Higher Octonions

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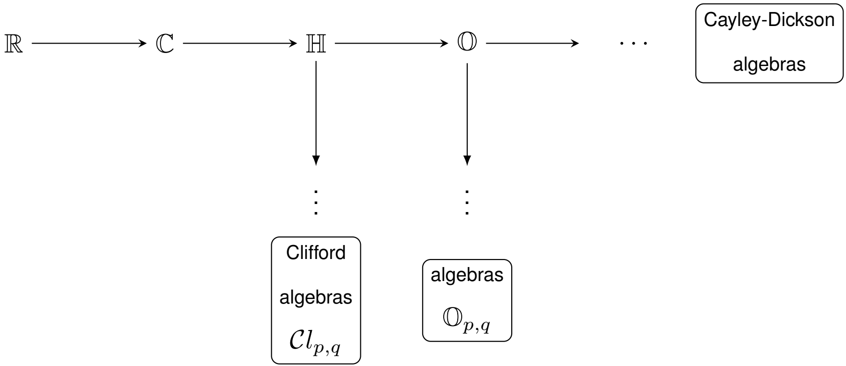
Algebras generalizing the Octonions



Algebra $\mathcal{C}l_{p,q}$

The algebra $\mathcal{C}l_{p,q}$ $(p+q=n\geq 0)$ is the unital associative algebra over the real numbers $\mathbb{R}$ generated by n elements $v_1,\ldots,v_n$ subject to the relations

$$(v_i)^2 = \begin{cases} +1 & \text{if} \quad 1 \leq i \leq p \\ -1 & \text{if} \quad p+1 \leq i \leq n \end{cases}$$

$$v_i \cdot v_j = -v_j \cdot v_i \qquad\qquad i \neq j$$

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The basis elements of $\mathcal{C}l_{p,q}$ are given by $\quad v_{i_1}\cdots v_{i_k}$

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Examples

$$1 \quad\longleftrightarrow\quad (0,\ldots,0)$$

$$v_i \quad\longleftrightarrow\quad (0,\ldots,0,1,0,\ldots,0) =: e_i$$

$$v_1\cdots v_n \quad\longleftrightarrow\quad (1,\ldots,1)$$

Algebra $\mathbb{O}_{p,q}$

The algebra $\mathbb{O}_{p,q}$ $(n = p + q \geq 3)$ is the 2^n-dimensional vector space on the real numbers $\mathbb{R}$ with the basis $\{u_x, x \in \mathbb{Z}_2^n\}$, equipped with the product

$$u_x \cdot u_y = (-1)^{f_{\mathbb{O}_{p,q}}(x,y)} u_{x+y}$$

for all $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{Z}_2^n$, where

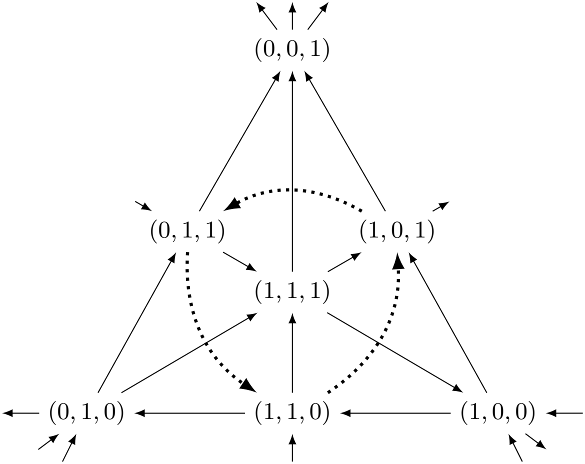
$$f_{\mathbb{O}_{p,q}}(x, y) = \sum_{1 \leq i < j < k \leq n} (x_i x_j y_k + x_i y_j x_k + y_i x_j x_k) + \sum_{1 \leq i \leq j \leq n} x_i y_j + \sum_{1 \leq i \leq p} x_i y_i.$$

[Ovsienko, Morier-Genoud, 2011]

Algebra $\mathbb{O}_{p,q}$

Example

$\mathbb{O}_{0,3} \simeq \mathbb{O}$



[Albuquerque, Majid, 1999]

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- Maximal associative subalgebra $\mathcal{C}l_{p,q-1} \subset \mathbb{O}_{p,q}$ (where $q > 0$)

  (select u_x where $x \in \mathbb{Z}_2^n$ such that $|x| \equiv 0$ mod 2)

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Applications

- Hurwitz square identities $\&$ combinatorics and number theory

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Theorem 1 [K., Morier-Genoud]

i) $\mathbb{O}_{p,q} \simeq \mathbb{O}_{q,p}$

ii) $\mathbb{O}_{p+4,q} \simeq \mathbb{O}_{p,q+4}$

$\Big\}$ $Cl_{p,q-1} \simeq Cl_{p',q'-1}$

iii) For $n \geq 5$, the algebras $\mathbb{O}_{0,n}$ and $\mathbb{O}_{n,0}$ are different

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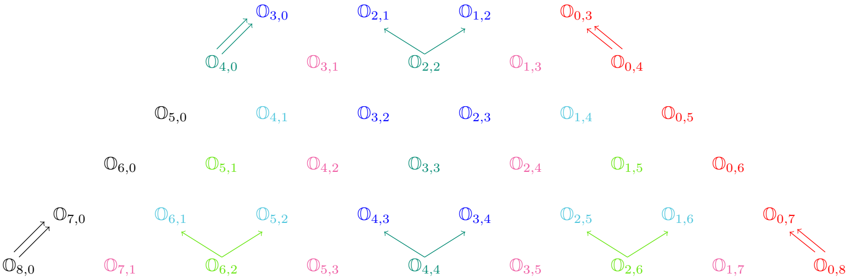

Summary table of classification of algebras $\mathbb{O}_{p,q}$

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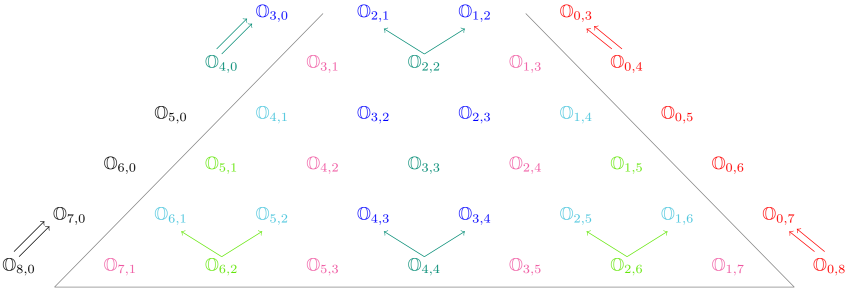

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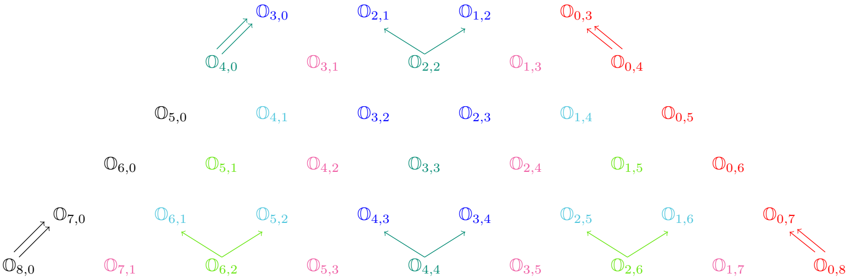

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Theorem 2 [K.]

$$i) \quad \mathbb{O}_{0,n+4} \simeq \mathcal{P}(\mathbb{O}_{0,n} \otimes \mathbb{O}_{5,0}) \simeq \mathcal{P}(\mathbb{O}_{n,0} \otimes \mathbb{O}_{0,5})$$

$$Cl_{0,q+2} \simeq Cl_{q,0} \otimes Cl_{0,2}$$

$$ii) \quad \mathbb{O}_{n+4,0} \simeq \mathcal{P}(\mathbb{O}_{n,0} \otimes \mathbb{O}_{5,0}) \simeq \mathcal{P}(\mathbb{O}_{0,n} \otimes \mathbb{O}_{0,5})$$

$$Cl_{p+2,0} \simeq Cl_{0,p} \otimes Cl_{2,0}$$

$$iii) \quad \mathbb{O}_{p+2,q+2} \simeq \mathcal{P}(\mathbb{O}_{p,q} \otimes \mathbb{O}_{2,3})$$

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where, for example, $\mathcal{P}(\mathbb{O}_{p,q} \otimes \mathbb{O}_{2,3})$ denotes the subalgebra of dimension 2^{n+4} made of linear combination of elements of the form

$$u_{(x_1,x_2,\ldots,x_n)} \otimes u_{(x_1,y_2,\ldots,y_5)}$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n$ and $(x_1, y_2, \ldots, y_5) \in \mathbb{Z}_2^5$.

Cubic form

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$$\beta(x,y) := f(x,y) + f(y,x)$$

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$$\phi(x, y, z) := f(y, z) + f(x + y, z) + f(x, y + z) + f(x, y) \quad \phi = df$$

The algebra is associative iff $\phi \equiv 0$ or equivalently, iff f is a 2-cocyle.

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Given a twisted group algebra $(\mathbb{R}[\mathbb{Z}_2^n], f)$, a function $\alpha : \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2$ is called a generating function if for all $x, y, z \in \mathbb{Z}_2^n$, we have

$$\begin{array}{llcl} (i) & f(x,x) & = & \alpha(x) \\ (ii) & \beta(x,y) & = & \alpha(x+y) + \alpha(x) + \alpha(y) \qquad \beta = d\alpha \\ (iii) & \phi(x,y,z) & = & \alpha(x+y+z) + \alpha(x+y) + \alpha(x+z) + \alpha(y+z) \\ & & & +\alpha(x) + \alpha(y) + \alpha(z) \end{array}$$

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Theorem [Ovsienko, Morier-Genoud, 2011]

(i) A twisted group algebra $(\mathbb{R}[\mathbb{Z}_2^n], f)$ has a generating function if and only if the function ϕ is symmetric.

(ii) The generating function α is a polynomial on $\mathbb{Z}_2^n$ of degree ≤ 3.

(iii) Given any polynomial α on $\mathbb{Z}_2^n$ of degree ≤ 3, there exists a unique twisted group algebra $(\mathbb{R}[\mathbb{Z}_2^n], f)$ having α as a generating function.

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Two functions $\alpha, \alpha' : \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2$ are equivalent if

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Corollary

Two twisted group algebras, $(\mathbb{R}[\mathbb{Z}_2^n], f)$ and $(\mathbb{R}[\mathbb{Z}_2^n], f')$ with equivalent generating functions α and α' are isomorphic.

Cubic (resp. quadratic) form of $\mathbb{O}_{p,q}$ (resp. $\mathcal{C}l_{p,q}$)

$$\alpha_{p,q}(x) = f_{\mathbb{O}_{p,q}}(x,x) = \sum_{1\leq i<j<k\leq n} x_i x_j x_k + \sum_{1\leq i\leq j\leq n} x_i x_j + \sum_{1\leq i\leq p} x_i$$

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Given a cubic form

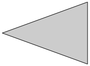
$$\alpha(x) = \sum_{1 \leq i<j<k \leq n} A_{ijk} x_i x_j x_k + \sum_{1 \leq i \leq j \leq n} B_{ij} x_i x_j + \sum_{1 \leq i \leq p} C_i x_i$$

the corresponding triangulated graph is as follows.

1. If x_i appears in α ($C_i = 1$) $\longleftrightarrow$ $\bullet$

   If x_i does not appear in α ($C_i = 0$) $\longleftrightarrow$ $\circ$

2. If $x_i x_j$ appears in α ($B_{ij} = 1$) $\longleftrightarrow$ —

3. If $x_i x_j x_k$ appears in α ($A_{ijk} = 1$) $\longleftrightarrow$ 

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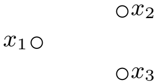
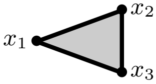
Examples

$$\alpha(x_1, x_2, x_3) \equiv 0 \qquad \longleftrightarrow$$

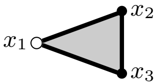


$$\alpha_{0,2}(x_1, x_2) = x_1x_2 + x_1 + x_2 \qquad \longleftrightarrow$$

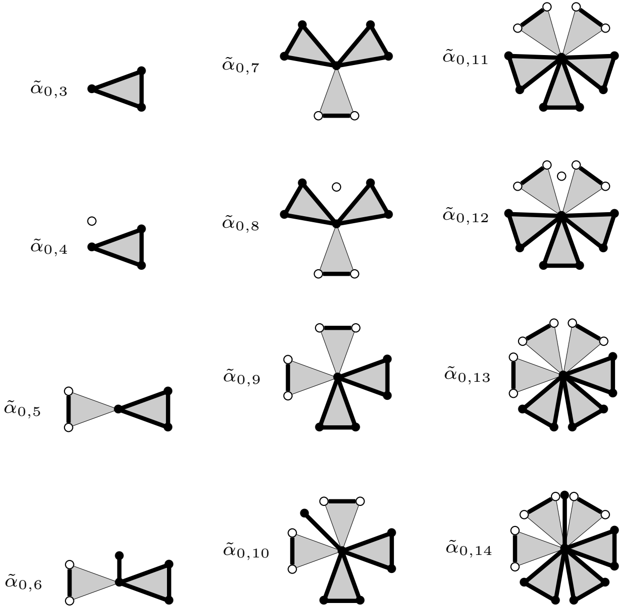
$$\begin{aligned} \alpha_{0,3}(x_1, x_2, x_3) &= x_1x_2x_3 + x_1x_2 + x_1x_3 \\ &\quad + x_2x_3 + x_1 + x_2 + x_3 \end{aligned} \qquad \longleftrightarrow$$



$$\begin{aligned} \alpha_{1,2}(x_1, x_2, x_3) &= x_1x_2x_3 + x_1x_2 + x_1x_3 \\ &\quad + x_2x_3 + x_2 + x_3 \end{aligned} \qquad \longleftrightarrow$$



Periodicity



Thank you for your attention

Marie KREUSCH

Open questions

- Arf invariant also called "democratic invariant"
- Use properties of triangulated graphs to deduce the ones on cubic forms
- Sperner's Lemma
- Find the algebra of derivation. It is already done for
  1. $n = 4$ (of dimension 28)
  2. $n = 5$ (of dimension 20)
  3. $n = 6$ (of dimension 31)
  4. $n = 7$ run out of memory
- Find the automorphism group for each algebra $\mathbb{O}_{0,n}$