# A Simple Deconvolution Algorithm

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**Abstract**

A very simple image deconvolution algorithm is described in three cases. It is first established in the case of noise-free data. Then, a method for taking noise into account is presented. Finally, a smoothing constraint is included in the algorithm.

## 1 Introduction

The imaging equation may be written, in the discrete case:

$$d_i = \sum_{j=1}^{M} S_{ij} f_j + n_i \tag{1}$$

where the sum is over all pixels $j = 1, \ldots, M$; $d_i$ is the observed signal, $S_{ij}$ the PSF (point spread function), $n_i$ the noise and $f_i$ the 'true' object.

The deconvolution problem may be summarized as following: given the data $d_i$ and the PSF $S_{ij}$, try to recover the true object $f_i$, solving Equation (1). It is immediately apparent that an exact solution of Equation (1) would require perfect knowledge of the noise $n_i$ in each pixel, which is, by definition, impossible. The noise has thus to be estimated in some way. Further, the deconvolution problem, as most inverse problems, is ill-posed: a small error on the data leads to very large errors on the solution (this is related to the near-singularity of the matrix $S$). Some method has thus to be found in order to 'stabilize' the solution.

## 2 Noise-Free Data

In the case of noise-free data, $n_i = 0$, the first of the above-mentioned problems disappears. Equation (1) reduces to

$$d_i = \sum_{j=1}^{M} S_{ij} f_j \tag{2}$$

whose direct solution

$$f_i = \sum_{j=1}^{M} (S^{-1})_{ij} d_j \tag{3}$$

leads to wild oscillations, due to the problem being badly posed. A classical way to reduce these oscillations is to avoid the direct solution (3) and rather solve Equation (2) in an approximate way, adding some smoothing constraint (e.g. maximum entropy of the object).

Another approach is to enforce positivity of $f_i$: no physical object being able to emit any negative amount of light, the condition $f_i \geq 0$ can be introduced in the problem. No negative value

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being allowed, the oscillations are automatically damped. Lucy's method [2] is such a method. It is a member of a class of recursive algorithms, of which we shall present another one, namely the 'quasi–inverse matrix' method [3].

The algorithm can be obtained by rewriting Equation (2) in the form

$$1 = \frac{d_i}{\sum_{j=1}^{M} S_{ij} f_j} \tag{4}$$

and multiplying both sides of the equation by $f_i$. Equation (4) can then be transformed into a recursive algorithm:

$$f_i^{(n+1)} = f_i^{(n)} \frac{d_i}{\sum_{j=1}^{M} S_{ij} f_j^{(n)}} \tag{5}$$

where $f_i^{(n)}$ is the $n^{th}$ estimate of the object $f_i$.

As a first estimate, the image itself can be used: $f_i^{(1)} = d_i$ (a constant value might also be used, but this just adds one iteration since $f_i^{(2)} = d_i$ if $f_i^{(1)} = constant$). It is easily seen that this algorithm ensures positivity of the object if the image itself is positive: $d_i \geq 0$ and $f_i^{(n)} \geq 0 \Rightarrow f_i^{(n+1)} \geq 0$, for any reasonable PSF.

Since this algorithm does not take noise into account, it is well suited for deconvolving high S/N images. See ref. [3] for an illustration.

## 3 Noisy Data

Manipulating Equation (1) in the same way as was Equation (2) in the previous section, we get

$$f_i^{(n+1)} = f_i^{(n)} \frac{d_i - n_i}{\sum_{j=1}^{M} S_{ij} f_j^{(n)}} \tag{6}$$

which is the quasi–inverse matrix algorithm applied to noisy data. However, $n_i$ being unknown, this algorithm cannot be applied as such: we first need to estimate the noise in each pixel.

Let us assume that the noise follows a normal distribution with standard deviation $\sigma_i$ in pixel $i$, with $\sigma_i$ known (e.g. in the case of CCD images, $\sigma_i \simeq \sqrt{R^2 + N_i}$ where $R$ is the readout noise and $N_i$ the number of electrons in pixel $i$).

Let us further assume that our estimate $f_i^{(n)}$ is close to the solution. Then, the difference between the data $d_i$ and the convolved object is approximately equal to the noise $n_i$. According to our first hypothesis, the quantity

$$\frac{n_i}{\sigma_i} \simeq \frac{d_i - \sum_{j=1}^{M} S_{ij} f_j^{(n)}}{\sigma_i} \tag{7}$$

should then follow a standard normal distribution (zero mean, unit standard deviation). Let $\nu_i$ $(i = 1, \ldots, M)$ be the theoretically expected values for M data following such a distribution (they can be easily computed following classical statistics textbooks). Let us then sort the $\nu_i$ and $n_i/\sigma_i$ by increasing values and replace each $n_i$ in Equation (6) by the corresponding $\sigma_i \nu_i$ (that is, the $k^{th}$ $n_i$ in the sorted list is replaced by the $k^{th}$ $\nu_i$ times the $\sigma_i$ for the corresponding pixel). We then get the following algorithm:

$$f_i^{(n+1)} = f_i^{(n)} \frac{d_i - \sigma_i \nu_i}{\sum_{j=1}^{M} S_{ij} f_j^{(n)}} \tag{8}$$

which can be applied in cases where the noise is not negligible and has been found to give very good results on some real astronomical images [1].

## 4 Smoothing Constraint

In this section, we briefly show how the quasi–inverse matrix method can be generalized to take a smoothing constraint into account.

In this case, one does not try to solve Equation (1) exactly, but rather minimize the sum of the square of the residuals (image minus fit), subject to some constraint $H(f_1,\ldots,f_M)=0$. This can be done by introducing a Lagrange parameter $\lambda$ and minimizing the sum

$$\mathcal{S} \equiv \sum_{i=1}^{M} \frac{1}{\sigma_i^2}[\sum_{j=1}^{M} S_{ij} f_j - d_i]^2 + \lambda H(f_1,\ldots,f_M) \tag{9}$$

The $f_k$'s can be found by solving a system of $M$ equations:

$$\frac{\partial \mathcal{S}}{\partial f_k} = 0 \quad (k=1,\ldots,M) \tag{10}$$

which can be written

$$\sum_{j=1}^{M} A_{kj} f_j = B_k - \frac{\lambda}{2}\frac{\partial H}{\partial f_k} \tag{11}$$

with

$$A_{kj} \equiv \sum_{i=1}^{M} \frac{1}{\sigma_i^2} S_{ik} S_{ij} \tag{12}$$

and

$$B_k \equiv \sum_{i=1}^{M} \frac{1}{\sigma_i^2} S_{ik} d_i \tag{13}$$

This equation has the same form as the imaging Equation (1) and can thus be, in principle, solved by a similar recursive algorithm:

$$f_k^{(n+1)} = f_k^{(n)} \frac{B_k - \frac{\lambda}{2}\frac{\partial H}{\partial f_k}}{\sum_{j=1}^{M} A_{kj} f_j^{(n)}} \tag{14}$$

This algorithm tends to give smoother solutions (even for $\lambda = 0$) then Equation (5), but convergence is slower. Also, depending on the form of the smoothing constraint $H$, convergence is not always guaranteed for large values of $\lambda$. In practice, the algorithm converges if the $B_k$ term dominates the numerator of Equation (14). This is generally not a serious limitation since smoothing is often quite effective even for very small values of $\lambda$.

## 5 Concluding Remarks

The convergence of the algorithms presented above is generally quite fast in the first iterations and then slows down as the solution is approached. They should thus be stopped after a few ($\sim$ 10) iterations.

Moreover, it should be pointed out that some care has to be taken not to introduce negative values of $d_i$ or $d_i - \sigma_i \nu_i$ in the right–hand–side of Equations (5), (8) or (14), in order to preserve the positivity of the solution. In this respect, one can note that Equation (8) is somewhat superior to Equation (5) as negative values of $d_i$ in the background will tend to correspond to negative values of $\nu_i$, in which case the numerator of Equation (8) might become positive. The few remaining negative values may then be set to zero, or any small positive quantity. One might also avoid negative values by adding a constant to the background, but this is not a good practice since the benefit of imposing positivity of the object would then be partially lost and 'ringing' might appear in the background around compact objects. For the same reason, it is wise to subtract the sky background prior to deconvolving.

## References

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