**CSMA 2015**
12e Colloque National en Calcul des Structures
18-22 Mai 2015, Presqu'île de Giens (Var)

# Bifurcation Analysis of Large-Scale Dynamical Systems Using the Harmonic Balance Method

T. Detroux, L. Renson, L. Masset, G. Kerschen

*Space Structures and Systems Laboratory (S3L), Structural Dynamics Research Group*
*Department of Aerospace and Mechanical Engineering, University of Liège, Belgium*
*{tdetroux,l.renson,luc.masset,g.kerschen}@ulg.ac.be*

**Abstract** — The harmonic balance (HB) method has been widely used in the past few years, as a numerical tool for the study of nonlinear industrial models. However, in its classical formulation the HB method is limited to the approximation of periodic solutions. The present paper proposes to extend the method to the detection and tracking of codimension-1 bifurcations in the system parameters space. As an application, the frequency response of a spacecraft and its bifurcations are studied, together with two nonlinear phenomena, namely quasiperiodic oscillations and detached resonance curves.
**Key words** — continuation of periodic solutions, bifurcation tracking, harmonic balance method, quasiperiodic oscillations, detached resonance curves.

## 1 Introduction

Because nowadays engineering structures are designed to be lighter and operate in more severe conditions, nonlinear phenomena such as amplitude jumps, modal interactions, limit cycle oscillations and quasiperiodic (QP) oscillations are expected to occur [4]. For most of these nonlinear systems, bifurcations of periodic solutions play a key role in the response dynamics; for example, fold bifurcations indicate a stability change for the solutions, while QP oscillations are encountered in the vicinity of Neimark-Sacker (NS) bifurcations. In that regard, it seems relevant to include stability analysis and bifurcation monitoring while performing parametric studies of the structures.

Time-domain methods such as shooting technique and orthogonal collocation, which deal with the resolution of a boundary value problem, usually prove accurate to analyze periodic solutions and bifurcations of low-dimensional structures. When applied to larger systems however, their computational burden become substantial. As an efficient alternative, most engineers apply a frequency-domain method, the so-called *harmonic balance* (HB) method (see, e.g., [3]). Indeed, through the approximation of the solutions with truncated Fourier series, the HB method involves algebraic equations with usually less unknowns than time-domain techniques. Nevertheless, in spite of its performance and accuracy, the HB method has never been extended to track bifurcations of mechanical structures. This is why the present paper proposes an extension of the methodology to study bifurcations in the system parameters space.

The first part of this paper is devoted to the HB theory and its formulation in the framework of a continuation algorithm for bifurcation tracking. In section 3, the methodology is validated with the analysis of the nonlinear dynamics of an industrial spacecraft.

## 2 Harmonic balance method and bifurcation tracking

### 2.1 Periodic solutions

The method will be applied to general non-autonomous nonlinear dynamical systems with $n$ degrees-of-freedom (DOFs) whose equations of motion are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}_{ext}(\omega,t) - \mathbf{f}_{nl}(\mathbf{x},\dot{\mathbf{x}}) = \mathbf{f}(\mathbf{x},\dot{\mathbf{x}},\omega,t) \quad (1)$$

where $\mathbf{M}$,$\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices respectively, $\mathbf{x}$ represents the displacements, the dots refer to the derivatives with respect to time $t$, $\mathbf{f}_{nl}$ represents the nonlinear forces and $\mathbf{f}_{ext}$

stands for the periodic external forces (harmonic excitation, for example) with frequency $\omega$. The term $\mathbf{f}$ gathers both the external and nonlinear forces.

The HB procedure consists in approximating $\mathbf{x}$ and $\mathbf{f}$ with their Fourier coefficients $\mathbf{z}$ and $\mathbf{b}$, respectively, and applying a Galerkin projection. Equations (1) then become

$$\mathbf{h}(\mathbf{z},\omega) \equiv \mathbf{A}(\omega)\mathbf{z} - \mathbf{b}(\mathbf{z},\omega) = \mathbf{0} \tag{2}$$

where $\mathbf{A}$ is the matrix describing the linear dynamics of the system (see [3] for further details). When embedded in a continuation procedure, solutions of (2) can be tracked with respect to the frequency $\omega$, which gives a frequency response of the system.

## 2.2 Stability analysis and detection of bifurcations

In the frequency domain, the stability of a periodic solution is usually assessed with Hill's method, which provides approximation of the Floquet exponents. In [1], it is shown that these exponents, here denoted $\tilde{\boldsymbol{\lambda}}$, can be found among the eigenvalues of $\tilde{\mathbf{B}}$, a matrix whose components are by-products of the HB method, i.e., depend only on $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{J}_z$, the jacobian of (2) with respect to $\mathbf{z}$. If at least one of the Floquet exponents has a positive real part, then the solution is unstable, otherwise it is asymptotically stable.

In this work, the detection of fold (F) and NS bifurcations along the frequency response is sought. This is achieved through the evaluation of test functions at each iteration of the branch, which have the property to pass through zero at a bifurcation. From the Floquet theory, it is known that a fold bifurcation is detected when one of the Floquet exponents $\tilde{\boldsymbol{\lambda}}$ crosses the imaginary axis through the real axis. As a consequence, the following test function has a root at a fold bifurcation:

$$\phi_F = \left|\tilde{\mathbf{B}}\right| \tag{3}$$

The NS bifurcations are characterized by a pair of Floquet exponents crossing the imaginary axis as complex conjugates. A dedicated test function is thus

$$\phi_{NS} = \left|\tilde{\mathbf{B}}_\odot\right| \tag{4}$$

where $\tilde{\mathbf{B}}_\odot$ stands for the bialternate matrix product of $\tilde{\mathbf{B}}$, which has the property to be singular when $\tilde{\mathbf{B}}$ has two complex-conjugate eigenvalues with no real part.

## 2.3 Tracking of bifurcations

In order to track bifurcations in a codimension-2 parameter space, one has to append one equation to (2), describing the bifurcation of interest:

$$\begin{cases} \mathbf{h} = 0 \\ \phi = 0 \end{cases} \tag{5}$$

Instead of considering $\phi = \phi_F$ and $\phi = \phi_{NS}$, which can lead to scaling problems for large matrices, in this paper the use of the so-called *bordering technique* [2] is proposed. The idea behind this technique is to replace the evaluation of the determinant of a given matrix $\mathbf{G}$ by the evaluation of a scalar function, herein denoted $g$, which vanishes as regular zero for the same system state and parameters as the determinant. It can be shown that a candidate for $g$ is obtained by solving the bordered system

$$\begin{bmatrix} \mathbf{G} & \mathbf{p} \\ \mathbf{q}^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ g \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \tag{6}$$

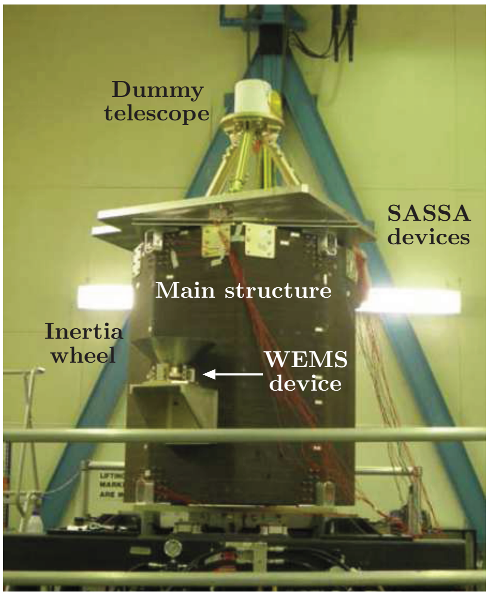
In this system, $^*$ denotes a conjugate transpose, and vectors $\mathbf{p}$ and $\mathbf{q}$ are chosen to ensure the nonsingularity of the bordered matrix. From (3-4), one then proposes the following equations for the extended system (5):

$$\begin{aligned} &\text{Fold bifurcations:} \quad && \phi = g_F \quad \text{with} \quad \mathbf{G}_F = \tilde{\mathbf{B}} \\ &\text{NS bifurcations:} \quad && \phi = g_{NS} \quad \text{with} \quad \mathbf{G}_{NS} = \tilde{\mathbf{B}}_\odot \end{aligned} \tag{7}$$

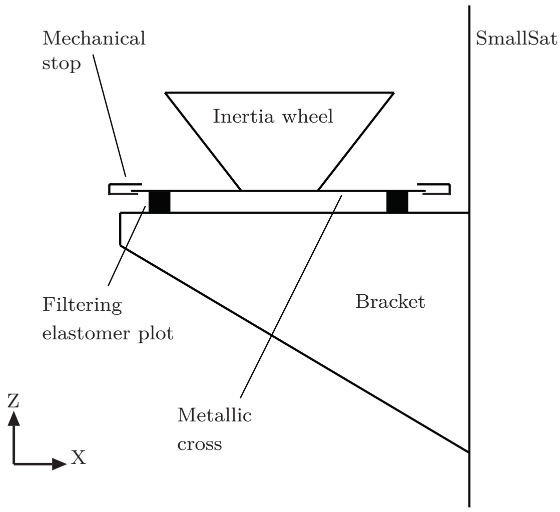
## 3 Validation of the method on the study of an industrial, complex model with strong nonlinearities: the SmallSat

The example studied is referred to as the *SmallSat*, a 64-kg structure represented in Figure 1(a) and which was conceived by EADS-Astrium as a platform for small satellites. As depicted in Figure 1(b), a support bracket connects to one of the eight walls the so-called *wheel elastomer mounting system* (WEMS) device which is loaded with an 8-kg dummy inertia wheel. The WEMS device is a mechanical filter which mitigates disturbances coming from the inertia wheel through the presence of a soft elastomeric interface between its mobile part, i.e. the inertia wheel and a supporting metallic cross, and its fixed part, i.e. the bracket and by extension the spacecraft. Moreover, eight mechanical stops limit the axial and lateral motions of the WEMS mobile part during launch, which gives rise to strongly nonlinear dynamical phenomena.

A reduced finite element model of the SmallSat with 37 DOFs is developed to conduct numerical experiments. Proportional damping is considered and the high dissipation in the elastomer components of the WEMS is described using lumped dashpots with coefficients $c_{ax}$ and $c_{lat}$ for axial (vertical) and lateral directions, respectively. Bilinear and trilinear springs are finally introduced to model the nonlinearities of the connections between the WEMS and the rest of the SmallSat. The DOF studied in this paper, referred to as NC1-$Z$, represents the vertical displacements of one of the nodes involved in the nonlinear connections.



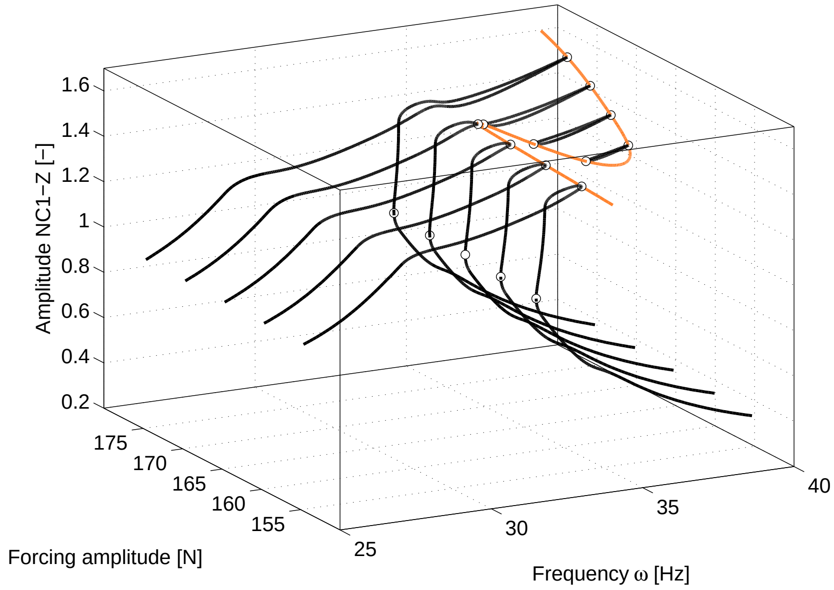

(a)



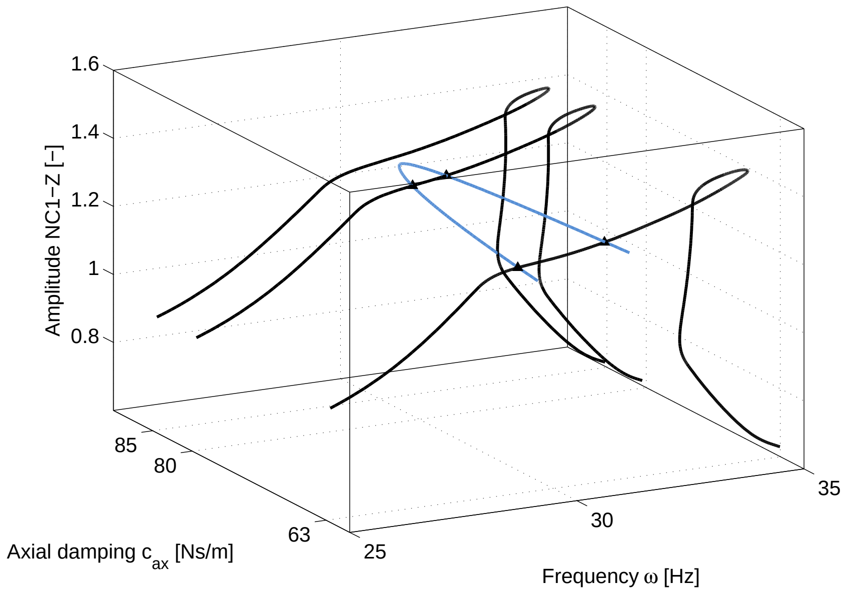

(b)

Figure 1: SmallSat spacecraft. (a) Photograph. (b) Schematic of the WEMS.

The responses of NC1-$Z$ to vertical harmonic excitations applied to the inertia wheel are given in Figures 2. In Figure 2(a), different forcing amplitudes $F$ are considered, with an axial damping value fixed to $c_{ax} = 63$ Ns/m. Fold bifurcations are then detected on the frequency responses, and using the outlined HB procedure the bifurcations related to the resonance peak are tracked with respect to $\omega$ and $F$. One verifies that the fold curve accurately follows the bifurcations close to the resonance peaks of the different frequency responses. Interestingly enough, the tracking also reveals the presence of detached resonance curves (DRCs), that are rarely discussed in the literature for such large systems. This example demonstrates the importance of a bifurcation monitoring: by only performing the continuation of periodic solutions, crucial information about the presence of the DRCs would be missed. Following a similar approach, Figure 2(b) represents forced responses computed for different axial damping values $c_{ax}$, with a forcing amplitude fixed to $F = 155$ N. In this case, one shows that increasing $c_{ax}$ up to 85 Ns/m leads to the elimination of the NS bifurcations, and hence to the elimination of undesirable QP oscillations.




(a)




(b)

Figure 2: Frequency responses (black lines) and bifurcation curves (orange and blue lines) of the NC1-$Z$ node of the SmallSat, for harmonic excitations applied vertically to the inertia wheel. (a) Tracking of fold bifurcations in the forcing amplitude $F$ - frequency $\omega$ parameters space, for a fixed value of axial damping $c_{ax} = 63\,\text{Ns/m}$. (b) Tracking of NS bifurcations in the axial damping $c_{ax}$ - frequency $\omega$ parameters space, for a fixed value of forcing amplitude $F = 155\,\text{N}$.

## References


[1] T. Detroux, L. Renson, G. Kerschen. *The harmonic balance method for advanced analysis and design of nonlinear mechanical systems*, Nonlinear Dynamics, vol. 2, Springer, 19-34, 2014.

[2] W. J. F. Govaerts. *Numerical methods for bifurcations of dynamical equilibria*, Siam, 2000.

[3] V. Jaumouillé, J.-J. Sinou, B. Petitjean. *An adaptive harmonic balance method for predicting the nonlinear dynamic responses of mechanical systems-Application to bolted structures*, Journal of Sound and Vibration, vol. 329, no. 19, Elsevier, 4048-4067, 2010.

[4] A. H. Nayfeh, D. T. Mook. *Nonlinear oscillations*, John Wiley & Sons, 2008.