# On some drawbacks and possible improvements of a Particle Finite Element Method for simulating incompressible flows

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My PhD focuses on the analysis and development of the PFEM for new applications involving free surfaces/interfaces

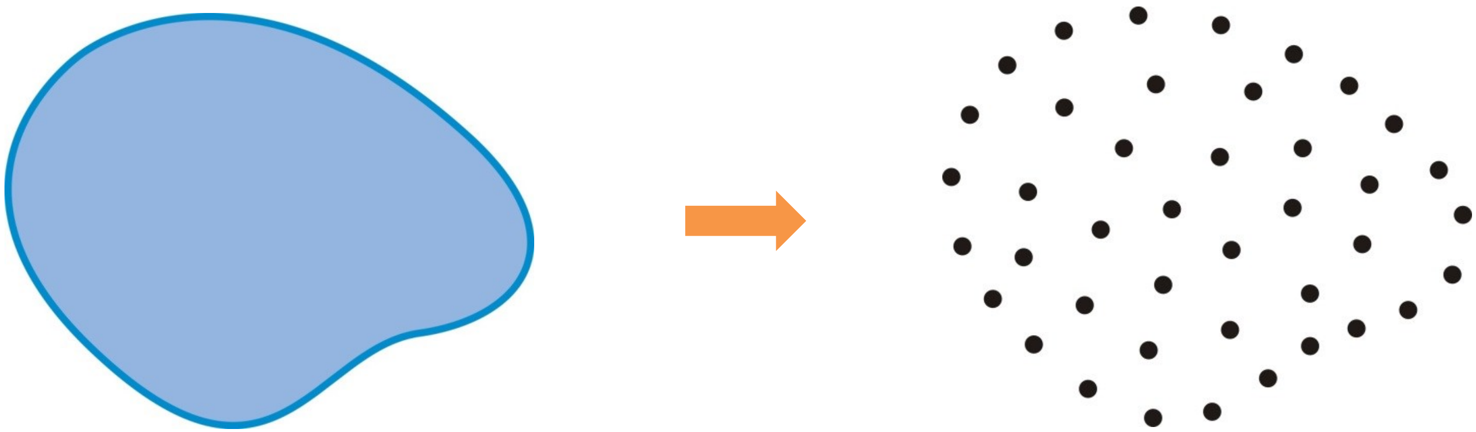
Bird strike experiment

# Presentation layout

- PFEM general ideas
- Correct formulation for incompressible free-surface flows
- PFEM issues
- Conclusions

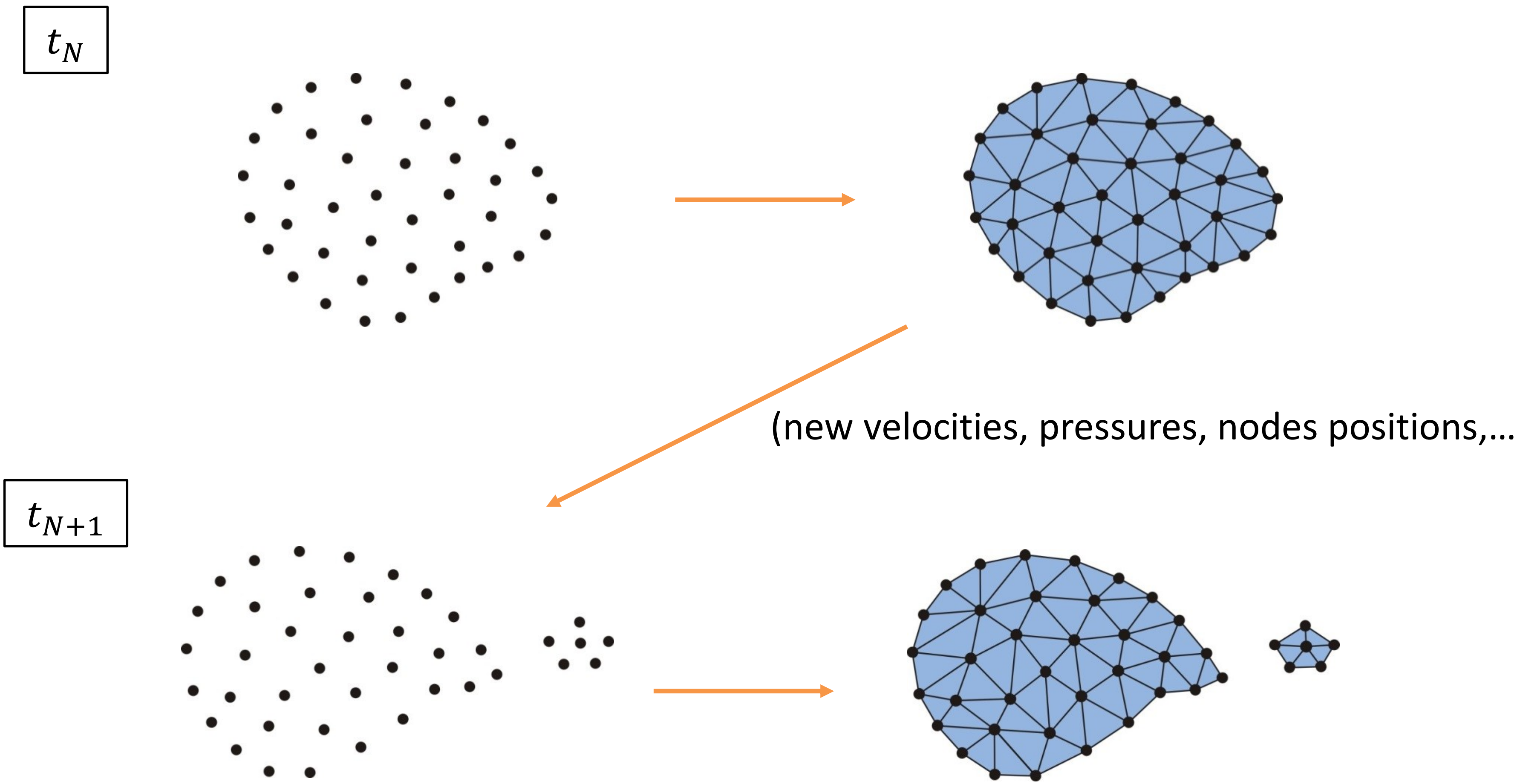
# PFEM general ideas

The first step in the PFEM is discretizing the continuum with some particles/nodes



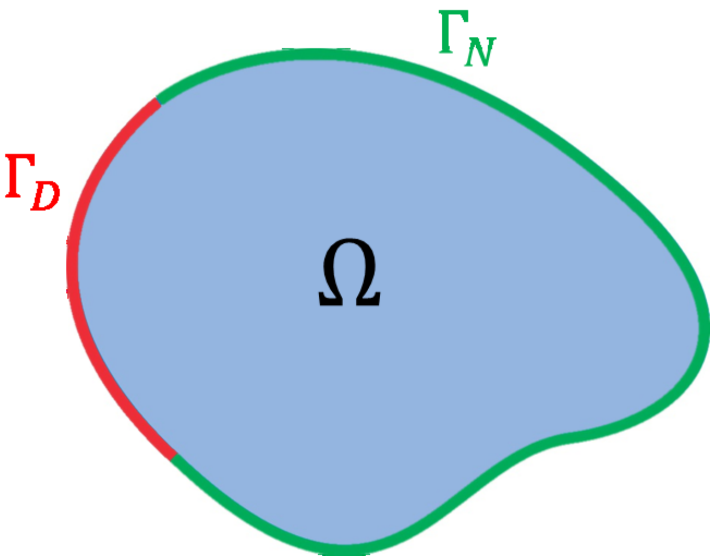
The particles carry all the physical and mathematical information (density, viscosity, velocity, pressure, …)

Then, particles are free to move and at each time step a new mesh is built in order to define the weak form



# Formulation for incompressible free-surface flows

The starting point are the equations of the continuum written in Lagrangian form and current configuration




$$
\begin{cases}
\rho \dfrac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \mathrm{div}\,\boldsymbol{\sigma} + \rho \boldsymbol{b} \quad \text{in } \boldsymbol{\Omega} \\
\dfrac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\mathrm{div}(\boldsymbol{u}) = 0 \quad \text{in } \boldsymbol{\Omega} \\
\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}
\end{cases}
$$

$$
\begin{cases}
\boldsymbol{u}(\boldsymbol{x},t) = \overline{\boldsymbol{u}}(\boldsymbol{x},t) \quad \forall \boldsymbol{x} \in \Gamma_D \\
\boldsymbol{\sigma}(\boldsymbol{x},t) \cdot \boldsymbol{n} = \bar{\boldsymbol{t}}(\boldsymbol{x},t) \quad \forall \boldsymbol{x} \in \Gamma_N
\end{cases}
$$

# From now on I will concentrate on Newtonian incompressible fluid flows

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{D}(\boldsymbol{u})\,,\ \ \mathbf{D}(\boldsymbol{u}) = \frac{1}{2}\left(\text{grad}(\boldsymbol{u}) + \text{grad}(\boldsymbol{u})^{\mathrm{T}}\right)$$

$$\begin{cases} \rho\dfrac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \text{div}\,\boldsymbol{\sigma} + \rho\boldsymbol{b} \quad \text{in } \boldsymbol{\Omega} \\ \dfrac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\text{div}(\boldsymbol{u}) = 0 \quad \text{in } \boldsymbol{\Omega} \end{cases}$$

$$\Rightarrow \begin{cases} \rho_0\dfrac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\text{div}(p\mathbf{I}) + \mu\,\text{div}\left(\text{grad}(\boldsymbol{u}) + \text{grad}(\boldsymbol{u})^{\mathrm{T}}\right) + \rho_0\boldsymbol{b} \quad \text{in } \boldsymbol{\Omega} \\ \text{div}(\boldsymbol{u}) = 0 \quad \text{in } \boldsymbol{\Omega} \end{cases}$$

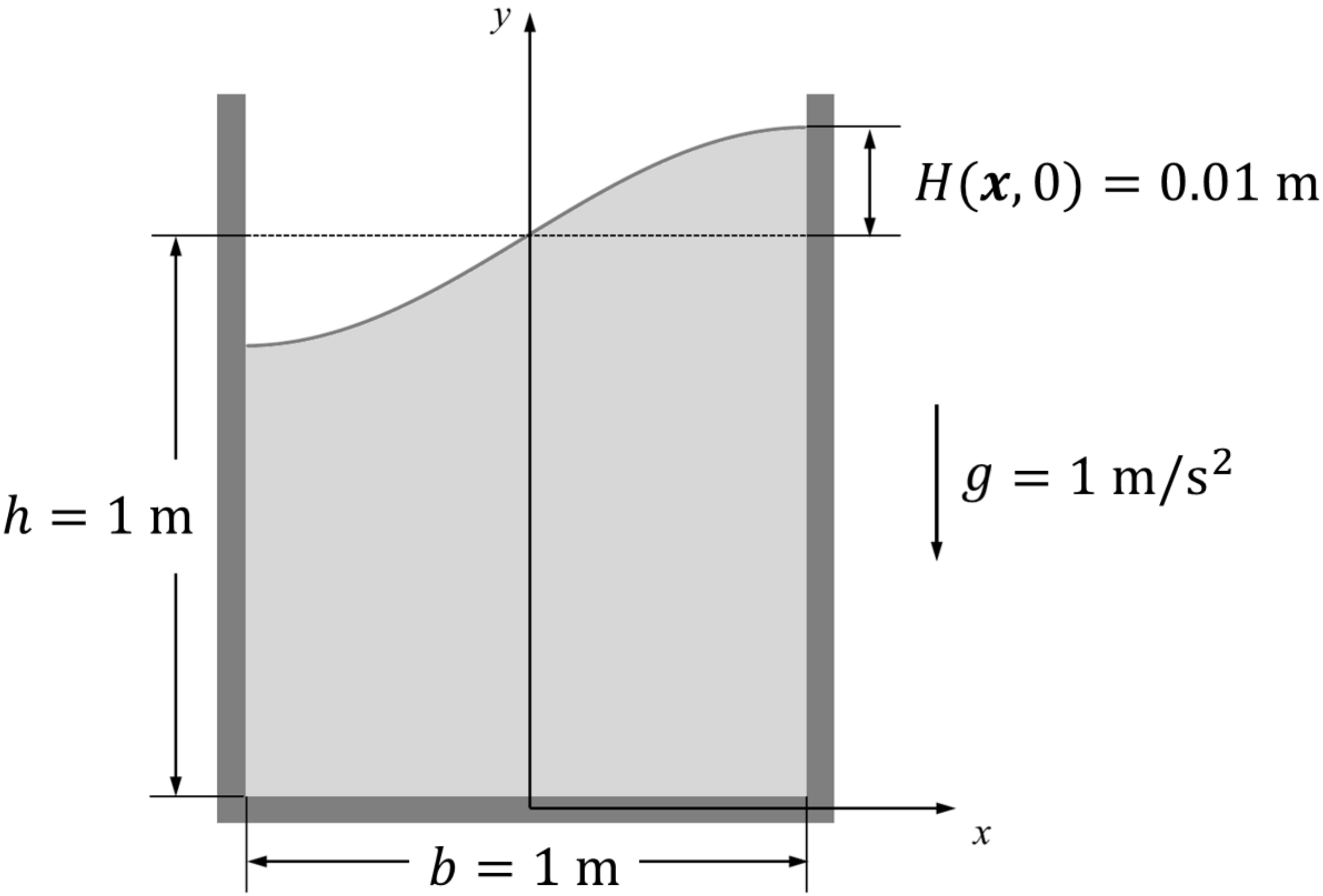
A stable weak form can be obtained by using a Galerking approach and a Petrov-Galerking stabilization for pressure

$$
\begin{cases}
\displaystyle\int_{\Omega} \rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot\boldsymbol{w}\,\mathrm{d}\Omega = \int_{\Omega} p\mathbf{I} : \mathrm{grad}(\boldsymbol{w})\,\mathrm{d}\Omega - \int_{\Omega} \mu\,\mathrm{grad}(\boldsymbol{u}) : \mathrm{grad}(\boldsymbol{w})\,\mathrm{d}\Omega\ + \\
\qquad\qquad - \displaystyle\int_{\Omega} \mu\,\mathrm{grad}(\boldsymbol{u})^{\mathrm{T}} : \mathrm{grad}(\boldsymbol{w})\,\mathrm{d}\Omega + \int_{\Omega} \rho_0\,\boldsymbol{b}\cdot\boldsymbol{w}\,\mathrm{d}\Omega + \int_{\Gamma_N} \bar{\boldsymbol{t}}\cdot\boldsymbol{w}\,\mathrm{d}\Gamma \\
\displaystyle\int_{\Omega} \mathrm{div}(\boldsymbol{u})q\,\mathrm{d}\Omega + \sum_{e=1}^{N_{el}} \int_{\Omega_0^e} \tau_{\mathrm{pspg}}^{e} \frac{1}{\rho_0}\mathrm{grad}(q)\left(\rho_0\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \mathrm{div}(p\mathbf{I}) - \mu\,\mathrm{div}\big(\mathrm{grad}(\boldsymbol{u}) + \mathrm{grad}(\boldsymbol{u})^{\mathrm{T}}\big) - \rho_0\boldsymbol{b}\right)
\end{cases}
$$

$$\forall \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)\,|\,\boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_D, \qquad \forall q \in L^2(\Omega)$$

$$
\begin{cases}
\mathbf{M}\dfrac{\boldsymbol{u}^{\mathrm{n}+1} - \boldsymbol{u}^{\mathrm{n}}}{\Delta t} + \mathbf{K}\boldsymbol{u} + \mathbf{D}^{\mathrm{T}}\boldsymbol{p} = \boldsymbol{B} \\
\mathbf{C}\dfrac{\boldsymbol{u}^{\mathrm{n}+1} - \boldsymbol{u}^{\mathrm{n}}}{\Delta t} + \mathbf{D}\boldsymbol{u} + \mathbf{L}\boldsymbol{p} = \boldsymbol{H}
\end{cases}
$$

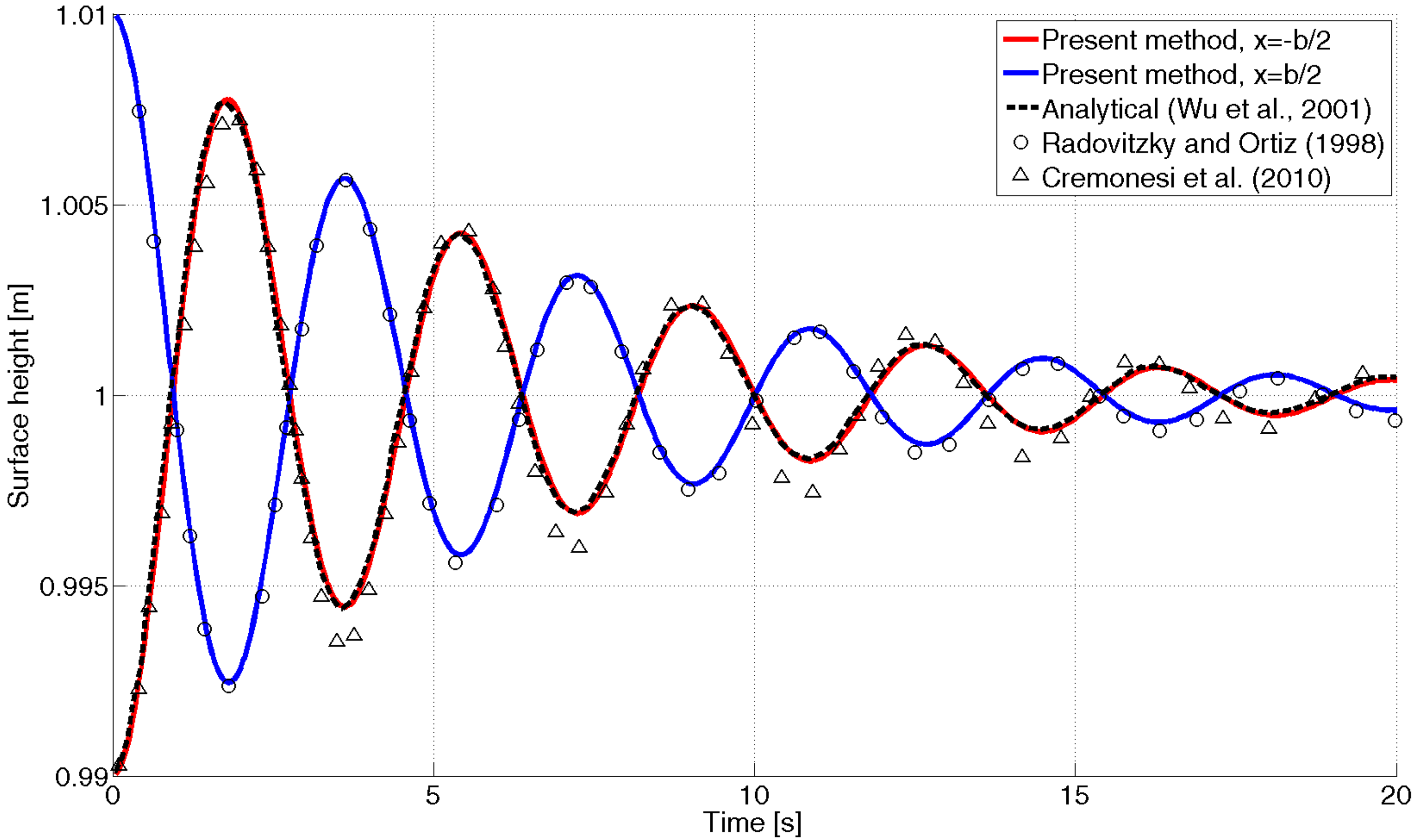
In order to validate the method we have tested it for a classical sloshing example, for which an analytical solution exists




$\rho_0 = 1$ kg/m$^3$

$\mu = 0.01$ kg/ms

# Our results perfectly agree with the analytical solution, better than those found by other authors



Careful! For free-surface flows some dangerous simplifications are often proposed in the literature

1. Strong imposition of the pressure at the free surface

neglected

$$\int_\Omega \rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \boldsymbol{w}\, \mathrm{d}\Omega \quad = (\dots) + \int_{\Gamma_N} \bar{\boldsymbol{t}} \cdot \boldsymbol{w}\, \mathrm{d}\Gamma \qquad \Rightarrow \qquad p = 0\,, \qquad \text{on } \Gamma_\mathrm{N}$$

2. Wrong definition of the boundary term

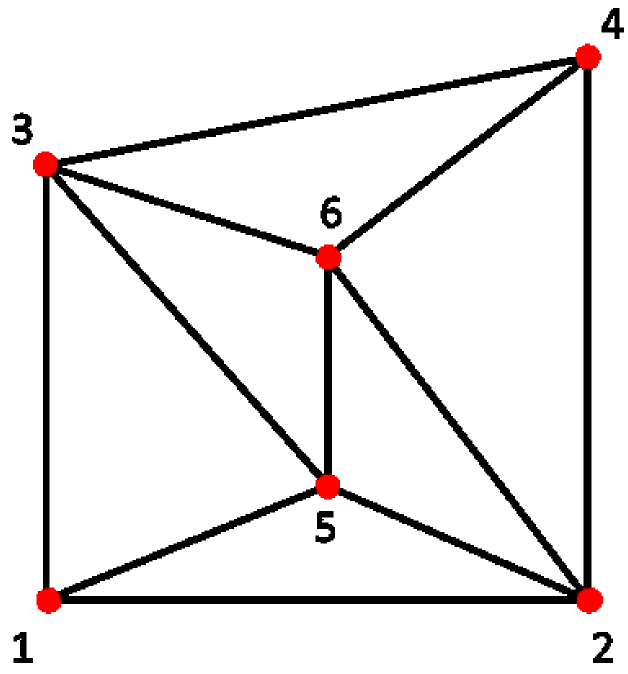
$$\mu\, \mathrm{div}\big(\mathrm{grad}(\boldsymbol{u}) + \mathrm{grad}(\boldsymbol{u})^\mathrm{T}\big) = \mu\, \Delta(\boldsymbol{u}), \qquad \text{for incompressible flows}$$

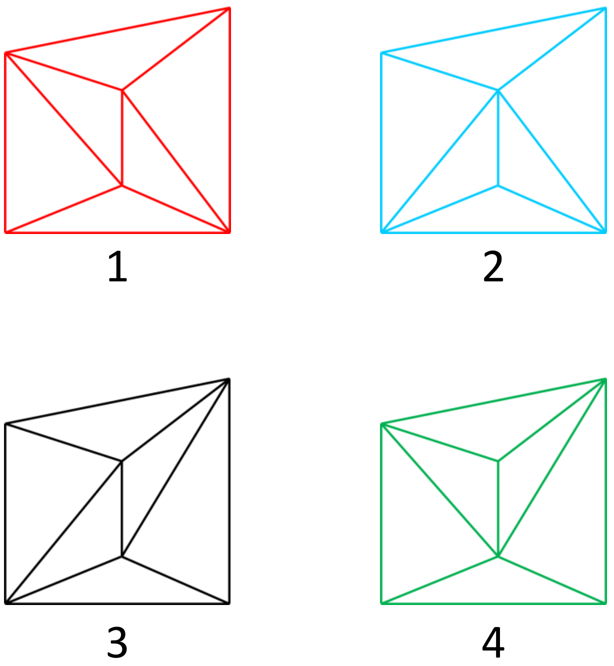
neglected

$$\int_\Omega \rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \boldsymbol{w}\, \mathrm{d}\Omega \quad = (\dots) - \int_\Omega \mu\, \mathrm{grad}(\boldsymbol{u}) : \mathrm{grad}(\boldsymbol{w})\, \mathrm{d}\Omega \quad + \int_{\Gamma_N} (\, \bar{\boldsymbol{t}} - \mu\, \mathrm{grad}(\boldsymbol{u})^\mathrm{T} \boldsymbol{n}\,) \cdot \boldsymbol{w}\, \mathrm{d}\Gamma$$

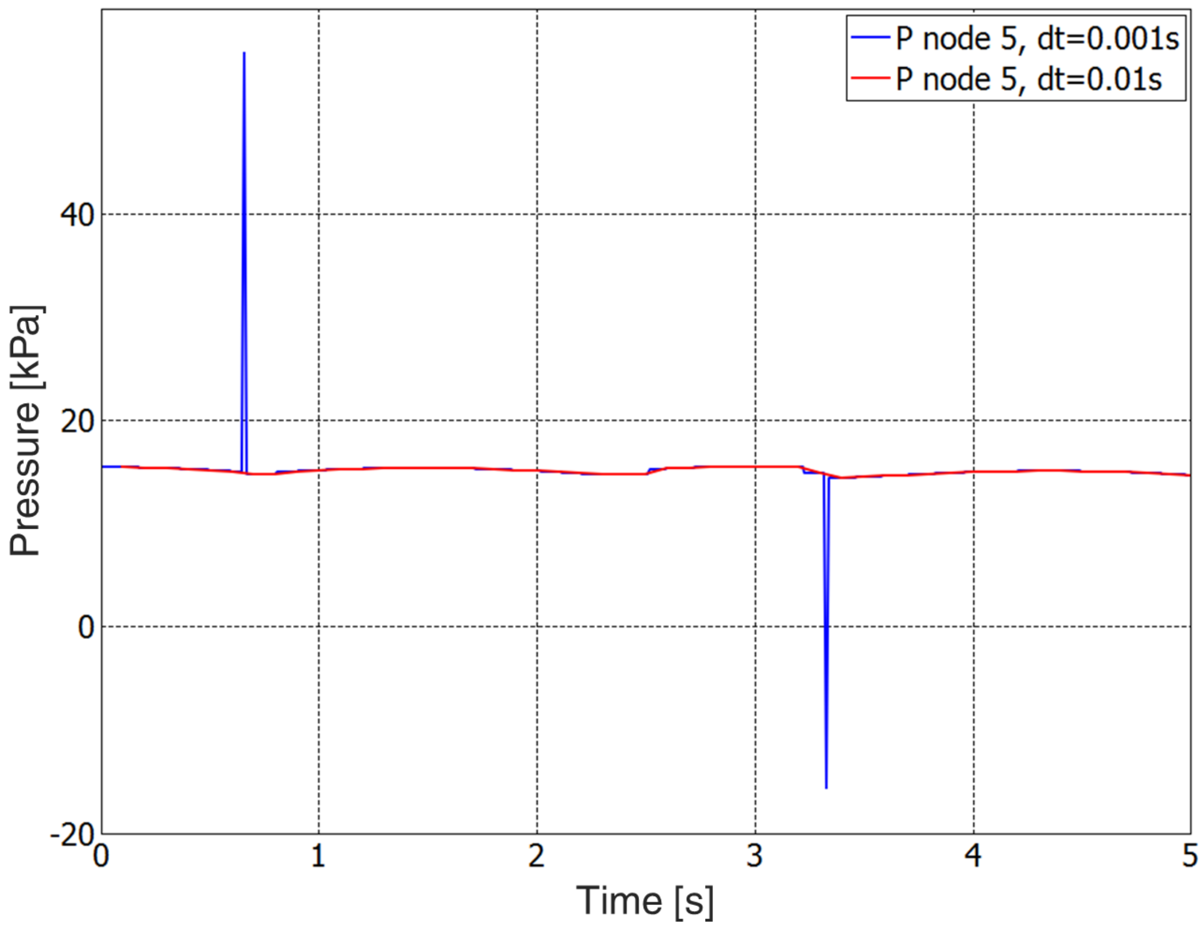
# PFEM issues

To introduce the problem, let's consider again a sloshing example, but with a very coarse discretization

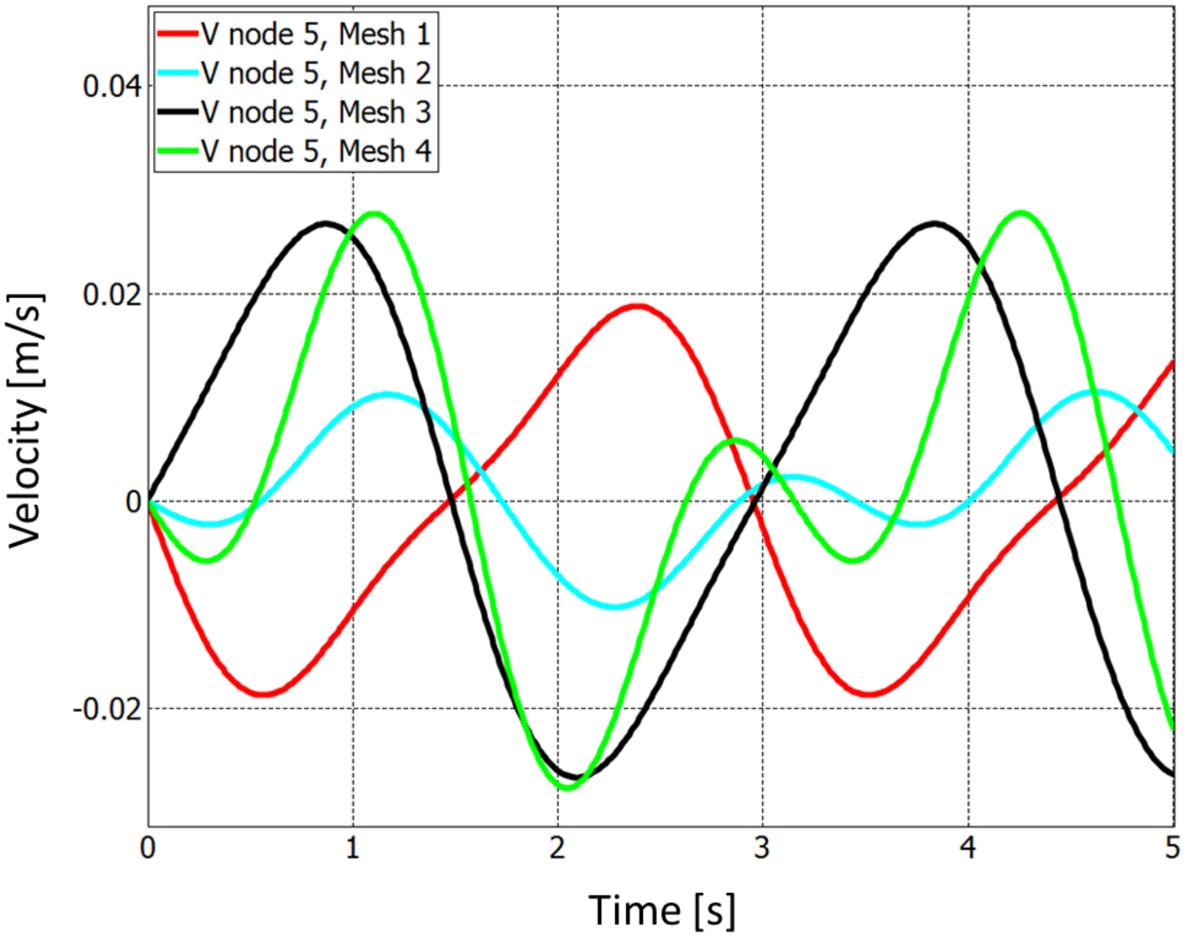


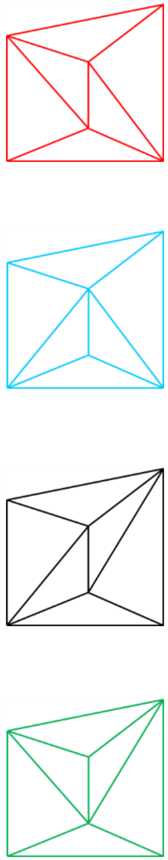



# Some odd oscillations in the pressure field appear when the time step is decreased

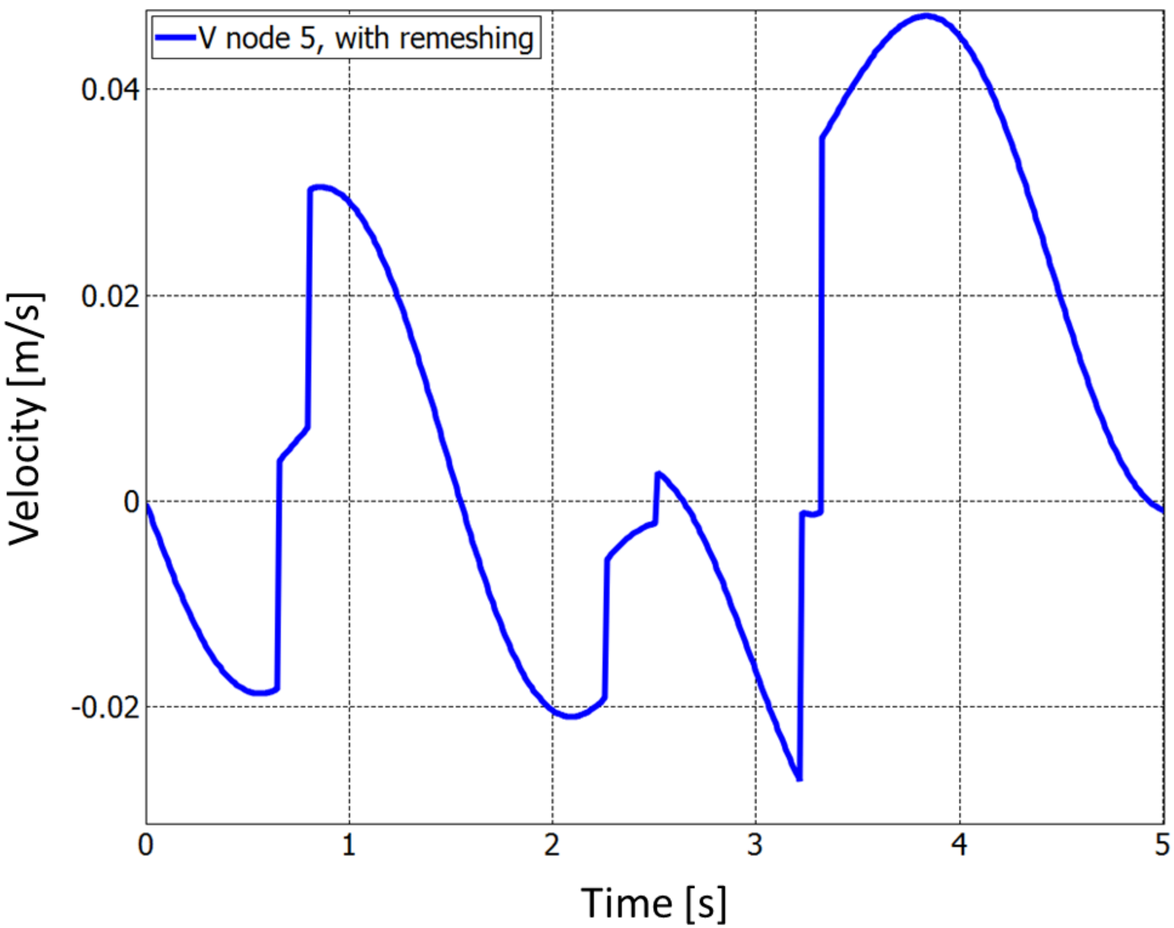


A first observation: the evolutions of the vertical velocity at node 5 for meshes 1 – 4, without performing any remeshing, are very different



# The remeshing introduces perturbations in the velocity field which have to be counter-balanced by the pressure gradient




$$\boldsymbol{u}^{\mathrm{n+1}} = \overline{\boldsymbol{u}}^{\mathrm{n+1}} + \delta \boldsymbol{u}$$

$$\boldsymbol{p}^{\mathrm{n+1}} = \overline{\boldsymbol{p}}^{\mathrm{n+1}} + \delta \boldsymbol{p}$$

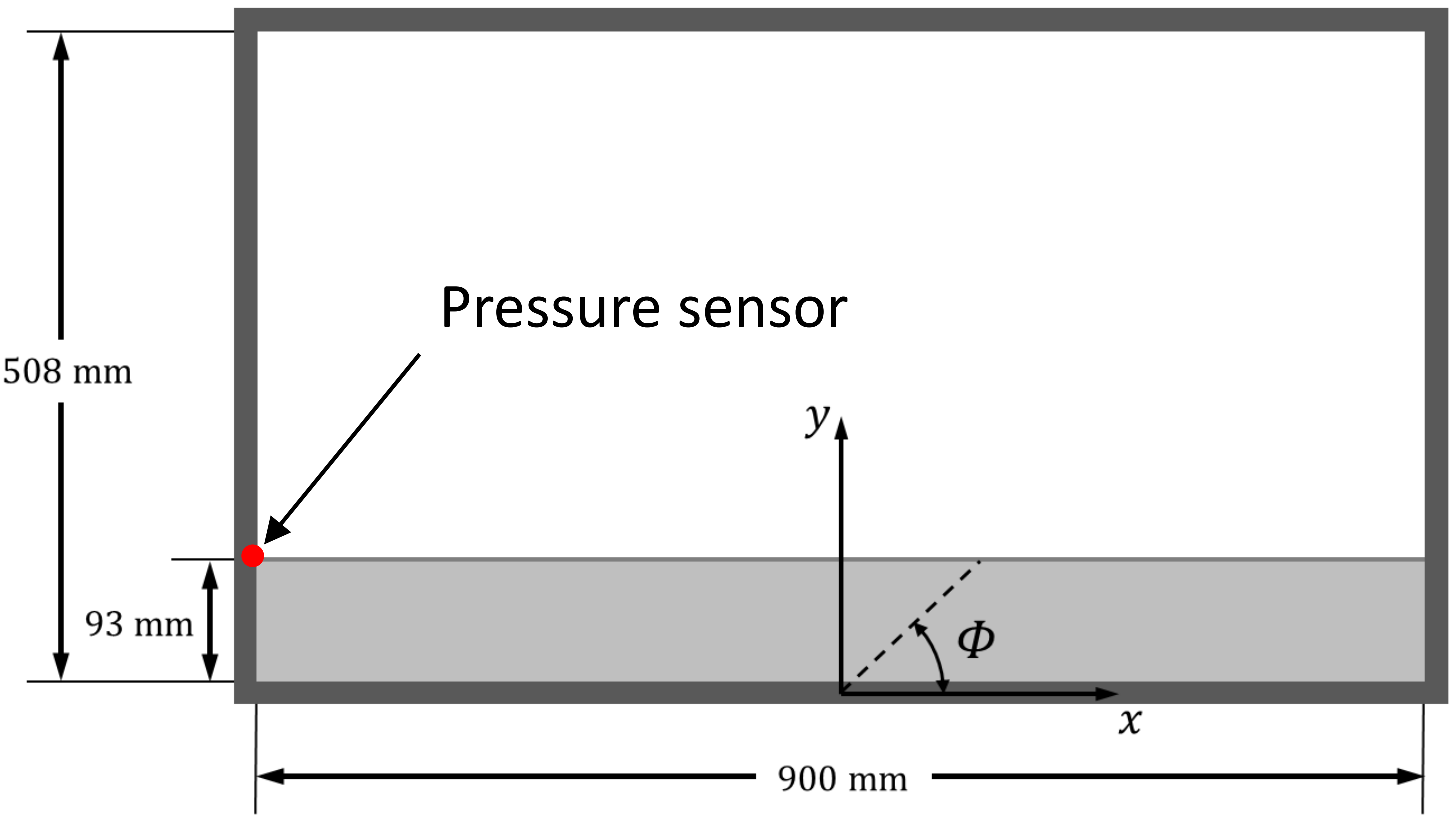
$$\mathbf{M}\frac{\overline{\boldsymbol{u}}^{\mathrm{n+1}} + \delta \boldsymbol{u} - \boldsymbol{u}^{\mathrm{n}}}{\Delta t}$$

$$+ \mathbf{K}(\overline{\boldsymbol{u}}^{\mathrm{n+1}} + \delta \boldsymbol{u})$$

$$+ \mathbf{D}^{\mathrm{T}}(\overline{\boldsymbol{p}}^{\mathrm{n+1}} + \delta \boldsymbol{p}) = \boldsymbol{B}$$

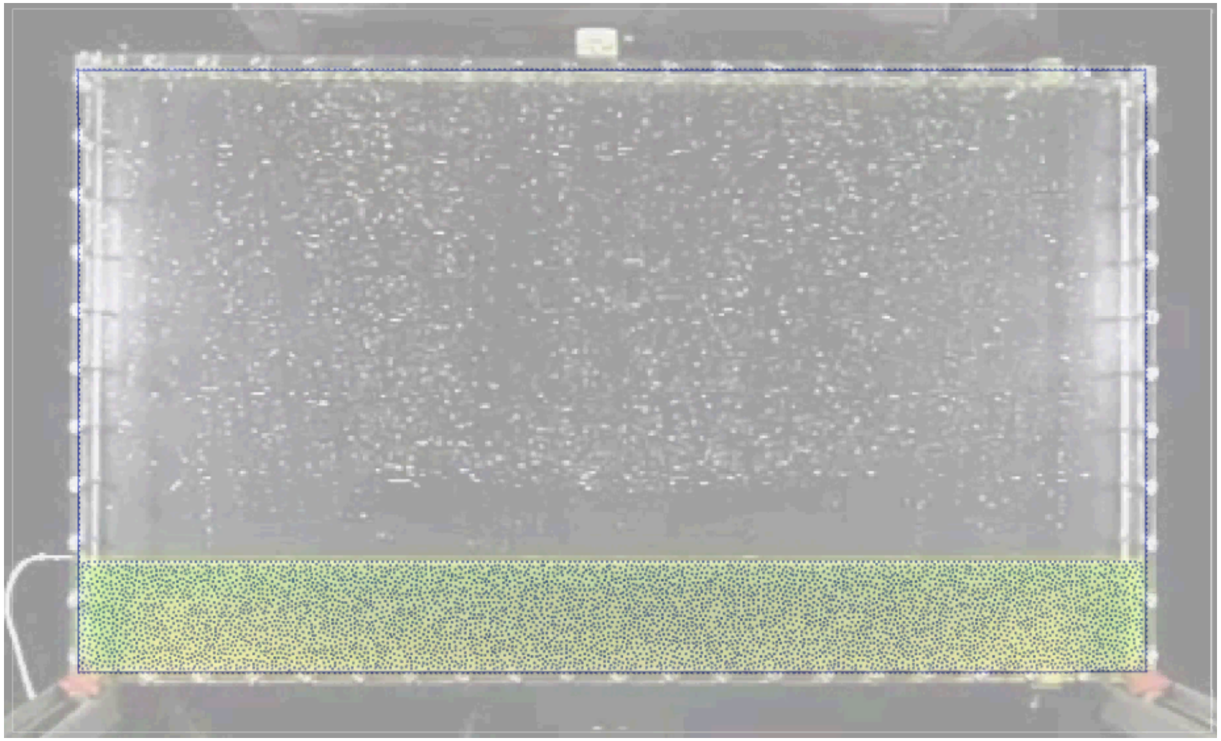
$$\mathbf{M}\frac{\delta \boldsymbol{u}}{\Delta t} + \mathbf{K}\delta \boldsymbol{u} + \mathbf{D}^{\mathrm{T}}\delta \boldsymbol{p} = \mathbf{0}$$

# Now, let's take a look at a more realistic problem…




[ Experimental results  available online on the SPHERIC community website: https://wiki.manchester.ac.uk/spheric ]

The present method can reproduce the global evolution of the phenomenon with very good accuracy

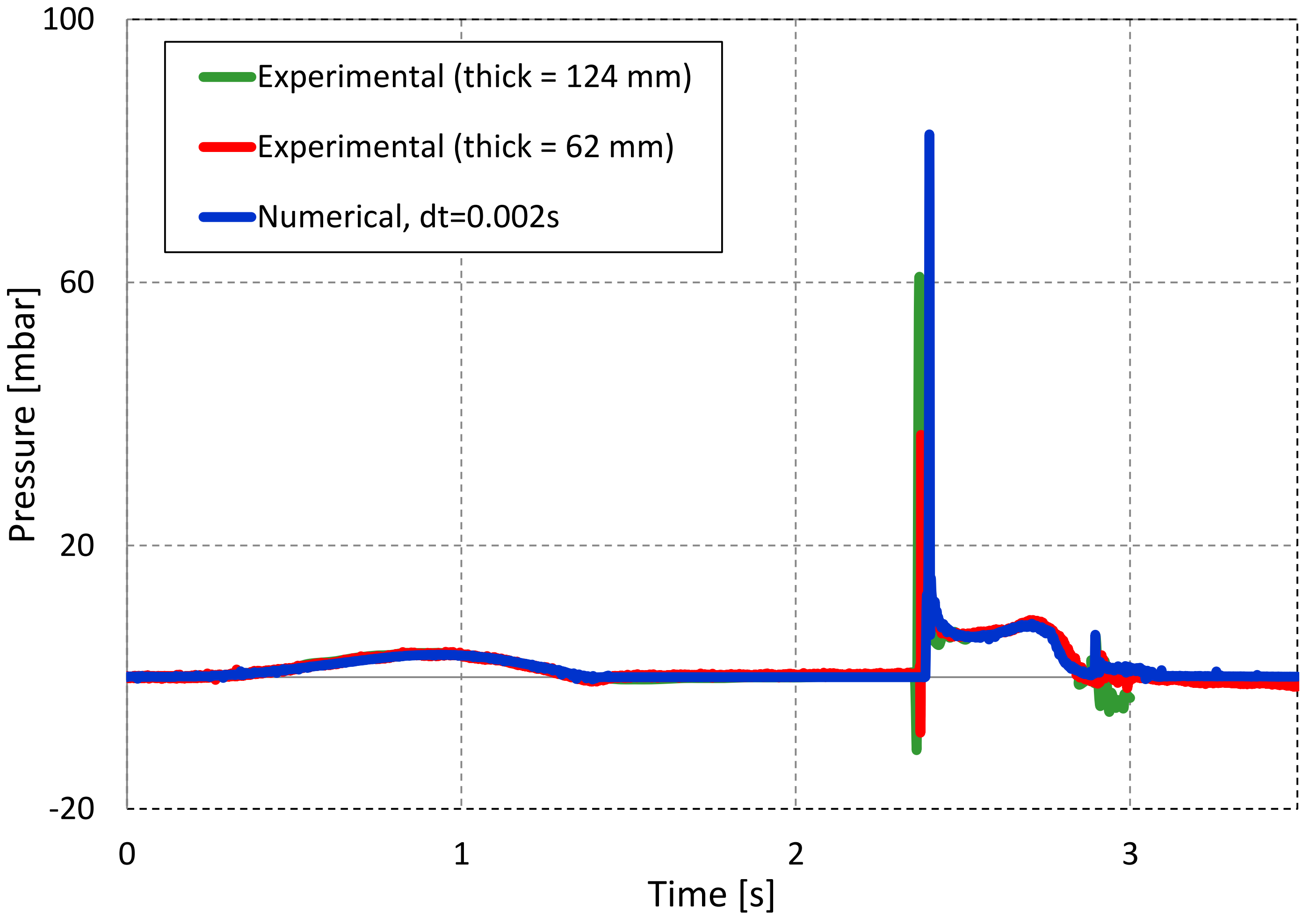


3.5s simulation

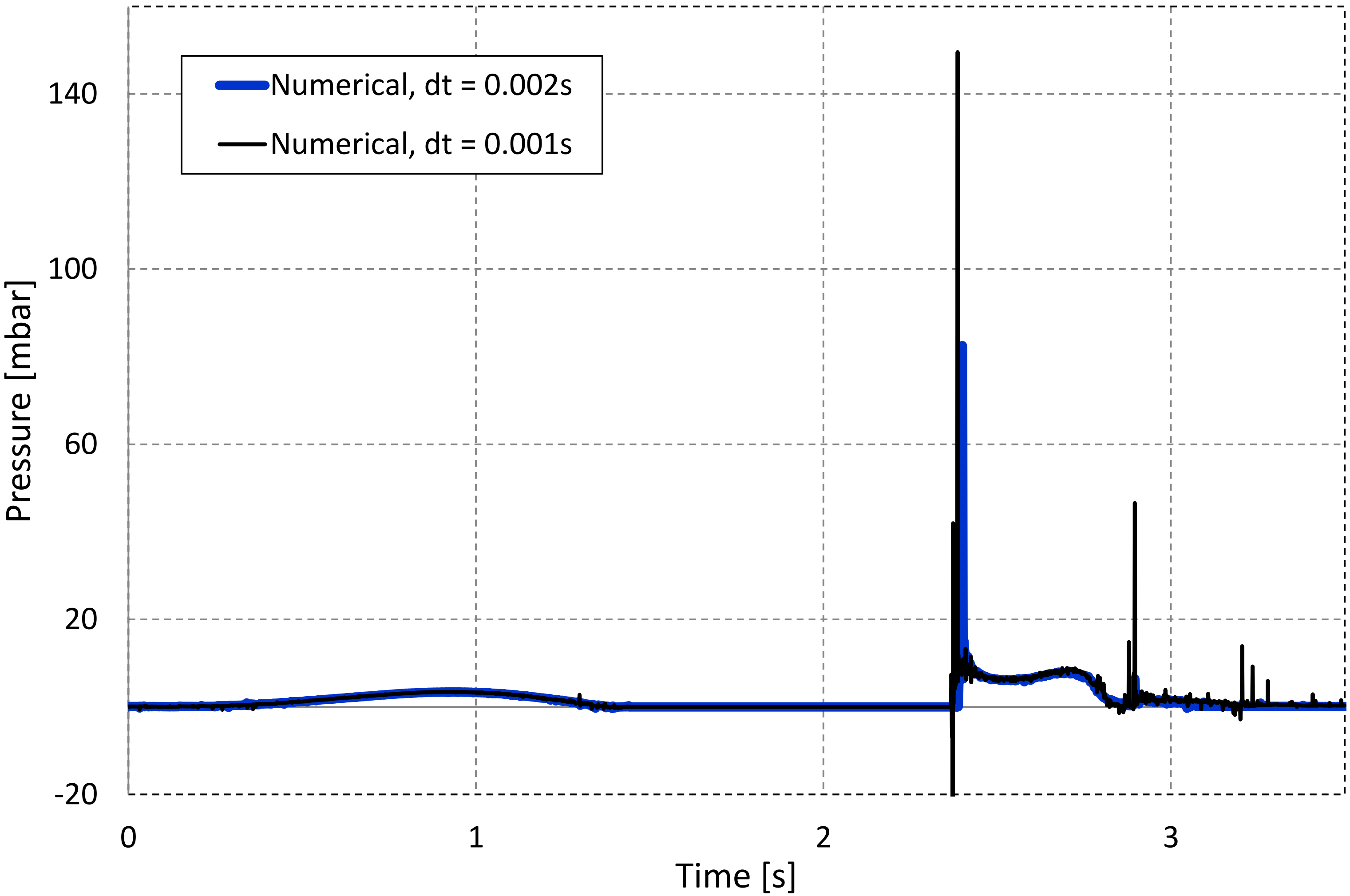
6000 particles

green: experimental
blue dots: numerical

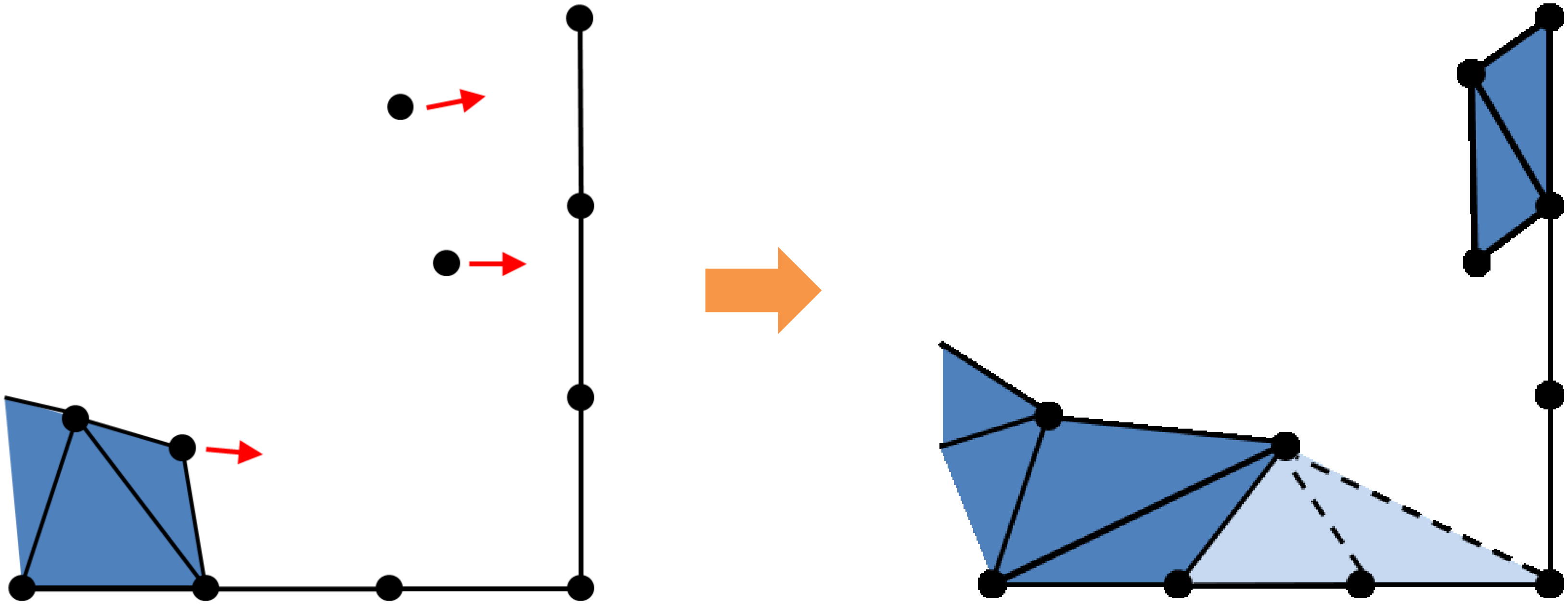
# If a reasonable discretizationis used pressure evolution appears to be very well reproduced



# Nevertheless, pressure oscillations are still present and become visible if the time step is slightly decreased



On fluid-solid boundaries higher gradients are present and/or the discretization can become too coarse: this is where pressure oscillations appear the most!



# Conclusions

Correct free-surface flows formulation:

- Avoid imposing pressure at the free surface
- Do not use so-called «pseudo-tractions»

Remeshing issues:

- Use large time steps
- Use fine discretizations
- Different fluid-solid contact definition

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# Some references


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