*) $\beta(x,y)=\alpha(x+y)+\alpha(x)+\alpha(y)$,

(*iii*) $\phi(x,y,z)=\alpha(x+y+z)+\alpha(x+y)+\alpha(x+z)+\alpha(y+z)+\alpha(x)+\alpha(y)+\alpha(z)$.

Theorem [M-G & O] :

- A twisted algebra $(\mathbb{R}[\mathbb{Z}_2^n],f)$ has a generating function iff ϕ is symmetric.
- The generating function α is a polynomial on $\mathbb{Z}_2^n$ of degree ≤ 3.
- Given any polynomial α on $\mathbb{Z}_2^n$ degree ≤ 3, there exists a unique (up to isomorphism) twisted group algebra $(\mathbb{R}[\mathbb{Z}_2^n],f)$ having α as generating function.

Algebras $\mathbb{O}_{p,q}$ and Clifford Algebras

The algebras $\mathbb{O}_{p,q}$ (as well as the Clifford algebras) have generating cubic forms (res. quadratic forms). They are given by

$$\alpha_{p,q}(x)=f_{\mathbb{O}_{p,q}}(x,x)=\alpha_n(x)+\sum_{1\leq i\leq p}x_i,$$

where

$$\alpha_n(x)=\sum_{1\leq i<j<k\leq n}x_ix_jx_k+\sum_{1\leq i<j\leq n}x_ix_j+\sum_{1\leq i\leq n}x_i,$$

for all $x=(x_1,\ldots,x_n)\in\mathbb{Z}_2^n$.

The *Hamming weight* of x is $|x|:=\#\{x_i\neq 0\}$, and one has

$$\alpha_n(x)=\begin{cases}0, & \text{if } |x|\equiv 0 \text{ mod } 4,\\ 1, & \text{otherwise.}\end{cases}$$

For $q\neq 0$, $\mathbb{O}_{p,q}$ contains a Clifford subalgebra $\mathcal{C}l_{p,q-1}\subset\mathbb{O}_{p,q}$ which basis is given by $\{u_x : x\in(\mathbb{Z}_2)^n,\ |x|\equiv 0 \text{ mod}2\}$

Classification

Theorem [1] :

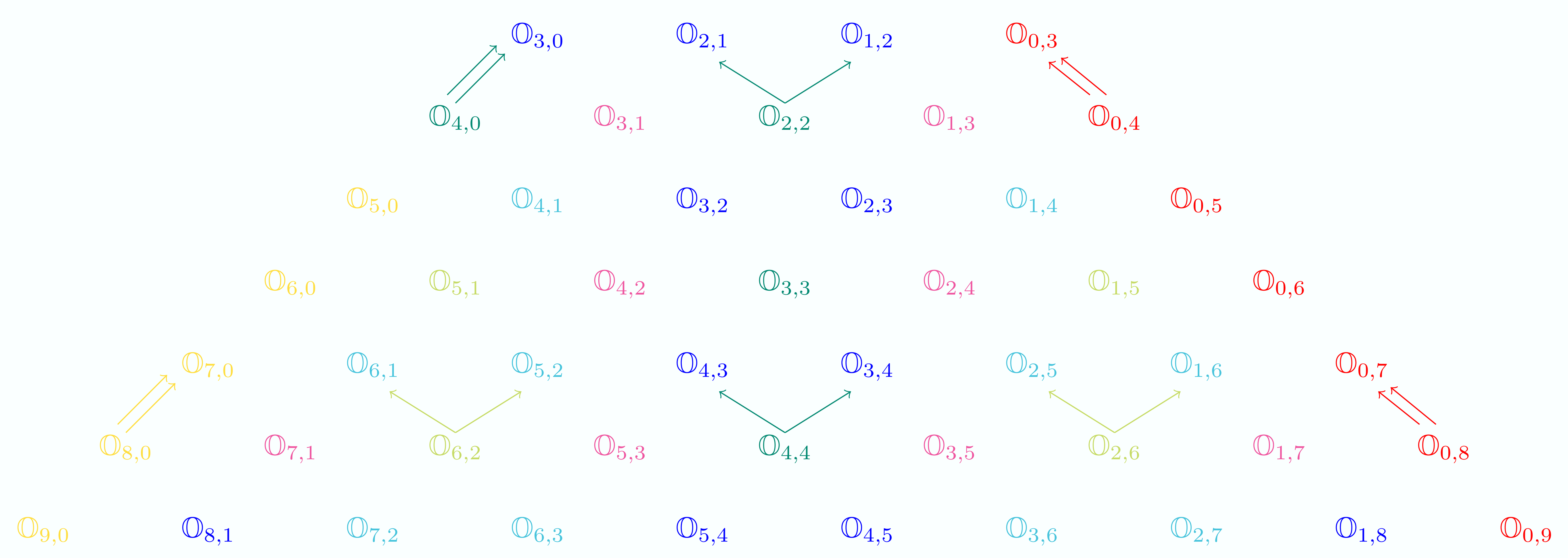
- If $pq\neq 0$, then one has
$$\left.\begin{array}{l}\mathbb{O}_{p,q}\simeq\mathbb{O}_{q,p}\\ \mathbb{O}_{p,q+4}\simeq\mathbb{O}_{p+4,q}\end{array}\right\}\Leftrightarrow\mathcal{C}l_{p,q-1}\simeq\mathcal{C}l_{p',q'-1}$$
- For $n\geq 5$, the algebras $\mathbb{O}_{n,0}$ and $\mathbb{O}_{0,n}$ are exceptional.

Bott Periodicity

Theorem [2] : If $n=p+q\geq 3$ and $pq\neq 0$, then

- $\mathbb{O}_{0,n+4}\simeq\mathbb{O}_{0,n}\otimes_{\mathbb{C}}\mathbb{O}_{0,5}$ $\qquad\mathrm{Cl}_{p,q+2}\simeq\mathrm{Cl}_{q,p}\otimes_{\mathbb{R}}\mathrm{Cl}_{0,2}$
  $\mathbb{O}_{n+4,0}\simeq\mathbb{O}_{n,0}\otimes_{\mathbb{R}\oplus\mathbb{R}}\mathbb{O}_{5,0}$ $\qquad\mathrm{Cl}_{p+2,q}\simeq\mathrm{Cl}_{q,p}\otimes_{\mathbb{R}}\mathrm{Cl}_{2,0}$
- $\mathbb{O}_{p+2,q+2}\simeq\mathbb{O}_{p,q}\otimes_{\mathbb{C}}\mathbb{O}_{2,3}$ $\qquad\mathrm{Cl}_{p+1,q+1}\simeq\mathrm{Cl}_{p,q}\otimes_{\mathbb{R}}\mathrm{Cl}_{1,1}$
  $\simeq\mathbb{O}_{p,q}\otimes_{\mathbb{R}\oplus\mathbb{R}}\mathbb{O}_{3,2}$

Summary Table of Classification of Algebras $\mathbb{O}_{p,q}$



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[1] Kreusch & Morier Genoud, *Classification of the algebras* $\mathbb{O}_{p,q}$
[2] Kreusch, *Bott type periodicity for higher octonions*