

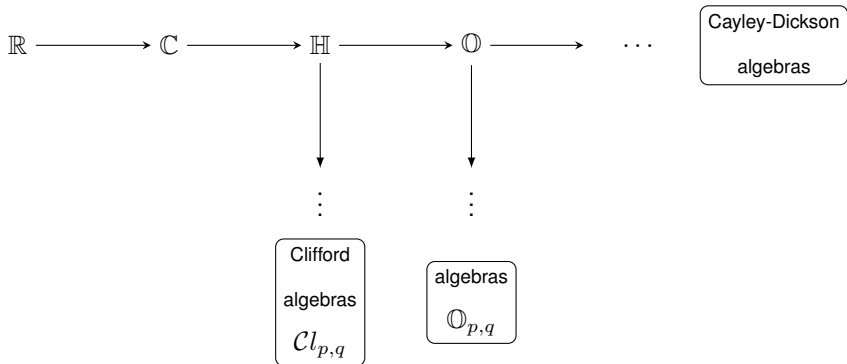
# Higher Octonions

11 June 2015

Marie Kreusch



# Algebras generalizing the Octonions



## Algebra $Cl_{p,q}$

The algebra  $Cl_{p,q}$  ( $p + q = n \geq 0$ ) is the unital associative algebra over the real numbers  $\mathbb{R}$  generated by  $n$  elements  $v_1, \dots, v_n$  subject to the relations

$$(v_i)^2 = \begin{cases} +1 & \text{if } 1 \leq i \leq p \\ -1 & \text{if } p + 1 \leq i \leq n \end{cases}$$

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The basis elements of  $\mathcal{Cl}_{p,q}$  are given by  $v_{i_1} \cdots v_{i_k}$

where  $1 \leq i_1 < \cdots < i_k \leq n$  and can be coded by an  $n$ -uplet of 0 or 1.

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Examples

$$1 \longleftrightarrow (0, \dots, 0)$$

$$v_i \longleftrightarrow (0, \dots, 0, 1, 0, \dots, 0) =: e_i$$

$$v_1 \cdots v_n \longleftrightarrow (1, \dots, 1)$$

## Algebra $\mathbb{O}_{p,q}$

The algebra  $\mathbb{O}_{p,q}$  ( $n = p + q \geq 3$ ) is the  $2^n$ -dimensional vector space on the real numbers  $\mathbb{R}$  with the basis  $\{u_x, x \in \mathbb{Z}_2^n\}$ , equipped with the product

$$u_x \cdot u_y = (-1)^{f_{\mathbb{O}_{p,q}}(x,y)} u_{x+y}$$

for all  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  in  $\mathbb{Z}_2^n$ , where

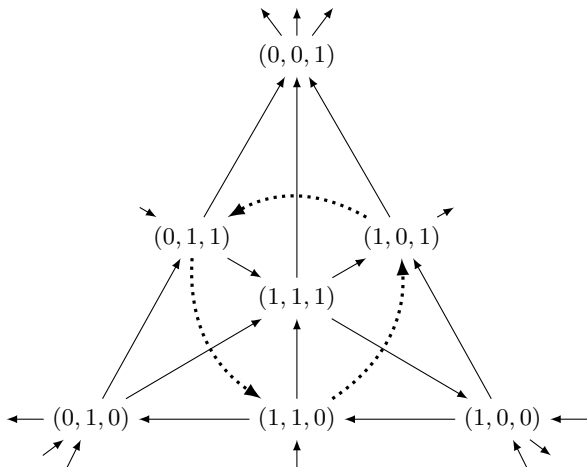
$$f_{\mathbb{O}_{p,q}}(x, y) = \sum_{1 \leq i < j < k \leq n} (x_i x_j y_k + x_i y_j x_k + y_i x_j x_k) + \sum_{1 \leq i \leq j \leq n} x_i y_j + \sum_{1 \leq i \leq p} x_i y_i.$$

[Ovsienko, Morier-Genoud, 2011]

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Example

$$\mathbb{O}_{0,3} \simeq \mathbb{O}$$



[Albuquerque, Majid, 1999]

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$$\mathbb{O}_{p,q} \simeq \mathbb{O}_{p',q'} \quad \text{where} \quad p + q = n = p' + q'$$

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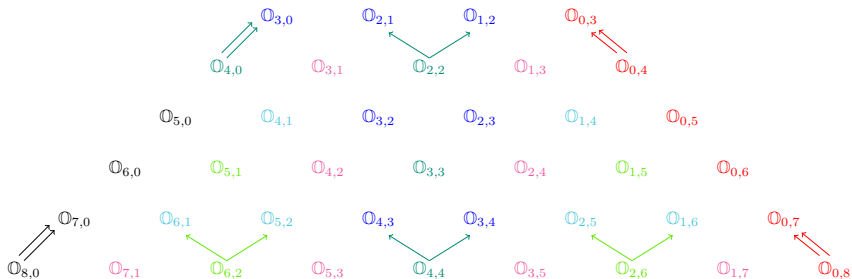
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Theorem 1 [K., Morier-Genoud]

- $i) \quad \mathbb{O}_{p,q} \simeq \mathbb{O}_{q,p}$
  - $ii) \quad \mathbb{O}_{p+4,q} \simeq \mathbb{O}_{p,q+4}$
  - $iii) \quad \text{For } n \geq 5, \text{ the algebras } \mathbb{O}_{0,n} \text{ and } \mathbb{O}_{n,0} \text{ are different}$
- $\left. \begin{array}{l} i) \\ ii) \end{array} \right\} \quad Cl_{p,q-1} \simeq Cl_{p',q'-1}$

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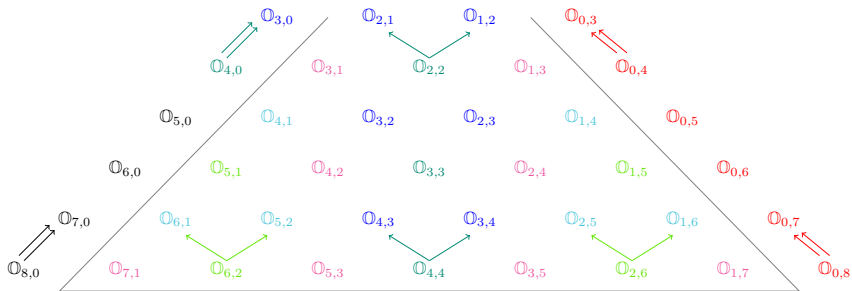
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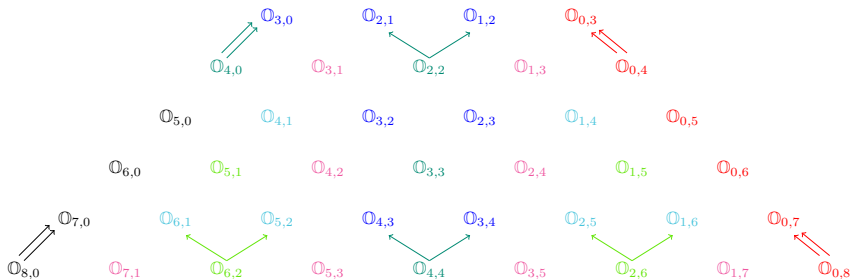
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Theorem 2 [K.]

$$\begin{array}{lll} i) & \mathbb{O}_{0,n+4} & \simeq \mathcal{P}(\mathbb{O}_{0,n} \otimes \mathbb{O}_{5,0}) \\ & & \simeq \mathcal{P}(\mathbb{O}_{n,0} \otimes \mathbb{O}_{0,5}) \\ & & Cl_{0,q+2} \simeq Cl_{q,0} \otimes Cl_{0,2} \\ ii) & \mathbb{O}_{n+4,0} & \simeq \mathcal{P}(\mathbb{O}_{n,0} \otimes \mathbb{O}_{5,0}) \\ & & \simeq \mathcal{P}(\mathbb{O}_{0,n} \otimes \mathbb{O}_{0,5}) \\ & & Cl_{p+2,0} \simeq Cl_{0,p} \otimes Cl_{2,0} \\ iii) & \mathbb{O}_{p+2,q+2} & \simeq \mathcal{P}(\mathbb{O}_{p,q} \otimes \mathbb{O}_{2,3}) \\ & & Cl_{p+1,q+1} \simeq Cl_{p,q} \otimes Cl_{1,1} \end{array}$$



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where, for example,  $\mathcal{P}(\mathbb{O}_{p,q} \otimes \mathbb{O}_{2,3})$  denotes the subalgebra of dimension  $2^{n+4}$  made of linear combination of elements of the form

$$u_{(x_1, x_2, \dots, x_n)} \otimes u_{(x_1, y_2, \dots, y_5)}$$

where  $(x_1, x_2, \dots, x_n) \in \mathbb{Z}_2^n$  and  $(x_1, y_2, \dots, y_5) \in \mathbb{Z}_2^5$ .

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The algebra is associative iff  $\phi \equiv 0$  or equivalently, iff  $f$  is a 2-cocycle.

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Given a twisted group algebra  $(\mathbb{R}[\mathbb{Z}_2^n], f)$ , a function  $\alpha : \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2$  is called a **generating function** if for all  $x, y, z \in \mathbb{Z}_2^n$ , we have

$$(i) \quad f(x, x) = \alpha(x)$$

$$(ii) \quad \beta(x, y) = \alpha(x + y) + \alpha(x) + \alpha(y) \quad \beta = d\alpha$$

$$(iii) \quad \phi(x, y, z) = \alpha(x + y + z) + \alpha(x + y) + \alpha(x + z) + \alpha(y + z) \\ + \alpha(x) + \alpha(y) + \alpha(z)$$

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**Theorem** [Ovsienko, Morier-Genoud, 2011]

- (i) A twisted group algebra  $(\mathbb{R}[\mathbb{Z}_2^n], f)$  has a generating function if and only if the function  $\phi$  is symmetric.
- (ii) The generating function  $\alpha$  is a polynomial on  $\mathbb{Z}_2^n$  of degree  $\leq 3$ .
- (iii) Given any polynomial  $\alpha$  on  $\mathbb{Z}_2^n$  of degree  $\leq 3$ , there exists a unique twisted group algebra  $(\mathbb{R}[\mathbb{Z}_2^n], f)$  having  $\alpha$  as a generating function.



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Two functions  $\alpha, \alpha' : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$  are **equivalent** if

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### Corollary

Two twisted group algebras,  $(\mathbb{R}[\mathbb{Z}_2^n], f)$  and  $(\mathbb{R}[\mathbb{Z}_2^n], f')$  with equivalent generating functions  $\alpha$  and  $\alpha'$  are isomorphic.

Cubic (resp. quadratic) form of  $\mathbb{O}_{p,q}$  (resp.  $\mathcal{C}l_{p,q}$ )

$$\alpha_{p,q}(x) = f_{\mathbb{O}_{p,q}}(x, x) = \sum_{1 \leq i < j < k \leq n} x_i x_j x_k + \sum_{1 \leq i \leq j \leq n} x_i x_j + \sum_{1 \leq i \leq p} x_i$$

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$$\alpha_{p,q}^{Cl}(x) = f_{\mathcal{C}l_{p,q}}(x, x) = \sum_{1 \leq i < j < k \leq n} x_i x_j x_k + \underbrace{\sum_{1 \leq i < j \leq n} x_i x_j}_{\alpha_n^{Cl}(x)} + \sum_{1 \leq i \leq p} x_i$$

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
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Given a cubic form

$$\alpha(x) = \sum_{1 \leq i < j < k \leq n} A_{ijk} x_i x_j x_k + \sum_{1 \leq i < j \leq n} B_{ij} x_i x_j + \sum_{1 \leq i \leq p} C_i x_i$$

the corresponding **triangulated graph** is as follows.

- |   |                       |   |
|---|-----------------------|---|
| 1. If $x_i$ appears in $\alpha$ ( $C_i = 1$ )             | $\longleftrightarrow$ | ●   |
| If $x_i$ does not appear in $\alpha$ ( $C_i = 0$ )        | $\longleftrightarrow$ | ○   |
| 2. If $x_i x_j$ appears in $\alpha$ ( $B_{ij} = 1$ )      | $\longleftrightarrow$ | —   |
| 3. If $x_i x_j x_k$ appears in $\alpha$ ( $A_{ijk} = 1$ ) | $\longleftrightarrow$ |  |



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Examples

$$\alpha(x_1, x_2, x_3) \equiv 0 \quad \longleftrightarrow \quad \begin{array}{c} \circ x_2 \\ x_1 \circ \\ \circ x_3 \end{array}$$

$$\alpha_{0,2}(x_1, x_2) = x_1x_2 + x_1 + x_2 \quad \longleftrightarrow \quad x_1 \bullet \text{---} \bullet x_2$$

$$\alpha_{0,3}(x_1, x_2, x_3) = x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1 + x_2 + x_3 \quad \longleftrightarrow \quad \begin{array}{c} x_2 \\ \bullet \\ \triangle \\ \bullet \\ x_3 \end{array} x_1 \bullet$$

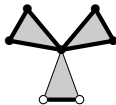
$$\alpha_{1,2}(x_1, x_2, x_3) = x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_2 + x_3 \quad \longleftrightarrow \quad \begin{array}{c} x_2 \\ \bullet \\ \triangle \\ \bullet \\ x_3 \end{array} x_1 \circ$$

# Periodicity

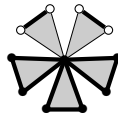
$\tilde{\alpha}_{0,3}$



$\tilde{\alpha}_{0,7}$



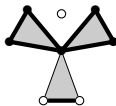
$\tilde{\alpha}_{0,11}$



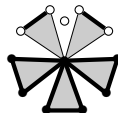
$\tilde{\alpha}_{0,4}$



$\tilde{\alpha}_{0,8}$



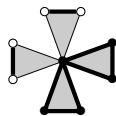
$\tilde{\alpha}_{0,12}$



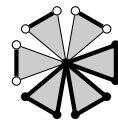
$\tilde{\alpha}_{0,5}$



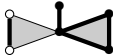
$\tilde{\alpha}_{0,9}$



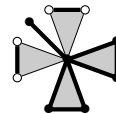
$\tilde{\alpha}_{0,13}$



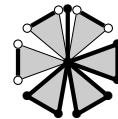
$\tilde{\alpha}_{0,6}$



$\tilde{\alpha}_{0,10}$



$\tilde{\alpha}_{0,14}$



ADDITIVE HISTORICAL MATHALYON ABTTOLE Red QUADRATIC IDENTITIES  
 ECONOMOLOGY teaching Biemann surfaces POSTER FROMOTISM  
 BOULEAN CUBICS FANO PLANE NONASSOCIATIVE dimension KRICHEVER-NOVIKOV TYPE  
 Module SPHERE HAMMING WEIGHT Sperner's Lemma  
 Work Extensions Equivalent Lie SUPERALGEBRA topology  
 CODE -GRADING Lie antialgebra cocycle ASSISTANT  
 PAI DYGEST Luxembourg trivial Exceptional  
 Central Octonions Santo Stephano del Sol tensor product BOTT PERIODICITY Canonical Line Bundle  
 tensor product BOTT PERIODICITY Canonical Line Bundle  
 POISSON ALGEBRA LUUVAIN-LA-NEUVE MTT180  
 Cycle Algebra GRENOBLE Anciens

Thank you for your attention

LYON COMPLEX BUTTERFLY GRAPH GEOMETRY RESEARCH  
 GRADED ALGEBRA Jordan Superalgebras COBOUNDARY DUAL PARIS classification  
 MEROMORPHIC WBI Noncommutative  
 CONFERENCE STATISTICS QUATERNIONS THETA CHARACTERISTICS  
 ZORN MATRIX PHD STUDENT MARSEILLE COMMUNICATION  
 STOCKHOLM vector bundle SRNI RETIREE  
 Automorphism Group CUBIC FORM Lille Quadratic Punctures  
 CLIFFORD ALGEBRAS  
 Virasoro Algebra LIÈGE FLIP MONASTIR PROVIDENCES  
 MATHÉMATICA SUMMER SCHOOL PHD NETWORK  
 Mangled Graph math a modeller

Marie KREUSCH

## Open questions

- ▶ Arf invariant also called "democratic invariant"
- ▶ Use properties of triangulated graphs to deduce the ones on cubic forms
- ▶ Sperner's Lemma
- ▶ Find the algebra of derivation. It is already done for
  1.  $n = 4$  (of dimension 28)
  2.  $n = 5$  (of dimension 20)
  3.  $n = 6$  (of dimension 31)
  4.  $n = 7$  run out of memory
- ▶ Find the automorphism group for each algebra  $\mathbb{O}_{0,n}$