A generalization of the S^ν spaces: getting rid of dyadic scales

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joint work with S. Nicolay

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Plan

1. **Introduction**
   - First definitions
   - The S^ν spaces
2. **A generalization of the S^ν spaces**
   - Definition
   - Topology
   - Link with generalized Besov spaces
3. **Conclusion**

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Definitions

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function.
The function f belongs to the **Hölder space** $\Lambda^s(x)$ ($s \geq 0$ and $x \in \mathbb{R}$) if there exist a constant $C > 0$ and a polynomial P of degree strictly less than s such that

$$|f(x+l) - P(l)| \leq C|l|^s$$

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The **spectrum of singularities** of f is defined by

$$d_f : h \geq 0 \mapsto \dim_{\mathcal{H}}(\{x : h_f(x) = h\}) \in [0,1] \cup \{-\infty\}$$

where $\dim_{\mathcal{H}}$ is the Hausdorff dimension.

Wavelets

There exist functions ϕ et ψ with "suitable" properties such that

$$\{\phi(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi(2^j \cdot - k) : j \in \mathbb{N},\ k \in \mathbb{Z}\}$$

is an orthogonal basis of $L^2(\mathbb{R})$, i.e. if $f \in L^2(\mathbb{R})$ then

$$f(x) = \sum_{k\in\mathbb{Z}} C_k \phi(x-k) + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} c_{jk} \psi(2^j x - k)$$

where

$$C_k = \int_{\mathbb{R}} f(x)\phi(x-k)dx \text{ et } c_{jk} = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k).$$

One can deduct a basis of $L^2([0,1])$ with the periodiziation operator

$$per[f] = \sum_{l\in\mathbb{Z}} f(x+l).$$

The associated wavelets are

$$\{per[\psi(2^j \cdot - k)] : j \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\}\}.$$

Definition

A function f belongs to the **Hölder-Zygmund space** Λ^ϵ ($\epsilon > 0$) if there exists $C > 0$ such that

$$|c_{jk}| \leq C 2^{-\epsilon j}$$

for all j, k. A function f is said **uniformly Hölder** if there exists $\epsilon > 0$ such that $f \in \Lambda^\epsilon$.

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Theorem

If f is uniformly Hölder and if ψ is "smooth enough", the Hölder exponent of f at x is

$$h_f(x) = \liminf_{j \to +\infty} \inf_k \frac{\log |c_{jk}|}{\log(2^{-j} + |k2^{-j} - x|)}.$$

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Definition

The **wavelet profil** of a function $f \in L^2([0,1])$ is defined by

$$\nu_f : \alpha \in \mathbb{R} \mapsto \lim_{\epsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \# E_j(1, \alpha + \epsilon)(f)}{\log 2^j}$$

where $E_j(C,\alpha)(f) = \{k : |c_{jk}| \geq C 2^{-\alpha j}\}$.

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- ν_f is increasing and right-continuous;
- there exist $\alpha_{\min} > 0$ such that $\nu_f(\alpha) = -\infty$ for all $\alpha < \alpha_{\min}$ and $\nu_f(\alpha) \in [0,1]$ for all $\alpha \geq \alpha_{\min}$;

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- ν_f is independent of the chosen wavelet basis (Jaffard, 2004 [6]);
- The strictly positive constant 1 appareaing in the definition of ν_f is arbitrarily (this fact is important for the implementation of this function - Kleyntssens, Esser, Nicolay, 2014 [7]).

Given a right-continuous increasing function ν and $\alpha_{\min} > 0$ such that $\nu(\alpha) = -\infty$ for all $\alpha < \alpha_{\min}$ and $\nu(\alpha) \in [0,1]$ for all $\alpha \geq \alpha_{\min}$.

Definition

The space S^ν is defined by

$$S^\nu = \{f \in L^2([0,1]) : \nu_f(\alpha) \leq \nu(\alpha) \text{ for all } \alpha \in \mathbb{R}\}.$$

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Theorem (Aubry, Bastin, Dispa, 2007 [3])

The set

$$\{f \in S^\nu : d_f(\alpha) = \begin{cases} \min\left\{\alpha \sup_{\alpha' \in]0,\alpha]} \frac{\nu_f(\alpha')}{\alpha'}, 1\right\} & \text{if } \alpha \leq \alpha_{max} \\ -\infty & \text{else} \end{cases} \}$$

are prevalent in S^ν.

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Theorem (Aubry, Bastin, 2010 [2])

The S^ν spaces are **Schwartz** but **not nuclear spaces**. Moreover, Ligaud's example [9] of a Schwartz pseudoconvex non p-convex space is a particular case of S^ν.

If $f \in S^\nu$ then, for all $\alpha \in \mathbb{R}$,

$$\lim_{\epsilon' \to 0^+} \limsup_{j \to +\infty} \frac{\log \# E_j(1, \alpha + \epsilon')(f)}{\log 2^j} \leq \nu(\alpha)$$

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i.e. for all $C > 0$, there exists $J > 0$ such that, for all $j \geq J$,

$$\frac{\log \# E_j(1, \alpha + \epsilon')(f)}{\log 2^j} \leq \nu(\alpha) + \epsilon$$

and

$$\begin{aligned} \# E_j(1, \alpha + \epsilon')(f) &= \# \{ k : |c_{jk}| \geq 2^{-(\alpha + \epsilon')j} \} \\ &\geq \# \{ k : |c_{jk}| \geq C 2^{-\alpha j} \} \end{aligned}$$

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i.e

$$\# E_j(C, \alpha)(f) \leq 2^{(\nu(\alpha)+\epsilon)j}.$$

Proposition (Aubry, Bastin, Dispa, Jaffard, 2006 [4])

One has

$$S^\nu = \{f \in L^2([0,1]) : \forall \alpha > 0 \ \forall \epsilon > 0 \ \forall C > 0 \\ \exists J > 0 \ \forall j \geq J, \ \#E_j(C,\alpha)(f) \leq 2^{(\nu(\alpha)+\epsilon)j}\}.$$

where $E_j(C,\alpha)(f) = \{k : |c_{jk}| \geq C2^{-\alpha j}\}$.

Seeing that $\alpha_{min} > 0$, the S^ν space is included in Hölder-Zygmund space Λ^ϵ, for all $\epsilon \in]0, \alpha_{\min}[$.

Proposition (Aubry, Bastin, Dispa, Jaffard, 2006 [4] - Jaffard, 2004 [6])

One has

$$S^\nu = \{\vec{c} : \forall \alpha > 0 \ \forall \epsilon > 0 \ \forall C > 0 \ \exists J > 0 \ \forall j \geq J, \ \# E_j(C,\alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\epsilon)j}\}.$$

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Recall : The function ν is a right-continuous increasing function and there exists $\alpha_{\min} > 0$ such that $\nu(\alpha) = -\infty$ for all $\alpha < \alpha_{\min}$ and $\nu(\alpha) \in [0,1]$ for all $\alpha \geq \alpha_{\min}$.

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Definition

For all $\alpha \in \mathbb{R}$, we note $\sigma^{(\alpha)}$ a sequence such that $\sigma_j^{(\alpha)} > 0$ and we define

$$S^{\nu,\sigma^{(\cdot)}} = \{\vec{c} : \forall \alpha \in \mathbb{R}\ \forall \epsilon > 0\ \forall C > 0\ \exists J > 0\ \forall j \geq J,\ \#E_j(C,\alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\epsilon)j}\}.$$

where $E_j(C, \sigma^{(\alpha)})(\vec{c}) = \{k : |c_{jk}| \geq C\sigma_j^{(\alpha)}\}$.

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Notation : $x_j \preceq_J y_j \equiv \forall j \geq J \ x_j \leq y_j$ and $x_j \preceq y_j \equiv \exists J > 0 \ x_j \preceq_J y_j$ two preorders in the reals sequences spaces.

Definition

Suppose that for all $\alpha < \alpha'$, we have $\sigma_j^{(\alpha')} \preceq \sigma_j^{(\alpha)}$. The **generalized profil** of $\vec{c}$ is defined by

$$\nu_{\vec{c},\sigma^{(\cdot)}} : \alpha \in \mathbb{R} \mapsto \lim_{\epsilon\to 0^+} \limsup_{j\to+\infty} \frac{\log \# E_j(1, \sigma^{(\alpha+\epsilon)})(\vec{c})}{\log 2^j}.$$

This function is a right-continuous increasing function and there exists $\alpha_{\min} \in \mathbb{R} \cup \{-\infty\}$ such that $\nu_{\vec{c},\sigma^{(\cdot)}}(\alpha) = -\infty$ for all $\alpha < \alpha_{\min}$ and $\nu(\alpha) \in [0,1]$ for all $\alpha \geq \alpha_{\min}$.

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Proposition

If for all $\alpha < \alpha'$, we have $\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0$ if $j \to +\infty$ then the strictly positive constant 1 apparearing in the definition of $\nu_{\vec{c},\sigma^{(\cdot)}}$ is arbitrarily and

$$S^{\nu,\sigma^{(\cdot)}} = \{\vec{c} : \nu_{\vec{c},\sigma^{(\cdot)}}(\alpha) \leq \nu(\alpha) \text{ for all } \alpha \in \mathbb{R}\}.$$

Plan

Definition

Let $\alpha \in \mathbb{R}$ and $\beta \in [0,+\infty[\cup \{-\infty\}$. A sequence $\vec{c}$ belongs to $E(\sigma^{(\alpha)}, \beta)$ if there exist $C, C' \geq 0$ such that

$$\#E_j(C, \sigma^{(\alpha)})(\vec{c}) \leq C' 2^{\beta j}$$

for all $j \in \mathbb{N}$. We define a metric on this space by

$$d_{\sigma^{(\alpha)},\beta}(\vec{c}, \vec{d}) = \inf\{C + C' : C, C' \geq 0, \\ \#E_j(C, \sigma^{(\alpha)})(\vec{c} - \vec{d}) \leq C' 2^{\beta j} \text{ for all } j \in \mathbb{N}\}.$$

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Proposition

- The $E(\sigma^{(\alpha)}, \beta)$ space is a vector space.
- The $E(\sigma^{(\alpha)}, \beta)$ space is complete. Besides, the sum is a continuous operation but the product is not necessarily continuous.

Theorem

If for all $\alpha < \alpha'$, we have $\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0$ if $j \to +\infty$ then for all sequence $(\alpha_n)_{n\in\mathbb{N}}$ dense in $\mathbb{R}$ and for all sequence $(\epsilon_m)_{m\in\mathbb{N}}$ of stricly positives reals which converges to 0, we have

$$S^{\nu,\sigma^{(\cdot)}} = \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} E(\sigma^{(\alpha_n)}, \nu(\alpha_n) + \epsilon_m).$$

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So, we can define a metric on $S^{\nu,\sigma^{(\cdot)}}$ which the topology is the weakest topology such that the identity

$$i_{n,m} : S^{\nu,\sigma^{(\cdot)}} \to E(\sigma^{(\alpha_n)}, \nu(\alpha_n) + \epsilon_m)$$

is continuous for all m, n.

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Besides, the topology is independant of sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\epsilon_m)_{m\in\mathbb{N}}$ and the $S^{\nu,\sigma^{(\cdot)}}$ space is complete topological vector space, thus it is a Baire space.

Plan

S^ν ⟷ (Aubry, Bastin, Dispa, Jaffard, 2006 [4]) ⟷ Besov spaces b^s_{pq}

$$S^\nu \subseteq \bigcap_{p>0} \bigcap_{\epsilon>0} b^{\frac{\eta(p)-\epsilon}{p}}_{p,\infty}$$

$=$ iff ν is concave

S^ν ⇣ $S^{\nu,\sigma^{(\cdot)}}$

Besov spaces b^s_{pq} ⇣ (Almeida, 2005 [1]) generalized Besov spaces b^θ_{pq}

$S^{\nu,\sigma^{(\cdot)}}$ ⟷ (???) ⟷ generalized Besov spaces b^θ_{pq}

Definition

A sequence σ of real positive numbers is called **admissible** if there exists a constant $C > 0$ such that

$$C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j$$

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We set

$$\underline{\Theta}_j = \inf_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k} \text{ and } \overline{\Theta}_j = \sup_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k}$$

and we define the lower and upper Boyd indices as follows,

$$\underline{s}(\sigma) = \lim_{j\to+\infty} \frac{\log \underline{\Theta}_j}{\log 2^j} \text{ and } \overline{s}(\sigma) = \lim_{j\to+\infty} \frac{\log \overline{\Theta}_j}{\log 2^j}.$$

In this case, for all $\epsilon > 0$, there exists a constant $C > 0$ such that

$$C^{-1}2^{j(\underline{s}(\sigma)-\epsilon)} \leq \frac{\sigma_{j+k}}{\sigma_k} \leq C2^{j(\overline{s}(\sigma)+\epsilon)} \quad \text{for all } j,k \in \mathbb{N}.$$

Definition

A sequence $\vec{c}$ belongs to the space $b^s_{p,\infty}$ if and only if

$$\sup_j 2^{(s-1/p)j} \left(\sum_k |c_{jk}|^p \right)^{1/p} < \infty.$$

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Definition

Let θ be an admissible sequence. A sequence $\vec{c}$ belongs to the space $b^\theta_{p,\infty}$ if and only if

$$\sup_j \theta_j 2^{-j/p} \left(\sum_k |c_{jk}|^p \right)^{1/p} < \infty.$$

Theorem

Suppose that for all $\alpha \in \mathbb{R}$, the sequence $\sigma^{(\alpha)}$ is an admissible sequence and for all $\alpha < \alpha'$, we have $\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0$ if $j \to +\infty$ and $\overline{s}(\sigma^{(\alpha)}) \to -\infty$ if $\alpha \to +\infty$. For all $p > 0$, we note $\theta^{(p)}$ an admissible sequence. The following propositions are equivalent:

- For all $p, \epsilon > 0$ and for all $\alpha \geq \alpha_{\min}$, there exists $C > 0$ such that
$$\theta_j^{(p)} 2^{-j\epsilon/p} \preceq C 2^{j/p} 2^{-j\nu(\alpha)/p} (\sigma_j^{(\alpha)})^{-1}.$$
- We have that
$$S^{\nu,\sigma^{(\cdot)}} \subseteq \bigcap_{p>0} \bigcap_{\epsilon>0} b_{p,\infty}^{(\theta_j^{(p)} 2^{-j\epsilon/p})_j}.$$

Remark : if $\theta_j \preceq \theta'_j$ then $b_{p,\infty}^{\theta'} \subset b_{p,\infty}^{\theta}$.

$$\theta_j^{(p)} 2^{-j\epsilon/p} \preceq C 2^{j/p} 2^{-j\nu(\alpha+\epsilon/2)/p} (\sigma_j^{(\alpha+\epsilon/2)})^{-1}$$

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i.e.

$$2^{(\nu(\alpha+\epsilon/2)-\epsilon)j} \preceq 2^j (\sigma_j^{(\alpha+\epsilon)} \theta_j^{(p)})^{-p}$$

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$$(\nu(\alpha+\epsilon/2)-\epsilon) \leq 1 - p\frac{\log(\sigma_j^{(\alpha+\epsilon)}\theta_j^{(p)})}{\log 2^j},$$

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$$(\nu(\alpha+\epsilon/2)-\epsilon) \leq 1 - p \frac{\log(\sigma_j^{(\alpha+\epsilon)} \theta_j^{(p)})}{\log 2^j},$$

i.e.

$$\nu(\alpha) \leq \lim_{\epsilon \to 0^+} \inf_{p>0} \limsup_{j \to +\infty} (1 - p \frac{\log(\sigma_j^{(\alpha+\epsilon)} \theta_j^{(p)})}{\log 2^j}) := \tilde{\nu}(\alpha)$$

for all $\alpha \geq \alpha_{min}$. We set $\tilde{\nu}(\alpha) = -\infty$ for all $\alpha < \alpha_{min}$.

$$\theta_j^{(p)} 2^{-j\epsilon/p} \preceq C 2^{j/p} 2^{-j\nu(\alpha+\epsilon/2)/p} (\sigma_j^{(\alpha+\epsilon/2)})^{-1}$$

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for all $\alpha \geq \alpha_{min}$. We set $\tilde{\nu}(\alpha) = -\infty$ for all $\alpha < \alpha_{min}$.
The function $\tilde{\nu}$ is increasing, right-continuous and positive on $[\alpha_{\min}, +\infty[$. This function is also smaller than 1 if, for example, for all $\alpha > \alpha_{\min}$ and for all $p > 0$, we have

$$\liminf_{j \to +\infty} \sigma_j^{(\alpha)} \theta_j^{(p)} > 0.$$

Proposition

Under the previous hypotheses, we have that

$$S^{\nu,\sigma^{(\cdot)}} \subseteq S^{\tilde{\nu},\sigma^{(\cdot)}}$$

and if for all $\alpha < \alpha_{\min}$, there exist $p, \epsilon > 0$ such that

$$\lim_{j \to +\infty} 2^{-j/p} \sigma_j^{(\alpha)} \theta_j^{(p)} 2^{-j\epsilon/p} = +\infty$$

then

$$\bigcap_{p>0} \bigcap_{\epsilon>0} b_{p,\infty}^{(\theta^{(p)} 2^{-j\epsilon/p})_j} \subseteq S^{\tilde{\nu},\sigma^{(\cdot)}}.$$

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If we have the previous inclusion, then, for all $\alpha < \alpha_{\min}$, there exist $p, \epsilon > 0$ such that

$$\limsup_{j\to+\infty} 2^{-j/p}\sigma_j^{(\alpha)}\theta_j^{(p)}2^{-j\epsilon/p} = +\infty.$$

Suppose that there exists $\alpha < \alpha_{\min}$ such that for all $p, \epsilon > 0$, we have

$$\limsup_{j \to +\infty} 2^{-j/p} \sigma_j^{(\alpha)} \theta_j^{(p)} 2^{-j\epsilon/p} < +\infty.$$

We set, for all $j \in \mathbb{N}$, $c_{jk} = \sigma_j^{(\alpha)}$ for one and only one k and $c_{jk} = 0$ for the other values of k.

- $\vec{c} \notin S^{\nu,\sigma^{(\cdot)}} \subset S^{\tilde{\nu},\sigma^{(\cdot)}}$ because

$$\nu_{\vec{c},\sigma^{(\cdot)}}(\alpha) = \lim_{\epsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \#\{k : |c_{jk}| \geq \sigma_j^{(\alpha+\epsilon)}\}}{\log 2^j} = 0 \not\leq \nu(\alpha) = -\infty$$

- $\vec{c} \in \bigcap_{p>0} \bigcap_{\epsilon>0} b_{p,\infty}^{(\theta_j^{(p)} 2^{-j\epsilon/p})_j}$ because, for all $p, \epsilon > 0$,

$$\sup_j \theta_j^{(p)} 2^{-j\epsilon/p} 2^{-j/p} \left(\sum_k |c_{jk}|^p \right)^{1/p} = \sup_j \theta_j^{(p)} 2^{-j\epsilon/p} 2^{-j/p} \sigma_j^{(\alpha)} < \infty.$$

Theorem

Under the previous hypotheses, if $\nu = \tilde{\nu}$ then we have that

$$S^{\nu,\sigma^{(\cdot)}} = \bigcap_{p>0} \bigcap_{\epsilon>0} b_{p,\infty}^{(\theta_j^{(p)} 2^{-j\epsilon/p})_j}.$$

Theorem

Under the previous hypotheses, if $\nu = \tilde{\nu}$ then we have that

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If we suppose that for all $p, \epsilon > 0$ and for all $\alpha \geq \alpha_{\min}$, there exists $C > 0$ such that

$$\theta_j^{(p)} 2^{-j\epsilon/p} \preceq C 2^{j/p} 2^{-j\tilde{\nu}(\alpha)/p} (\sigma_j^{(\alpha)})^{-1}$$

then the previous theorem becomes an equivalence.

The case of the S^ν spaces

We have $\sigma_j^{(\alpha)} = 2^{-\alpha j}$. We must find a sequence $\theta_j^{(p)}$ such that for all $p, \epsilon > 0$ and for all $\alpha \geq \alpha_{\min}$, there exists $C > 0$ such that

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So, we can take $\theta_j^{(p)} 2^{-\epsilon/p} = 2^{j(\eta(p)-\epsilon)/p}$ where

$$\eta(p) = \inf_{\alpha \geq \alpha_{\min}} \{\alpha p - \nu(\alpha) + 1\}.$$

So, we have that $\tilde{\nu}(\alpha) = \inf_{p>0}\{1 + \alpha p - \eta(p)\}$ for all $\alpha \geq \alpha_{\min}$ and the previous hypothesis on $\theta_j^{(p)}$ is always verified for $\tilde{\nu}$.

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So, we obtain that $S^\nu = \bigcap_{p>0} \bigcap_{\epsilon>0} b_{p,\infty}^{(\theta_j^{(p)} 2^{-j\epsilon/p})_j}$ if and only if $\nu = \tilde{\nu}$, i.e. ν is concave.

Plan

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Conclusion

- With suitable hypotheses on $\sigma^{(\alpha)}$, the new spaces $S^{\nu,\sigma^{(\cdot)}}$ have similar topology properties as the S^ν spaces;
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- I also study a link with the generalized Hölder-Zygmund spaces (Kreit, Nicolay, 2013 [8]);
- Several examples are being studied, such as $\sigma_j^{(\alpha)} = (2^{-j}|\log|\log 2^{-j}||)^\alpha$ often used with Brownian motions;

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- This new theory is being implemented and the first results are conclusive. We have a better characterization of the fractal nature of real-life signals.

Thank you for your attention !

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