Algebras generalizing the Octonions

Marie Kreusch


Overview

Cayley-Dickson Algebras

\[ \mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O} \rightarrow \mathbb{O}_2 \rightarrow \mathbb{O}_3 \rightarrow \mathbb{O}_4 \rightarrow \mathbb{O}_5 \rightarrow \mathbb{O}_6 \rightarrow \mathbb{O}_7 \rightarrow \mathbb{O}_8 \rightarrow \mathbb{O}_9 \]

Clifford Algebras

\[ \mathcal{C}_{p,q} \]

Algebras generalizing the Octonions

\[ \mathcal{O}_{p,q} \]

Applications to the classical Hurwitz problem of "Sums of Squares".

Main Definition

The algebra \( \mathcal{O}_{p,q} \) (\( p + q = n \geq 3 \)) is the 2\(^2\)-dimensional vector space on \( \mathbb{R} \) with the basis \( \{ u_x : x \in \mathbb{Z}_2^n \} \), and equipped with the product

\[ u_x \cdot u_y = (-1)^{\langle x,y \rangle} u_{x+y} \]

for all \( x, y \in \mathbb{Z}_2^n \), where

\[ f_{\lambda_{\mu}}(x,y) = \sum_{\substack{1 \leq i,j \leq n \in \mathbb{Z}_2}} (x_i y_i + x_j y_j) + \sum_{\substack{1 \leq i \leq n \in \mathbb{Z}_2}} x_i y_i + \sum_{\substack{1 \leq j \leq n \in \mathbb{Z}_2}} x_j y_j \]

In particular, the algebra \( \mathcal{O}_{p,q} \) is a twisted group algebra, noted \( (\mathbb{Z}_2^n, f_{\lambda_{\mu}}) \).

Generating Cubic Form

The defect of commutativity and associativity of a twisted algebra \( (\mathbb{Z}_2^n, f) \) is measured by a symmetric function \( f : Z_2^n \times Z_2^n \to Z_2 \), and a function \( \phi : Z_2^n \times Z_2^n \times Z_2^n \to Z_2 \), respectively

\[ u_x \cdot u_y = (-1)^{\phi(x,y)} u_{x+y} \]

A function \( \alpha : Z_2^n \to Z_2 \) is called a generating function for \( (\mathbb{Z}_2^n, f) \), if for \( x, y \) and \( z \) in \( Z_2^n \),

(i) \( f(x,x) = \alpha(x) \),
(ii) \( \phi(x,y) = \alpha(x+y) + \alpha(x) + \alpha(y) \),
(iii) \( \phi(x,y,z) = \alpha(x+y+z) + \alpha(x+y) + \alpha(x+z) + \alpha(y+z) + \alpha(x+y) + \alpha(y) + \alpha(z) \).

Theorem [M & O]:

- A twisted algebra \( (\mathbb{Z}_2^n, f) \) has a generating function if \( \phi \) is symmetric.
- The generating function \( \alpha \) is a polynomial on \( Z_2^n \) of degree \( \leq 3 \).
- Given any polynomial \( \alpha \) on \( Z_2^n \) degree \( \leq 3 \), there exists a unique (up to isomorphism) twisted group algebra \( (\mathbb{Z}_2^n, f) \) having \( \alpha \) as generating function.

Classification

Theorem [1]:

- If \( pq \neq 0 \), then one has
  \[ \mathcal{O}_{p,q} \cong \mathcal{O}_{p,q} \]
  \[ \mathcal{O}_{p,q} \cong \mathcal{O}_{p-q,0} \]
  \[ \mathcal{O}_{p,q} \cong \mathcal{O}_{p+q,0} \]

- For \( n \geq 5 \), the algebras \( \mathcal{O}_{0,2} \) and \( \mathcal{O}_{2,0} \) are exceptional.

Bott Periodicity

Theorem [2]: If \( p + q \geq 3 \) and \( pq \neq 0 \), then

- \( \mathcal{O}_{0,n} \cong \mathcal{O}_{0,0} \cong \mathcal{O}_{0,5} \)
  \[ \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \]

- \( \mathcal{O}_{n,0} \cong \mathcal{O}_{0,0} \cong \mathcal{O}_{0,5} \)
  \[ \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \]

- \( \mathcal{O}_{p,q} \cong \mathcal{O}_{q,p} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \)
  \[ \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \]

- \( \mathcal{O}_{p,q} \cong \mathcal{O}_{q,p} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \)
  \[ \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \cong \mathcal{C}_{p,q} \]

Summary Table of Classification of Algebras \( \mathcal{O}_{p,q} \)