Batch Mode Reinforcement Learning based on the Synthesis of Artificial Trajectories

R. Fonteneau[(1),(2)]

Joint work with Susan A. Murphy[(3)] , Louis Wehenkel[(2)] and Damien Ernst[(2)]

[(1)] Inria Lille – Nord Europe, France
[(2)] University of Liège, Belgium
[(3)] University of Michigan, USA

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Outline

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  - Batch Mode Reinforcement Learning
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- **Estimating the Performances of Policies**
  - Model-free Monte Carlo Estimation
  - The MFMC Algorithm
  - Theoretical Analysis
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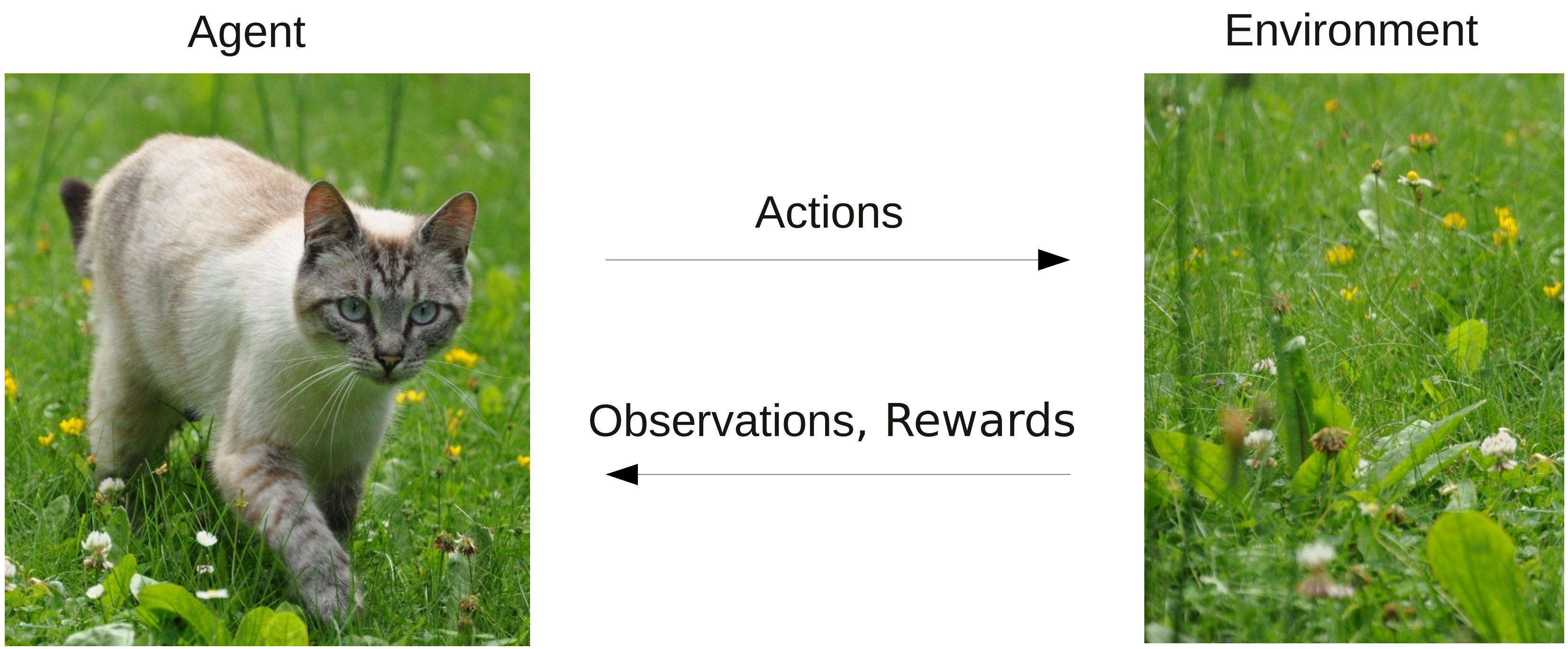
- **Conclusions**

Batch Mode Reinforcement Learning

Reinforcement Learning



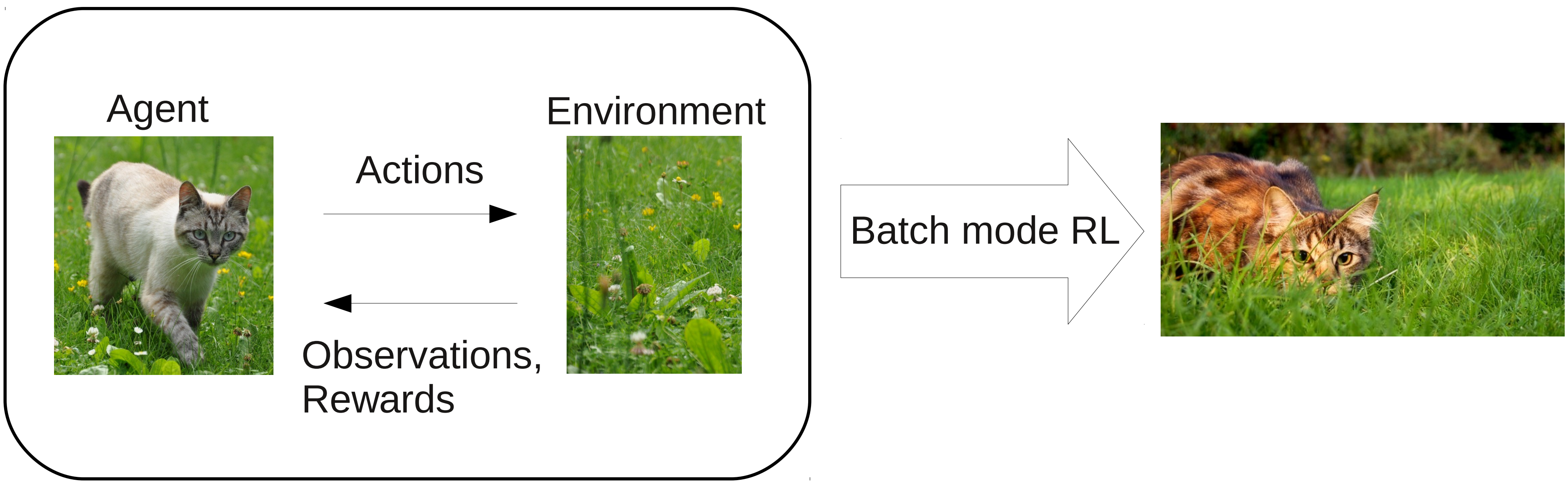
Examples of rewards:




- Reinforcement Learning (RL) aims at **finding a policy maximizing received rewards** by **interacting** with the environment

Batch Mode Reinforcement Learning

- All the available information is contained in a **batch collection of data**
- Batch mode RL aims at computing a (near-)optimal policy from this collection of data

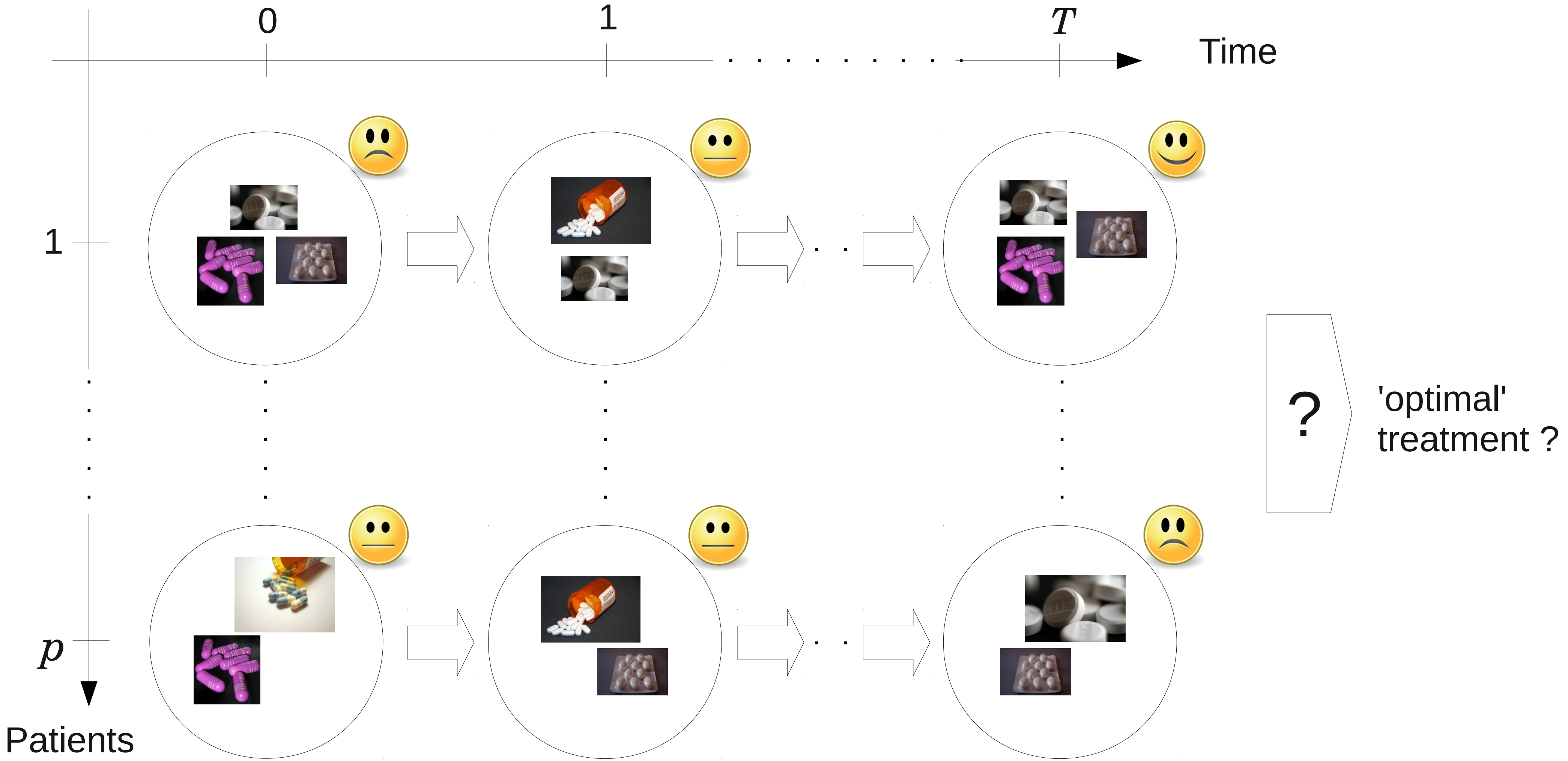


Finite collection of trajectories of the agent

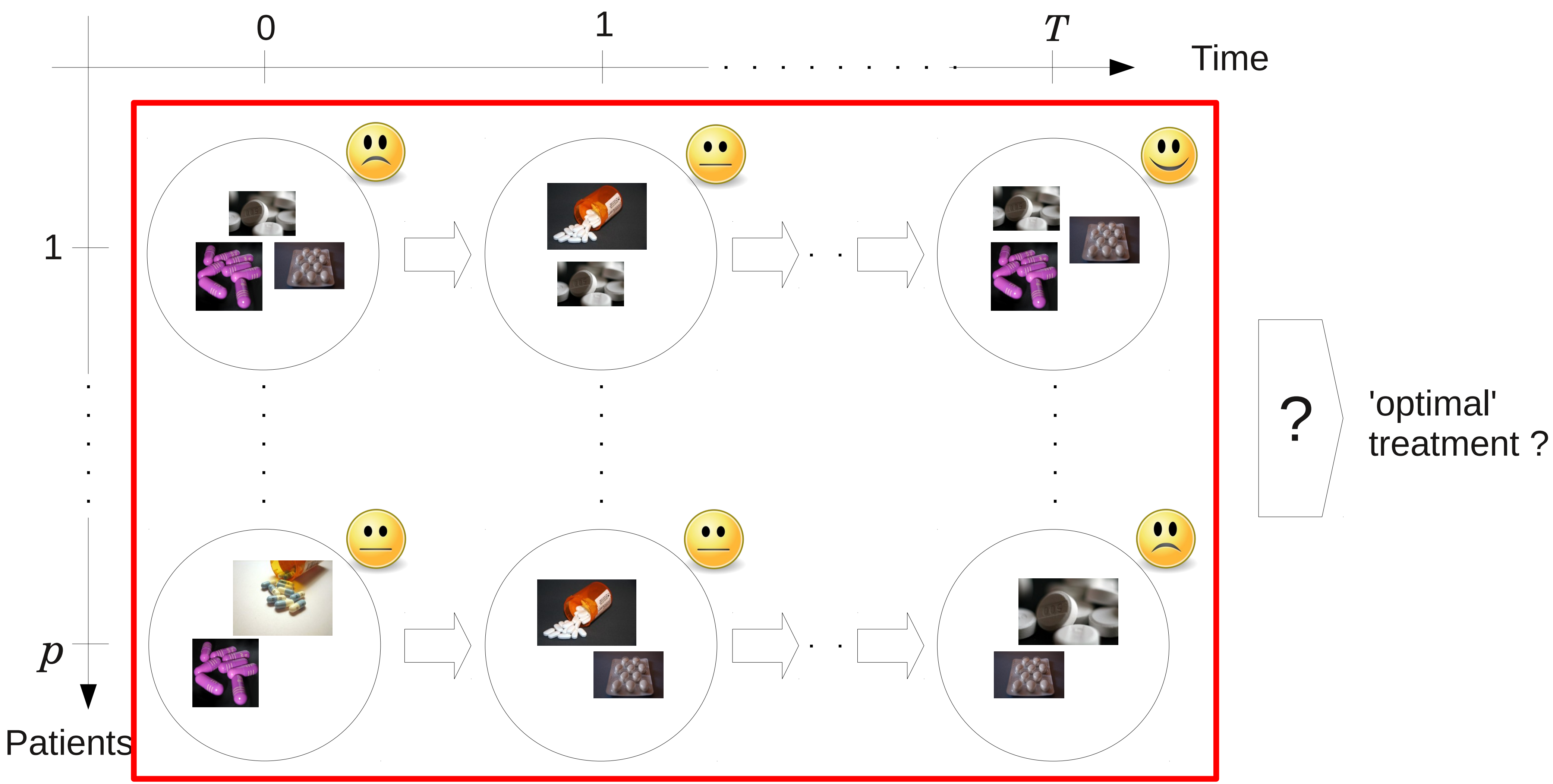
Near-optimal decision strategy

- Examples of BMRL problems: dynamic treatment regimes (inferred from clinical data), marketing optimization (based on customers histories), finance, etc...

Batch Mode Reinforcement Learning



Batch Mode Reinforcement Learning



Batch collection of trajectories of patients

Objectives

- Main goal: **Finding a "good" policy**

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  - Computing safe policies
  - Choosing how to generate additional transitions
  - ...

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- Usual Approach:
  - To **combine dynamic programming with function approximators** (neural networks, regression trees, SVM, linear regression over basis functions, etc)
  - Function approximators have two main roles:
    - To offer a **concise representation** of state-action value function for deriving value / policy iteration algorithms
    - To **generalize information** contained in the finite sample

Remaining Challenges

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- **A New Approach: Synthesizing Artificial Trajectories**

A New Approach: Synthesizing Artificial Trajectories

Formalization

Reinforcement learning

- System dynamics: $x_{t+1} = f\left(x_t, u_t, w_t\right)$

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$t \in \{0, \ldots, T-1\}$ $x_t \in \mathcal{X} \subset \mathbb{R}^d$ $u_t \in \mathcal{U}$ $w_t \in \mathcal{W}$ $w_t \sim p_{\mathcal{W}}(\cdot)$

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- Reward function: $r_t = \rho\left(x_t, u_t, w_t\right)$

- Performance of a policy $h : \{0, \ldots, T-1\} \times \mathcal{X} \to \mathcal{U}$

$$J^h(x_0) = \mathbb{E}\left[R^h(x_0, w_0, \ldots, w_{T-1})\right]$$

where

$$R^h(x_0, w_0, \ldots, w_{T-1}) = \sum_{t=0}^{T-1} \rho(x_t, h(t, x_t), w_t)$$

$$x_{t+1} = f(x_t, h(t, x_t), w_t)$$

Formalization

Batch mode reinforcement learning

- The system dynamics, reward function and disturbance probability distribution are **unknown**

Formalization

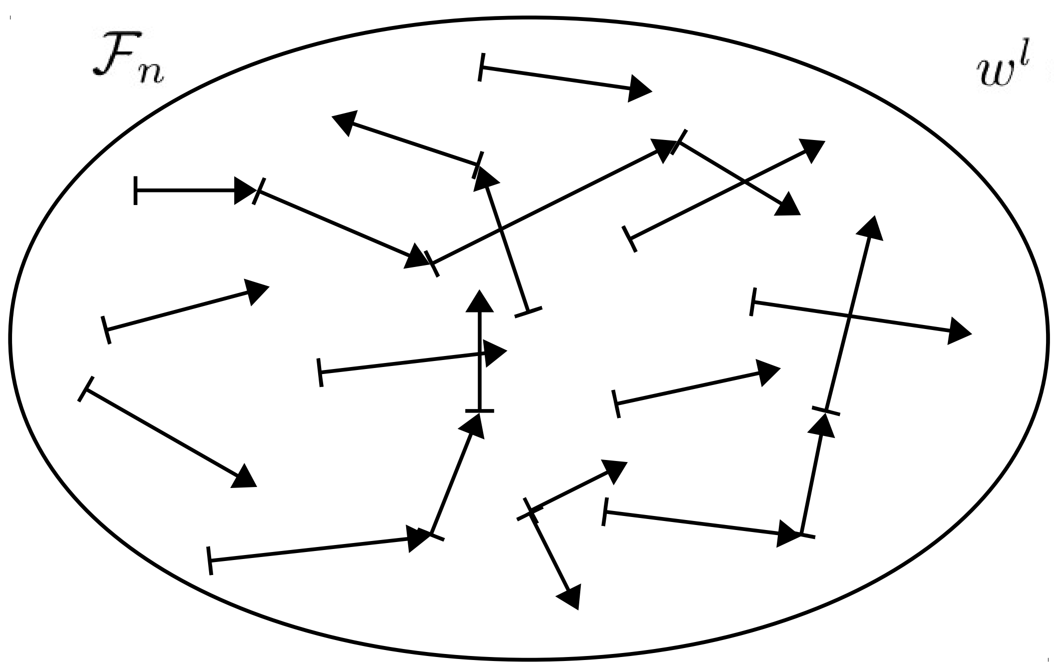
Batch mode reinforcement learning

- The system dynamics, reward function and disturbance probability distribution are **unknown**
- Instead, we have access to a **sample of one-step system transitions**:

$$\mathcal{F}_n = \left\{ \left(x^l, u^l, r^l, y^l \right) \right\}_{l=1}^{n} \qquad \forall l \in \{1, \ldots, n\}, \qquad r^l = \rho\left(x^l, u^l, w^l\right)$$

$$y^l = f\left(x^l, u^l, w^l\right)$$

$$w^l \sim p_{\mathcal{W}}(\cdot)$$



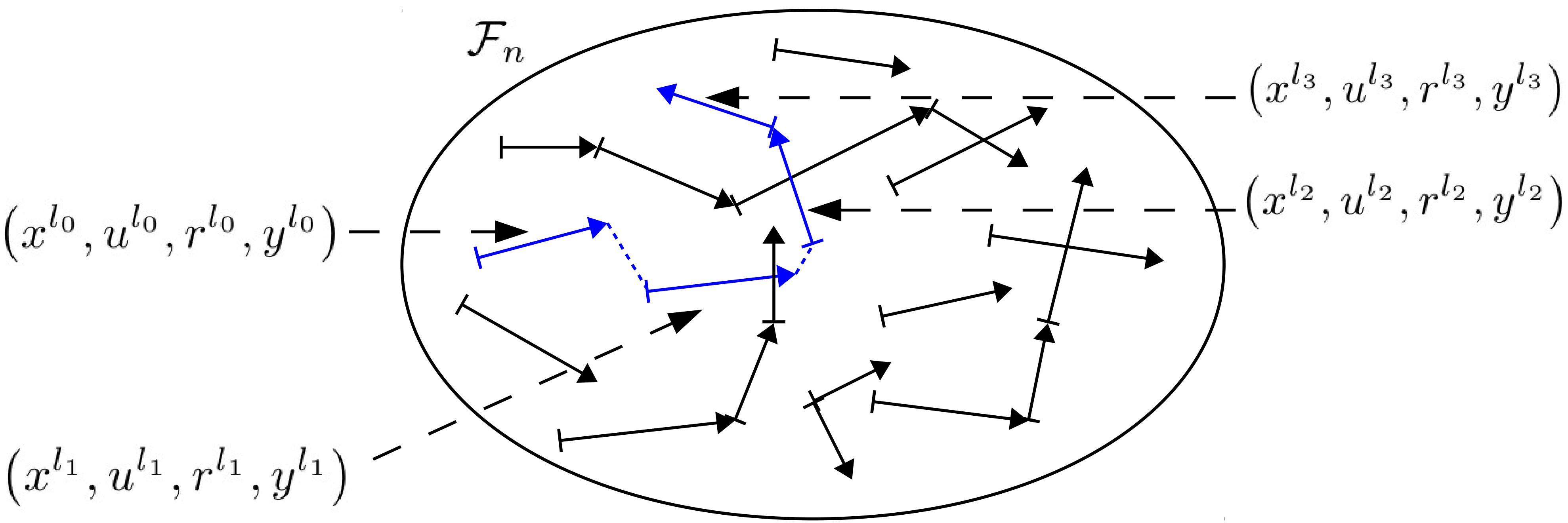
Formalization

Artificial trajectories

- Artificial trajectories are **(ordered) sequences of elementary pieces of trajectories:**

$$\left[(x^{l_0}, u^{l_0}, r^{l_0}, y^{l_0}), \ldots, (x^{l_{T-1}}, u^{l_{T-1}}, r^{l_{T-1}}, y^{l_{T-1}}) \right] \in \mathcal{F}_n^T$$

$$l_t \in \{1, \ldots, n\} , \qquad \forall t \in \{0, \ldots, T-1\}$$



Artificial Trajectories: What For?

- Artificial trajectories can help for:
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Estimating the Performances of Policies

Model-free Monte Carlo Estimation

- If the system dynamics and the reward function were accessible to simulation, then **Monte Carlo estimation** would allow estimating the performance of h

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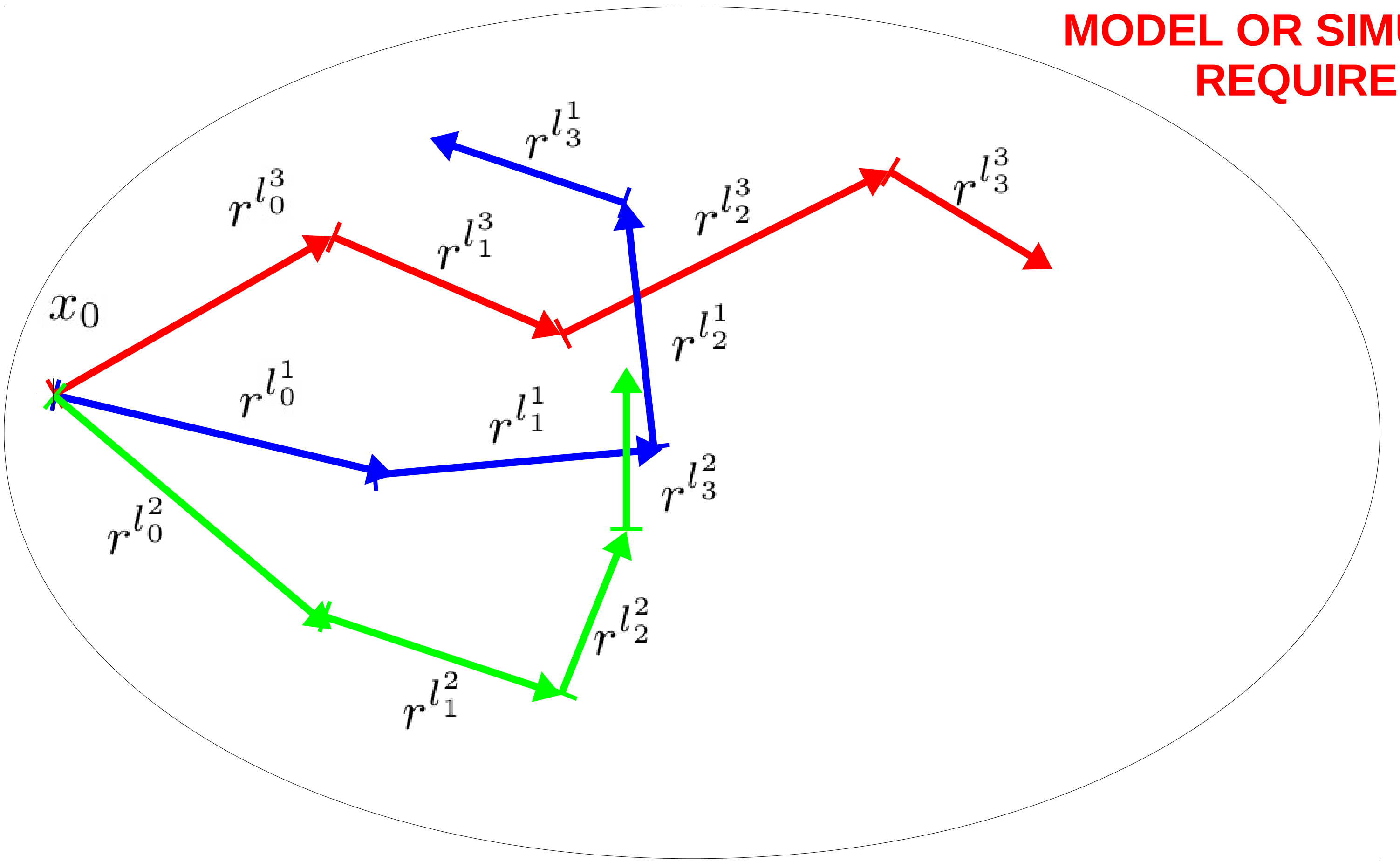

MODEL OR SIMULATOR REQUIRED!

$$\mathbb{M}_3^h(x_0) = \frac{\left(r^{l_0^1} + r^{l_1^1} + r^{l_2^1} + r^{l_3^1}\right) + \left(r^{l_0^2} + r^{l_1^2} + r^{l_2^2} + r^{l_3^2}\right) + \left(r^{l_0^3} + r^{l_1^3} + r^{l_2^3} + r^{l_3^3}\right)}{3}$$

Model-free Monte Carlo Estimation

- If the system dynamics and the reward function were accessible to simulation, then **Monte Carlo (MC) estimation** would allow estimating the performance of h
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- These artificial trajectories are built so as to **minimize the discrepancy (using a distance metric Δ) with a classical MC sample** that could be obtained by simulating the system with the policy h; each one step transition is used **at most once**

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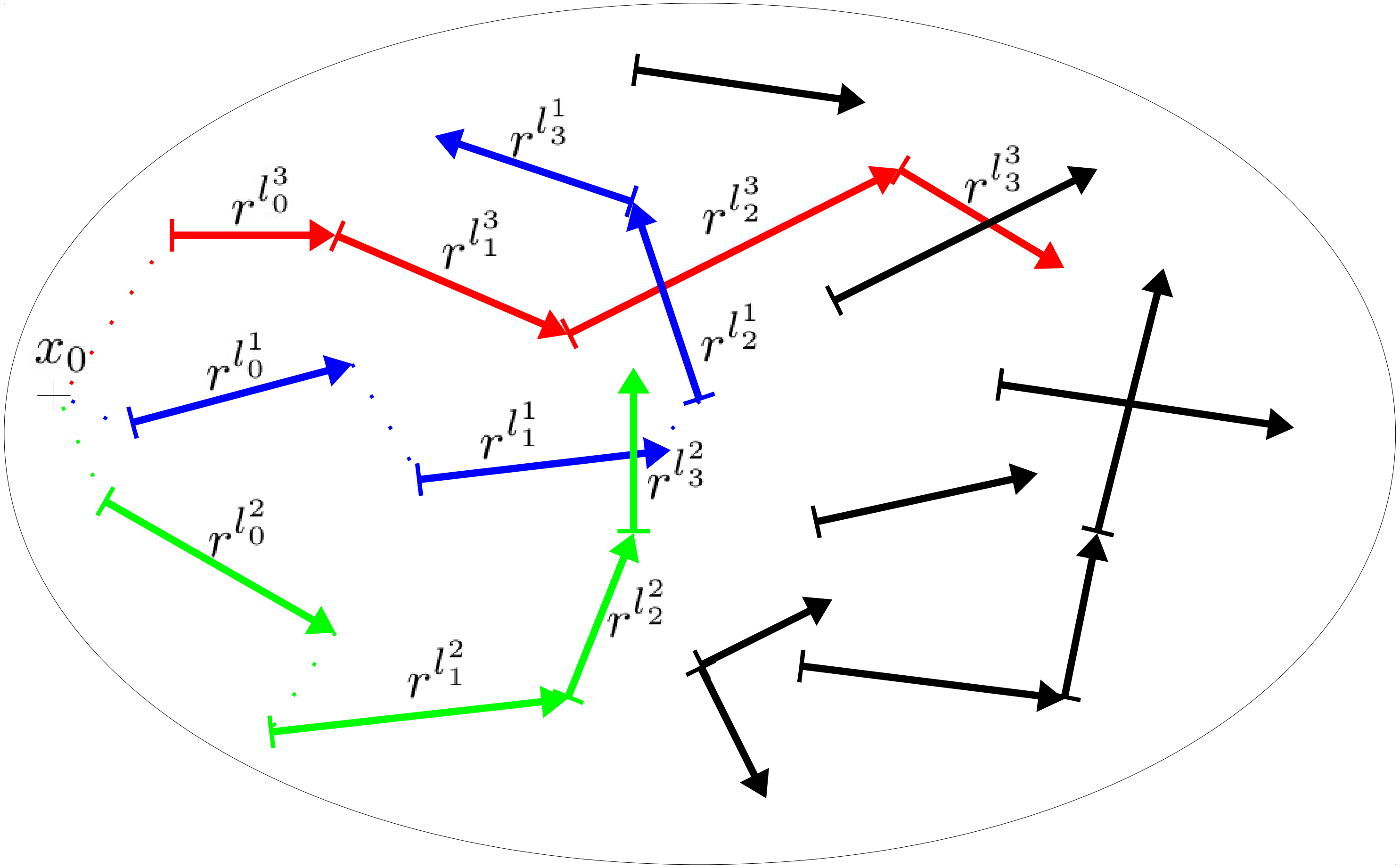
- We propose an approach that mimics MC estimation by rebuilding p **artificial trajectories** from one-step system transitions
- These artificial trajectories are built so as to **minimize the discrepancy (using a distance metric Δ) with a classical MC sample** that could be obtained by simulating the system with the policy h; each one step transition is used **at most once**
- We average the cumulated returns over the p artificial trajectories to obtain the **Model-free Monte Carlo estimator** (MFMC) of the expected return of h:

$$\mathfrak{M}_p^h(\mathcal{F}_n, x_0) = \frac{1}{p} \sum_{i=1}^{p} \sum_{t=0}^{T-1} r^{l_t^i}$$

Model-free Monte Carlo Estimation

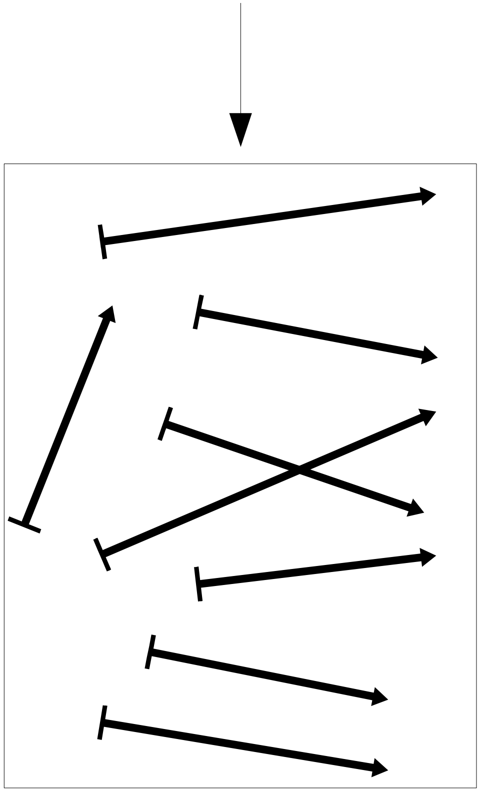


$$\mathfrak{M}_3^h\left(\mathcal{F}_n, x_0\right) = \frac{\left(r^{l_0^1} + r^{l_1^1} + r^{l_2^1} + r^{l_3^1}\right) + \left(r^{l_0^2} + r^{l_1^2} + r^{l_2^2} + r^{l_3^2}\right) + \left(r^{l_0^3} + r^{l_1^3} + r^{l_2^3} + r^{l_3^3}\right)}{3}$$

The MFMC algorithm

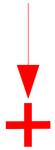
Example with T = 3, p = 2, n = 8

$$\mathcal{F}_n = \left\{ \left(x^l, u^l, r^l, y^l \right) \in \mathcal{X} \times \mathcal{U} \times \mathbb{R} \times \mathcal{X} \right\}_{l=1}^{n}$$



The MFMC algorithm

$(x_0, h(0, x_0))$



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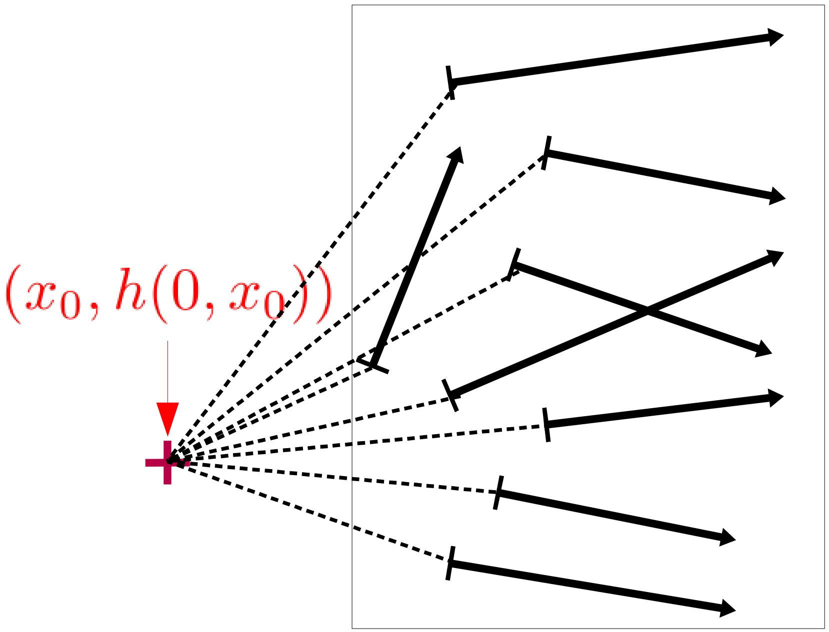
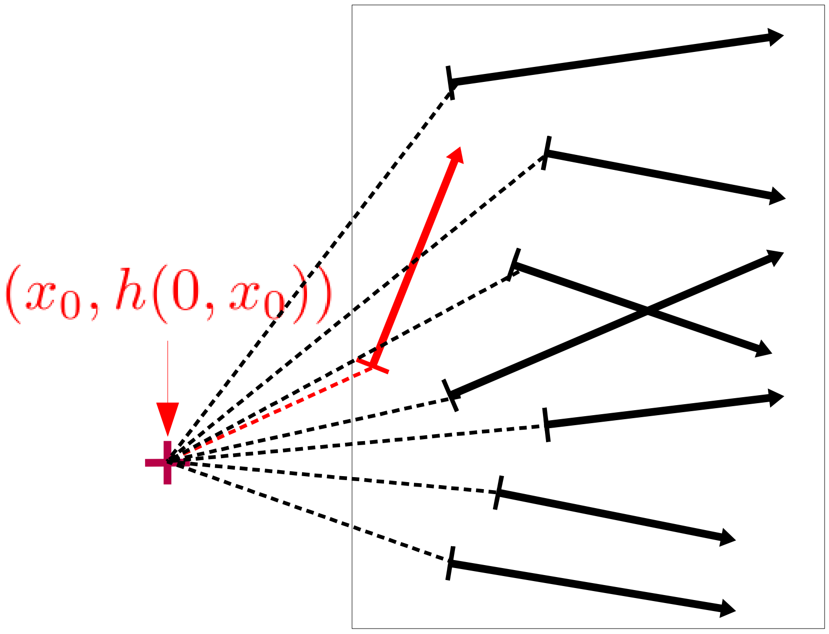


$\mathcal{G} = \mathcal{F}_n$

The MFMC algorithm



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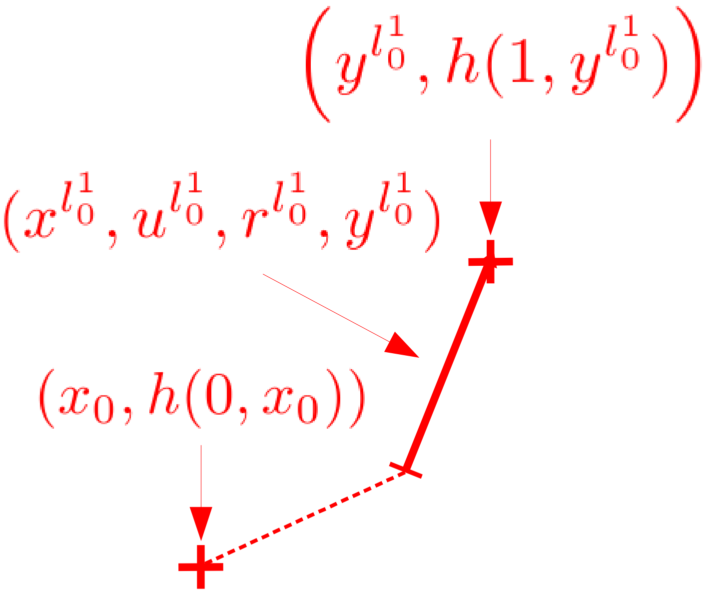
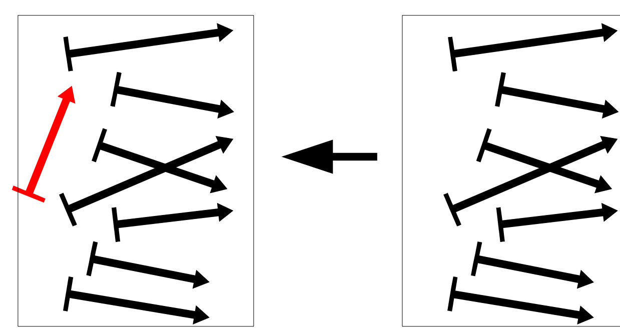


$$\mathcal{G} = \mathcal{G} \setminus \{(x^{l_0^1}, u^{l_0^1}, r^{l_0^1}, y^{l_0^1})\}$$



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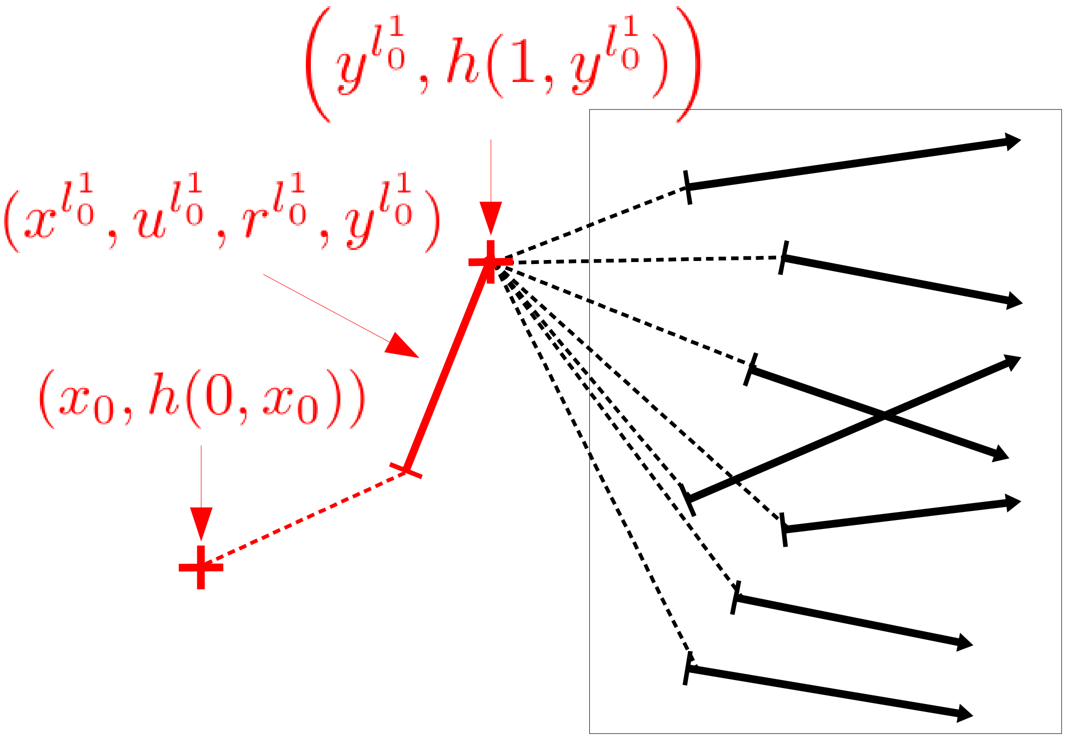

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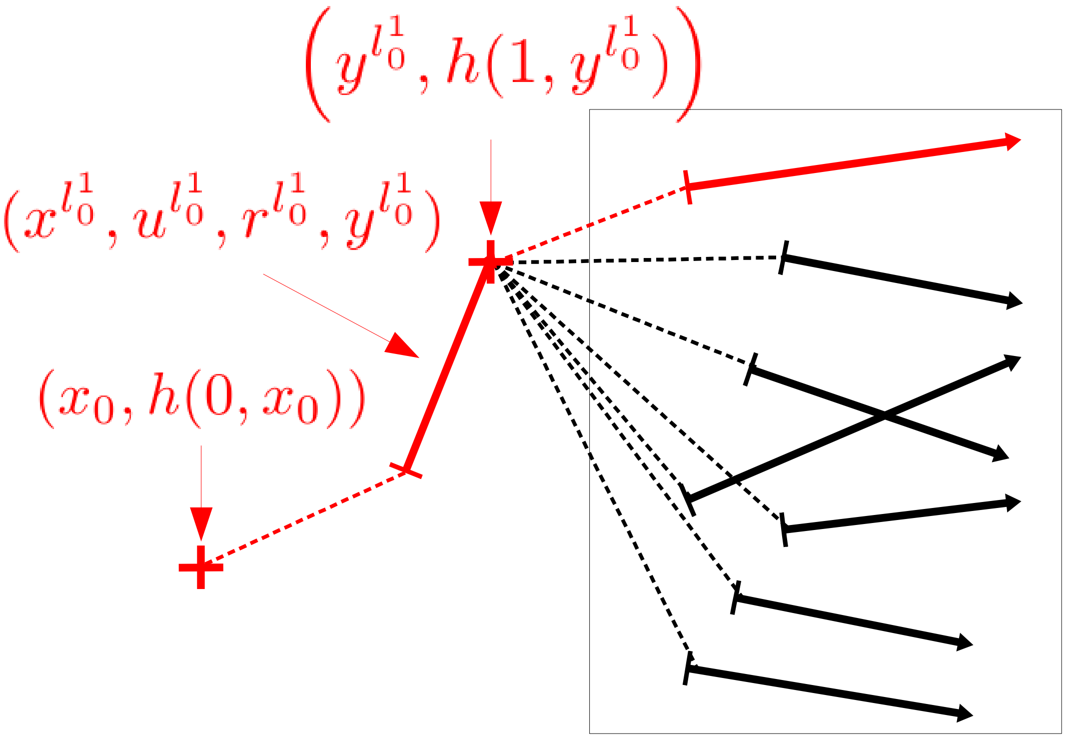

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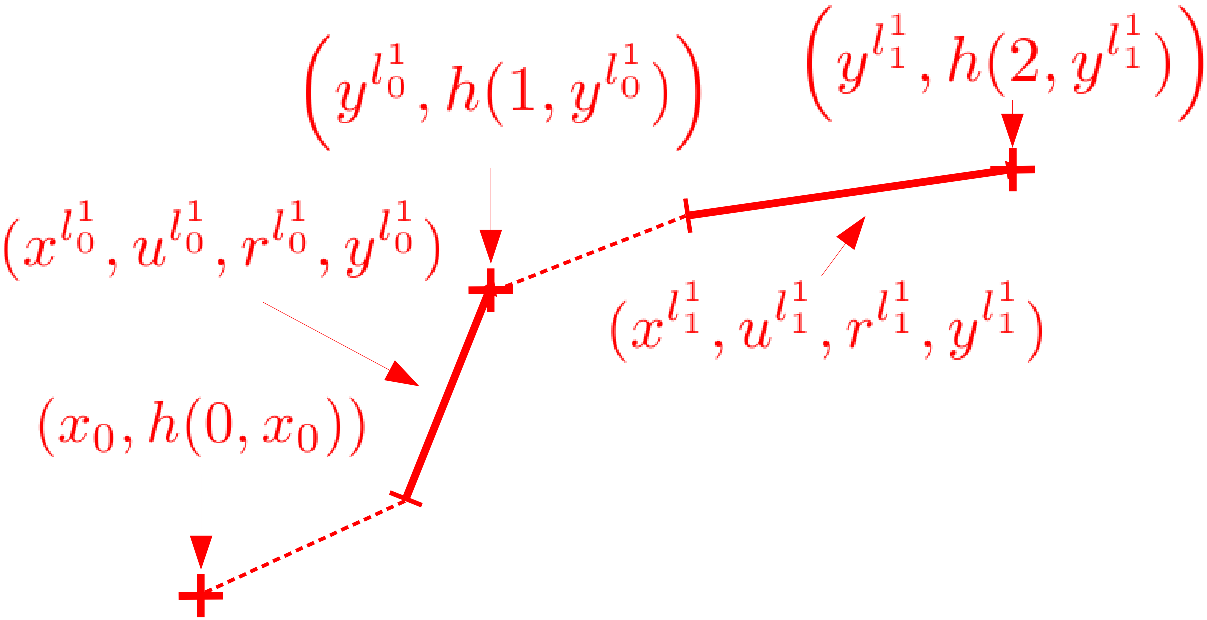

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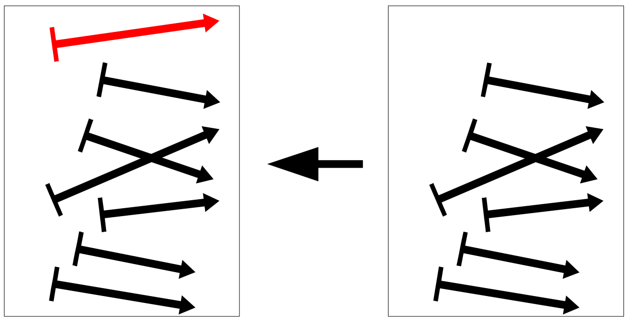

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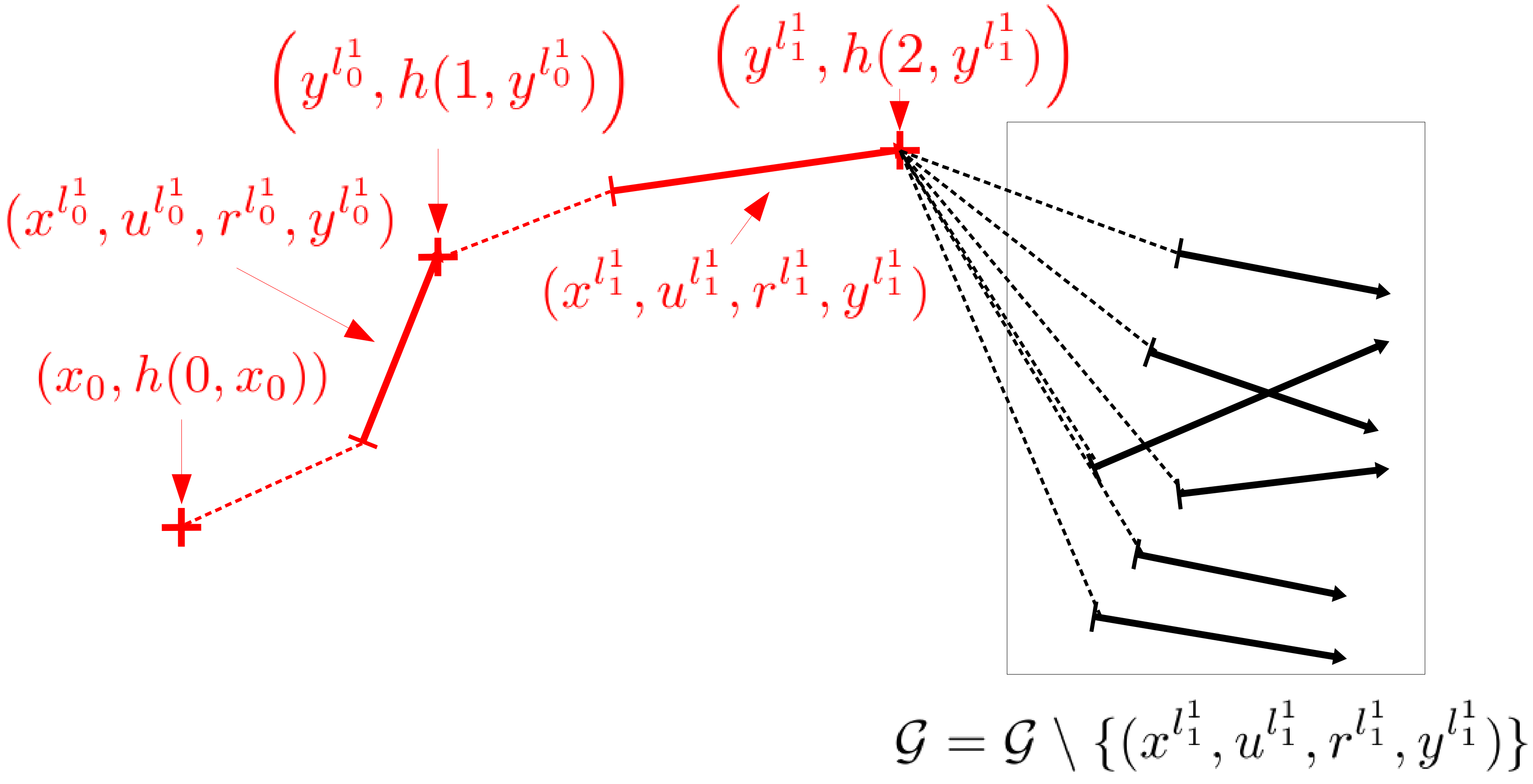

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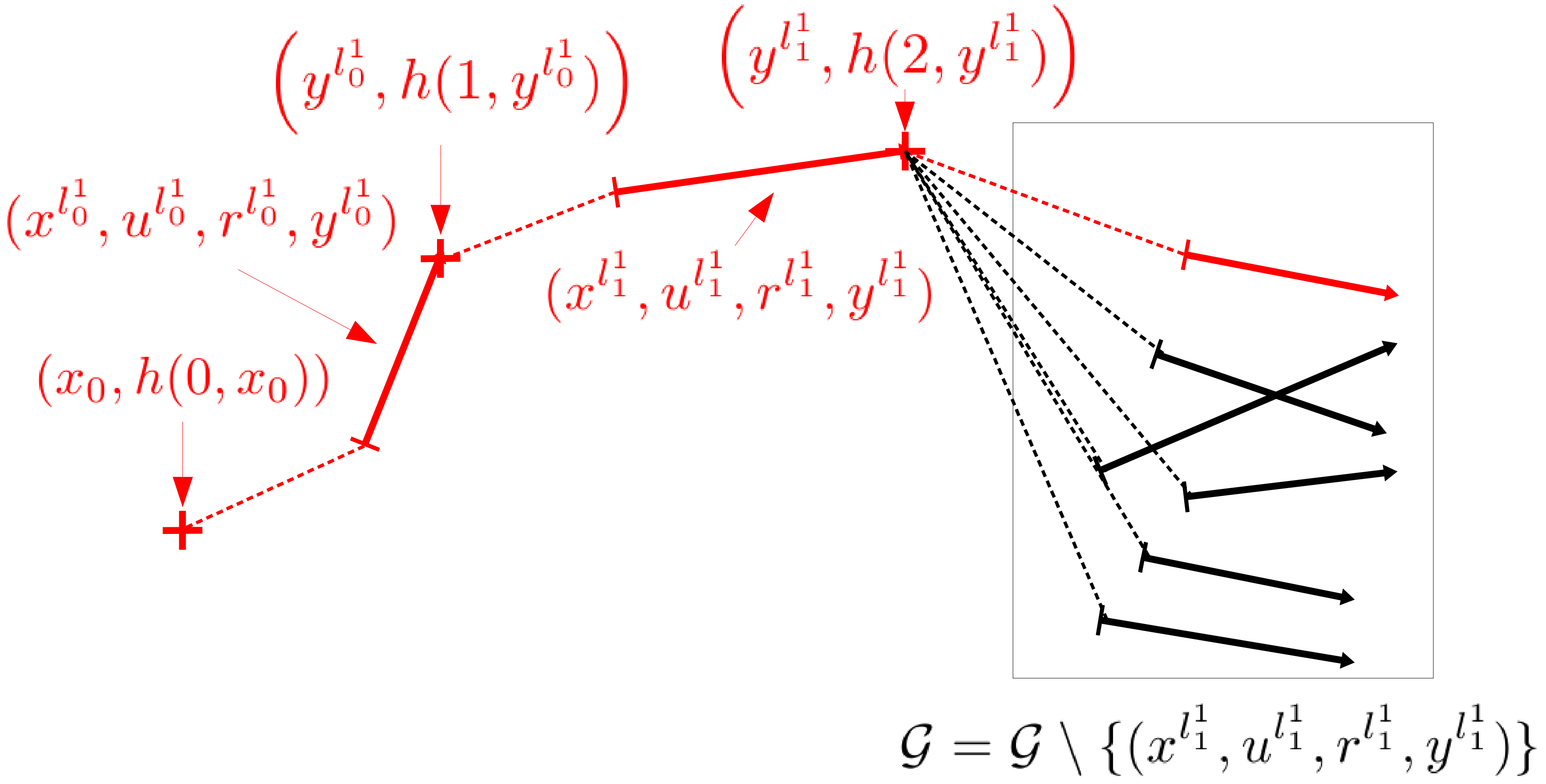

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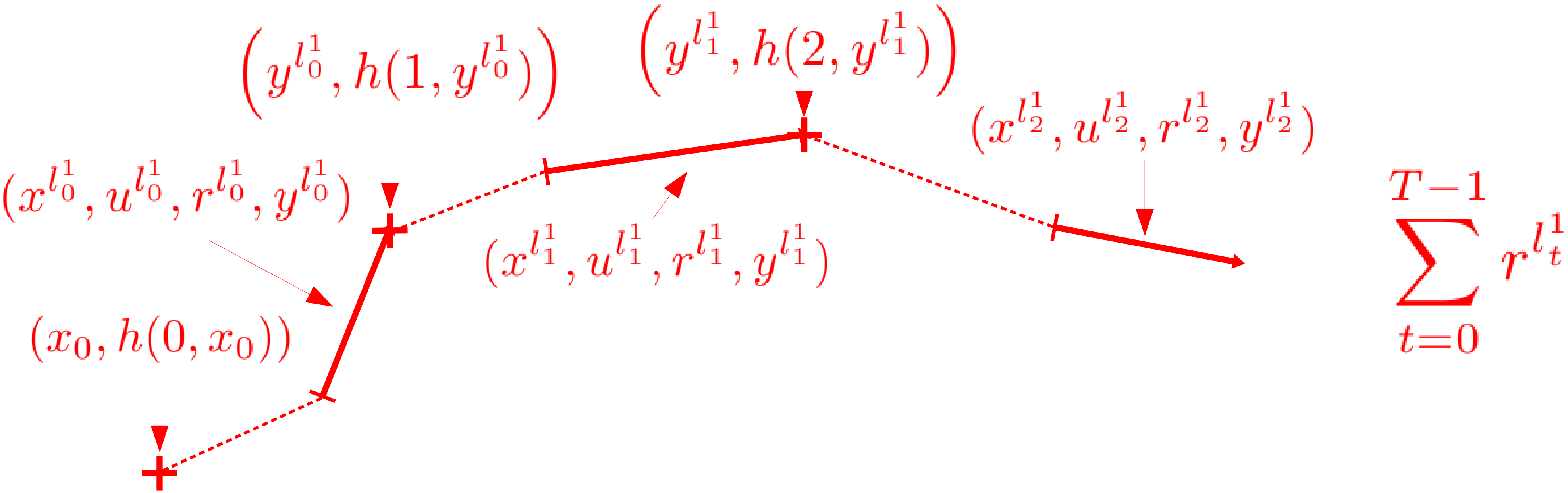

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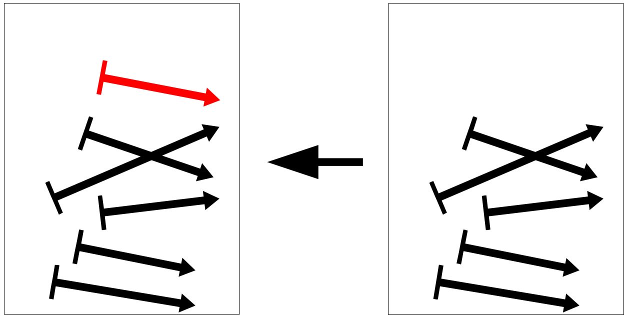

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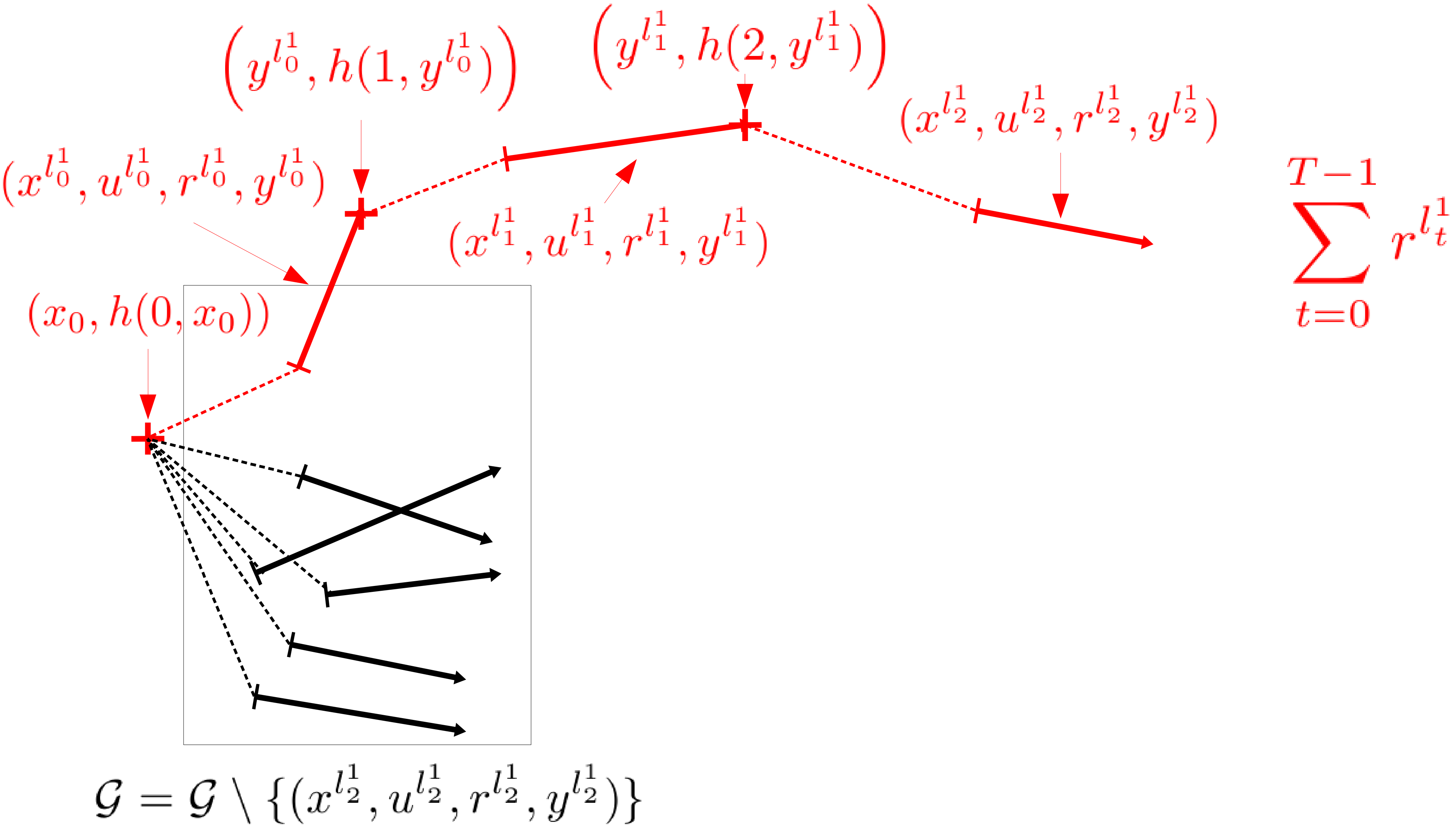

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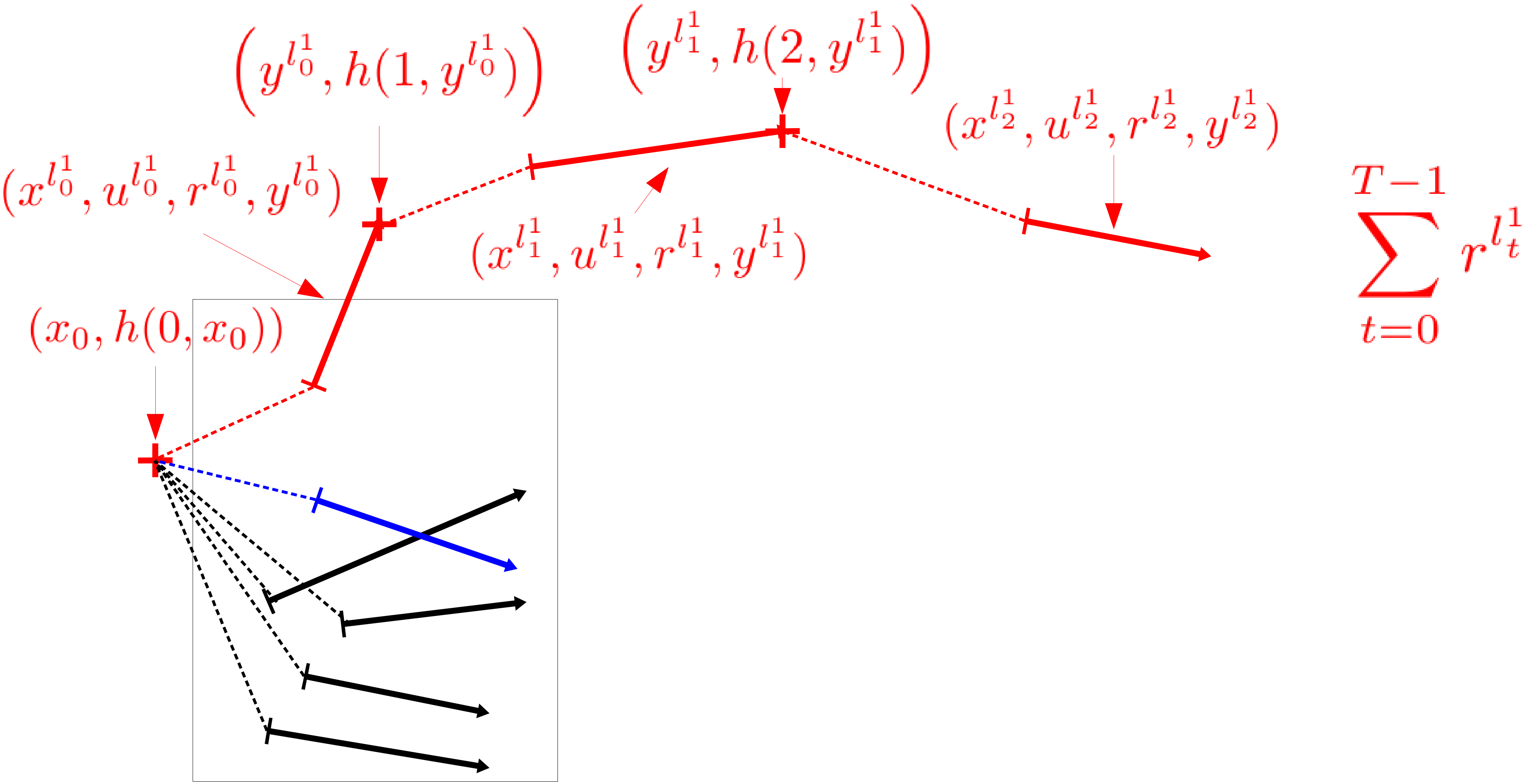

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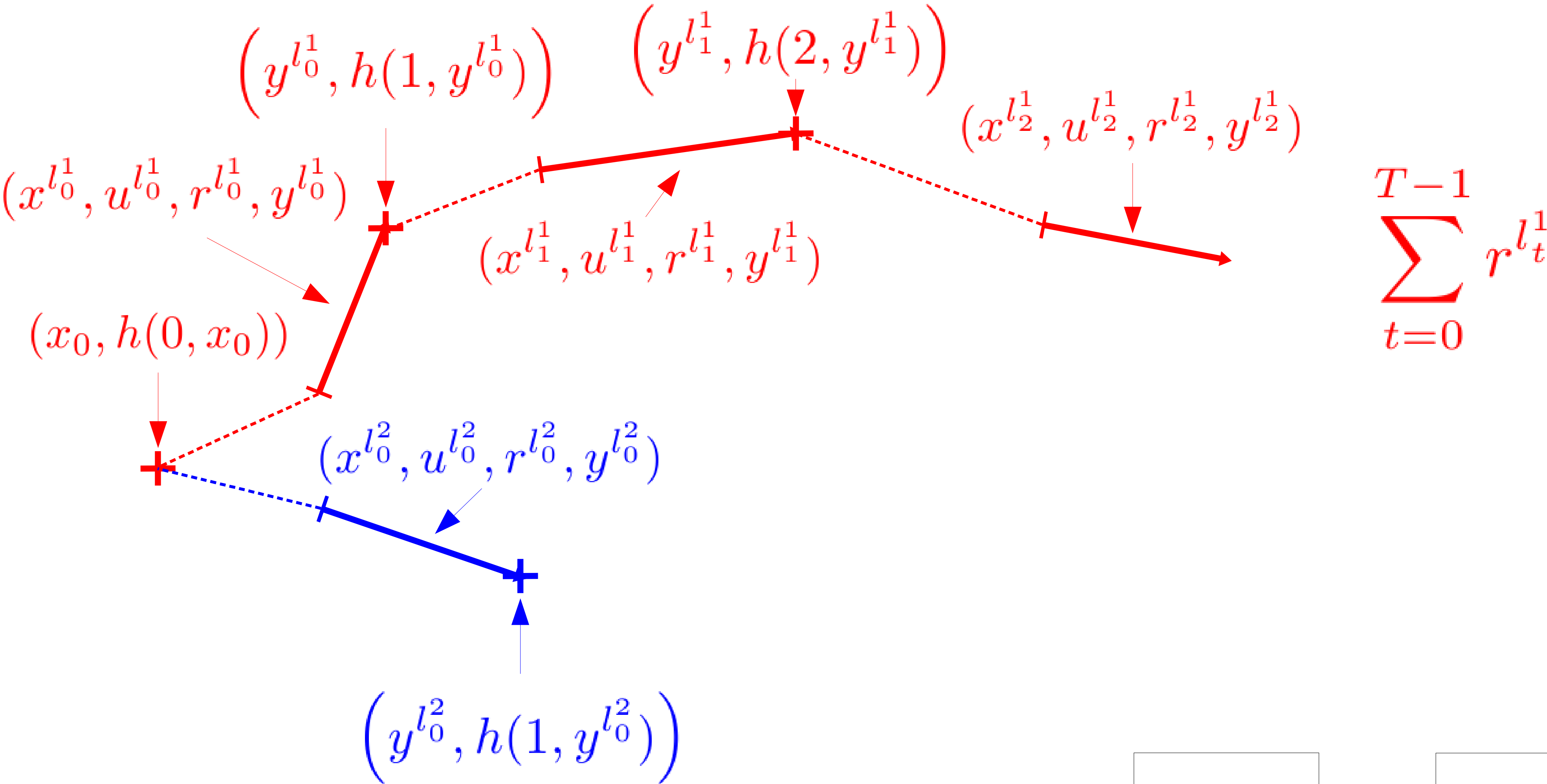

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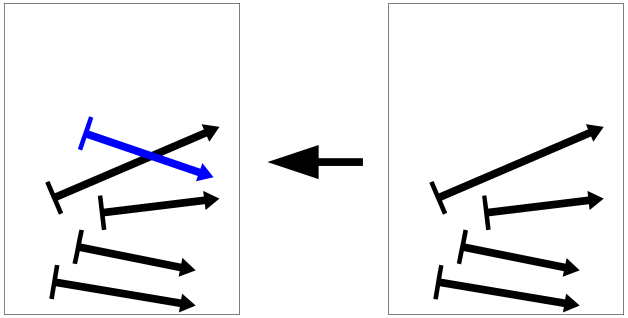

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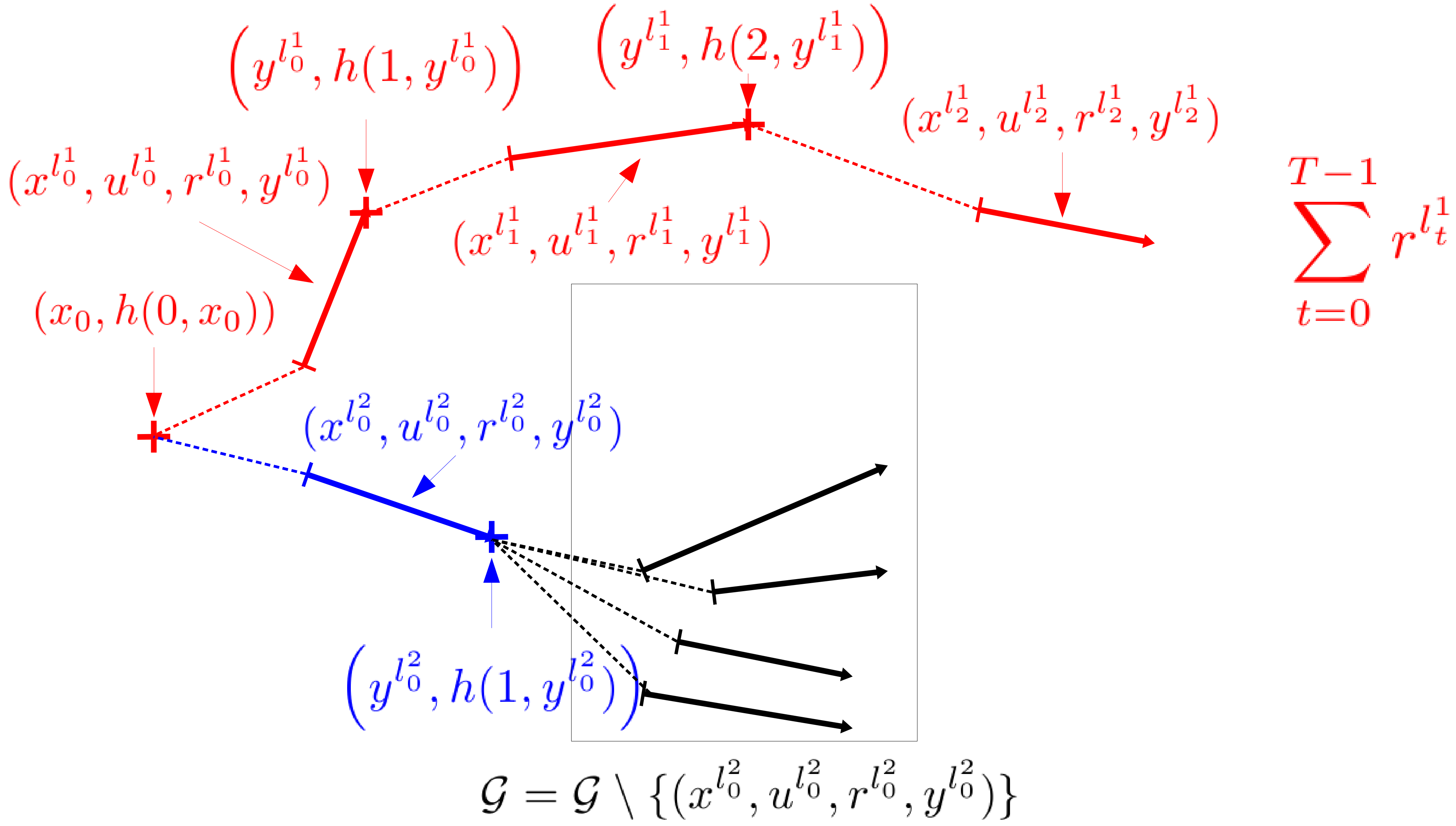

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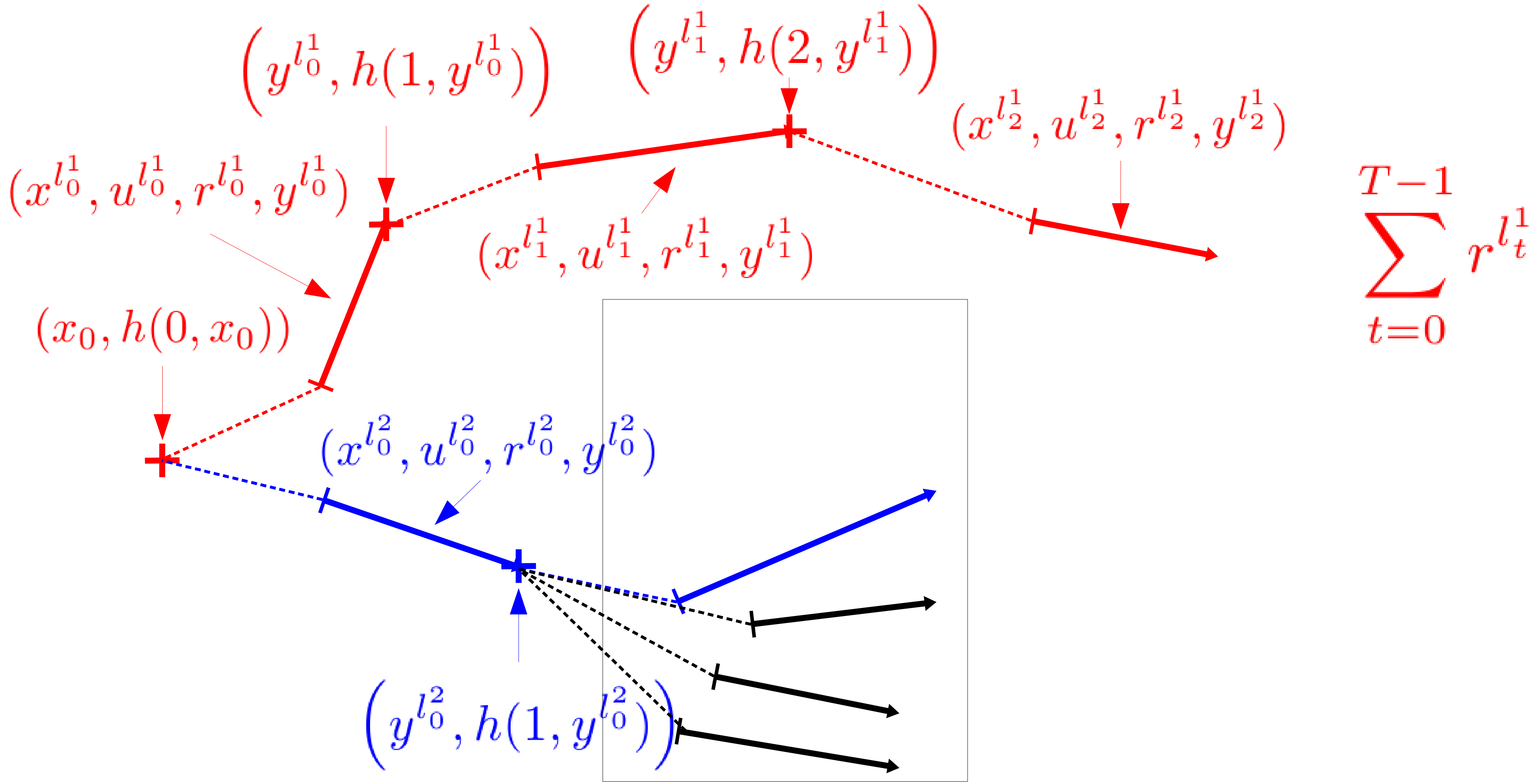

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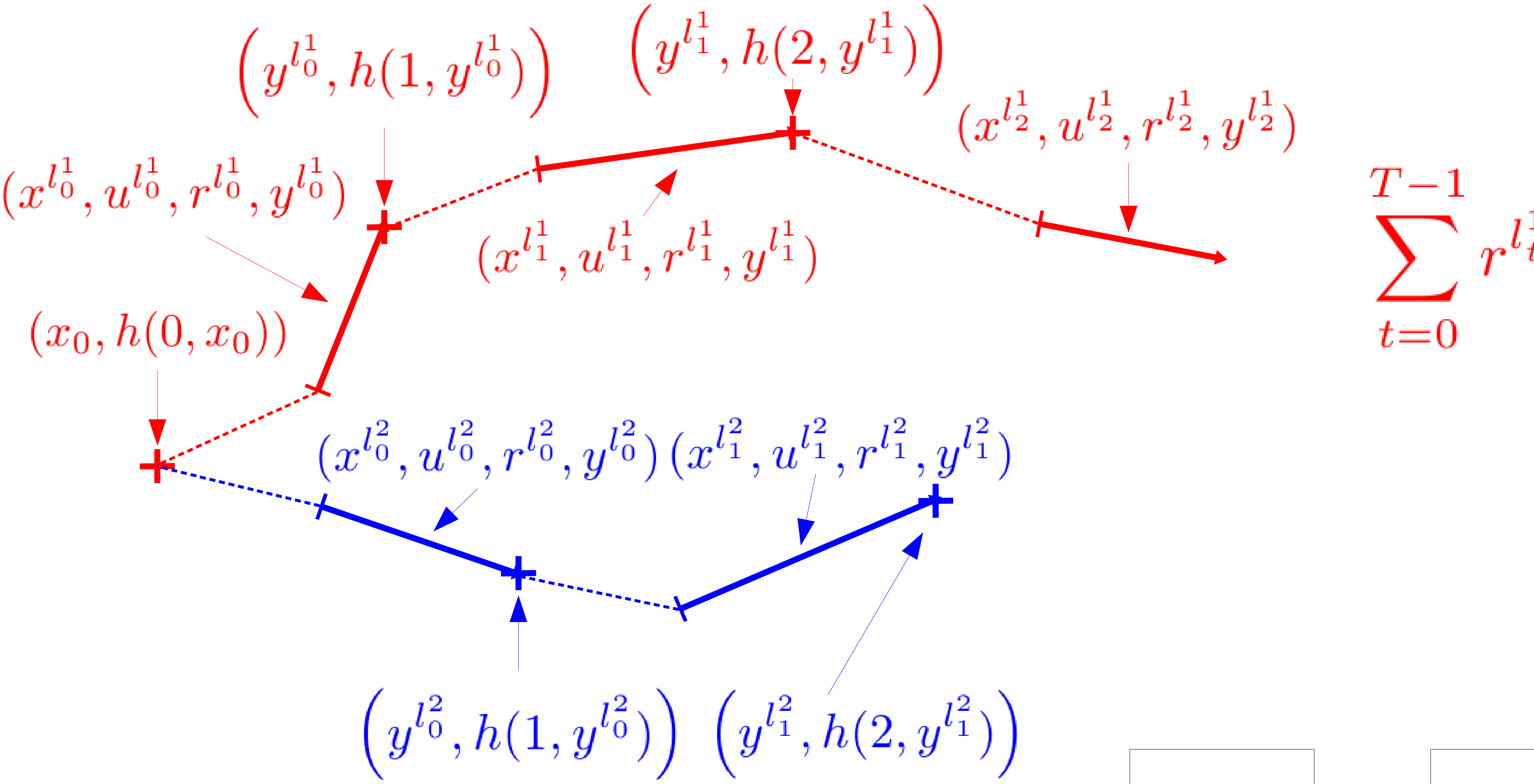

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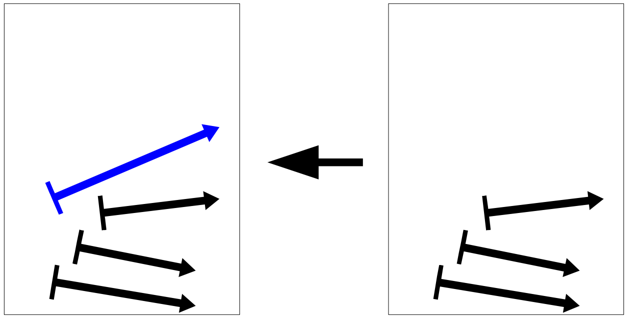

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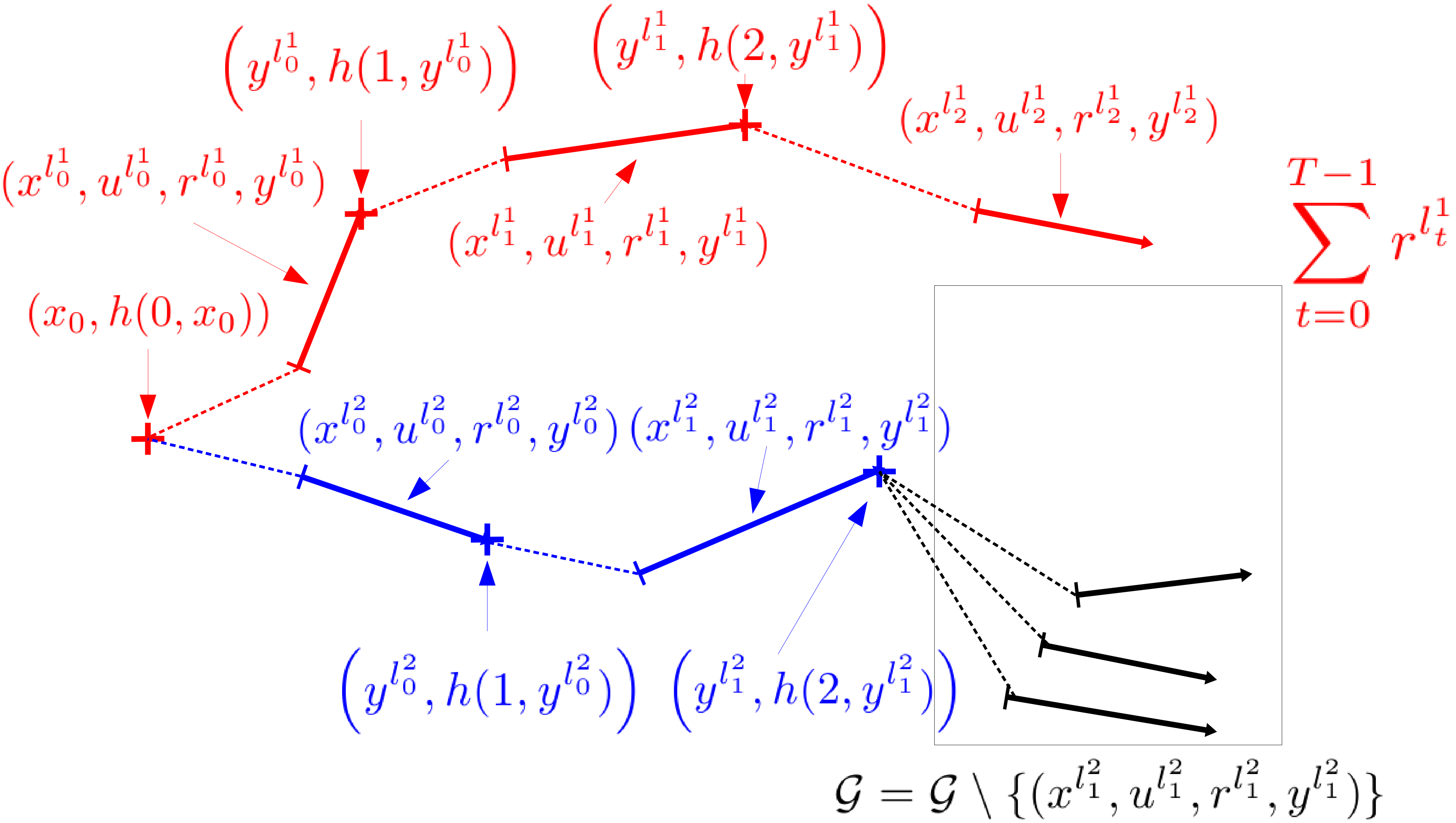

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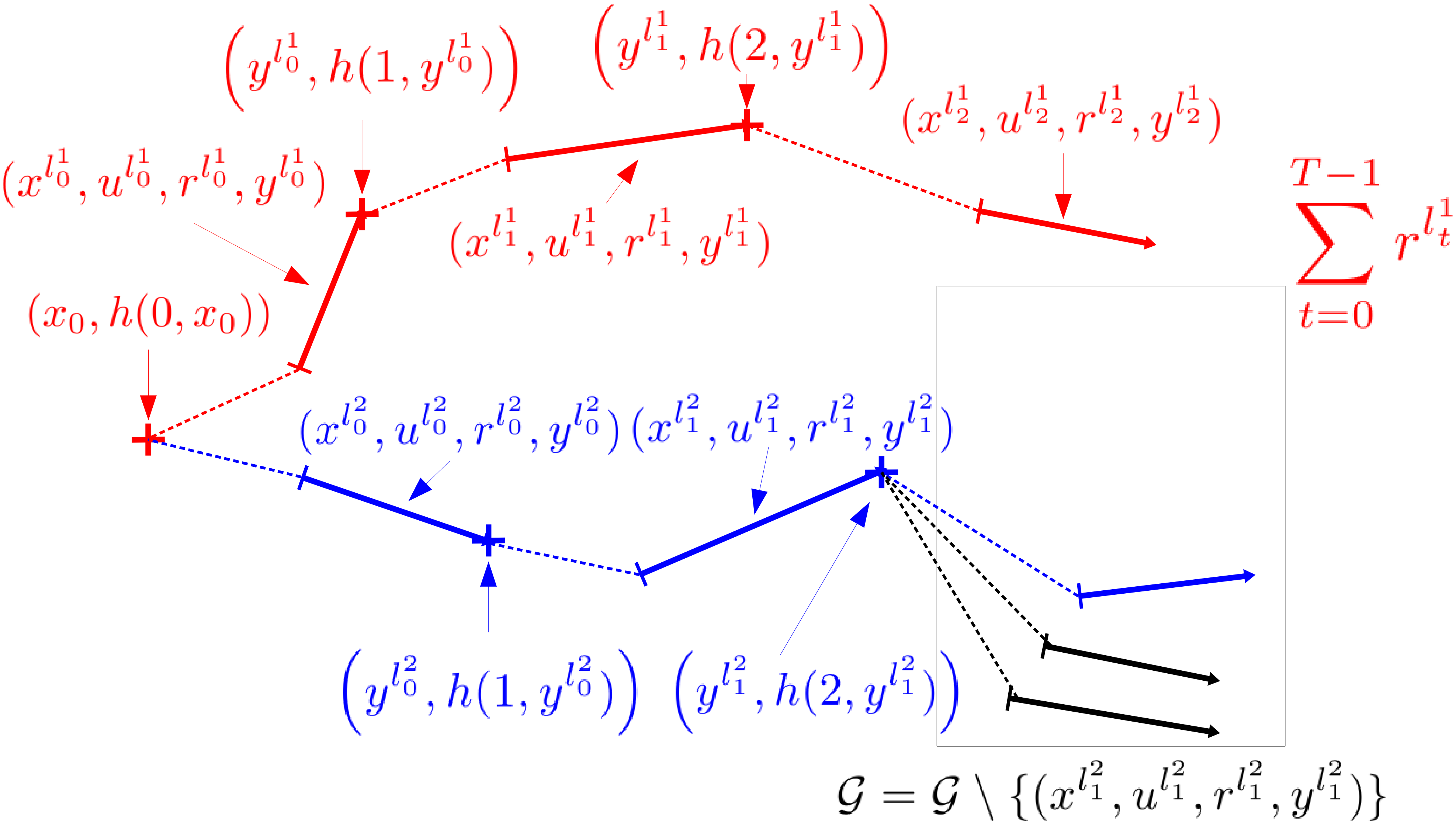

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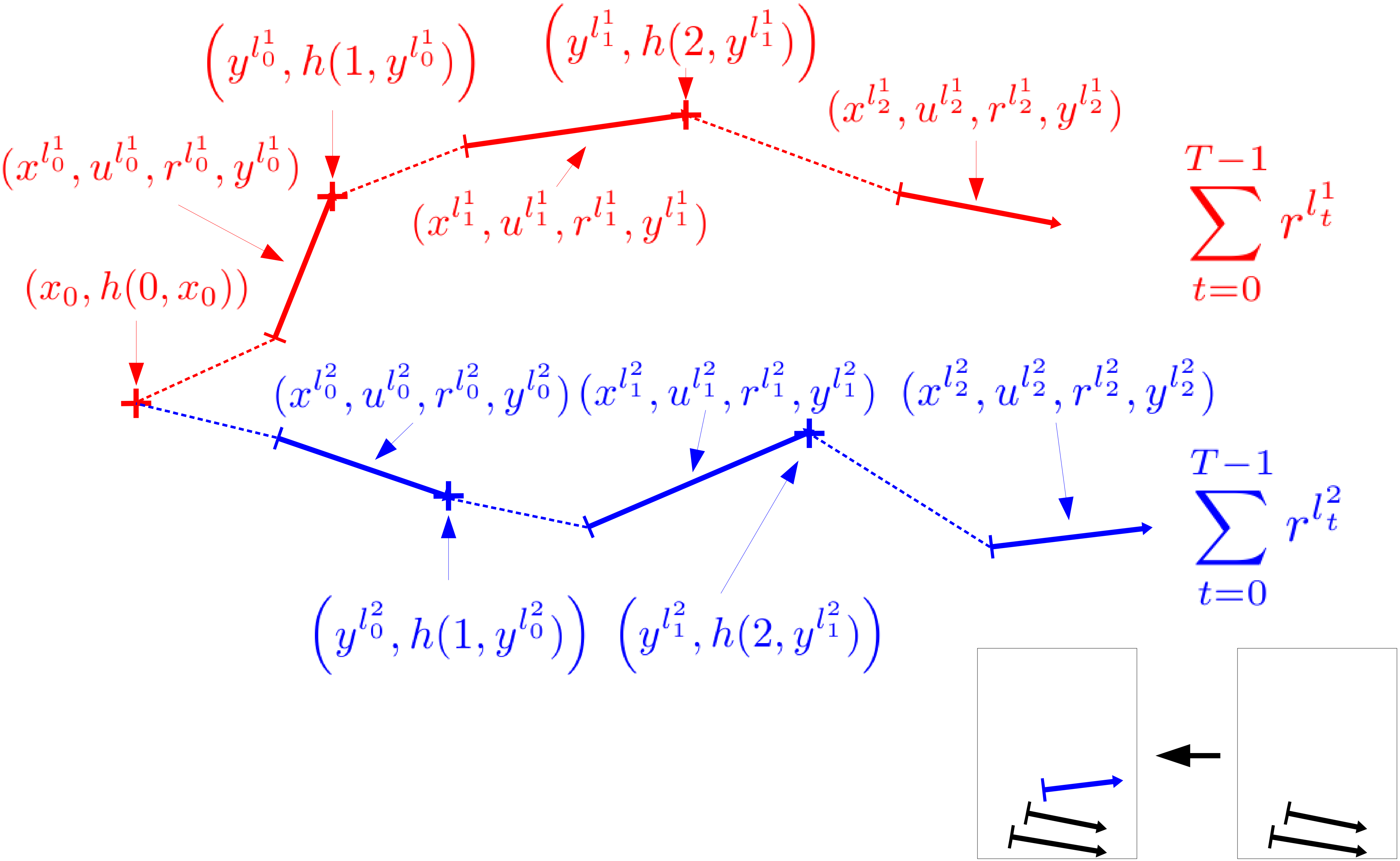

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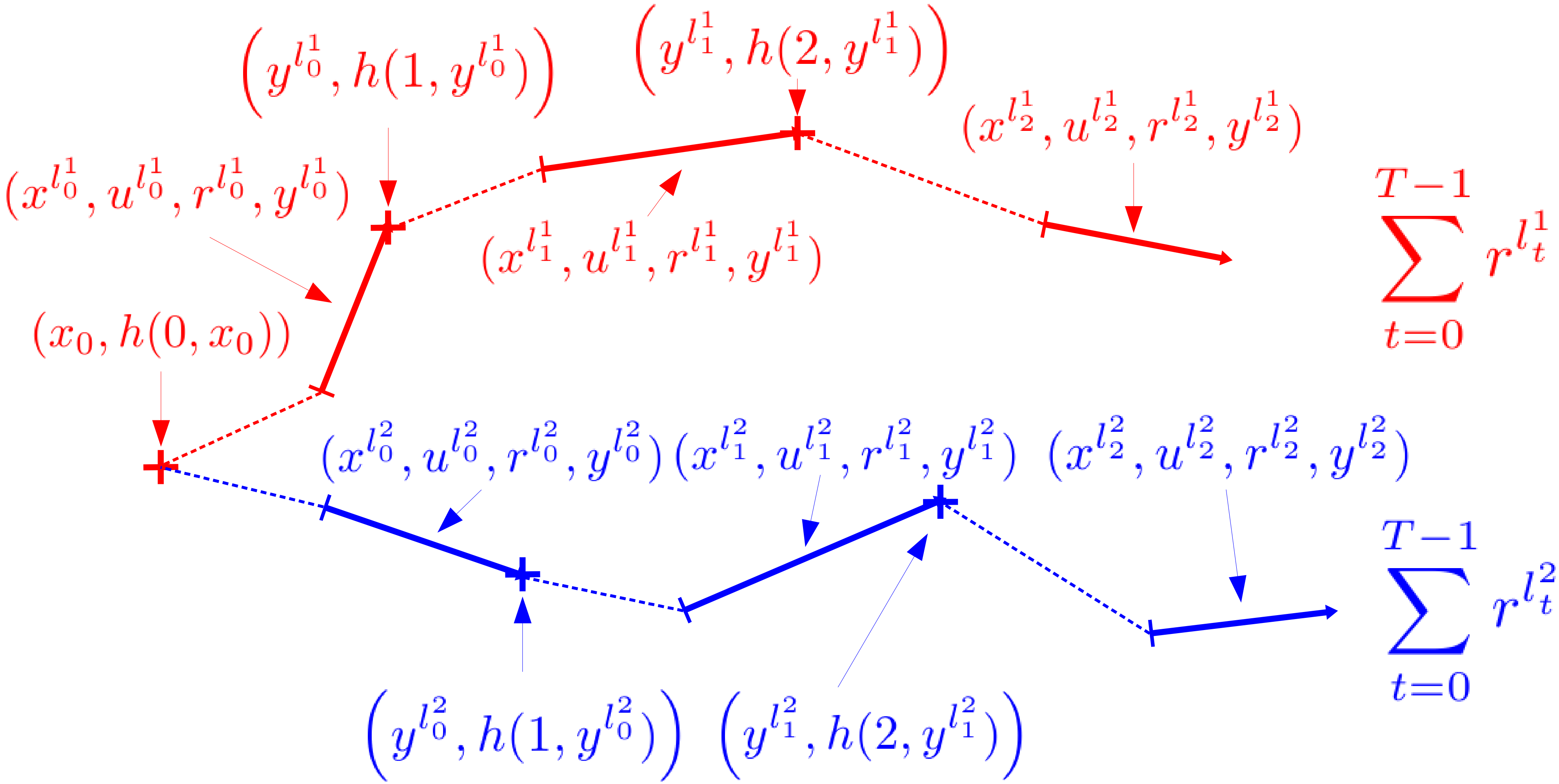


$$\mathfrak{M}_2^h(\mathcal{F}_8, x_0) = \frac{1}{2}\left(\sum_{t=0}^{2} r^{l_t^1} + \sum_{t=0}^{2} r^{l_t^2}\right)$$

Theoretical Analysis

Assumptions

- Lipschitz continuity assumptions:

$$\exists L_f, L_\rho, L_h \in \mathbb{R}^+ : \forall\, (x, x', u, u', w) \in \mathcal{X}^2 \times \mathcal{U}^2 \times \mathcal{W},$$

$$\|f(x,u,w) - f(x',u',w)\|_{\mathcal{X}} \leq L_f(\|x - x'\|_{\mathcal{X}} + \|u - u'\|_{\mathcal{U}}),$$

$$|\rho(x,u,w) - \rho(x',u',w)| \leq L_\rho(\|x - x'\|_{\mathcal{X}} + \|u - u'\|_{\mathcal{U}}),$$

$$\forall t \in [\![0, T-1]\!], \|h(t,x) - h(t,x')\|_{\mathcal{U}} \leq L_h \|x - x'\|_{\mathcal{X}}$$

Theoretical Analysis

Assumptions

- Distance metric Δ

$$\forall (x, x', u, u') \in \mathcal{X}^2 \times \mathcal{U}^2,$$
$$\Delta((x,u),(x',u')) = (\|x - x'\|_{\mathcal{X}} + \|u - u'\|_{\mathcal{U}})$$

- k-sparsity

$$\alpha_k(\mathcal{P}_n) = \sup_{(x,u)\in\mathcal{X}\times\mathcal{U}} \{\Delta_k^{\mathcal{P}_n}(x,u)\}$$

- $\Delta_k^{\mathcal{P}_n}(x,u)$ denotes the distance of (x,u) to its k-th nearest neighbor (using the distance Δ) in the sample

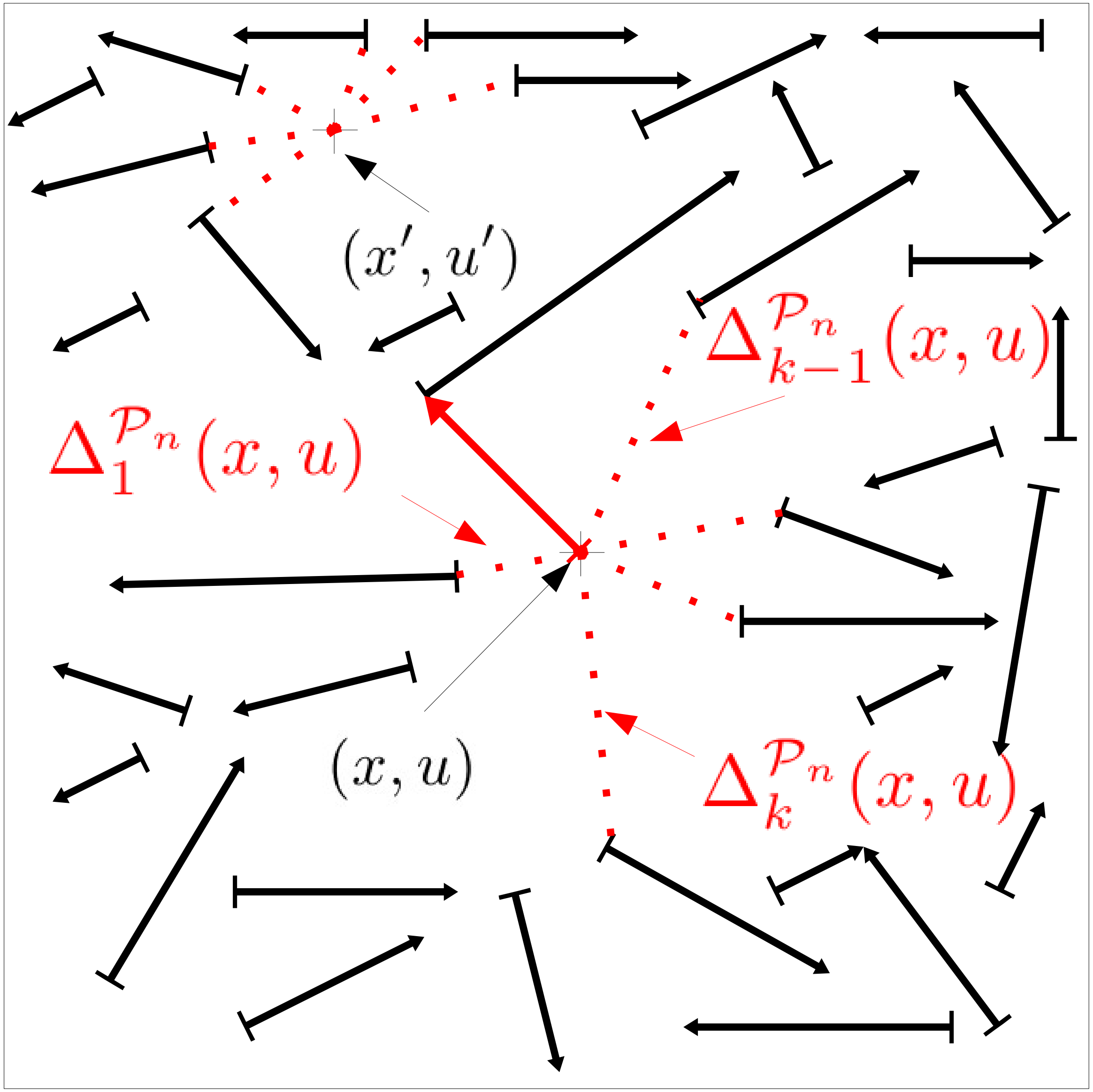
$$\mathcal{P}_n = [(x^l, u^l)]_{l=1}^n$$

Theoretical Analysis

Assumptions

- The k-sparsity can be seen as the smallest radius such that all Δ-balls in X×U contain at least k elements from

$$\mathcal{P}_n = [(x^l, u^l)]_{l=1}^n$$



Theoretical Analysis

Theoretical results

- **Expected value of the MFMC estimator**

$$E^h_{p,\mathcal{P}_n}(x_0) = \underset{w^1,\ldots,w^n \sim p_{\mathcal{W}}(.)}{\mathbb{E}} \left[\mathfrak{M}^h_p \left(\tilde{\mathcal{F}}_n \left(\mathcal{P}_n, w^1, \ldots, w^n \right), x_0 \right) \right]$$

Theoretical Analysis

Theoretical results

- **Expected value of the MFMC estimator**

$$E^h_{p,\mathcal{P}_n}(x_0) = \underset{w^1,\ldots,w^n \sim p_{\mathcal{W}}(.)}{\mathbb{E}} \left[\mathfrak{M}^h_p \left(\tilde{\mathcal{F}}_n \left(\mathcal{P}_n, w^1, \ldots, w^n \right), x_0 \right) \right]$$

- **Theorem**

$$\left| J^h(x_0) - E^h_{p,\mathcal{P}_n}(x_0) \right| \leq C \alpha_{pT} \left(\mathcal{P}_n \right)$$

with

$$C = L_\rho \sum_{t=0}^{T-1} \sum_{i=0}^{T-t-1} \left(L_f (1 + L_h) \right)^i$$

Theoretical Analysis

Theoretical results

- **Variance of the MFMC estimator**

$$V^h_{p,\mathcal{P}_n}(x_0) = \underset{w^1,\ldots,w^n \sim p_{\mathcal{W}}(.)}{\mathbb{E}} \left[\left(\mathfrak{M}^h_p \left(\tilde{\mathcal{F}}_n \left(\mathcal{P}_n, w^1, \ldots, w^n \right), x_0 \right) - E^h_{p,\mathcal{P}_n}(x_0) \right)^2 \right]$$

Theoretical Analysis

Theoretical results

- **Variance of the MFMC estimator**

$$V^h_{p,\mathcal{P}_n}(x_0) = \underset{w^1,\ldots,w^n \sim p_{\mathcal{W}}(.)}{\mathbb{E}}\left[\left(\mathfrak{M}^h_p\left(\tilde{\mathcal{F}}_n\left(\mathcal{P}_n, w^1, \ldots, w^n\right), x_0\right) - E^h_{p,\mathcal{P}_n}(x_0)\right)^2\right]$$

- **Theorem**

$$V^h_{p,\mathcal{P}_n}(x_0) \leq \left(\frac{\sigma_{R^h}(x_0)}{\sqrt{p}} + 2C\alpha_{pT}\left(\mathcal{P}_n\right)\right)^2$$

with

$$C = L_\rho \sum_{t=0}^{T-1} \sum_{i=0}^{T-t-1} \left(L_f(1 + L_h)\right)^i$$

Experimental Illustration

Benchmark

- Dynamics:

$$x_{t+1} = \sin\left(\frac{\pi}{2}(x_t + u_t + w_t)\right)$$

- Reward function:

$$\rho(x_t, u_t, w_t) = \frac{1}{2\pi} e^{-\frac{1}{2}(x_t^2 + u_t^2)} + w_t$$

- Policy to evaluate:

$$h(t, x) = -\frac{x}{2}, \qquad \forall x \in \mathcal{X}, \forall t \in \{0, \ldots, T-1\}$$

- Other information:

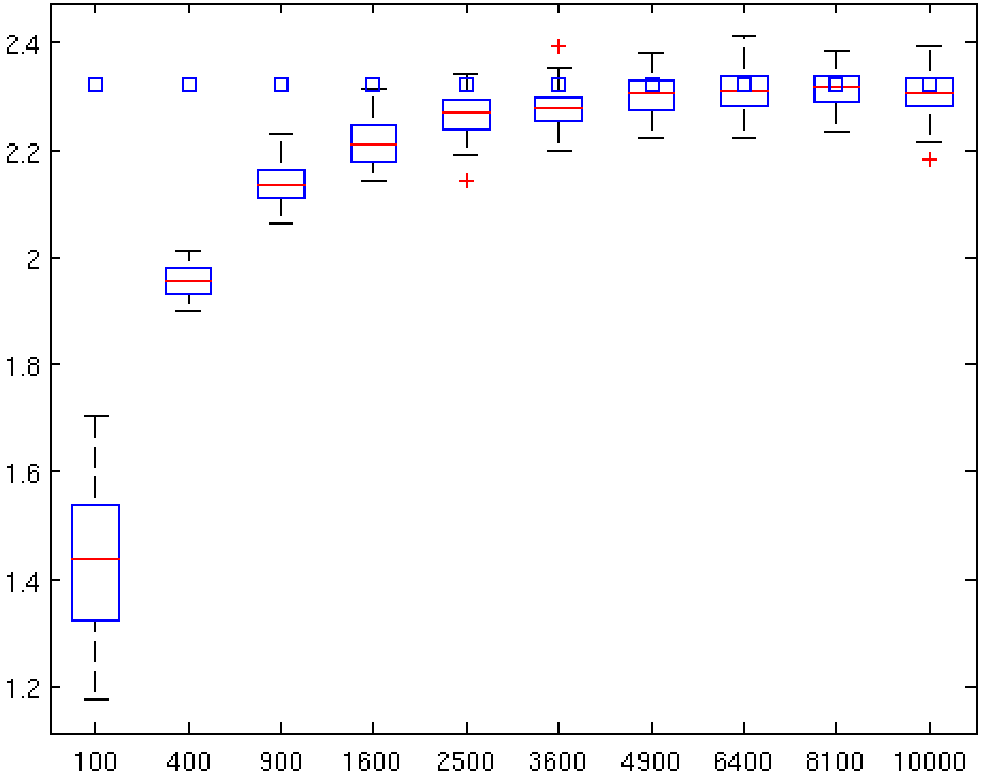
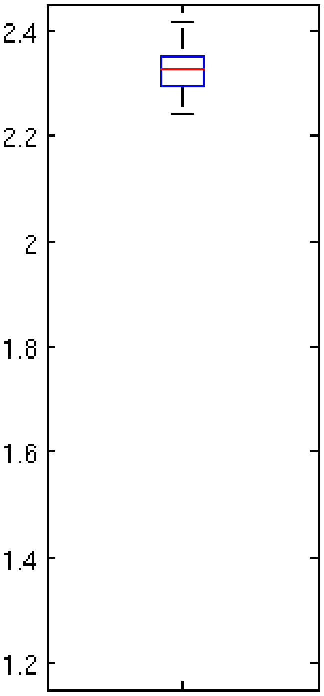
$$\mathcal{X} = [-1, 1], \mathcal{U} = [-\tfrac{1}{2}, \tfrac{1}{2}], \mathcal{W} = [-\tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}] \text{ with } \epsilon = 0.1$$

$p_W(.)$ is uniform,

Experimental Illustration

Influence of n

- Simulations for p = 10, n = 100 … 10 000, uniform grid, T = 15, x_0 = - 0.5 .

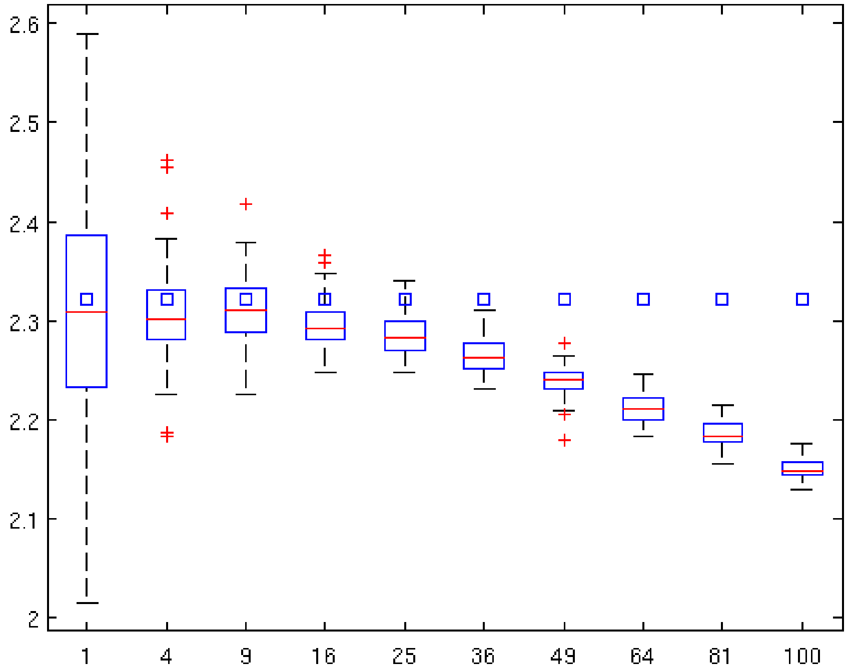
Model-free Monte Carlo estimator	Monte Carlo estimator
 n = 100 … 10 000, p = 10	 p = 10

Experimental Illustration

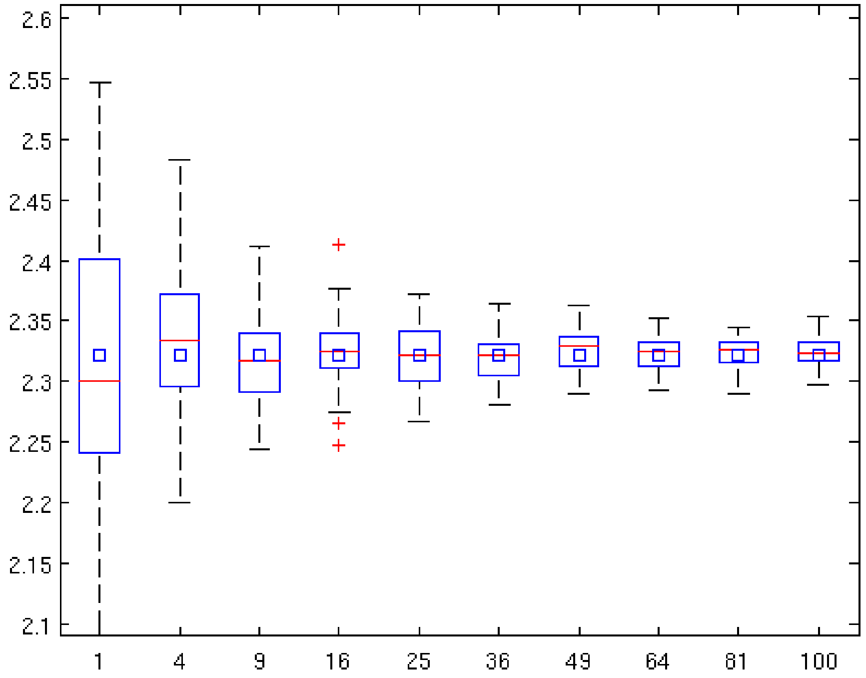
Influence of p

- Simulations for p = 1 … 100, n = 10 000 , uniform grid, T = 15, x_0 = - 0.5 .

Model-free Monte Carlo estimator	Monte Carlo estimator



p = 1 … 100, n=10 000

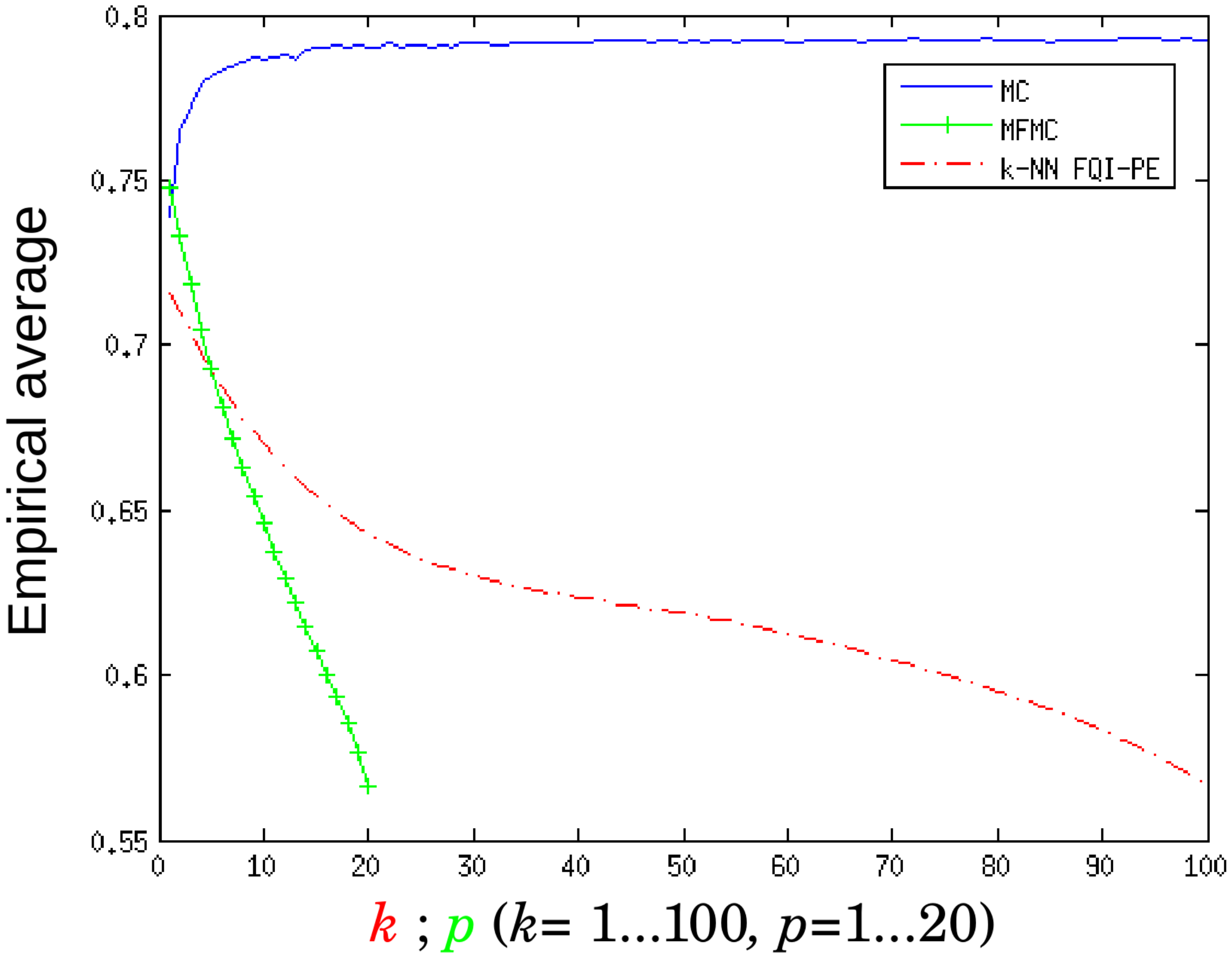


p = 1 … 100

Experimental Illustration

MFMC vs FQI-PE

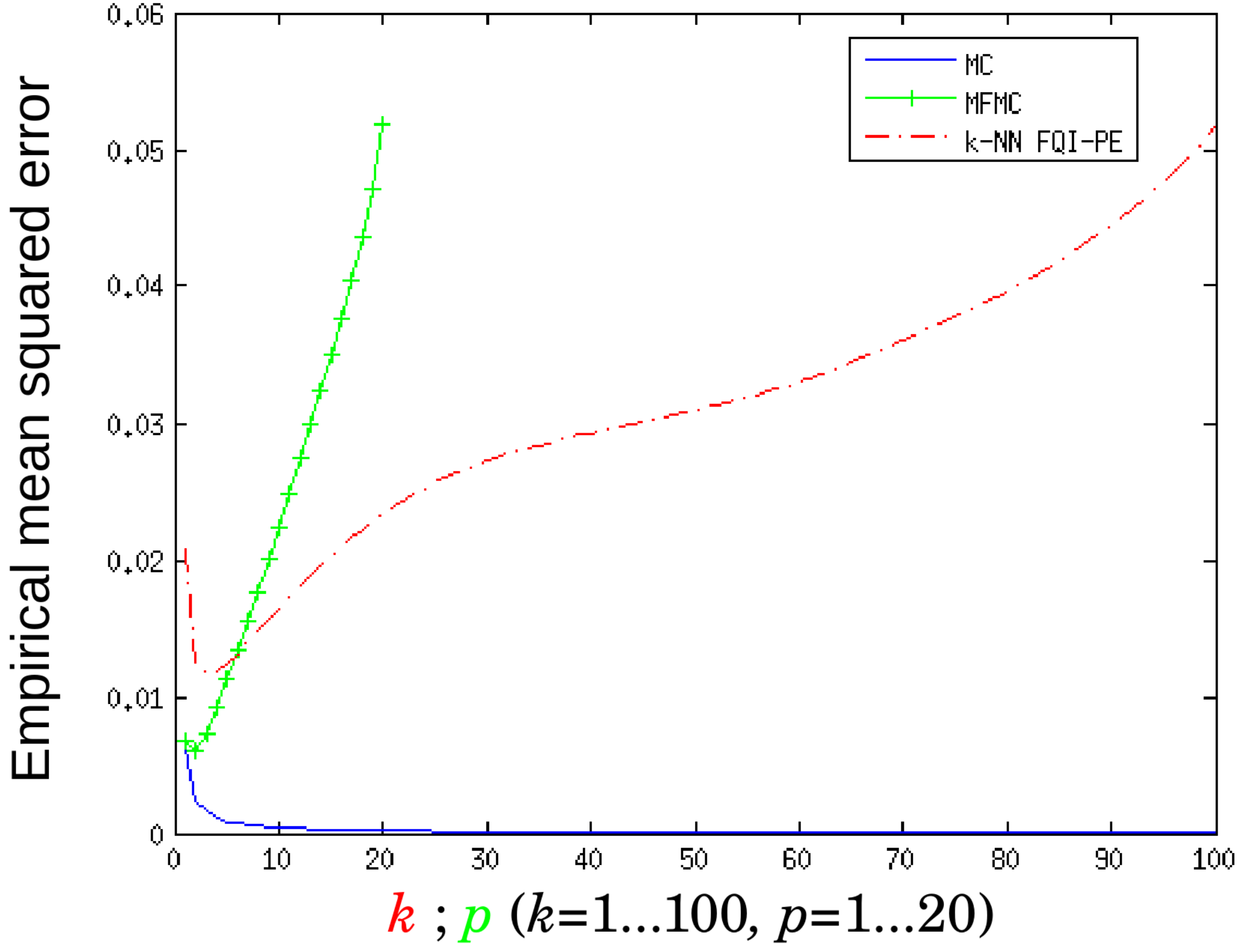
- Comparison with the FQI-PE algorithm using k-NN, n=100, T=5 .



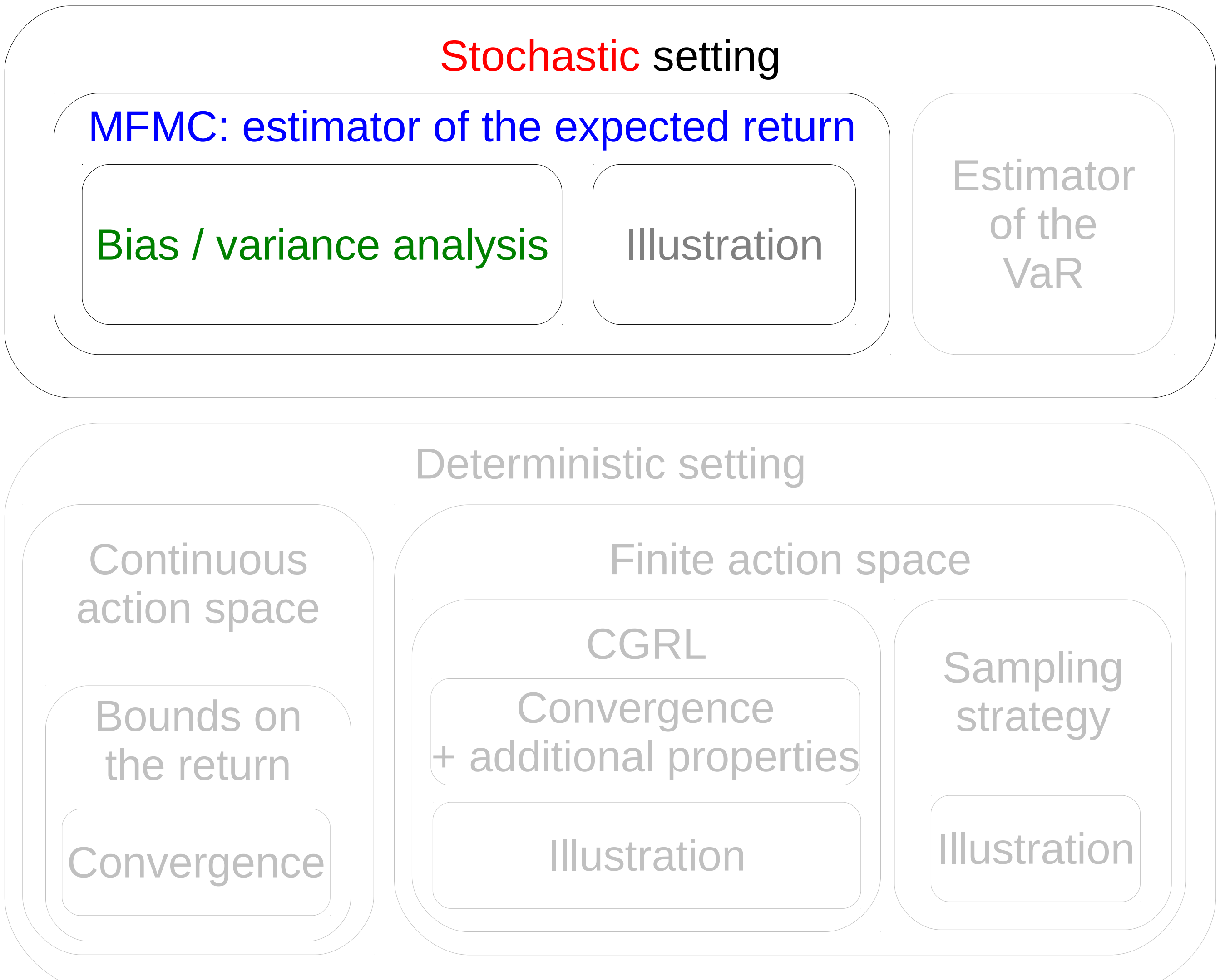
Experimental Illustration

MFMC vs FQI-PE

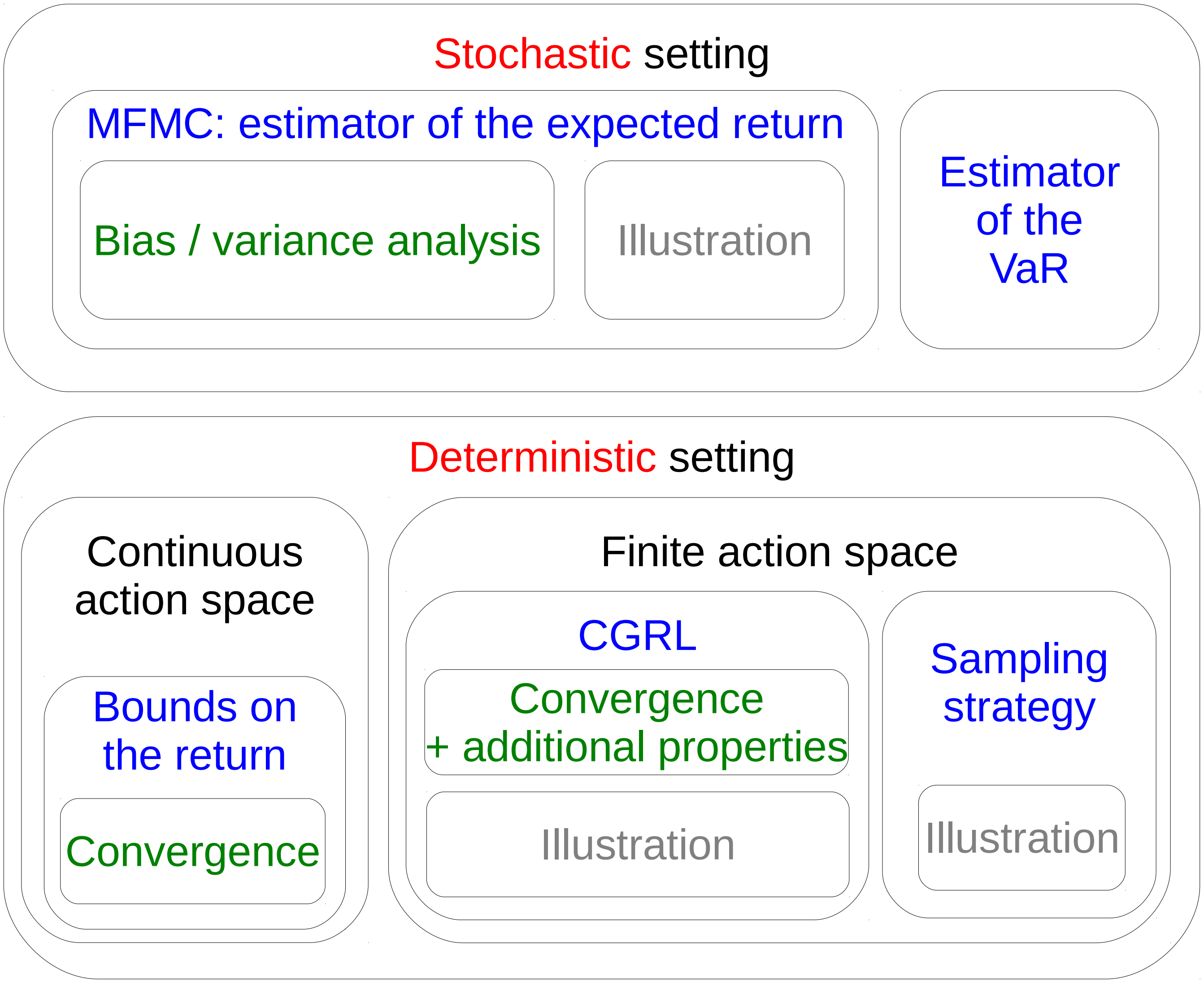
- Comparison with the FQI-PE algorithm using k-NN, n=100, T=5 .



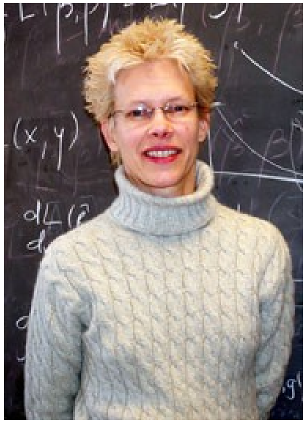
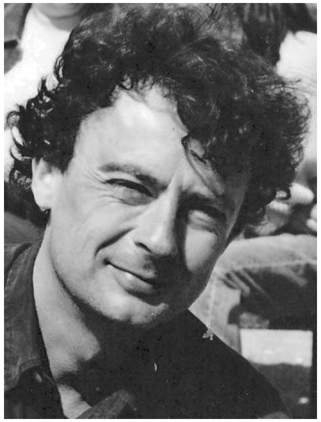
Conclusions



Conclusions



References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Appendix

Estimating the Performances of Policies

Risk-sensitive criterion

- Consider again the *p* artificial trajectories that were rebuilt by the MFMC estimator
- The Value-at-Risk of the policy *h*

$$J_{RS}^{h,(b,c)}(x_0) = \begin{cases} -\infty & \text{if } P\left(R^h(x_0, w_0, \ldots, w_{T-1}) < b\right) > c \\ J^h(x_0) & \text{otherwise} \end{cases}$$

can be straightforwardly estimated as follows:

$$\tilde{J}_{RS}^{h,(b,c)}(x_0) = \begin{cases} -\infty & \text{if } \frac{1}{p}\sum_{i=1}^{p} \mathbb{I}_{\{\mathbf{r}^i < b\}} > c\ , \\ \mathfrak{M}^h(\mathcal{F}_n, x_0) & \text{otherwise} \end{cases}$$

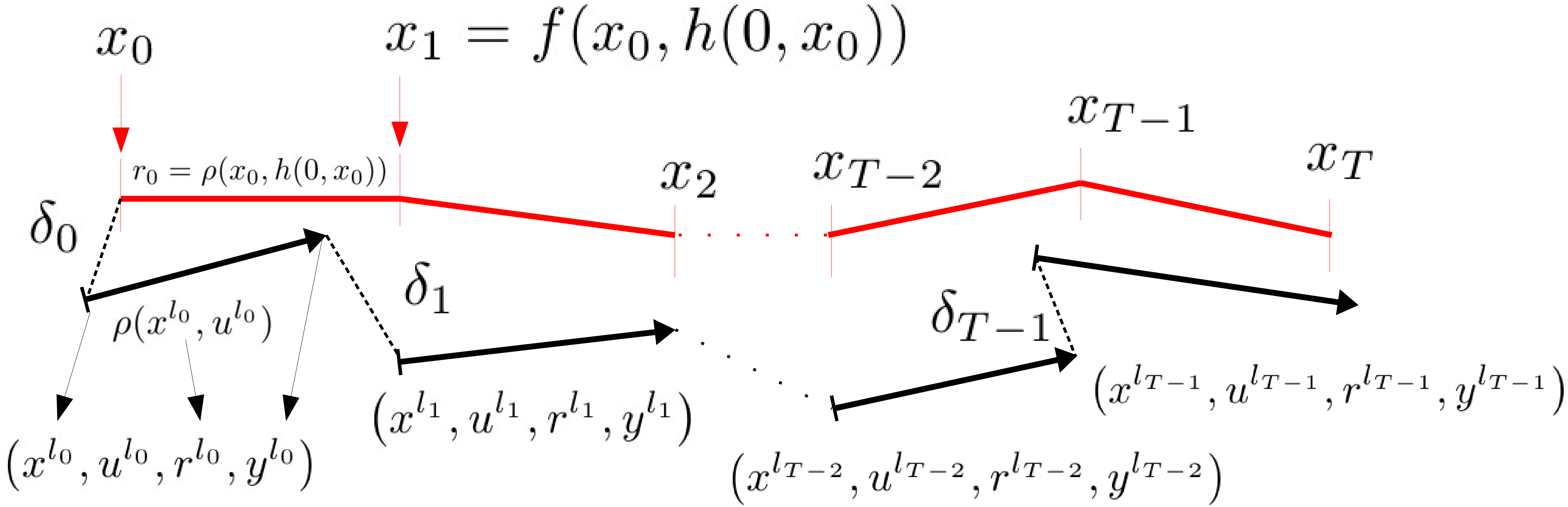
with $\mathbf{r}^i = \sum_{t=0}^{T-1} r^{l_t^i}$

$c \in [0, 1[\quad b \in \mathbb{R}$

Deterministic Case: Computing Bounds

Bounds from a Single Trajectory

- Given an artificial trajectory : $\tau = \left[\left(x^{l_t}, u^{l_t}, r^{l_t}, y^{l_t} \right) \right]_{t=0}^{T-1}$



$$\delta_0 = \left\| x^{l_0} - x_0 \right\|_{\mathcal{X}} + \left\| u^{l_0} - h(0, x_0) \right\|_{\mathcal{U}} ,$$

$$\delta_1 = \left\| y^{l_0} - x^{l_1} \right\|_{\mathcal{X}} + \left\| u^{l_1} - h(1, y^{l_0}) \right\|_{\mathcal{U}} \quad \ldots$$

Deterministic Case: Computing Bounds

Bounds from a Single Trajectory

- **Proposition:**

Let $\left[\left(x^{l_t}, u^{l_t}, r^{l_t}, y^{l_t}\right)\right]_{t=0}^{T-1}$ be an artificial trajectory. Then,

$$J^h(x_0) \geq \sum_{t=0}^{T-1} r^{l_t} - \sum_{t=0}^{T-1} L_{Q_{T-t}} \Delta\left((y^{l_{t-1}}, h(t, y^{l_{t-1}})), (x^{l_t}, u^{l_t})\right)$$

with

$$L_{Q_{T-t}} = L_\rho \sum_{i=0}^{T-t-1} \left(L_f\left(1 + L_h\right)\right)^i$$

$$y^{l_{-1}} = x_0$$

Deterministic Case: Computing Bounds

Maximal Bounds

- **Maximal lower and upper-bounds**

$$L^h(\mathcal{F}_n, x_0) = \max_{[(x^{l_t}, u^{l_t}, r^{l_t}, y^{l_t})]_{t=0}^{T-1} \in \mathcal{F}_n^T} \sum_{t=0}^{T-1} r^{l_t} - \sum_{t=0}^{T-1} L_{Q_{T-t}} \Delta\left((y^{l_{t-1}}, h(t, y^{l_{t-1}})), (x^{l_t}, u^{l_t})\right)$$

$$U^h(\mathcal{F}_n, x_0) = \min_{[(x^{l_t}, u^{l_t}, r^{l_t}, y^{l_t})]_{t=0}^{T-1} \in \mathcal{F}_n^T} \sum_{t=0}^{T-1} r^{l_t} + \sum_{t=0}^{T-1} L_{Q_{T-t}} \Delta\left((y^{l_{t-1}}, h(t, y^{l_{t-1}})), (x^{l_t}, u^{l_t})\right)$$

Deterministic Case: Computing Bounds

Tightness of Maximal Bounds

- **Proposition:**

$$\exists C_b > 0 : \quad J^h(x_0) - L^h(\mathcal{F}_n, x_0) \leq C_b \alpha_1(\mathcal{P}_n)$$
$$U^h(\mathcal{F}_n, x_0) - J^h(x_0) \leq C_b \alpha_1(\mathcal{P}_n)$$

Inferring Safe Policies

From Lower Bounds to Cautious Policies

- Consider the set of open-loop policies:

$$\Pi = \{\pi : \{0, \ldots, T-1\} \to \mathcal{U}\}$$

- For such policies, bounds can be computed in a similar way
- We can then search for a specific policy for which the associated lower bound is maximized:

$$\hat{\pi}^*_{\mathcal{F}_n, x_0} \in \arg\max_{\pi \in \Pi} \quad L^{\pi}(\mathcal{F}_n, x_0)$$

- A O($T\, n^2$) algorithm for doing this: the CGRL algorithm (Cautious approach to Generalization in RL)

Inferring Safe Policies

Convergence

- **Theorem**

Let $\mathfrak{J}^*(x_0)$ *be the set of optimal open-loop policies:*

$$\mathfrak{J}^*(x_0) = \arg\max_{\pi \in \Pi} \; J^\pi(x_0) \; ,$$

and let us suppose that $\mathfrak{J}^*(x_0) \neq \Pi$ *(if* $\mathfrak{J}^*(x_0) = \Pi$*, the search for an optimal policy is indeed trivial). We define*

$$\epsilon(x_0) = \min_{\pi \in \Pi \setminus \mathfrak{J}^*(x_0)} \left\{ \left(\max_{\pi' \in \Pi} J^{\pi'}(x_0) \right) - J^\pi(x_0) \right\} \; .$$

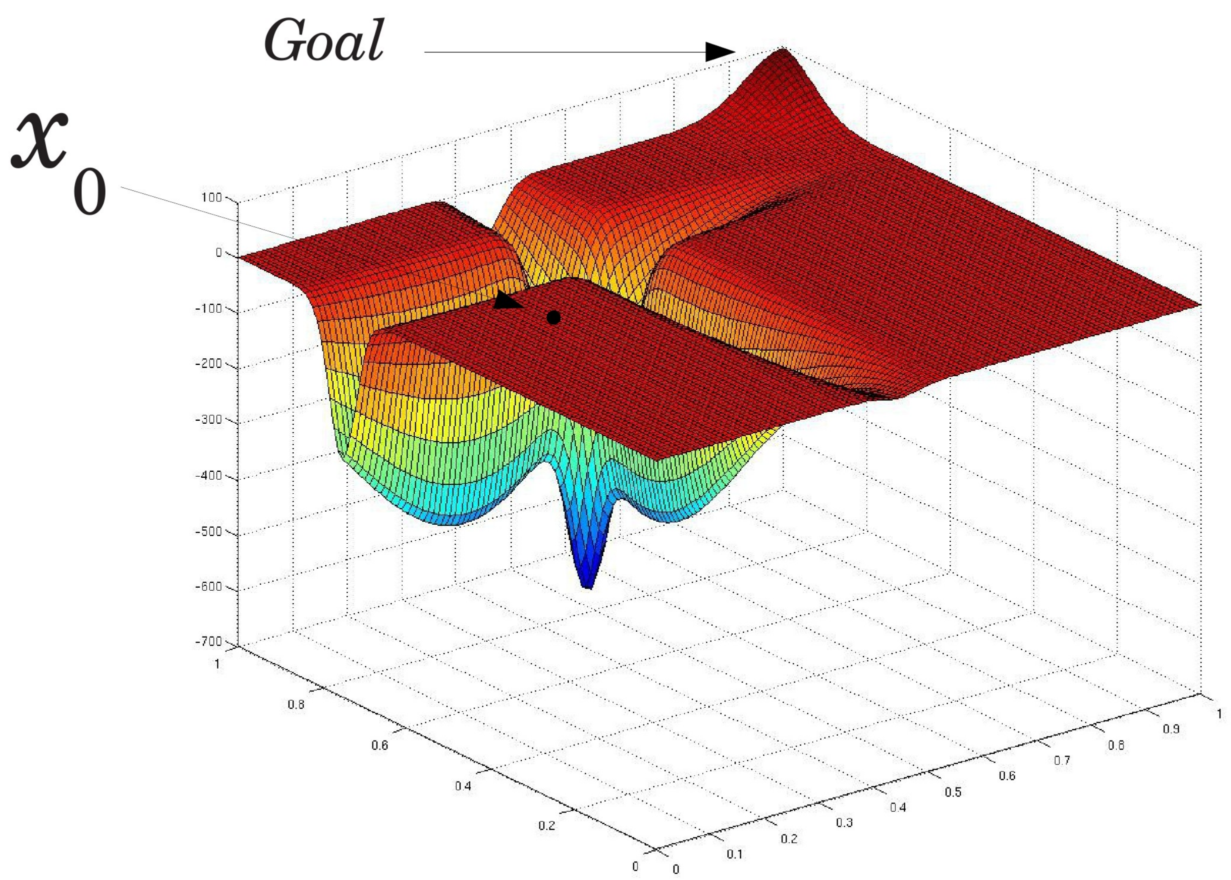
Then,

$$\left(C_b' \alpha^*(\mathcal{P}_n) < \epsilon(x_0) \right) \implies \hat{\pi}^*_{\mathcal{F}_n, x_0} \in \mathfrak{J}^*(x_0) \; .$$

Inferring Safe Policies

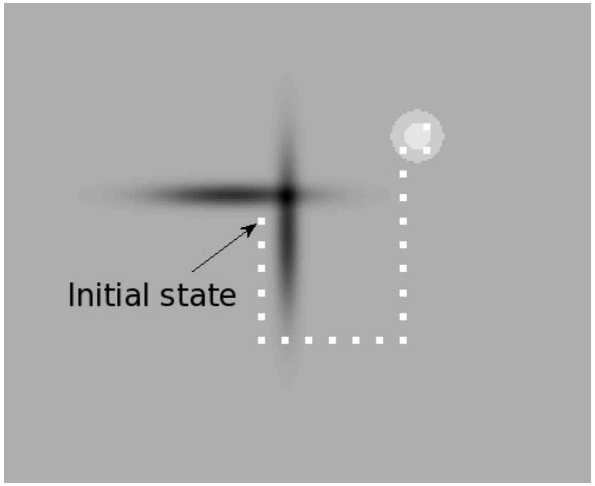
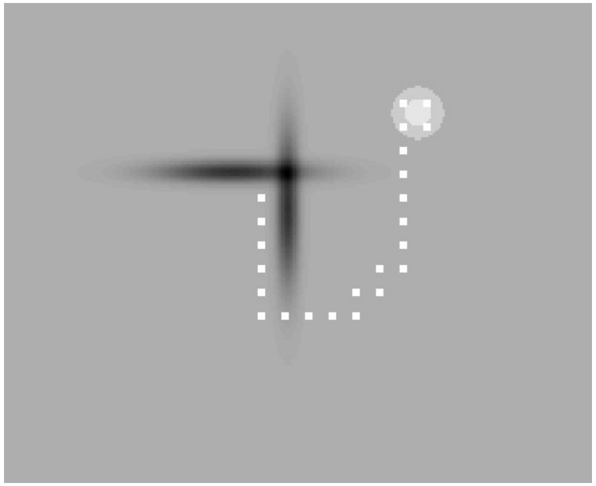
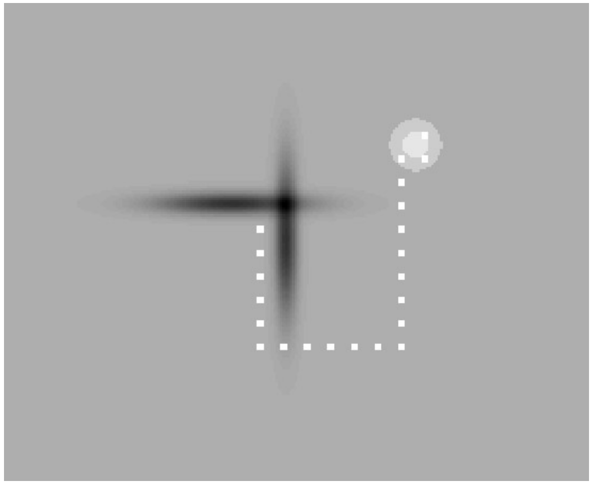
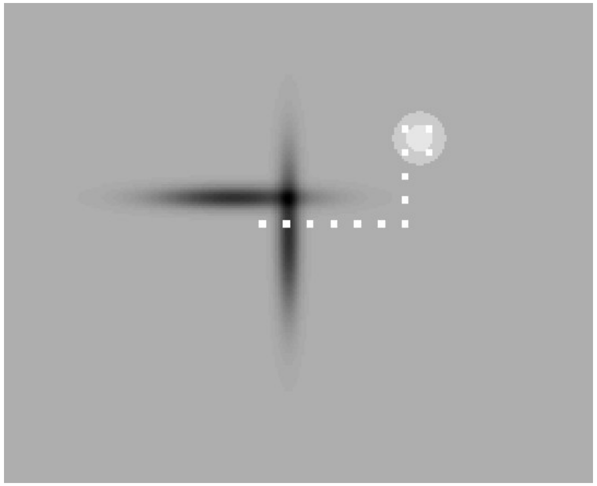
Experimental Results

- The puddle world benchmark



Inferring Safe Policies

Experimental Results

	CGRL	FQI (Fitted Q Iteration)
The state space is uniformly covered by the sample		
Information about the Puddle area is removed		

Inferring Safe Policies

Bonus

- **Theorem**

Let $\pi^*_{x_0} \in \mathfrak{J}^*(x_0)$ *be an optimal open-loop policy. Let us assume that one can find in* $\mathcal{F}_n$ *a sequence of* T *one-step system transitions*

$$\left[\left(x^{l_0}, u^{l_0}, r^{l_0}, x^{l_1}\right), \left(x^{l_1}, u^{l_1}, r^{l_1}, x^{l_2}\right), \ldots, \left(x^{l_{T-1}}, u^{l_{T-1}}, r^{l_{T-1}}, x^{l_T}\right)\right] \in \mathcal{F}_n^T$$

such that

$$x^{l_0} = x_0 \ ,$$
$$u^{l_t} = \pi^*_{x_0}(t) \qquad \forall t \in \{0, \ldots, T-1\} \ .$$

Let $\hat{\pi}^*_{\mathcal{F}_n, x_0}$ *be such that*

$$\hat{\pi}^*_{\mathcal{F}_n, x_0} \in \underset{\pi \in \Pi}{\arg\max} \qquad L^\pi(\mathcal{F}_n, x_0) \ .$$

Then,

$$\hat{\pi}^*_{\mathcal{F}_n, x_0} \in \mathfrak{J}^*(x_0) \ .$$

Sampling Strategies

An Artificial Trajectories Viewpoint

- Given a sample of system transitions

$$\mathcal{F}_n = \left\{ \left(x^l, u^l, r^l, y^l \right) \in \mathcal{X} \times \mathcal{U} \times \mathbb{R} \times \mathcal{X} \right\}_{l=1}^{n}$$

How can we determine where to sample additional transitions ?

- We define the set of candidate optimal policies:

$$\Pi(\mathcal{F}, x_0) = \left\{ \pi \in \Pi \quad | \quad \forall \pi' \in \Pi, U^{\pi}(\mathcal{F}, x_0) \geq L^{\pi'}(\mathcal{F}, x_0) \right\}$$

- A transition $(x, u, r, y) \in \mathcal{X} \times \mathcal{U} \times \mathbb{R} \times \mathcal{X}$ is said compatible with $\mathcal{F}$ if

$$\forall (x^l, u^l, r^l, y^l) \in \mathcal{F}, \quad \left(u^l = u \right) \implies \begin{cases} \left| r - r^l \right| & \leq L_\rho \| x - x^l \|_{\mathcal{X}} \\ \left\| y - y^l \right\|_{\mathcal{X}} & \leq L_f \| x - x^l \|_{\mathcal{X}} \end{cases}$$

and we denote by $\mathcal{C}(\mathcal{F})$ set of all such compatible transitions.

Sampling Strategies

An Artificial Trajectories Viewpoint

- Iterative scheme:

$$(x^{m+1}, u^{m+1}) \in \arg\min_{(x,u) \in \mathcal{X} \times \mathcal{U}} \left\{ \max_{\substack{(r,y) \in \mathbb{R} \times \mathcal{X} \;\; s.t. (x,u,r,y) \in \mathcal{C}(\mathcal{F}_m) \\ \pi \in \mathit{\Pi}(\mathcal{F}_m \cup \{(x,u,r,y)\}, x_0)}} \delta^\pi(\mathcal{F}_m \cup \{(x,u,r,y)\}, x_0) \right\}$$

with

$$\delta^\pi(\mathcal{F}, x_0) = U^\pi(\mathcal{F}, x_0) - L^\pi(\mathcal{F}, x_0)$$

- Conjecture:

$$\exists m_0 \in \mathbb{N} \setminus \{0\} : \forall m \in \mathbb{N}, \left(m \geq m_0\right) \implies \mathit{\Pi}(\mathcal{F}_m, x_0) = \mathfrak{J}^*(x_0)$$

Sampling Strategies

Illustration

- Action space: $\mathcal{U} = \{-0.20, -0.10, 0, +0.10, +0.20\}$
- Dynamics and reward function:

$$f(x,u) = x + u$$

$$\rho(x,u) = x + u$$

- Horizon: $T = 3$
- Initial sate:

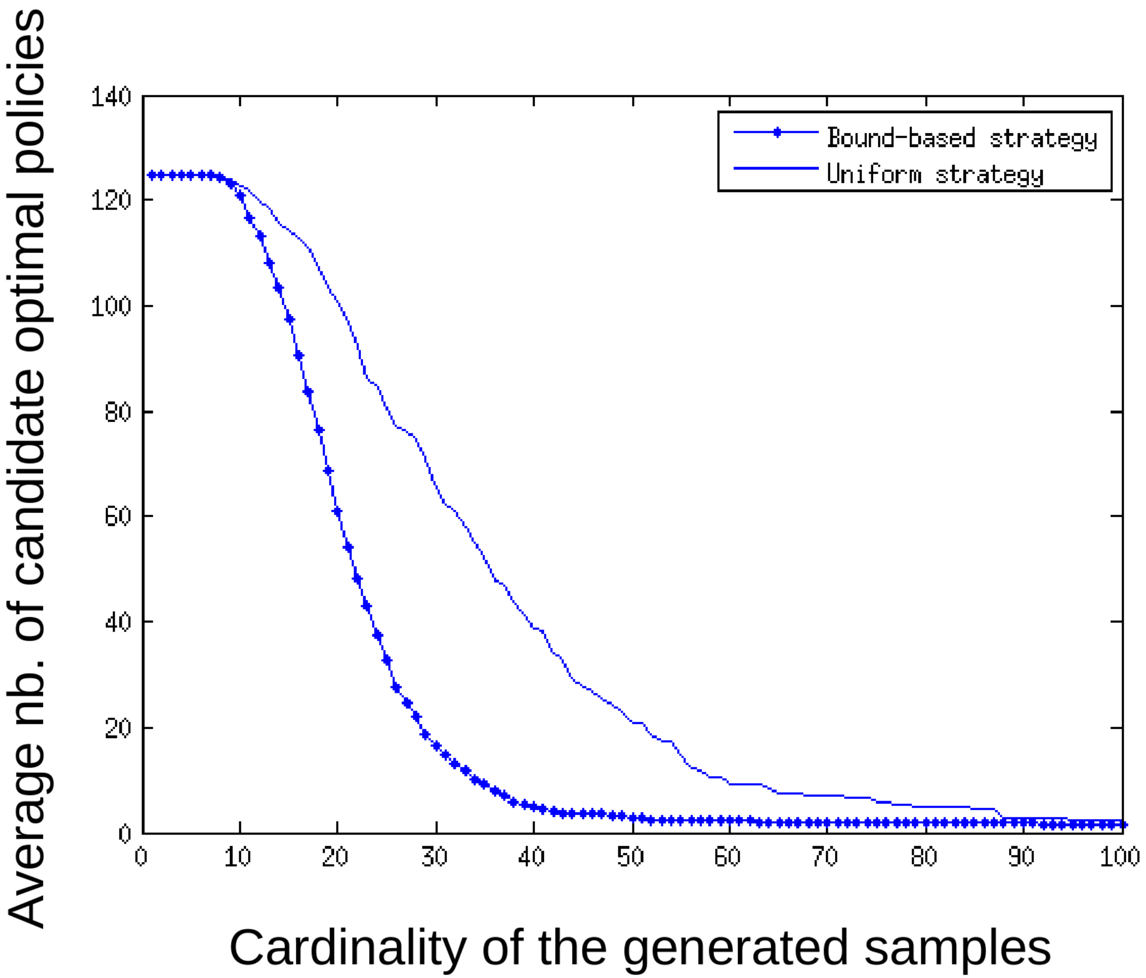
$$x_0 = -0.65$$

- Total number of policies:

$$5^3 = 125$$

- Number of transitions needed for discriminating:

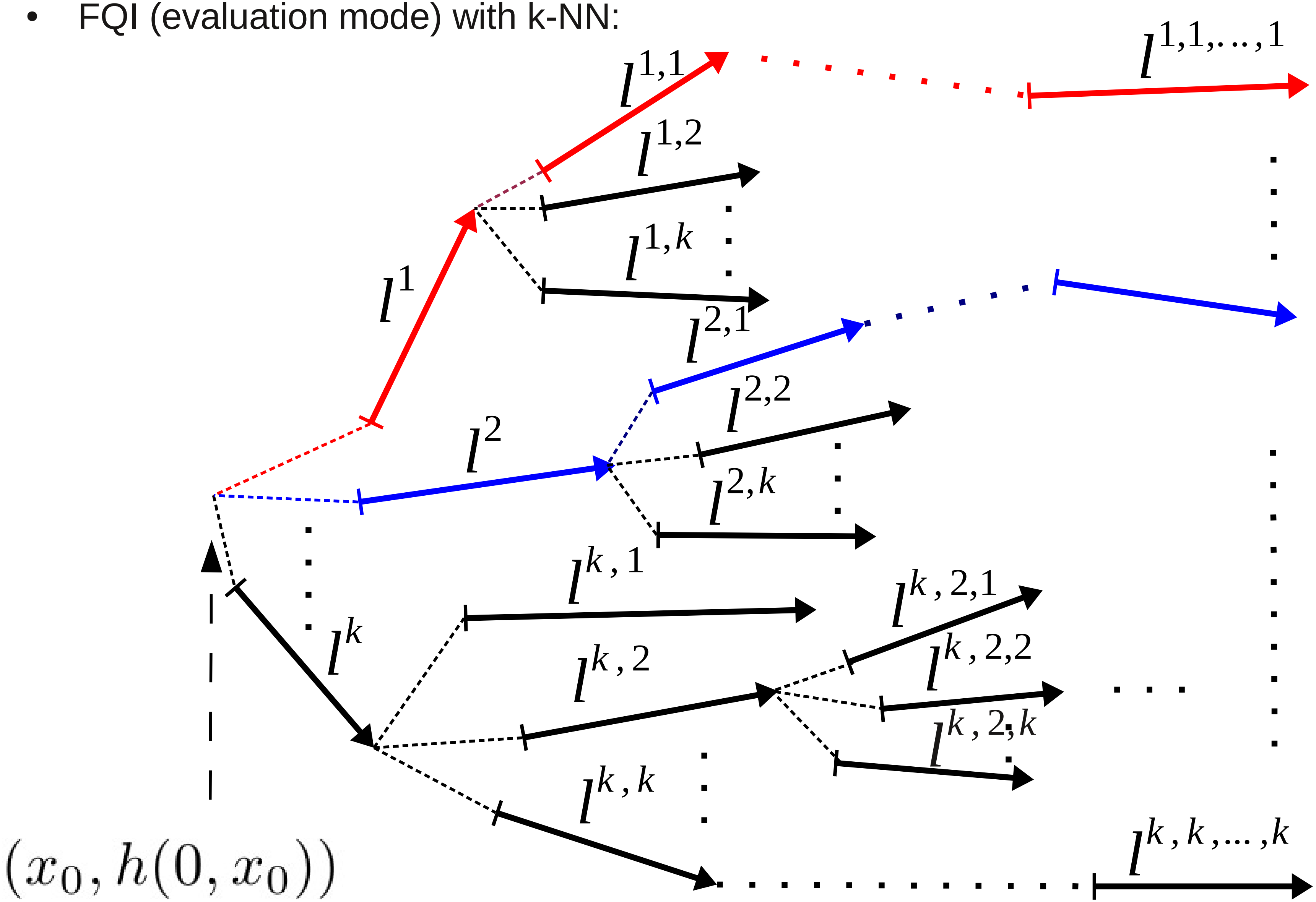
$$5 + 25 + 125 = 155$$



Connexion to Classic Batch Mode RL

Towards a New Paradigm for Batch Mode RL

- FQI (evaluation mode) with k-NN:



Connexion to Classic Batch Mode RL

Towards a New Paradigm for Batch Mode RL

- The k-NN FQI-PE algorithm:

- $\forall (x,u) \in \mathcal{X} \times \mathcal{U},$

$$\hat{Q}_0^h(x,u) = 0 \ ,$$

- For $t = T-1 \ldots 0$ $, \forall (x,u) \in \mathcal{X} \times \mathcal{U},$

$$\hat{Q}_{T-t}^h(x,u) = \frac{1}{k}\sum_{i=1}^{k}\left(r^{l_i(x,u)} + \hat{Q}_{T-t-1}^h\left(y^{l_i(x,u)}, h\left(t+1, y^{l_i(x,u)}\right)\right)\right)$$

- The k-NN FQI-PE estimator:

$$\hat{J}_{FQI}^h(\mathcal{F}_n, x_0) = \hat{Q}_T^h\left(x_0, h(0,x_0)\right)$$