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HÖLDER CONTINUITY AND WAVELETS

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Abstract

There exist a lot of continuous nowhere differentiable functions, but these functions do not have the same irregularity. Hölder continuity, and more precisely Hölder exponent, allow to quantify this irregularity. If the Hölder exponent of a function takes several values, the function is said multifractal. In the first part of this thesis, we study in details the regularity and the multifractality of some functions: the Darboux function, the Cantor bijection and a generalization of the Riemann function.

The theory of wavelets notably provides a tool to investigate the Hölder continuity of a function. Wavelets also take part in other contexts. In the second part of this thesis, we consider a nonstationary version of the classical theory of wavelets. More precisely, we study the nonstationary orthonormal bases of wavelets and their construction from a nonstationary multiresolution analysis. We also present the nonstationary continuous wavelet transform.

For some irregular functions, it is difficult to determine its Hölder exponent at each point. In order to get some information about this one, new function spaces based on wavelet leaders have been introduced. In the third and last part of this thesis, we present these new spaces and their first properties. We also define a natural topology on them and we study some properties.

Résumé

Il existe beaucoup de fonctions continues et nulle part dérivables, mais ces fonctions n'ont pas toutes la même irrégularité. La continuité höldérienne et plus précisément l'exposant de Hölder permettent de quantifier cette irrégularité. Lorsque l'exposant de Hölder d'une fonction prend plusieurs valeurs, cette fonction est dite multifractale. Dans la première partie de cette thèse, nous étudions en détail la régularité et la multifractalité de quelques fonctions : la fonction de Darboux, la bijection de Cantor et une généralisation de la fonction de Riemann.

La théorie des ondelettes fournit notamment un outil pour examiner la continuité höldérienne d'une fonction. Les ondelettes interviennent également dans d'autres contextes. Dans la deuxième partie de cette thèse, nous considérons une version non-stationnaire de la théorie classique des ondelettes. Plus précisément, nous étudions les bases orthonormées d'ondelettes non-stationnaires et leur construction à partir d'une analyse multirésolution non-stationnaire. Nous présentons aussi la transformée continue en ondelette non-stationnaire.

Pour certaines fonctions irrégulières, il est difficile de déterminer son exposant de Hölder en chaque point. Afin d'obtenir tout de même des informations sur celui-ci, de nouveaux espaces de fonctions basés sur les coefficients d'ondelettes dominants ont été introduits. Dans la troisième et dernière partie de cette thèse, nous présentons ces nouveaux espaces et leurs premières propriétés. Nous définissons une topologie naturelle sur ceux-ci et nous en étudions quelques propriétés.

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Introduction

Continuous but nowhere differentiable functions? Mathematicians of the early 19$^{\text{th}}$ century thought they did not exist. Moreover, Ampère [2] tried to prove that any continuous function is differentiable, except possibly at a finite number of points. In 1872, Weierstrass [121] showed that

$$x \mapsto \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x)$$

where $a \in (0,1)$ and b is an odd integer such that $ab > 1 + 3\pi/2$ is a continuous nowhere differentiable function. A lot of such functions were then constructed (see [114] for some examples). The mathematical community was extremely astonished about this discovery (see Sections 5.7 and 6.8 in [77]). Some mathematicians, as Hermite and Poincaré, even rejected the relevance of such functions, which they called "monsters" (see page 132 in [102]).

Such functions are irregular, but they can behave in many different ways. Hölder continuity, and more precisely Hölder exponent, allows to quantify the irregularity (see [116]). This notion provides a tool to analyse whether some regularity occurs in the irregularity of a function. On the one hand, the Hölder exponent of a function can be the same everywhere, which means that this function has the same irregularity at every point. On the other hand, the Hölder exponent of a function can also be irregular. In this case, the function is said to be multifractal and its behaviour is completely erratic.

Many mathematicians have been interested in the Hölder continuity and in the multifractality of irregular functions. From the Weierstrass function (see [49, 65, 121]) to Eisenstein series (see [100]) recently, through the Takagi function (see [110, 113]) and the Riemann function (see [49,55,61]), many other functions have been investigated (see also [62] for other examples and [70] for some space-filling maps).

A tool to study the Hölder continuity of a function is given by the theory of wavelets (see [33, 55, 59–61, 68, 92, 115]). The behaviour of its wavelet coefficients (that are its coefficients in an orthonormal basis of wavelets) or the behaviour of its continuous wavelet transform allows to obtain its Hölder continuity. Actually, Hölder continuity can be completely characterized by wavelet coefficients or by continuous wavelet transform. This technique established by Jaffard and Meyer has already proven its worth in the study of the regularity of some functions (see [55,61,100] for some examples).

The theory of wavelets takes also part in other contexts. In the nineties, the notion of "nonstationarity" appeared in the classical theory of orthonormal basis of wavelets (see [16,35,40,41, 98,119]). In the nonstationary setting, orthonormal bases of wavelets using Exponential-Splines have been obtained in [35]. The problem of the construction of regular compactly supported orthonormal bases of wavelets in the general context of Sobolev spaces have been studied in [15,16]. Moreover, infinitely differentiable orthonormal bases of wavelets with compact support have been considered in [41].

Typically, an orthonormal basis of wavelets of $L^2(\mathbb{R})$ is an orthonormal basis of $L^2(\mathbb{R})$ of type

$$2^{j/2}\psi(2^j\cdot -k), \quad j,k\in\mathbb{Z},$$

where $\psi \in L^2(\mathbb{R})$. The nonstationary version of this definition consists in introducing a dependence on the parameter j for the function ψ. More precisely, a nonstationary orthonormal basis of wavelets of $L^2(\mathbb{R})$ is an orthonormal basis of $L^2(\mathbb{R})$ of type

$$2^{j/2}\psi^{(j)}(2^j\cdot -k), \quad j,k\in\mathbb{Z},$$

where $\psi^{(j)} \in L^2(\mathbb{R})$ for $j\in\mathbb{Z}$.

As in the classical case, it is possible to construct such a basis from a procedure called multiresolution analysis, with some adaptations to the nonstationary case. A family of scaling functions can lead to a nonstationary multiresolution analysis (see [**16**, **35**, **98**]).

The present thesis is concerned with the Hölder continuity of functions and the theory of wavelets. This is the explanation of the title. It is mainly based on the papers [**14**,**17**,**18**,**96**,**97**]. It is divided into three parts.

Part I studies the Hölder continuity of several functions. After some recalls about pointwise and uniform Hölder continuity in Chapter 1, we first determine the Hölder exponent of the Darboux function. Chapter 2 focuses on a well-known space-filling function, called Cantor's bijection. We explore the multifractal nature of this one-to-one correspondence between the unit segment $[0,1]$ and the unit square $[0,1]^2$. Moreover, in the appendix, we construct another bijection between $[0,1]$ and $[0,1]^2$ inspired by an idea of Cantor. Finally, in Chapter 4, we study the uniform Hölder continuity of a generalization of the Riemann function. To do so, we use the known characterization of Hölder continuity with continuous wavelet transform formulated in Chapter 3. We also analyse the behaviour of this generalized Riemann function according to its parameters.

Part II mainly focuses on the theory of wavelets. We investigate the classical notions of orthonormal basis of wavelets and of continuous wavelet transform in a nonstationary setting. Firstly, in Chapter 5, we consider the construction of a nonstationary orthonormal basis of wavelets in $L^2(\mathbb{R})$ from a nonstationary multiresolution analysis. Under some additional asymptotic assumption, we present a necessary and sufficient condition about such a procedure. We notably illustrate the results on the example of Exponential-Splines. Secondly, we propose a nonstationary version of the continuous wavelet transform of a square integrable function in Chapter 6. After having given some examples, we study the reconstruction of a square integrable function from its nonstationary continuous wavelet transform.

Part III studies new spaces first introduced in the context of multifractal analysis. These spaces provide a tool to investigate the regularity (and more precisely some information about the Hölder exponent) of a function from its wavelet leaders, that is to say from quantities using the coefficients of the function in an orthonormal basis of wavelets. Since these new spaces do not depend on the chosen orthonormal basis of wavelets, they can be considered as sequence spaces. We present these new sequence spaces and their first properties in Chapter 7. Then, in Chapter 8, we study them from the functional analysis point of view. We define a natural topology on these spaces and study some of its properties.

Let us end this introduction with some explanations about this thesis. Except for the beginning of Chapter 1, we have included the proofs of new results and the proofs of known results for which we have not found a proof in the literature. If a result is given without a proof, at least one reference is mentioned to find the result and a proof of the latter.

The notations of this thesis are classical. The symbol $\mathbb{N}$ denotes the set of strictly positive natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Both $\hat{f}$ and $\mathcal{F}^- f$ designate the (negative) Fourier transform of the function f. For $f \in L^1(\mathbb{R})$, we have

$$\hat{f}(\xi) = \mathcal{F}^-_\xi f = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx, \quad \xi \in \mathbb{R}\,.$$

A list of symbols classified by section is given at the end of this thesis.

Part I

Hölder Continuity of Particular Functions

Chapter 1

Continuous Nowhere Differentiable Functions and Hölder Continuity

There exist a lot of functions which are continuous, but nowhere differentiable (see [4,56,91]). The most famous example of such functions is certainly the Weierstrass function W defined by

$$W(x) := \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x), \quad x \in \mathbb{R} \tag{1.1}$$

where $a \in (0,1)$ and $b > 0$ with $ab > 1$ (see [49,121]). Another well-known continuous nowhere differentiable function is the Takagi function T defined by

$$T(x) := \sum_{n=0}^{+\infty} \frac{1}{2^n} \operatorname{dist}(2^n x, \mathbb{Z}), \quad x \in \mathbb{R}$$

(see [113]). The graphics of W and T are represented in Figure 1.1. Amazingly, W and T are not the first constructions of continuous nowhere differentiable functions. In fact, Bolzano and also Cellérier earlier built such a function, without publishing their discovery (see [57] for some historical information).

The Hölder spaces allow to define a notion of smoothness or regularity for a function and, in particular, they roughly provide an "intermediate level" between continuity and differentiability. In this chapter, we first give the definition of Hölder spaces and Hölder continuity in this context. The general definition is also considered in the pointwise case. We then introduce the notion of Hölder exponent. We finish with the Hölder continuity of a detailed first example: the Darboux function.

1.1 Hölder Continuity and Hölder Spaces

1.1.1 Pointwise Hölder Continuity

Let us begin with the definition of pointwise Hölder continuity (see [33,65,88,95,115]).

Definition 1.1.1. Let $\alpha \in (0,1]$ and $x_0 \in \mathbb{R}$. The function f is *Hölder continuous of order α at x_0* if there exist $C, \delta > 0$ such that

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha \tag{1.2}$$

for all $x \in (x_0 - \delta, x_0 + \delta)$. We denote by $\Lambda^\alpha(x_0)$ the space of Hölder continuous functions of order α at x_0 and it is called *Hölder space of order α at x_0*.

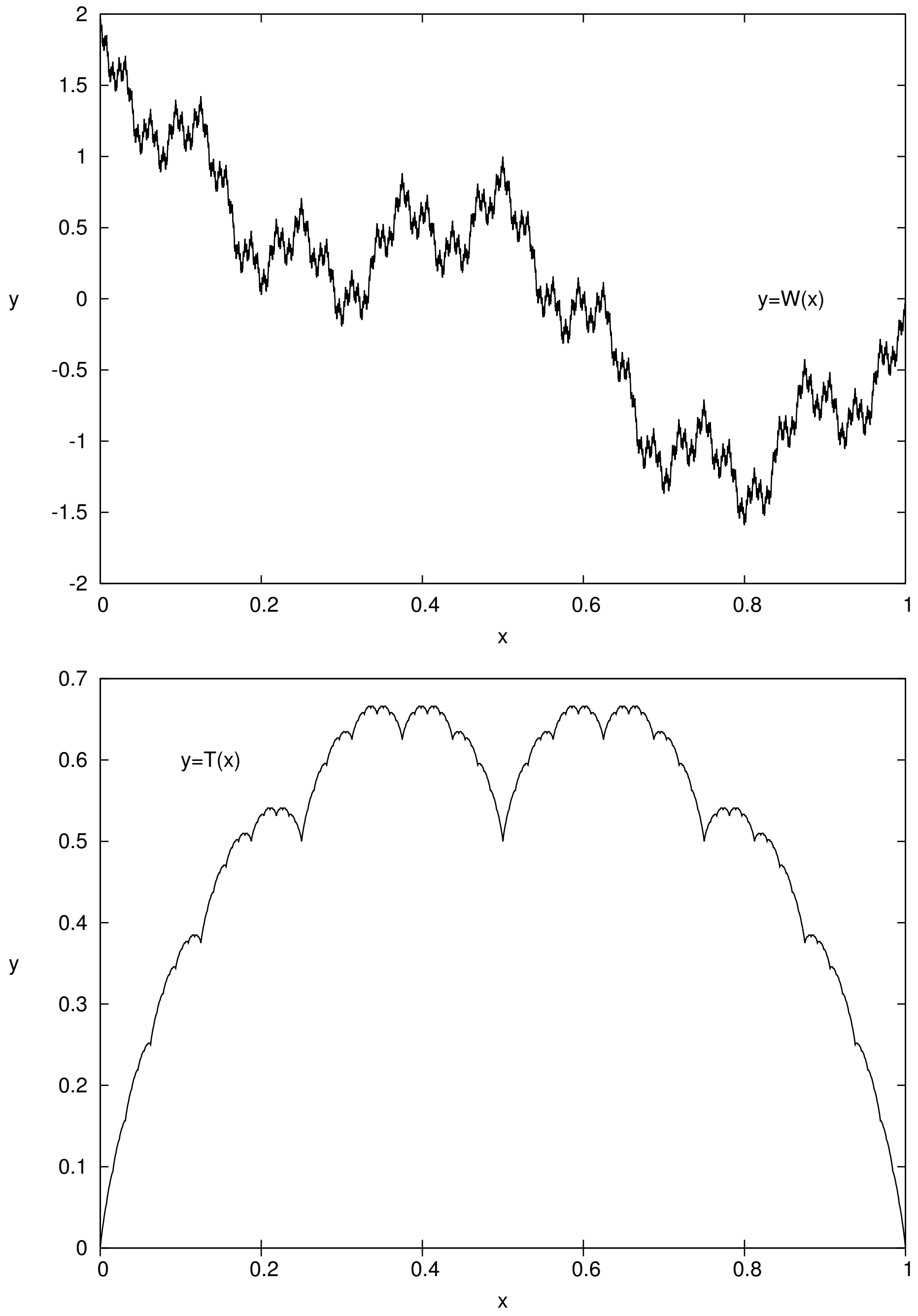

Figure 1.1. Graphical representations of W (with $a = 1/2$ and $b = 4$) and of T.

The definition implies that f is bounded in a neighbourhood of x_0 if $f \in \Lambda^\alpha(x_0)$ for some $\alpha \in (0,1]$. Incidentally, if we consider the case $\alpha = 0$, $\Lambda^0(x_0)$ would simply be the space of bounded functions in a neighbourhood of x_0. The case $\alpha = 1$ corresponds to the space of Lipschitz functions at x_0.

Hölder spaces are clearly embedded: if $\alpha, \beta \in (0,1]$ such that $\alpha > \beta$, we have $\Lambda^\alpha(x_0) \subset \Lambda^\beta(x_0)$ for all $x_0 \in \mathbb{R}$. This property will be proved in a more general case (see Proposition 1.1.12).

The following proposition investigates the links between differentiability, Hölder continuity and continuity at a point.

Proposition 1.1.2. *Let $x_0 \in \mathbb{R}$.*

(a) If $f \in \Lambda^\alpha(x_0)$ for some $\alpha \in (0,1]$, then f is continuous at x_0.

(b) If f is differentiable at x_0, then $f \in \Lambda^\alpha(x_0)$ for all $\alpha \in (0,1]$.

Proof. The first item is evident and let us prove the second item. By hypothesis, there exists $\delta \in (0,1)$ such that

$$\left| (Df)(x_0) - \frac{f(x) - f(x_0)}{x - x_0} \right| \leq 1$$

and then

$$|f(x) - f(x_0)| \leq (1 + |(Df)(x_0)|)\, |x - x_0|$$

for all $x \in (x_0 - \delta, x_0 + \delta)$. So $f \in \Lambda^1(x_0)$, which suffices using the embedding of pointwise Hölder spaces. ■

The converse of each item of the previous proposition is false. On the one hand, the function $x \mapsto -\chi_{(0,1)}(x)/\log(x)$ is continuous at 0, but there exists no $\alpha \in (0,1]$ such that it belongs to $\Lambda^\alpha(0)$. On the other hand, the function $x \mapsto |x|$ belongs to $\Lambda^\alpha(0)$ for all $\alpha \in (0,1]$, but is not differentiable at 0.

1.1.2 Uniform Hölder Continuity

Let us go on with the uniform Hölder continuity (see [33, 80, 92, 95, 115]). Before that, let us make the following remark about Definition 1.1.1.

Remark 1.1.3. If f is moreover bounded on $\mathbb{R}$ in Definition 1.1.1, Condition (1.2) holds everywhere. Indeed, for $x \in \mathbb{R}$ such that $|x - x_0| \geq \delta$, we have

$$|f(x) - f(x_0)| \leq 2 \sup_{y \in \mathbb{R}} |f(y)| \leq \frac{2}{\delta^\alpha} \sup_{y \in \mathbb{R}} |f(y)| |x - x_0|^\alpha.$$

Then, the bounded function f belongs to $\Lambda^\alpha(x_0)$ if and only if there exists $C > 0$ such that

$$|f(x) - f(x_0)| \leq C |x - x_0|^\alpha$$

for all $x \in \mathbb{R}$.

Definition 1.1.4. Let $\alpha \in (0,1]$ and f be a bounded function on $\mathbb{R}$. The function f is *uniformly Hölder continuous of order α (on $\mathbb{R}$)* if there exists $C > 0$ such that

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha$$

for all $x, x_0 \in \mathbb{R}$. We denote by $\Lambda^\alpha(\mathbb{R})$ the space of uniformly Hölder continuous functions of order α (on $\mathbb{R}$) and it is called *uniform Hölder space of order α (on $\mathbb{R}$)*.

In comparison with Definition 1.1.1, the constant C does not depend here on x_0. If we consider the case $\alpha = 0$, $\Lambda^0(\mathbb{R})$ would be the space of bounded functions. The case $\alpha = 1$ corresponds to the space of uniformly Lipschitz functions.

Remark 1.1.5. If f is uniformly Hölder continuous (of order $\alpha \in (0,1]$) on $\mathbb{R}$, then f is clearly Hölder continuous (of order α) at each point in $\mathbb{R}$. The reverse is false. For example, the function f defined by

$$f(x) := \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is Hölder continuous of order 1 at each point in $\mathbb{R}$, but is not uniformly Hölder continuous of order 1. Indeed, it is easy to check that $f \in \Lambda^1(0)$. If $x_0 > 0$, there exists $\delta > 0$ such that $x_0 - \delta > 0$ and

$$\begin{aligned} |f(x) - f(x_0)| &\leq |x - x_0| \left|\sin\left(\frac{1}{x_0}\right)\right| + |x| \left|\sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x_0}\right)\right| \\ &\leq |x - x_0| + |x| \left|\int_{x_0}^{x} \frac{-1}{t^2} \cos\left(\frac{1}{t}\right) dt\right| \\ &\leq |x - x_0| + |x| \left|\frac{1}{x_0} - \frac{1}{x}\right| \\ &\leq \left(1 + \frac{1}{|x_0|}\right) |x - x_0| \end{aligned}$$

for all $x \in (x_0 - \delta, x_0 + \delta)$. The case $x_0 < 0$ is similar. Then, $f \in \Lambda^1(x_0)$ for all $x_0 \in \mathbb{R}$. Let us now show that $f \notin \Lambda^1(\mathbb{R})$. Let $C > 0$ and let us set

$$x_n := \frac{1}{\pi(n + \frac{1}{2})}, \quad n \in \mathbb{N}.$$

There exists $N \in \mathbb{N}$ such that $2(2n+1) > C$ for all $n \geq N$. For such n, we have

$$|f(x_{2n}) - f(x_{2n+1})| = \frac{2}{\pi} \frac{2n+1}{(2n + \frac{1}{2})(2n + \frac{3}{2})} > C|x_{2n} - x_{2n+1}|,$$

hence the conclusion.

Uniform Hölder spaces are also embedded, and this comes from the hypothesis of boundedness in the definition of uniform Hölder continuity. This is the object of the following proposition.

Proposition 1.1.6. *If $\alpha, \beta \in (0,1]$ such that $\alpha > \beta$, we have $\Lambda^\alpha(\mathbb{R}) \subset \Lambda^\beta(\mathbb{R})$.*

Proof. Let $f \in \Lambda^\alpha(\mathbb{R})$. By hypothesis, there exists $C > 0$ such that

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha$$

for all $x, x_0 \in \mathbb{R}$. If $|x - x_0| \leq 1$, we have

$$|f(x) - f(x_0)| \leq C|x - x_0|^\beta$$

and if $|x - x_0| > 1$, we have

$$|f(x) - f(x_0)| \leq 2 \sup_{y \in \mathbb{R}} |f(y)| \, |x - x_0|^\beta$$

because f is bounded. With $C' := \max\{C, 2\sup_{y\in\mathbb{R}} |f(y)|\}$, we thus have

$$|f(x) - f(x_0)| \leq C'|x - x_0|^\beta$$

for all $x, x_0 \in \mathbb{R}$ and $f \in \Lambda^\beta(\mathbb{R})$. ■

Let us investigate the links between differentiability, uniform Hölder continuity and (uniform) continuity.

Proposition 1.1.7. *Let f be a bounded function.*

(a) If $f \in \Lambda^\alpha(\mathbb{R})$ for some $\alpha \in (0, 1]$, then f is uniformly continuous (and so continuous) on $\mathbb{R}$.

(b) If f is differentiable on $\mathbb{R}$ and if Df is bounded, then $f \in \Lambda^\alpha(\mathbb{R})$ for all $\alpha \in (0, 1]$.

Proof. (a) By hypothesis, there exists $C > 0$ such that

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha$$

for all $x, x_0 \in \mathbb{R}$. Let $\varepsilon > 0$ and let $\eta := (\varepsilon/C)^{1/\alpha}$. We have $|f(x) - f(x_0)| \leq \varepsilon$ for all $x, x_0 \in \mathbb{R}$ such that $|x - x_0| \leq \eta$ and so, f is uniformly continuous on $\mathbb{R}$.

(b) By Proposition 1.1.6, it suffices to show that $f \in \Lambda^1(\mathbb{R})$. For all $x, x_0 \in \mathbb{R}$, we have

$$|f(x) - f(x_0)| = \left| \int_{x_0}^x Df(t)\, dt \right| \leq \sup_{t \in \mathbb{R}} |Df(t)| \, |x - x_0|$$

because Df is bounded. Hence the conclusion. ■

Remark 1.1.8. The condition "Df is bounded" is also necessary. More precisely, if f is differentiable and uniformly Hölder of order 1, then Df is bounded. Indeed, there exists $C > 0$ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq C$$

for all $x, x_0 \in \mathbb{R}$ with $x \neq x_0$ and taking the limit for $x \to x_0$, we have $|Df(x_0)| \leq C$ for all $x_0 \in \mathbb{R}$.

1.1.3 Extension

Let us now consider Hölder continuity of order strictly bigger than 1. Before that, let us make the following remark.

Remark 1.1.9. Let $\alpha > 1$ and f be a function defined on $\mathbb{R}$. If there exists $C > 0$ such that

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha$$

for all $x, x_0 \in \mathbb{R}$, then f is constant on $\mathbb{R}$. Indeed, for $x, x_0 \in \mathbb{R}$ with $x \neq x_0$, we have

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} \leq C|x - x_0|^{\alpha-1}.$$

Consequently, f is differentiable and $Df = 0$ on $\mathbb{R}$, hence the conclusion.

Let us give the general definition of Hölder continuity (see [65, 95, 115]).

Definition 1.1.10. Let $\alpha > 0$ and $x_0 \in \mathbb{R}$. The function f is *Hölder continuous of order α at x_0* if there exist $C, \delta > 0$ and a polynomial P of degree strictly smaller than α such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha \tag{1.3}$$

for all $x \in (x_0 - \delta, x_0 + \delta)$. We still denote by $\Lambda^\alpha(x_0)$ the set of Hölder continuous functions of order α at x_0 and this set is called *Hölder space of order α at x_0*.

Definition 1.1.10 is clearly a generalization of Definition 1.1.1. Indeed, taking $x = x_0$ in Inequality (1.3), we directly have $P(0) = f(x_0)$ and so, the independent term of P is $f(x_0)$.

Remark 1.1.11. In the following, we write the polynomial P of Definition 1.1.10 as

$$P(x) := \sum_{k=0}^{\underline{\alpha}} p_k x^k, \quad x \in \mathbb{R}$$

where $\underline{\alpha}$ is the greatest natural number strictly smaller than α and $p_k \in \mathbb{C}$ for $k \in \{0, \ldots, \underline{\alpha}\}$ (which eventually depend on x_0). We already know that $p_0 = f(x_0)$. Moreover, P is unique. To show that, let us assume that there exists a polynomial Q of degree strictly smaller than α such that

$$|f(x) - Q(x - x_0)| \leq C|x - x_0|^\alpha$$

for all $x \in (x_0 - \delta, x_0 + \delta)$. Let us write

$$Q(x) := \sum_{k=0}^{\underline{\alpha}} q_k x^k,\ x \in \mathbb{R}$$

with $q_k \in \mathbb{C}$ for $k \in \{0, \ldots, \underline{\alpha}\}$. For $x \in (x_0 - \delta, x_0 + \delta)$, we have

$$|P(x - x_0) - Q(x - x_0)| \leq 2C|x - x_0|^\alpha.$$

Taking $x = x_0$, we directly have $q_0 = Q(0) = P(0) = p_0$. For $x \neq x_0$, we first have

$$\left|\sum_{k=1}^{\underline{\alpha}} (p_k - q_k)(x - x_0)^{k-1}\right| = \left|\frac{P(x - x_0) - Q(x - x_0)}{x - x_0}\right| \leq 2C|x - x_0|^{\alpha-1}$$

and then $q_1 = p_1$ taking the limit for $x \to x_0$. Step by step, we get $q_k = p_k$ for $k \in \{2, \ldots, \underline{\alpha}\}$ since $\underline{\alpha} < \alpha$.

These pointwise Hölder spaces remain embedded.

Proposition 1.1.12. *If $\alpha > \beta > 0$, we have $\Lambda^\alpha(x_0) \subset \Lambda^\beta(x_0)$ for all $x_0 \in \mathbb{R}$.*

Proof. Let $f \in \Lambda^\alpha(x_0)$. There then exist $C, \delta > 0$ and a polynomial P of degree strictly smaller than α such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

for all $x \in (x_0 - \delta, x_0 + \delta)$. Let us set

$$P'(x) := \sum_{k=0}^{\underline{\beta}} p_k x^k, \quad x \in \mathbb{R}\,.$$

If $\underline{\beta} = \underline{\alpha}$, it is evident since $P' = P$ on $\mathbb{R}$ and $|x - x_0|^\alpha \leq |x - x_0|^\beta$ for all $x \in (x_0 - 1, x_0 + 1)$. If $\underline{\beta} > \underline{\alpha}$, we have

$$|f(x) - P'(x - x_0)| \leq |f(x) - P(x - x_0)| + \sum_{k=\underline{\beta}+1}^{\underline{\alpha}} |p_k||x - x_0|^k \leq \left(C + \sum_{k=\underline{\beta}+1}^{\underline{\alpha}} |p_k|\right) |x - x_0|^\beta$$

for $x \in (x_0 - \delta', x_0 + \delta')$ with $\delta' := \min\{\delta, 1\}$. Hence $f \in \Lambda^\beta(x_0)$. ■

We know that if a function is differentiable at $x_0 \in \mathbb{R}$, then it belongs to $\Lambda^\alpha(x_0)$ for $\alpha \in (0, 1]$. The following result shows that a Hölder continuous function of order strictly bigger than 1 at x_0 is differentiable at x_0.

Proposition 1.1.13. *Let $x_0 \in \mathbb{R}$. If $f \in \Lambda^\alpha(x_0)$ for some $\alpha > 1$, then f is differentiable at x_0.*

Proof. By hypothesis, there exist $C, \delta > 0$ and a polynomial P of degree strictly smaller than α such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

and then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - p_1 \right| \leq C|x - x_0|^{\alpha - 1} + \sum_{k=2}^{m} |p_k||x - x_0|^{k-1}$$

for all $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, with the notations of Remark 1.1.11. Consequently, f is differentiable at x_0 and $(Df)(x_0) = p_1$. ■

We know that $p_0 = f(x_0)$. With the previous proof, we see that $p_1 = (Df)(x_0)$. In fact, if $f \in \Lambda^\alpha(x_0)$ is $\underline{\alpha}$ times continuously differentiable on a neighbourhood of x_0, we can show that the polynomial P in Definition 1.1.10 is the Taylor's polynomial of degree $\underline{\alpha}$ at x_0. This is the object of the following proposition.

Proposition 1.1.14. *Let $x_0 \in \mathbb{R}$, $\varepsilon > 0$, $p \in \mathbb{N}$ and $\alpha > 0$.*

(a) If f is p times continuously differentiable on $(x_0 - \varepsilon, x_0 + \varepsilon)$, then $f \in \Lambda^p(x_0)$. In particular, the polynomial in Definition 1.1.10 is the Taylor's polynomial of degree $p - 1$ at x_0.

(b) If $f \in \Lambda^\alpha(x_0)$ is $\underline{\alpha}$ times continuously differentiable on $(x_0 - \varepsilon, x_0 + \varepsilon)$, then the polynomial in Definition 1.1.10 is the Taylor's polynomial of degree $\underline{\alpha}$ at x_0.

Proof. (a) By Taylor's formula, for all $x \in (x_0 - \varepsilon/2, x_0 + \varepsilon/2)$, there exists θ between x and x_0 such that

$$f(x) = \sum_{k=0}^{p-1} \frac{(D^k f)(x_0)}{k!}(x - x_0)^k + \frac{(D^p f)(\theta)}{p!}(x - x_0)^p$$

and then

$$\left| f(x) - \sum_{k=0}^{p-1} \frac{(D^k f)(x_0)}{k!}(x - x_0)^k \right| \leq \frac{1}{p!} \sup_{y \in [x_0 - \varepsilon/2, x_0 + \varepsilon/2]} |(D^p f)(y)| \, |x - x_0|^p.$$

Consequently, $f \in \Lambda^p(x_0)$ and, by the uniqueness of polynomial in Definition 1.1.10 (see Remark 1.1.11), we have the conclusion.

(b) By hypothesis, there exist $C, \delta > 0$ and a polynomial P of degree strictly smaller than α such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

for all $x \in (x_0 - \delta, x_0 + \delta)$. Let us use the same notations of Remark 1.1.11. By the previous item and the uniqueness of polynomial in Definition 1.1.10, we have

$$P(x - x_0) = p_{\underline{\alpha}}(x - x_0)^{\underline{\alpha}} + \sum_{k=0}^{\underline{\alpha}-1} \frac{(D^k f)(x_0)}{k!}(x - x_0)^k$$

for $x \in \mathbb{R}$ and it only remains to show that $p_{\underline{\alpha}} = (D^{\underline{\alpha}} f)(x_0)/\underline{\alpha}!$. Let us set $\eta := \min\{\varepsilon, \delta\}$. By Taylor's formula, for all $x \in (x_0 - \eta, x_0 + \eta) \setminus \{x_0\}$, there exists θ between x and x_0 such that

$$f(x) - P(x - x_0) = \left(\frac{(D^{\underline{\alpha}} f)(\theta)}{\underline{\alpha}!} - p_{\underline{\alpha}} \right) (x - x_0)^{\underline{\alpha}}$$

and then

$$\left| \frac{(D^{\underline{\alpha}} f)(\theta)}{\underline{\alpha}!} - p_{\underline{\alpha}} \right| \leq C|x - x_0|^{\alpha - \underline{\alpha}}.$$

Since $\alpha - \underline{\alpha} > 0$, we have the conclusion. ■

Uniform Hölder continuity can also be defined for order strictly greater than 1 (see [**80**, **92**, **95**]). We will not need it in this thesis and therefore, we will not consider the general definition.

1.2 Hölder Exponent

The embedding of Hölder spaces allows to define a notion of regularity, known as Hölder exponent.

Definition 1.2.1. The *Hölder exponent of the function f at $x_0 \in \mathbb{R}$* is

$$h_f(x_0) := \sup\{\alpha > 0 : f \in \Lambda^\alpha(x_0)\}.$$

Following this definition and the previous section, if f is differentiable at x_0, then $h_f(x_0) \geq 1$. Moreover, $h_f(x_0) < 1$ implies that f is not differentiable at x_0 and $h_f(x_0) > 1$ implies that f is differentiable at x_0. However, there exist functions which are not differentiable at x_0 and with an Hölder exponent at x_0 equal to 1; the function $x \mapsto |x|$ with the point 0 is a trivial example.

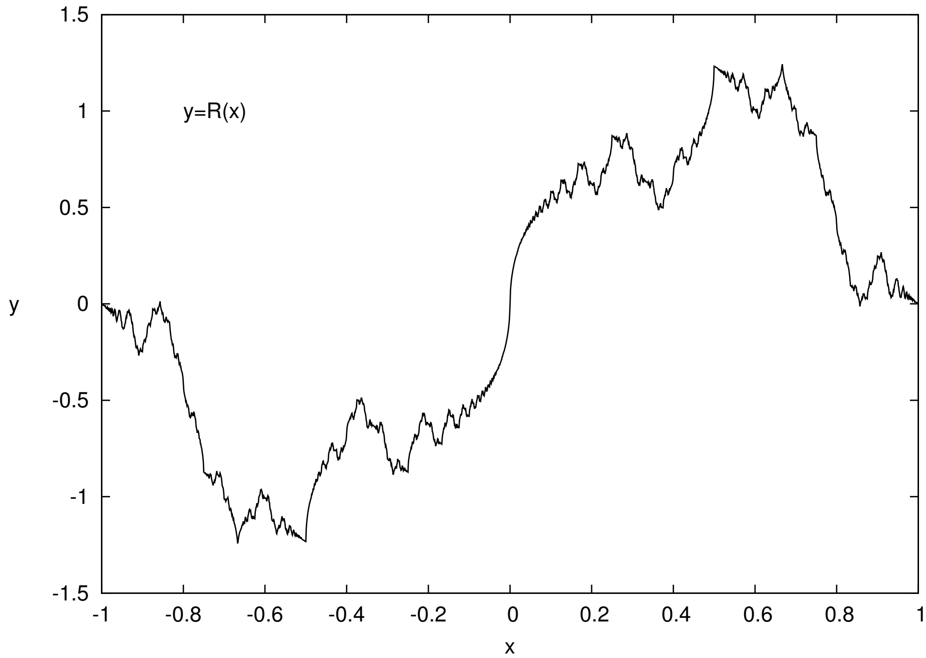

Figure 1.2. Graphical representation of R.

Let us note that the Hölder exponent of a function at a point can be infinite. This is the case for infinitely continuously differentiable functions. By convention, we set $h_f(x_0) := 0$ if there exists no $\alpha > 0$ such that $f \in \Lambda^\alpha(x_0)$.

Let us mention the cases of the Weierstrass function and the Takagi function. On the one hand, W belongs to $\Lambda^w(x)$ and $h_W(x) = w$ for all $x \in \mathbb{R}$ where $w := -\log(a)/\log(b)$ (see [65]). Let us remark that it shows directly that W is a continuous and nowhere differentiable function since $w < 1$ (in fact, $b > 1/a > 1$ with the hypotheses on a and b, see Expression (1.1)). On the other hand, T belongs to $\Lambda^1(x)$ and $h_T(x) = 1$ for all $x \in \mathbb{R}$ (see [110]). In comparison with W, it does not imply that T is nowhere differentiable. We can note that the Hölder exponent of W or T remains the same at each point. The Weierstrass function and the Takagi function are then monofractal functions.

Definition 1.2.2. The function f is *monofractal* if there exists $h > 0$ such that $h_f(x) = h$ for all $x \in \mathbb{R}$. Otherwise, f is *multifractal*.

Let us now consider the Riemann function R defined by

$$R(x) := \sum_{n=1}^{+\infty} \frac{\sin(\pi n^2 x)}{n^2}, \quad x \in \mathbb{R}\,.$$

The graphic of R is represented in Figure 1.2. We know that $h_R(0) = 1/2$ and $h_R(1) = 3/2$ (see [61, 68] for the complete result) and so, R is a multifractal function. More information about R is given in Chapter 4, where we study the uniform Hölder continuity of generalized Riemann function.

The Hölder exponent of a continuous nowhere differentiable function is everywhere smaller (or equal) than 1. Therefore, in this context, we consider rather the restricted pointwise and uniform Hölder exponent.

Definition 1.2.3. (a) The *restricted Hölder exponent of the function f at $x_0 \in \mathbb{R}$* is

$$H_f(x_0) := \sup\{\alpha \in (0,1] : f \in \Lambda^\alpha(x_0)\}.$$

(b) The *restricted uniform Hölder exponent of the bounded function f (on $\mathbb{R}$)* is

$$H_f(\mathbb{R}) := \sup\{\alpha \in (0,1] : f \in \Lambda^\alpha(\mathbb{R})\}.$$

We clearly have $h_f(x_0) \geq H_f(x_0) \geq H_f(\mathbb{R})$ for all $x_0 \in \mathbb{R}$. Moreover, $h_f(x_0) = H_f(x_0)$ if $x_0 \in \mathbb{R}$ with $h_f(x_0) \in (0,1]$. For example, $H_W(\mathbb{R}) = -\log(a)/\log(b)$ (see [65]), $H_T(\mathbb{R}) = 1$ (see [110]) and $H_R(\mathbb{R}) = 1/2$ (see [55]).

A way to calculate restricted pointwise Hölder exponent is given by the following formula (see [69] for example). Other methods to determine Hölder exponent will be exposed later.

Proposition 1.2.4. *Let $x_0 \in \mathbb{R}$ and let $f \in \Lambda^\alpha(x_0)$ for some $\alpha \in (0,1]$. We have*

$$H_f(x_0) = \liminf_{x \to x_0} \frac{\log|f(x) - f(x_0)|}{\log|x - x_0|}. \tag{1.4}$$

Proof. By hypothesis, there exist $C > 0$ and $\delta \in (0,1)$ such that

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha$$

and then

$$\frac{\log|f(x) - f(x_0)|}{\log|x - x_0|} \geq \frac{\log(C)}{\log|x - x_0|} + \alpha$$

for all $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. Consequently, we have

$$\inf_{x \in (x_0 - \delta, x_0 + \delta)\setminus\{x_0\}} \frac{\log|f(x) - f(x_0)|}{\log|x - x_0|} \geq \frac{\log(C)}{\log(\delta)} + \alpha$$

and

$$\liminf_{x \to x_0} \frac{\log|f(x) - f(x_0)|}{\log|x - x_0|} \geq H_f(x_0).$$

Let us show that this inequality is an equality. By contradiction, let us assume that there exists $\alpha \in (0,1]$ such that

$$\liminf_{x \to x_0} \frac{\log|f(x) - f(x_0)|}{\log|x - x_0|} > \alpha > H_f(x_0).$$

Then, there exists $\delta \in (0,1)$ such that

$$|f(x) - f(x_0)| \leq |x - x_0|^\alpha$$

for all $x \in (x_0 - \delta, x_0 + \delta)$ and so $f \in \Lambda^\alpha(x_0)$. Hence a contradiction since $\alpha > H_f(x_0)$. ∎

Remark 1.2.5. Since the function f defined by

$$f(x) := \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous at 0, there exists no $\alpha \in (0,1]$ such that $f \in \Lambda^\alpha(0)$ and then $H_f(0) = 0$ by convention. However, we have

$$\liminf_{x \to 0} \frac{\log|f(x) - f(0)|}{\log|x - 0|} = -1$$

and thus, Equality (1.4) is not verified.

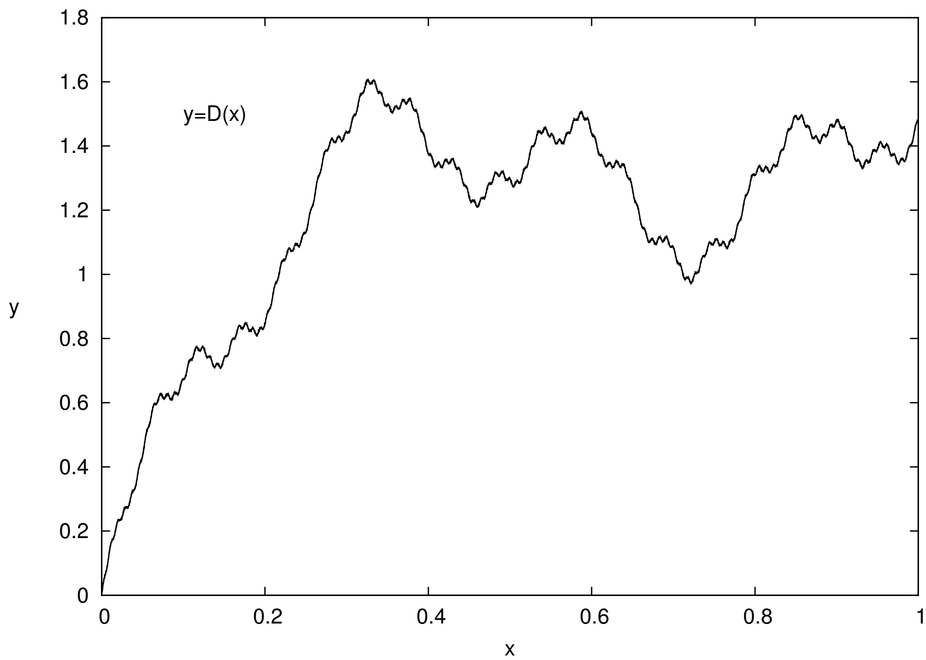

Figure 1.3. Graphical representation of D.

Let us end this chapter by the investigation of the Hölder continuity of the Darboux function. In next chapters, we will study other functions: the Cantor's bijection is considered in Chapter 2 and the generalized Riemann function in Chapter 4. Other examples can be found in [62, 71].

1.3 A First Detailed Example: the Darboux Function

Darboux [31, 32] showed that the function D defined by

$$D(x) := \sum_{n=0}^{+\infty} \frac{\sin((n+1)!\,x)}{n!}, \quad x \in \mathbb{R},$$

is continuous, but nowhere differentiable on $\mathbb{R}$. The graphic of D is represented in Figure 1.3. Let us prove that D and T have the same Hölder exponent, which is everywhere equal to 1.

Proposition 1.3.1. *We have $D \in \Lambda^{1-2\theta}(\mathbb{R})$ for all $\theta \in (0, 1/2)$ and then $H_D(\mathbb{R}) = 1$. Moreover, $h_D(x) = 1$ for all $x \in \mathbb{R}$ and D is a monofractal function.*

Proof. Let us fix $x, x_0 \in \mathbb{R}$. We have

$$\begin{aligned} |D(x) - D(x_0)| &\leq \sum_{n=0}^{N} (n+1) \left| \int_{x_0}^{x} \cos((n+1)!t)\, dt \right| + 2 \sum_{n=N+1}^{+\infty} \frac{1}{n!} \\ &\leq |x - x_0| \sum_{n=0}^{N} (n+1) + 2 \sum_{n=N+1}^{+\infty} 2^{-n+1} \\ &\leq 3N^2 |x - x_0| + 2^{2-N} \end{aligned}$$

for all $N \in \mathbb{N}$. Let us also fix $\theta \in (0, 1/2)$. There exists $\delta \in (0, 1)$ such that

$$2^{-\frac{1}{|x-x_0|^\theta}} \leq |x - x_0|$$

for $x, x_0 \in \mathbb{R}$ with $0 < |x - x_0| \leq \delta$ because $t^{1/\theta} 2^{-t} \to 0$ if $t \to +\infty$. For such x and x_0, there exists a unique $N \in \mathbb{N}$ such that

$$N \leq \frac{1}{|x - x_0|^\theta} < N + 1$$

and then

$$2^{2-N} \leq 8 \, 2^{-\frac{1}{|x-x_0|^\theta}} \leq 8|x - x_0|.$$

So, we obtain

$$|D(x) - D(x_0)| \leq 3|x - x_0|^{1-2\theta} + 8|x - x_0| \leq |x - x_0|^{1-2\theta} \left(3 + 8\delta^{2\theta}\right)$$

if $|x - x_0| \leq \delta$. Moreover, since D is bounded on $\mathbb{R}$, we directly have

$$|D(x) - D(x_0)| \leq \left(\frac{2}{\delta^{1-2\theta}} \sup_{t \in \mathbb{R}} |D(t)| \right) |x - x_0|^{1-2\theta}$$

if $|x - x_0| > \delta$. Finally, we have shown that for all $\theta \in (0, 1/2)$, there exists $C > 0$ such that

$$|D(x) - D(x_0)| \leq C|x - x_0|^{1-2\theta}$$

for all $x, x_0 \in \mathbb{R}$. Consequently, $D \in \Lambda^{1-2\theta}(\mathbb{R})$ for all $\theta \in (0, 1/2)$ and hence $H_D(\mathbb{R}) = 1$.

Since D is nowhere differentiable on $\mathbb{R}$, $h_D(x) \leq 1$ for all $x \in \mathbb{R}$. We know that $h_D(x) \geq H_D(\mathbb{R}) = 1$ for all $x \in \mathbb{R}$. Thus, we obtain $h_D(x) = 1$ for all $x \in \mathbb{R}$. ■

Chapter 2

Cantor's Bijection(s)

At the end of the 19$^{\text{th}}$ century, Cantor spent a lot of his time on proving the existence of one-to-one mappings between sets. In particular, as borne out by the epistolary relation with Dedekind (see [**34**, **38**]), he was concerned about finding such a correspondence between the set of natural numbers and the set of positive real numbers. Even if, following Dedekind, this work was only of theoretical interest, Cantor [**24**] showed in 1874 that there does not exist any bijection between the set of all natural numbers and the unit interval. Such a result was the precursor of the notion of cardinality and paved the way for the set theory.

Once this problem solved, Cantor addressed to Dedekind a question that can be resumed as follows: "Can a surface (e.g. the unit square) be put into relation with a curve (e.g. the unit segment)?" (see [**34**, **38**]). At the time, such a question was surprising and even considered as an absurdity, because mathematicians were convinced that two (independent) variables cannot be reduced to one.

In 1877, Cantor [**25**] proved that there exists a one-to-one correspondence between the points of the unit line segment $[0,1]$ and the points of the unit square $[0,1]^2$. About this discovery, he wrote to Dedekind (see [**34**, **38**, **46**, **120**]): "Je le vois, mais je ne le crois pas !" ("I see it, but I don't believe it!"). With such a result, the notion of dimension had to be reconsidered and this helped to clarify the confusion between dimension and cardinality.

The bijection between $[0,1]$ and $[0,1]^2$ constructed by Cantor is defined via continued fractions. It is therefore challenging to have any intuition about its regularity. When looking at its definition or at the graphical representation of each component, it is not hard to convince oneself that the behaviour of such a function is necessarily "erratic". It is well known that most of the "historical" space-filling functions are monofractal with Hölder exponent equal to $1/2$ (see [**70**, **71**]). Is it still the case of Cantor's bijection?

In this chapter, after some preliminaries about the space of sequences of natural numbers and the theory of continued fractions, we first recall the construction of Cantor's bijection based on continued fractions and give a graphical representation of the two components of this map. We then investigate the regularity (continuity and Hölder continuity) of this application. In particular, we explore its multifractal nature showing that its Hölder exponent lies in an interval which contains $1/2$. We finish by an appendix with another construction of a bijection between $[0,1]$ and $[0,1]^2$, also based on a idea of Cantor. The results presented in this chapter are mainly from [**96**, **97**].

2.1 Some Preliminaries

In this chapter, we set $E := [0,1]$, $D := E \cap \mathbb{Q}$ and $I := E \setminus D$.

2.1.1 The Space of Sequences of Natural Numbers

The space of the (infinite) sequences of natural numbers is denoted by $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$. Since this space is a countable product of metric spaces, we define the usual distance d by

$$d(\boldsymbol{a}, \boldsymbol{b}) := \sum_{j=1}^{\infty} 2^{-j} \frac{|a_j - b_j|}{|a_j - b_j| + 1}$$

for two elements $\boldsymbol{a} := (a_j)_{j\in\mathbb{N}}$ and $\boldsymbol{b} := (b_j)_{j\in\mathbb{N}}$ of $\mathcal{N}$. We will implicitly consider that $\mathcal{N}$ is equipped with this distance and that E, D and I are endowed with the Euclidean distance.

Remark 2.1.1. Considering $\boldsymbol{a}$ and $\boldsymbol{b}$ as two infinite words on the alphabet $\mathbb{N}$ (see [85]), we can also use the following ultrametric distance on $\mathcal{N}$: for $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{N}$, let $\boldsymbol{a} \wedge \boldsymbol{b}$ denote the longest common prefix of $\boldsymbol{a}$ and $\boldsymbol{b}$, so that the length $|\boldsymbol{a} \wedge \boldsymbol{b}|$ of this prefix is equal to the lowest natural number j such that $a_j \neq b_j$ minus 1. A distance between $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$d'(\boldsymbol{a}, \boldsymbol{b}) := \begin{cases} 0 & \text{if } \boldsymbol{a} = \boldsymbol{b} \\ 2^{-|\boldsymbol{a}\wedge\boldsymbol{b}|} & \text{if } \boldsymbol{a} \neq \boldsymbol{b} \end{cases} .$$

The distances d et d' are equivalent. More precisely, we have the following inequalities.

Proposition 2.1.2. *We have*

$$\frac{1}{4} d' \leq d \leq d'.$$

Proof. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{N}$. If $\boldsymbol{a} = \boldsymbol{b}$, we clearly have $d(\boldsymbol{a}, \boldsymbol{b}) = d'(\boldsymbol{a}, \boldsymbol{b})$. Let us assume that $\boldsymbol{a} \neq \boldsymbol{b}$ and let us set $J := |\boldsymbol{a} \wedge \boldsymbol{b}|$. We then have $a_j = b_j$ for all $j \in \{1, \dots, J\}$ and $a_{J+1} \neq b_{J+1}$. On the one hand, we have

$$d(\boldsymbol{a}, \boldsymbol{b}) \leq \sum_{j=J+1}^{+\infty} 2^{-j} = d'(\boldsymbol{a}, \boldsymbol{b})$$

and on the other hand, we have

$$d(\boldsymbol{a}, \boldsymbol{b}) \geq 2^{-(J+1)} \frac{|a_{J+1} - b_{J+1}|}{1 + |a_{J+1} - b_{J+1}|} \geq \frac{1}{4} d'(\boldsymbol{a}, \boldsymbol{b}).$$

∎

For the sake of completeness, let us recall the following result (see [75]).

Proposition 2.1.3. *The space $(\mathcal{N}, d)$ is a separable complete metric space.*

2.1.2 Continued Fractions

Let us recall the basic facts about the continued fractions (see [23, 76, 112]). Here, we state the results for E, but they can be easily extended to the whole real line.

If $n \in \mathbb{N}$, let $\boldsymbol{a} := (a_j)_{j\in\{1,\dots,n\}}$ be a finite sequence of strictly positive real numbers. Let us set

$$[a_1] := \frac{1}{a_1} \quad \text{and} \quad [a_1,\dots,a_m] := \frac{1}{a_1 + [a_2,\dots,a_m]},$$

for any $m \in \{2,\dots,n\}$. In the following and unless stated otherwise (as in Proposition 2.1.14 for example), we will only consider the case where the elements of $\boldsymbol{a}$ are natural numbers.

Definition 2.1.4. A *continued fraction* is an expression of the form

$$[a_1, a_2, \dots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

where $a_1, a_2, \dots, a_n \in \mathbb{N}$ and $n \in \mathbb{N}$.

Proposition 2.1.5. *For any $\boldsymbol{a} \in \mathbb{N}^n$ with $n \in \mathbb{N}$, $[a_1,\dots,a_n]$ belongs to $D \setminus \{0\}$. Conversely, for any $x \in D \setminus \{0\}$, there exist $n \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{N}^n$ such that $x = [a_1,\dots,a_n]$.*

The representation of a rational number as a continued fraction is not unique, as shown by the following remark. This will be used in the proof of Proposition 2.3.6.

Remark 2.1.6. If $\boldsymbol{a} \in \mathbb{N}^n$ with $n \in \mathbb{N}$ is such that $a_n > 1$, we have

$$[a_1,\dots,a_n] = [a_1,\dots,a_n - 1, 1].$$

Let us now define the notion of convergent. For all $\boldsymbol{a} \in \mathbb{N}^n$ with $n \in \mathbb{N}$ and for each integer $j \in \{-1,\dots,n\}$, let us define recursively the quantities $p_j(\boldsymbol{a})$ and $q_j(\boldsymbol{a})$ as follows: we set $p_{-1}(\boldsymbol{a}) := 1$, $q_{-1}(\boldsymbol{a}) := 0$, $p_0(\boldsymbol{a}) := 0$, $q_0(\boldsymbol{a}) := 1$ and

$$\begin{cases} p_j(\boldsymbol{a}) := a_j p_{j-1}(\boldsymbol{a}) + p_{j-2}(\boldsymbol{a}) \\ q_j(\boldsymbol{a}) := a_j q_{j-1}(\boldsymbol{a}) + q_{j-2}(\boldsymbol{a}) \end{cases} \tag{2.1}$$

for $j \in \mathbb{N}$.

Definition 2.1.7. For $\boldsymbol{a} \in \mathbb{N}^n$ with $n \in \mathbb{N}$ and $j \in \{1,\dots,n\}$, the quotient $p_j(\boldsymbol{a})/q_j(\boldsymbol{a})$ is called the *convergent of order j of $\boldsymbol{a}$*.

Convergents are closely related to the continued fractions.

Proposition 2.1.8. *Let $\boldsymbol{a} \in \mathbb{N}^n$ with $n \in \mathbb{N}$. For $j \in \{1,\dots,n\}$, we have*

$$\frac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})} = [a_1,\dots,a_j].$$

Furthermore, we have

$$\begin{cases} q_j(\boldsymbol{a})p_{j-1}(\boldsymbol{a}) - p_j(\boldsymbol{a})q_{j-1}(\boldsymbol{a}) = (-1)^j & \text{for } j \in \{1,\dots,n\} \\ q_j(\boldsymbol{a})p_{j-2}(\boldsymbol{a}) - p_j(\boldsymbol{a})q_{j-2}(\boldsymbol{a}) = (-1)^{j-1}a_j & \text{for } j \in \{2,\dots,n\} \end{cases}.$$

As a consequence, we also have

$$\begin{cases} \dfrac{p_{j-1}(\boldsymbol{a})}{q_{j-1}(\boldsymbol{a})} - \dfrac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})} = \dfrac{(-1)^j}{q_j(\boldsymbol{a})q_{j-1}(\boldsymbol{a})} & \text{for } j \in \{2,\dots,n\} \\ \dfrac{p_{j-2}(\boldsymbol{a})}{q_{j-2}(\boldsymbol{a})} - \dfrac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})} = \dfrac{(-1)^{j-1}a_j}{q_j(\boldsymbol{a})q_{j-2}(\boldsymbol{a})} & \text{for } j \in \{3,\dots,n\} \end{cases}.$$

Of course, we can define the numbers $p_j(\boldsymbol{a})$ and $q_j(\boldsymbol{a})$ for an element $\boldsymbol{a}$ of $\mathcal{N}$. The convergents allow to introduce the notion of infinite continued fraction, thanks to the following result, which is simply a consequence of the previous proposition and of the property:

$$\lim_{j\to+\infty} q_j(\boldsymbol{a}) = +\infty. \tag{2.2}$$

Corollary 2.1.9. *For any $\boldsymbol{a} \in \mathcal{N}$, we have*

$$\frac{p_2(\boldsymbol{a})}{q_2(\boldsymbol{a})} < \frac{p_4(\boldsymbol{a})}{q_4(\boldsymbol{a})} < \dots < \frac{p_{2j}(\boldsymbol{a})}{q_{2j}(\boldsymbol{a})} < \frac{p_{2j-1}(\boldsymbol{a})}{q_{2j-1}(\boldsymbol{a})} < \dots < \frac{p_3(\boldsymbol{a})}{q_3(\boldsymbol{a})} < \frac{p_1(\boldsymbol{a})}{q_1(\boldsymbol{a})}$$

for all $j \in \mathbb{N}$. As a consequence, the sequence

$$\left(\frac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})}\right)_{j\in\mathbb{N}}$$

converges.

Definition 2.1.10. If $\boldsymbol{a} \in \mathcal{N}$, we say that

$$[a_1,\dots] := \lim_{n\to+\infty}[a_1,\dots,a_n]$$

is an *infinite continued fraction*.

If $\boldsymbol{a}$ is an element of $\mathcal{N}$ or $\mathbb{N}^n$ with $n \in \mathbb{N}$, we will sometimes simply write $[\boldsymbol{a}]$ instead of $[a_1,\dots]$ or $[a_1,\dots,a_n]$ respectively.

We know that the rational numbers (of $E \setminus \{0\}$) can be represented by a finite continued fraction. The following result considers the case of irrational numbers (of $E \setminus \{0\}$).

Theorem 2.1.11. *We have $x \in I$ if and only if x is represented by an infinite continued fraction, i.e. there exists $\boldsymbol{a} \in \mathcal{N}$ such that $x = [\boldsymbol{a}]$. Moreover, this infinite continued fraction is unique.*

If the real number $x \in E \setminus \{0\}$ is equal to the continued fraction $[\boldsymbol{a}]$, we say that $[\boldsymbol{a}]$ is a continued fraction corresponding to x. We know that if $x \in I$, then $\boldsymbol{a} \in \mathcal{N}$ and $[\boldsymbol{a}]$ is the unique continued fraction corresponding to x. If $x \in D \setminus \{0\}$, then $\boldsymbol{a} \in \mathbb{N}^n$ with $n \in \mathbb{N}$ and $[\boldsymbol{a}]$ is not the single continued fraction corresponding to x (see Remark 2.1.6).

Let us mention the quite particular case of ultimately periodic continued fraction (see [**23**, **76**]).

Definition 2.1.12. A sequence $\boldsymbol{a} \in \mathcal{N}$ is *ultimately periodic of period* $k \in \mathbb{N}$ if there exists $J \in \mathbb{N}$ such that $a_{j+k} = a_j$ for any $j \geq J$. In this case, the corresponding continued fraction $[\boldsymbol{a}]$ is also called ultimately periodic of period k.

The quadratic numbers (of E), i.e. the numbers (of E) which are zeros of a polynomial with integer coefficients, are characterized by their particular corresponding continued fractions. This is the object of the following result.

Theorem 2.1.13. *An element of I is a quadratic number if and only if the corresponding continued fraction is ultimately periodic.*

Let us now give a brief introduction of the notion of the metric theory of continued fractions (see [76, 112]). Let us first recall the following result.

Proposition 2.1.14. *If $x \in E \setminus \{0\}$ can be written as $x = [a_1, \dots, a_n, r_{n+1}]$ with $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{N}$ and $r_{n+1} \in [1, +\infty)$, the following relation holds:*

$$x = \frac{p_n(\boldsymbol{a}) r_{n+1} + p_{n-1}(\boldsymbol{a})}{q_n(\boldsymbol{a}) r_{n+1} + q_{n-1}(\boldsymbol{a})}$$

where $\boldsymbol{a} := (a_j)_{j \in \{1,\dots,n\}}$.

For any $\boldsymbol{a} \in \mathcal{N}$, we know that $[\boldsymbol{a}]$ corresponds to an irrational number $x \in I$. For each $j \in \mathbb{N}$, the term a_j can be so considered as a function of x: $a_j := a_j(x)$. Let us fix $j \in \mathbb{N}$ and write $x = [a_1, \dots, a_{j-1}, r_j]$ with $r_j \in [1, +\infty)$. It is easy to check that, for any $k \in \mathbb{N}$, we have

$$a_j = k \quad \text{if and only if} \quad \frac{1}{k+1} < r_j \leq \frac{1}{k}$$

if j is odd and

$$a_j = k \quad \text{if and only if} \quad k \leq r_j < k+1$$

if j is even. Thus, a_j is a piecewise constant function. Moreover, a_j is non-increasing if j is odd and non-decreasing if j is even. The functions a_1 and a_2 are represented in Figure 2.1.

Let $x = [\boldsymbol{a}]$ be an irrational number. For $n \in \mathbb{N}$, we set

$$I_n(x) := \{y \in I : \exists \boldsymbol{b} \in \mathcal{N} \text{ such that } y = [\boldsymbol{b}] \text{ and } b_j = a_j \, \forall j \in \{1, \dots, n\}\}.$$

We will say that $I_n(x)$ is an *interval of rank* n. For any $n \in \mathbb{N}$, $I_{n+1}(x) \subset I_n(x) \subset I$ and

$$\bigcap_{n \in \mathbb{N}} I_n(x) = \{x\}.$$

Indeed, using Proposition 2.1.14 with $r_{n+1} = 1$ and $r_{n+1} \to +\infty$, we get

$$I_n(x) = \left(\frac{p_n(\boldsymbol{a})}{q_n(\boldsymbol{a})}, \frac{p_n(\boldsymbol{a}) + p_{n-1}(\boldsymbol{a})}{q_n(\boldsymbol{a}) + q_{n-1}(\boldsymbol{a})} \right) \cap I,$$

if n is even (if n is odd, the endpoints of the interval are reversed). Every interval of rank n is partitioned into a countable infinite number of intervals of rank $n+1$. We will denote by $|I_n(x)|$ the Lebesgue measure of $I_n(x)$. Using Proposition 2.1.8, we have

$$|I_n(x)| = \frac{1}{q_n(\boldsymbol{a})(q_n(\boldsymbol{a}) + q_{n-1}(\boldsymbol{a}))}. \tag{2.3}$$

Thanks to Property (2.2), we directly obtain

$$\lim_{n \to +\infty} |I_n(x)| = 0. \tag{2.4}$$

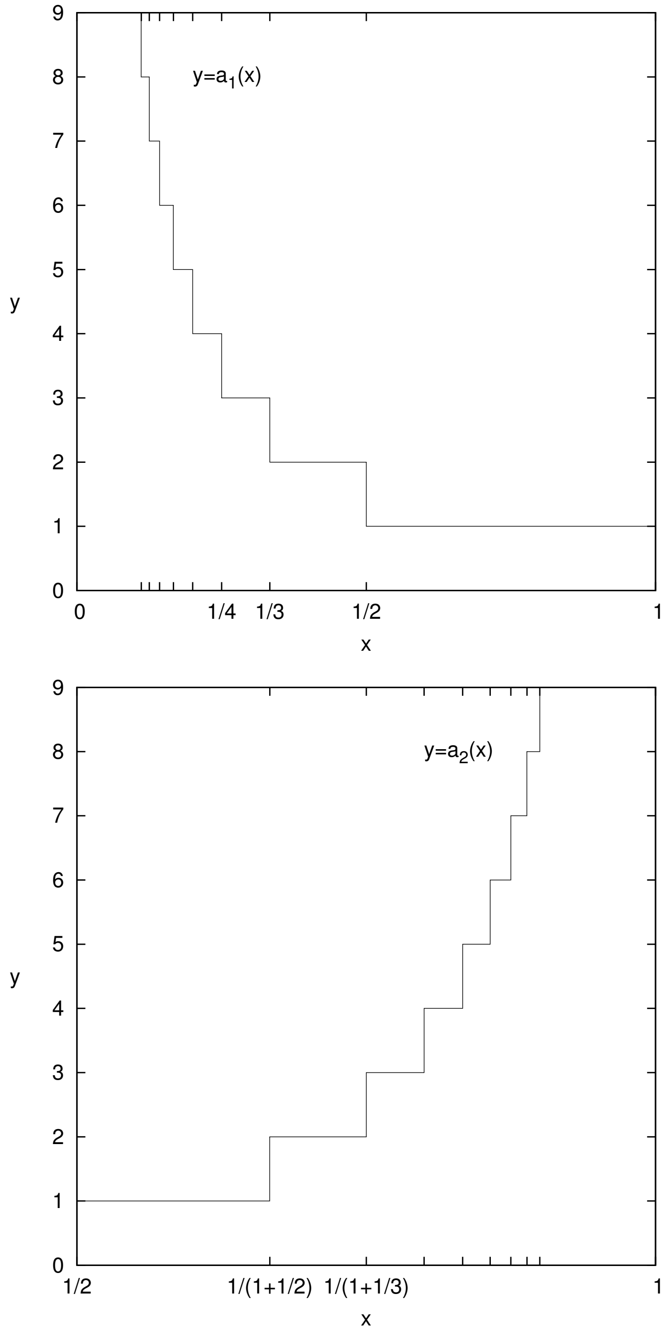

Figure 2.1. The graphics of functions $x \mapsto a_1(x)$ and $x \mapsto a_2(x)$ if $a_1(x) = 1$. This illustrates the fact that $I_1(x)$ is partitioned into a countable infinite number of intervals of rank 2; in this case, $I_2(x) \subset [1/2, 1] \cap I$, since $a_1(x) = 1$ if and only if $x \in [1/2, 1] \cap I$.

2.2 Cantor's Bijection

Cantor's bijection on I (see [25]) is a one-to-one mapping from I onto I^2. It is constructed as follows. If $x \in I$, let $[\boldsymbol{a}]$ be the corresponding continued fraction and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the applications defined by

$$\mathcal{C}_1(x) := [a_1, a_3, \ldots, a_{2j+1}, \ldots] \quad \text{and} \quad \mathcal{C}_2(x) := [a_2, a_4, \ldots, a_{2j}, \ldots].$$

These applications are represented in Figure 2.2. Theorem 2.1.11 implies that the application

$$\mathcal{C} : I \to I^2;\ x \mapsto (\mathcal{C}_1(x), \mathcal{C}_2(x))$$

is a one-to-one mapping. It is called Cantor's bijection. If Q denotes the quadratic numbers of I, $\mathcal{C}$ is a one-to-one mapping from Q onto Q^2 by Theorem 2.1.13. Since the cardinals of E and I are equal, $\mathcal{C}$ can be extended to a one-to-one mapping from E onto E^2.

2.3 Continuity of Cantor's Bijection

Let us study the continuity of Cantor's bijection on I.

Proposition 2.3.1. *Cantor's bijection $\mathcal{C}$ is continuous on I.*

Proof. For any $n \in \mathbb{N}$ and any $x \in I$, $\mathcal{C}_1$ maps the interval $I_n(x)$ onto $I_m(\mathcal{C}_1(x))$ where $m = n/2$ if n is even and $m = (n+1)/2$ if n is odd. This shows that $\mathcal{C}_1$ is a continuous function on I. Indeed, let $x_0 \in I$ and $\varepsilon > 0$. With Property (2.4), there exists $M \in \mathbb{N}$ such that $|I_M(\mathcal{C}_1(x_0))| \leq \varepsilon$. If $x \in I_{2M}(x_0)$, we have $|x - x_0| \leq |I_{2M}(x_0)|$ and

$$|\mathcal{C}_1(x) - \mathcal{C}_1(x_0)| \leq |I_M(\mathcal{C}_1(x_0))| \leq \varepsilon.$$

Obviously, the same argument can be applied to $\mathcal{C}_2$ and we have the conclusion. ∎

In fact, Cantor's bijection is even an homeomorphism between I and I^2. To show that, we first define a map from I onto $\mathcal{N}$. For $x \in I$, we write $\sigma(x) := \boldsymbol{a}$ if $\boldsymbol{a} \in \mathcal{N}$ satisfies $x = [\boldsymbol{a}]$. The application σ is clearly a bijection from I onto $\mathcal{N}$ by Theorem 2.1.11.

Proposition 2.3.2. *The application σ is an homeomorphism from I onto $\mathcal{N}$.*

Proof. Let $x_0 \in I$ and $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $2^{-N} \leq \varepsilon$. For $x \in I_N(x_0)$, we have $|x - x_0| \leq |I_N(x_0)|$ and

$$d(\sigma(x), \sigma(x_0)) \leq d'(\sigma(x), \sigma(x_0)) \leq 2^{-N} \leq \varepsilon.$$

So, σ is continuous on I.

Conversely, let $\boldsymbol{a}_0 \in \mathcal{N}$ and $\varepsilon > 0$. With Property (2.4), there exists $N \in \mathbb{N}$ such that $|I_N(\sigma^{-1}(\boldsymbol{a}_0))| \leq \varepsilon$. For $\boldsymbol{a} \in \mathcal{N}$ such that $d'(\boldsymbol{a}, \boldsymbol{a}_0) \leq 2^{-N}$, we have $\sigma^{-1}(\boldsymbol{a}) \in I_N(\sigma^{-1}(\boldsymbol{a}_0))$ and

$$|\sigma^{-1}(\boldsymbol{a}) - \sigma^{-1}(\boldsymbol{a}_0)| \leq |I_N(\sigma^{-1}(\boldsymbol{a}_0))| \leq \varepsilon.$$

So, σ^{-1} is continuous on $\mathcal{N}$. ∎

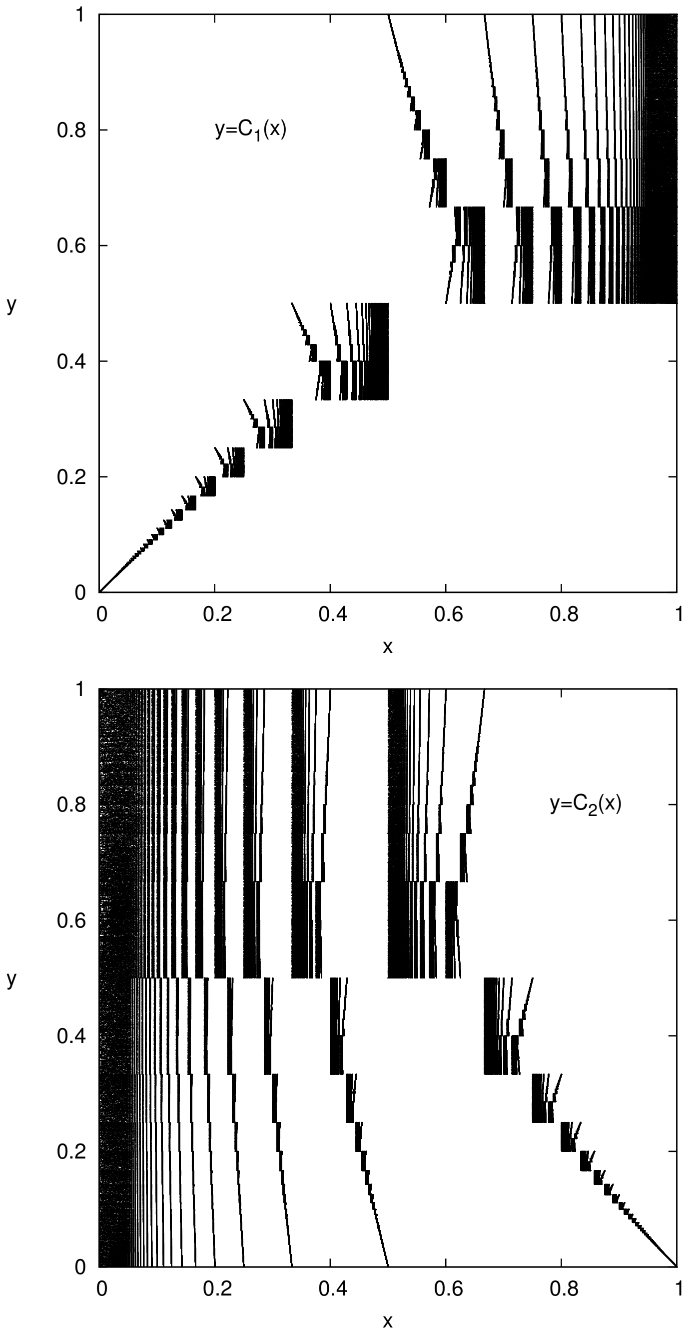

Figure 2.2. Graphical representations of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Remark 2.3.3. We obviously have $[\cdot] = \sigma^{-1}$ on $\mathcal{N}$.

Since $(\mathcal{N}, d)$ is a separable complete metric space (see Proposition 2.1.3), we have reobtained the following well-known result (see [75]).

Corollary 2.3.4. *The space I is a Polish space.*

Proposition 2.3.5. *Cantor's bijection $\mathcal{C}$ is an homeomorphism between I and I^2.*

Proof. Since the application

$$(\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{N} \times \mathcal{N} \mapsto \boldsymbol{c} := (c_j)_{j \in \mathbb{N}} \in \mathcal{N}$$

where

$$c_j := \begin{cases} a_{(j+1)/2} & \text{if } j \text{ is odd} \\ b_{j/2} & \text{if } j \text{ is even} \end{cases}$$

is an homeomorphism, we have the conclusion, using Proposition 2.3.2. ∎

Netto's theorem (see [108]) guarantees that such a function $\mathcal{C}$ can not be extended to a continuous function from E to E^2. The following result gives additional information.

Proposition 2.3.6. *Any extension of Cantor's bijection to E is discontinuous at any rational number.*

Proof. Let $x \in D \setminus \{0\}$. There exists $k \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{N}^k$ with $a_k > 1$ such that

$$x = [a_1, \ldots, a_k] = [a_1, \ldots, a_k - 1, 1].$$

Let $\boldsymbol{b} \in \mathcal{N}$. For $n \in \mathbb{N}$, let us set $x_n := [a_1, \ldots, a_k, r_n]$, $y_n := [a_1, \ldots, a_k - 1, 1, r_n]$ with $r_n := n + [\boldsymbol{b}]$. By construction, x_n and y_n are irrational numbers for all $n \in \mathbb{N}$ and

$$\lim_{n \to +\infty} r_n = +\infty.$$

By Proposition 2.1.14 and Proposition 2.1.8, we have

$$\lim_{n \to +\infty} x_n = \lim_{n \to +\infty} \frac{r_n p_k(\sigma(x_n)) + p_{k-1}(\sigma(x_n))}{r_n q_k(\sigma(x_n)) + q_{k-1}(\sigma(x_n))} = \frac{p_k(\sigma([a_1, \ldots, a_k]))}{q_k(\sigma([a_1, \ldots, a_k]))} = x \tag{2.5}$$

since $p_k(\sigma(x_n)) = p_k(\sigma([a_1, \ldots, a_k]))$ and $q_k(\sigma(x_n)) = q_k(\sigma([a_1, \ldots, a_k]))$ for all $n \in \mathbb{N}$. Similarly, we have

$$\lim_{n \to +\infty} y_n = \lim_{n \to +\infty} \frac{r_n p_{k+1}(\sigma(y_n)) + p_k(\sigma(y_n))}{r_n q_{k+1}(\sigma(y_n)) + q_k(\sigma(y_n))} = \frac{p_{k+1}(\sigma([a_1, \ldots, a_k - 1, 1]))}{q_{k+1}(\sigma([a_1, \ldots, a_k - 1, 1]))} = x.$$

Let us assume that k is odd, the other case is similar. We have

$$\begin{aligned} \lim_{n \to +\infty} \mathcal{C}(x_n) &= \lim_{n \to +\infty} ([a_1, a_3, \ldots, a_k, b_1, b_3, \ldots], [a_2, a_4, \ldots, a_{k-1}, n, b_2, b_4, \ldots]) \\ &= ([a_1, \ldots, a_k, b_1, b_3, \ldots], [a_2, \ldots, a_{k-1}]) \end{aligned}$$

and

$$\begin{aligned} \lim_{n \to +\infty} \mathcal{C}(y_n) &= \lim_{n \to +\infty} ([a_1, a_3, \ldots, a_k - 1, n, b_2, b_4, \ldots], [a_2, a_4, \ldots, a_{k-1}, 1, b_1, b_3, \ldots]) \\ &= ([a_1, a_3, \ldots, a_k - 1], [a_2, a_4, \ldots, a_{k-1}, 1, b_1, b_3, \ldots]), \end{aligned}$$

using a similar development as Expression (2.5). Thus, these two limits are not equal, while both sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ converge to x. Hence the conclusion. ∎

2.4 Hölder Continuity of Cantor's Bijection

In this section, we give some preliminary results about the Hölder continuity of Cantor's bijection.

Theorem 2.4.1. *Let $x = [\boldsymbol{a}]$ be an element of I and $y \in I_n(x) \setminus I_{n+1}(x)$ with $n \in \mathbb{N}$. We have*

$$\frac{\frac{1}{n}\sum_{j=1}^{\lceil n/2\rceil}\log(a_{2j-1})}{\frac{1}{n}\sum_{j=1}^{n+3}\log(a_j+1)+\frac{1}{n}C_1(n)} \leq \frac{\log|\mathcal{C}_1(x)-\mathcal{C}_1(y)|}{\log|x-y|} \leq \frac{\frac{1}{n}\sum_{j=1}^{\lceil n/2\rceil+3}\log(a_{2j-1}+1)+\frac{1}{2n}C_2(n)}{\frac{1}{n}\sum_{j=1}^{n}\log(a_j)}$$

where

$$C_1(n) := \frac{\log(2)}{2} + \log\left(\max\left\{\frac{a_{n+2}+2}{a_{n+2}+1}, \frac{a_{n+3}+2}{a_{n+3}+1}\right\}\right)$$

and

$$C_2(n) := \frac{\log(2)}{2} + \log\left(\max\left\{\frac{a_{2\lceil n/2\rceil+3}+2}{a_{2\lceil n/2\rceil+3}+1}, \frac{a_{2\lceil n/2\rceil+5}+2}{a_{2\lceil n/2\rceil+5}+1}\right\}\right).$$

Proof. By hypothesis, we have

$$y = [a_1, \ldots, a_n, b_{n+1}, b_{n+2}, \ldots]$$

with $b_{n+1} \neq a_{n+1}$. Let us suppose that n is even, the other case is similar. We will bound $|x-y|$ and $|\mathcal{C}_1(x) - \mathcal{C}_1(y)|$ with terms depending on $\boldsymbol{a}$ and n only.

Since $y \in I_n(x)$, we have $|x-y| \leq |I_n(x)|$ and

$$|x-y| \leq |I_n(x)| = \frac{1}{q_n^2(\boldsymbol{a})}\frac{1}{1+\frac{q_{n-1}(\boldsymbol{a})}{q_n(\boldsymbol{a})}} \leq \frac{1}{q_n^2(\boldsymbol{a})} \tag{2.6}$$

using Equality (2.3). Moreover, since

$$\begin{aligned} q_n(\boldsymbol{a}) &= a_n q_{n-1}(\boldsymbol{a}) + q_{n-2}(\boldsymbol{a}) \geq a_n q_{n-1}(\boldsymbol{a}) \\ &\geq a_n(a_{n-1}q_{n-2}(\boldsymbol{a}) + q_{n-3}(\boldsymbol{a})) \geq a_n \cdots a_3(a_2 q_1(\boldsymbol{a}) + q_0(\boldsymbol{a})) \\ &\geq a_n \cdots a_1 \end{aligned}$$

thanks to Equality (2.1), we get

$$|x-y| \leq \frac{1}{a_1^2 \cdots a_n^2}. \tag{2.7}$$

The same reasoning can be applied to

$$\mathcal{C}_1(x) = [a_1, a_3, \ldots, a_{n-1}, a_{n+1}, \ldots]$$

and

$$\mathcal{C}_1(y) = [a_1, a_3, \ldots, a_{n-1}, b_{n+1}, b_{n+3}, \ldots]$$

to obtain

$$|\mathcal{C}_1(x) - \mathcal{C}_1(y)| \leq |I_{n/2}(\mathcal{C}_1(x))| \leq \frac{1}{a_1^2 a_3^2 \cdots a_{n-1}^2}. \tag{2.8}$$

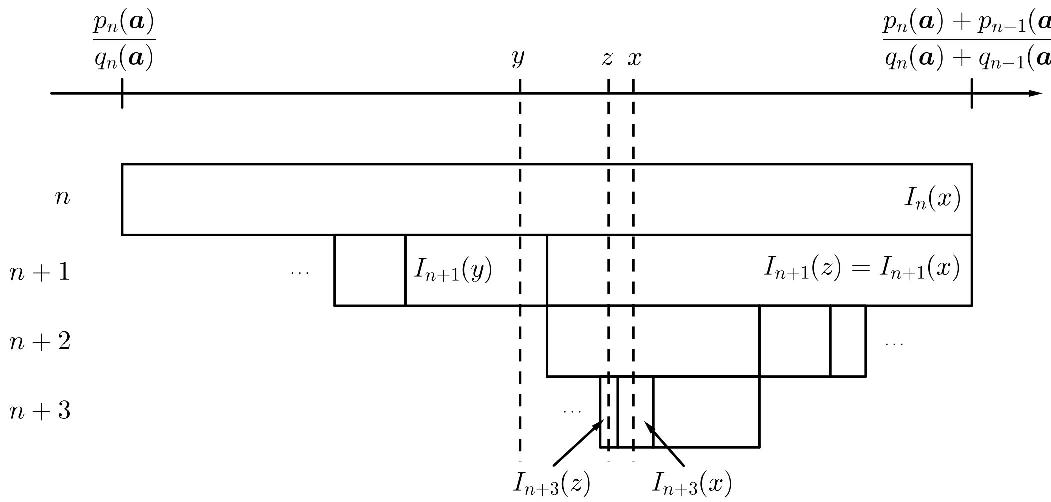

Figure 2.3. Illustration of the choice of z with $I_{n+1}(z) = I_{n+1}(x) \neq I_{n+1}(y)$ in the case $y < x$.

For the lower bound of $|x-y|$, let us remark that $I_{n+1}(x) \cap I_{n+1}(y) = \emptyset$, but the distance between $I_{n+1}(x)$ and $I_{n+1}(y)$ can be zero. However, for any fixed $j \in \mathbb{N}$, there exists a countable infinite number of intervals of rank $n+1+j$ in between $I_{n+1+j}(x)$ and $I_{n+1+j}(y)$, i.e. there exists a countable infinite number of $z \in I$ such that $z' \in I_{n+1+j}(z)$ implies $x < z' < y$ or $y < z' < x$. If $z = [\boldsymbol{c}]$ is such an element, we have

$$|x-y| \geq |I_{n+3}(z)| \geq \frac{1}{q_{n+3}(\boldsymbol{c})(q_{n+3}(\boldsymbol{c}) + q_{n+2}(\boldsymbol{c}))} \geq \frac{1}{2q_{n+3}^2(\boldsymbol{c})}. \tag{2.9}$$

The relations

$$\begin{aligned} q_{n+3}(\boldsymbol{c}) &= c_{n+3}q_{n+2}(\boldsymbol{c}) + q_{n+1}(\boldsymbol{c}) \leq (c_{n+3}+1)q_{n+2}(\boldsymbol{c}) \\ &\leq (c_{n+3}+1)(c_{n+2}q_{n+1}(\boldsymbol{c}) + q_n(\boldsymbol{c})) \leq (c_{n+3}+1)\cdots(c_1+1) \end{aligned}$$

lead to

$$|I_{n+3}(z)| \geq \frac{1}{2(c_1+1)^2\cdots(c_{n+3}+1)^2}.$$

Now let

$$j_0 := \begin{cases} n+2 & \text{if } x < y \\ n+3 & \text{if } y < x \end{cases}$$

and we can choose z such that $c_j := a_j$ for any $j \in \mathbb{N}$ except for the index j_0 for which $c_{j_0} := a_{j_0} + 1$, so that $z > x$ in case $x < y$ and $z < x$ in case $y < x$. Moreover, $I_{n+1}(z) = I_{n+1}(x) \neq I_{n+1}(y)$, so that $x < z < y$ in case $x < y$ and $y < z < x$ in case $y < x$. Figure 2.3 gives a sketch of this last situation. We therefore have

$$|x-y| \geq |I_{n+3}(z)| \geq \frac{1}{2(a_1+1)^2\cdots(a_{n+2}+1)^2(a_{n+3}+2)^2} \tag{2.10}$$

or

$$|x-y| \geq |I_{n+3}(z)| \geq \frac{1}{2(a_1+1)^2\cdots(a_{n+2}+2)^2(a_{n+3}+1)^2}$$

depending on the value of j_0. Without loss of generality, let us assume that j_0 corresponds to the largest integer in such inequalities, i.e. $n+3$ here.

There also exists $w = [\boldsymbol{d}]$ such that $I_{n/2+3}(w)$ lies between $I_{n/2+3}(\mathcal{C}_1(x))$ and $I_{n/2+3}(\mathcal{C}_1(y))$. Moreover, we can choose w such that $d_j := a_{2j-1}$ for any j except for one index $j_0' \in \{n/2+2, n/2+3\}$, for which $d_{j_0'} := a_{2j_0'-1}+1$. Without loss of generality again, let us suppose that j_0' is equal to $n/2+3$. We thus have

$$|\mathcal{C}_1(x) - \mathcal{C}_1(y)| \geq |I_{n/2+3}(w)| \geq \frac{1}{2(a_1+1)^2(a_3+1)^2 \cdots (a_{n+3}+1)^2(a_{n+5}+2)^2}. \tag{2.11}$$

Putting Inequalities (2.7), (2.8), (2.10) and (2.11) together and taking the logarithm, we get

$$\frac{-2\displaystyle\sum_{j=1}^{n/2} \log(a_{2j-1})}{-\log(2) - 2\displaystyle\sum_{j=1}^{n+3}\log(a_j+1) - 2\log\left(\dfrac{a_{n+3}+2}{a_{n+3}+1}\right)} \leq \frac{\log|\mathcal{C}_1(x)-\mathcal{C}_1(y)|}{\log|x-y|}$$

and

$$\frac{\log|\mathcal{C}_1(x)-\mathcal{C}_1(y)|}{\log|x-y|} \leq \frac{-\log(2) - 2\displaystyle\sum_{j=1}^{n/2+3}\log(a_{2j-1}+1) - 2\log\left(\dfrac{a_{n+5}+2}{a_{n+5}+1}\right)}{-2\displaystyle\sum_{j=1}^{n}\log(a_j)},$$

which are the desired results. ■

Of course, the same reasoning can be applied to $\mathcal{C}_2$, leading to the same result.

Theorem 2.4.2. *Let $x = [\boldsymbol{a}]$ be an element of I and $y \in I_n(x) \setminus I_{n+1}(x)$ with $n \in \mathbb{N}$. We have*

$$\frac{\dfrac{1}{n}\displaystyle\sum_{j=1}^{\lfloor n/2\rfloor}\log(a_{2j})}{\dfrac{1}{n}\displaystyle\sum_{j=1}^{n+3}\log(a_j+1) + \dfrac{1}{n}C_1(n)} \leq \frac{\log|\mathcal{C}_2(x)-\mathcal{C}_2(y)|}{\log|x-y|} \leq \frac{\dfrac{1}{n}\displaystyle\sum_{j=1}^{\lfloor n/2\rfloor+3}\log(a_{2j}+1) + \dfrac{1}{n}C_2(n)}{\dfrac{1}{n}\displaystyle\sum_{j=1}^{n}\log(a_j)}$$

where $C_1(n)$ is defined as in Theorem 2.4.1 and

$$C_2(n) := \frac{\log(2)}{2} + \log\left(\max\left\{\frac{a_{2\lfloor n/2\rfloor+4}+2}{a_{2\lfloor n/2\rfloor+4}+1}, \frac{a_{2\lfloor n/2\rfloor+6}+2}{a_{2\lfloor n/2\rfloor+6}+1}\right\}\right).$$

To obtain a generic result about the regularity of Cantor's bijection, we need a direct consequence of the ergodic theorem on continued fractions (see [107]). We say that a property P concerning sequences of $\mathcal{N}$ holds almost everywhere if for almost every $x \in I$ (with respect to the Lebesgue measure), the sequence $\boldsymbol{a} \in \mathcal{N}$ such that $x = [\boldsymbol{a}]$ satisfies P. The following result can be obtained from the main theorem of [94].

Theorem 2.4.3. *Let $k \in \mathbb{N}_0$. For almost every sequence $\boldsymbol{a} \in \mathcal{N}$, we have*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}\log(a_j+k) = \lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}\log(a_{2j}+k) = \lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}\log(a_{2j-1}+k) = \log(\kappa_k)$$

where κ_k is defined by

$$\kappa_k := \prod_{j=1}^{\infty}\left(1+\frac{1}{j(j+2)}\right)^{\frac{\log(j+k)}{\log(2)}}.$$

The result $\frac{1}{n}\sum_{j=1}^{n}\log(a_j)\to\log(\kappa_0)$ if $n\to+\infty$ was proved in [76] and the constant κ_0 is called the Khintchine's constant. Here, we will be interested in the values

$$\log(\kappa_0)\approx 0.987849056\cdots \qquad \text{and} \qquad \log(\kappa_1)\approx 1.409785988\cdots$$

Using Theorem 2.4.1 and Theorem 2.4.2 as n goes to infinity (or equivalently as y tends to x), Theorem 2.4.3 and Proposition 1.2.4, we get the following result.

Corollary 2.4.4. *For almost every $x\in I$, we have*

$$h_{\mathcal{C}_1}(x), h_{\mathcal{C}_2}(x)\in\left[\frac{\log(\kappa_0)}{2\log(\kappa_1)},\frac{\log(\kappa_1)}{2\log(\kappa_0)}\right].$$

Thus, the Hölder exponent of $\mathcal{C}_1$ and $\mathcal{C}_2$ lies between 0.35 and 0.72 almost everywhere. In fact, thanks to Theorem 2.4.1 (and Theorem 2.4.2), we can exactly determine the Hölder exponent of $\mathcal{C}_1$ (and of $\mathcal{C}_2$) at some points of I. For example, let $\boldsymbol{a}^{(1)},\boldsymbol{a}^{(2)},\boldsymbol{a}^{(3)}\in\mathcal{N}$ be the sequences defined by

$$a_j^{(1)} := \begin{cases} 2^j & \text{if } j \text{ is even} \\ 1 & \text{if } j \text{ is odd}\end{cases}, \quad a_j^{(2)} := 2^j \quad \text{and} \quad a_j^{(3)} := \begin{cases} 1 & \text{if } j \text{ is even} \\ 2^j & \text{if } j \text{ is odd}\end{cases}$$

for any $j\in\mathbb{N}$. Using Theorem 2.4.1, it is easy to check that

$$h_{\mathcal{C}_1}([\boldsymbol{a}^{(1)}])=0,\quad h_{\mathcal{C}_1}([\boldsymbol{a}^{(2)}])=\frac{1}{2}\quad\text{and}\quad h_{\mathcal{C}_1}([\boldsymbol{a}^{(3)}])=1.$$

We then obtain the following corollary.

Corollary 2.4.5. *The functions $\mathcal{C}_1$ and $\mathcal{C}_2$ are multifractal. Consequently, $\mathcal{C}$ is multifractal.*

Let us finish this section with some improvements of Corollary 2.4.4 under some conditions. Actually, we can refine the bounds of Theorem 2.4.1 and Theorem 2.4.2. Indeed, taking the notations and conventions of the proof of Theorem 2.4.1, we have

$$\frac{1}{2q_{n+3}^2(\boldsymbol{c})}\le|x-y|\le\frac{1}{q_n^2(\boldsymbol{a})}$$

and

$$\frac{1}{2q_{n/2+3}^2(\boldsymbol{d})}\le|\mathcal{C}_1(x)-\mathcal{C}_1(y)|\le\frac{1}{q_{n/2}^2(\boldsymbol{a}')}$$

with Inequalities (2.6) and (2.9), where $\boldsymbol{a}':=(a_{2j-1})_{j\in\mathbb{N}}$. We then have

$$\frac{2\log(q_{n/2}(\boldsymbol{a}'))}{\log(2)+2\log(q_{n+3}(\boldsymbol{c}))}\le\frac{\log|\mathcal{C}_1(x)-\mathcal{C}_1(y)|}{\log|x-y|}\le\frac{\log(2)+2\log(q_{n/2+3}(\boldsymbol{d}))}{2\log(q_n(\boldsymbol{a}))}. \tag{2.12}$$

Of course, we have similar inequalities for $\mathcal{C}_2$. What happens when taking the limit as $n\to+\infty$? Is it possible to obtain the Hölder exponent of $\mathcal{C}_1$ (and of $\mathcal{C}_2$) at x? On the one hand, we have the following result (see [76, 84, 101]).

Theorem 2.4.6. *For almost every sequence $\boldsymbol{b} \in \mathcal{N}$, we have*

$$\lim_{n\to+\infty} \frac{1}{n}\log(q_n(\boldsymbol{b})) = \frac{\pi^2}{12\log(2)}.$$

The real number $\pi^2/(12\log(2))$ is called the Lévy's constant. On the other hand, since $q_{n+3}(\boldsymbol{a}) \leq q_{n+3}(\boldsymbol{c}) \leq 2q_{n+3}(\boldsymbol{a})$ (using the definition of $\boldsymbol{c}$ and Equality (2.1)), we have

$$\lim_{n\to+\infty} \frac{1}{n+3}\log(q_{n+3}(\boldsymbol{c})) = \lim_{n\to+\infty} \frac{1}{n}\log(q_n(\boldsymbol{a})) \tag{2.13}$$

and similarly, we also have

$$\lim_{n\to+\infty} \frac{1}{\frac{n}{2}+3}\log(q_{n/2+3}(\boldsymbol{d})) = \lim_{n\to+\infty} \frac{2}{n}\log(q_{n/2}(\boldsymbol{a}')) \tag{2.14}$$

(if all these limits exist). It only remains to compare Expressions (2.13) and (2.14), which is not evident. In any case, from Inequality (2.12) and from the above, we have the following proposition.

Proposition 2.4.7. *Let $x = [\boldsymbol{a}]$ be an element of I and let $\boldsymbol{a}' := (a_{2j-1})_{j\in\mathbb{N}}$. If we assume that*

$$\lim_{n\to+\infty} \frac{1}{n}\log(q_n(\boldsymbol{a})) = \lim_{n\to+\infty} \frac{1}{n}\log(q_n(\boldsymbol{a}')) = \frac{\pi^2}{12\log(2)}, \tag{2.15}$$

then we have

$$h_{\mathcal{C}_1}(x) = \frac{1}{2}.$$

There is of course a similar result for $\mathcal{C}_2$. With Theorem 2.4.6, we can hope that Equality (2.15) is satisfied for almost every sequence $\boldsymbol{a} \in \mathcal{N}$ and thus we can make the following conjecture.

Conjecture 2.4.8. *For almost every $x \in [0,1]$, we have*

$$h_{\mathcal{C}_1}(x) = h_{\mathcal{C}_2}(x) = \frac{1}{2}.$$

Let us give an idea to attempt to prove Equality (2.15) and then Conjecture 2.4.8.

Let τ be the left shift operator on $\mathcal{N}$, i.e. the application defined by

$$\tau((b_j)_{j\in\mathbb{N}}) := (b_{j+1})_{j\in\mathbb{N}}$$

We denote by τ^m the m^{th} iterate of τ for $m \in \mathbb{N}$ and by τ^0 the identity. The next lemma based on the properties of the convergents of a sequence can be useful (see the proof of Theorem 8.3 in [112] for example).

Lemma 2.4.9. *For all $\boldsymbol{b} \in \mathcal{N}$ and $n \in \mathbb{N}$, we have*

$$\log(q_n(\boldsymbol{b})) = -\sum_{j=0}^{n-1}\log\left(\frac{p_{n-j}(\tau^j(\boldsymbol{b}))}{q_{n-j}(\tau^j(\boldsymbol{b}))}\right).$$

Using this formula, we then have

$$\frac{1}{n}\log(q_n(\boldsymbol{b})) = -\frac{1}{n}\sum_{j=0}^{n-1}\log([\tau^j(\boldsymbol{b})]) + R_n(\boldsymbol{b}) \tag{2.16}$$

for all $\boldsymbol{b} \in \mathcal{N}$ and $n \in \mathbb{N}$, where

$$R_n(\boldsymbol{b}) := \frac{1}{n}\sum_{j=0}^{n-1}\left(\log([\tau^j(\boldsymbol{b})]) - \log\left(\frac{p_{n-j}(\tau^j(\boldsymbol{b}))}{q_{n-j}(\tau^j(\boldsymbol{b}))}\right)\right).$$

The limit of $R_n(\boldsymbol{b})$ as $n \to +\infty$ is given by the following lemma (see again the proof of Theorem 8.3 in [112] for example).

Lemma 2.4.10. *For all $\boldsymbol{b} \in \mathcal{N}$, we have*

$$\lim_{n\to+\infty} R_n(\boldsymbol{b}) = 0.$$

Let $x = [\boldsymbol{a}] \in I$. By definition, we have $\mathcal{C}_1(x) = [\boldsymbol{a}']$ where $\boldsymbol{a}' := (a_{2j-1})_{j\in\mathbb{N}}$. Using Equality (2.16) with $\boldsymbol{a}'$, we obtain

$$\begin{aligned}\frac{1}{n}\log(q_n(\boldsymbol{a}')) &= -\frac{1}{n}\sum_{j=0}^{n-1}\log([\tau^j(\boldsymbol{a}')]) + R_n(\boldsymbol{a}')\\ &= -\frac{1}{n}\sum_{j=0}^{n-1}\log([\tau^{2j}(\boldsymbol{a})]) + S_n(\boldsymbol{a}) + R_n(\boldsymbol{a}')\end{aligned} \tag{2.17}$$

where

$$S_n(\boldsymbol{a}) := \frac{1}{n}\sum_{j=0}^{n-1}\left(\log([\tau^{2j}(\boldsymbol{a})]) - \log([\tau^j(\boldsymbol{a}')])\right) = \frac{1}{n}\sum_{j=0}^{n-1}\log\left(\frac{[\tau^{2j}(\boldsymbol{a})]}{\mathcal{C}_1([\tau^{2j}(\boldsymbol{a})])}\right).$$

Thanks to Lemma 2.4.10, we know that

$$\lim_{n\to+\infty} R_n(\boldsymbol{a}') = 0.$$

We also have the following theorem, which is a consequence of the main result of [94].

Theorem 2.4.11. *For almost all $\boldsymbol{b} \in \mathcal{N}$, we have*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log([\tau^{2j}(\boldsymbol{b})]) = \frac{1}{\log(2)}\int_0^1\frac{\log(t)}{t+1}\,dt = -\frac{\pi^2}{12\log(2)}$$

From Equality (2.17), we then have

$$\lim_{n\to+\infty}\frac{1}{n}\log(q_n(\boldsymbol{a}')) = \frac{\pi^2}{12\log(2)} + \lim_{n\to+\infty} S_n(\boldsymbol{a})$$

and it only remains to show that

$$\lim_{n\to+\infty} S_n(\boldsymbol{a}) = 0, \tag{2.18}$$

which is not evident. In fact, it is difficult to reasonably compare $[\tau^{2j}(\boldsymbol{a})]$ and $\mathcal{C}_1([\tau^{2j}(\boldsymbol{a})])$. The sequences which define these two continued fractions only have the first element in common. Unfortunately, this observation is not sufficient to obtain Equality (2.18) and then Conjecture 2.4.8.

2.5 Appendix: Another Cantor's Bijection

Actually, the application $\mathcal{C}$ (with the use of continued fractions) was not the first idea of CANTOR to construct a one-to-one mapping between $[0,1]$ and $[0,1]^2$. In 1877 (the same year as the construction of $\mathcal{C}$), CANTOR first proposed the following example, based on the (unique) proper decimal expansion of the real numbers. If x and y both belong to the unit segment $[0,1)$, let us write

$$x := \sum_{k=1}^{+\infty} \frac{x_k}{10^k} = 0.x_1x_2\cdots \qquad \text{and} \qquad y := \sum_{k=1}^{+\infty} \frac{y_k}{10^k} = 0.y_1y_2\cdots$$

(where $x_k, y_k \in \{0,\ldots,9\}$ for $k \in \mathbb{N}$) with proper expansions (i.e. there does not exist $k_0 \in \mathbb{N}$ such that $x_k = 9$ for all $k > k_0$). Let $\mathscr{C}$ be the map defined as

$$\mathscr{C} : [0,1)^2 \to [0,1);\ (x,y) \mapsto \sum_{k=1}^{+\infty} \frac{x_k}{10^{2k-1}} + \sum_{k=1}^{+\infty} \frac{y_k}{10^{2k}} = 0.x_1y_1x_2y_2x_3y_3\cdots.$$

DEDEKING objected that such a function is not surjective, since a number of the form

$$z := \sum_{k=1}^{l} \frac{z_k}{10^k} + \sum_{k=1}^{+\infty} \frac{9}{10^{l+2k-1}} + \sum_{k=1}^{+\infty} \frac{z_{l+2k}}{10^{l+2k}} = 0.z_1z_2\cdots z_l 9 z_{l+2} 9 z_{l+4} 9\cdots$$

(where $z_k \in \{0,\ldots,9\}$ for $k \in \mathbb{N}$) with $l \in \mathbb{N}$ has no preimage under $\mathscr{C}$: if l is even, there is no x such that $\mathscr{C}(x,y) = z$ and if l is odd, there is no y such that $\mathscr{C}(x,y) = z$. CANTOR then overcame this problem by replacing the decimal expansion in $\mathscr{C}$ with the expansion in terms of continued fractions. His work was published in [25], with a praragraph explaining why his first idea could not work and omitting any reference to DEDEKIND (see [38] for some historical references).

In this last section, we go back on Cantor's first idea. We start from the map $\mathscr{C}$ relying on the decimal expansion and use the Schröder-Bernstein theorem to define the desired bijection between $[0,1]^2$ and $[0,1]$. This theorem was first conjectured by CANTOR and independently proved by BERNSTEIN and SCHRÖDER in 1896 (see [19, 27, 109], let us also notice that other names, such as DEDEKIND, should be added to this list). In other words, Cantor's first idea could have led to the craved mapping, but he did not have such a result at the time he was working on the topic. It would be conjectured by himself a few years later in [26]. Before building the bijective map, we recall the Schröder-Bernstein theorem and give a classical proof that will be used in the sequel.

2.5.1 A "Practical" Proof of Schröder-Bernstein Theorem

There exist several proofs of Schröder-Bernstein theorem (see [53]): the most classical ones use Tarski's fixed point theorem, or follow the idea of DEDEKIND [36] or KÖNIG [78]. The advantage of the one we present below (which is largely inspired by ideas of [20, 104]) is that it explicitly shows how to build a bijection between two non-empty sets, starting from injections between these sets.

Theorem 2.5.1 (Schröder-Bernstein). *Let A and B be non-empty sets. If there exist an injection from A to B and an injection from B to A, then there exists a bijection from A onto B.*

Proof. Let f be an injection from A to B and g be an injection from B to A. We define the sequences $(A_n)_{n\in\mathbb{N}_0}$ of subsets of A and $(B_n)_{n\in\mathbb{N}_0}$ of subsets of B as follows:

$$\begin{cases} A_0 := A \setminus g(B) & \\ B_n := f(A_n), & \text{for } n \in \mathbb{N}_0 \\ A_n := g(B_{n-1}), & \text{for } n \in \mathbb{N} \end{cases} . \tag{2.19}$$

If $A_0 = \emptyset$, then $g(B) = A$ and thus g is surjective. The application g^{-1} is then a bijection from A onto B. Therefore, we can assume that A_0 is not empty.

None of the elements of the sequences $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are empty and thus

$$\bigcup_{n\in\mathbb{N}_0} A_n \neq \emptyset, \quad \bigcup_{n\in\mathbb{N}_0} B_n \neq \emptyset \quad \text{and} \quad f\left(\bigcup_{n\in\mathbb{N}_0} A_n\right) \neq \emptyset.$$

Moreover, we have

$$f\left(\bigcup_{n\in\mathbb{N}_0} A_n\right) \subset \bigcup_{n\in\mathbb{N}_0} B_n$$

and the restriction $\tilde{f}$ of f to $\bigcup_{n\in\mathbb{N}_0} A_n$ is clearly a bijection from $\bigcup_{n\in\mathbb{N}_0} A_n$ onto $\bigcup_{n\in\mathbb{N}_0} B_n$.

If $B = \bigcup_{n\in\mathbb{N}_0} B_n$, then $A = \bigcup_{n\in\mathbb{N}_0} A_n$ because f is injective and thus $\tilde{f}$ is a bijection from A to B.

Let us now assume that $B \setminus \bigcup_{n\in\mathbb{N}_0} B_n \neq \emptyset$. Since g is injective, we have

$$g\left(B \setminus \bigcup_{n\in\mathbb{N}_0} B_n\right) \subset A \setminus \bigcup_{n\in\mathbb{N}_0} A_n$$

and $A \setminus \bigcup_{n\in\mathbb{N}_0} A_n \neq \emptyset$. Let us denote by $\tilde{g}$ the restriction of g to $B \setminus \bigcup_{n\in\mathbb{N}_0} B_n$ and show that $\tilde{g}$ is a bijection from $B \setminus \bigcup_{n\in\mathbb{N}_0} B_n$ onto $A \setminus \bigcup_{n\in\mathbb{N}_0} A_n$. It is clear that $\tilde{g}$ is injective. Since

$$\begin{aligned} A \setminus \bigcup_{n\in\mathbb{N}_0} A_n &= (A \setminus A_0) \cap \left(\bigcap_{n\in\mathbb{N}} (A \setminus A_n)\right) \\ &= g(B) \cap \left(\bigcap_{n\in\mathbb{N}} (A \setminus g(B_{n-1}))\right) \\ &= g(B) \cap \left(A \setminus g\left(\bigcup_{n\in\mathbb{N}_0} B_n\right)\right), \end{aligned}$$

$\tilde{g}$ is also surjective.

It only remains to put the pieces together in order to construct a bijection from A onto B. Since $\tilde{f}$ is a bijection from $\bigcup_{n\in\mathbb{N}_0} A_n$ onto $\bigcup_{n\in\mathbb{N}_0} B_n$ and $\tilde{g}^{-1}$ is a bijection from $A \setminus \bigcup_{n\in\mathbb{N}_0} A_n$ onto $B \setminus \bigcup_{n\in\mathbb{N}_0} B_n$, the application h defined by

$$h(a) := \begin{cases} \tilde{f}(a) & \text{if } a \in \bigcup_{n\in\mathbb{N}_0} A_n \\ \tilde{g}^{-1}(a) & \text{if } a \in A \setminus \bigcup_{n\in\mathbb{N}_0} A_n \end{cases}$$

is a bijection from A onto B, hence the conclusion. ■

Remark 2.5.2. Let us note that the definition of the map h given above is non-constructive (see [**117**]): there is no general method to decide whether or not an element of A belongs to $\bigcup_{n\in\mathbb{N}_0} A_n$ in a finite number of steps. However, in the specific case we will consider, the problem becomes simpler.

2.5.2 A Bijection between the Unit Square and the Unit Segment Based on the Decimal Expansion

Let us build a bijection between the unit square $[0,1]^2$ and the unit segment $[0,1]$ starting from the function $\mathscr{C}$ (see [**97**]). Since the construction is entirely based on the proof of the previous theorem, we will use the notations of this proof.

Let us set $A := [0,1]^2$, $B := [0,1]$ and let f be the function defined by

$$f(x,y) := \begin{cases} \sum_{k=1}^{+\infty} \frac{x_k}{10^{2k-1}} + \sum_{k=1}^{+\infty} \frac{y_k}{10^{2k}} = 0.x_1y_1x_2y_2x_3y_3\cdots & \text{if } (x,y) \in [0,1)^2 \\ \sum_{k=1}^{+\infty} \frac{9}{10^{2k-1}} + \sum_{k=1}^{+\infty} \frac{y_k}{10^{2k}} = 0.9y_19y_29y_3\cdots & \text{if } (x,y) \in \{1\}\times[0,1) \\ \sum_{k=1}^{+\infty} \frac{x_k}{10^{2k-1}} + \sum_{k=1}^{+\infty} \frac{9}{10^{2k}} = 0.x_19x_29x_3\cdots & \text{if } (x,y) \in [0,1)\times\{1\} \\ 1 & \text{if } (x,y) = (1,1) \end{cases}$$

where $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ are the proper decimal expansions of the real numbers x and y of $[0,1)$. In fact, we have $f(x,y) = \mathscr{C}(x,y)$ for $(x,y) \in [0,1)^2$, so that f is simply an extension of $\mathscr{C}$ to $[0,1]^2$. Let g be the function defined by $g(t) := (t,0)$ for $t \in B$. It easy to check that both f and g are injective.

Let us construct the sequences $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ step by step as in Expression (2.19). For $n = 0$, we have

$$A_0 = A \setminus g(B) = [0,1] \times (0,1]$$

and

$$B_0 = f(A_0) = \{1\} \cup \{t \in [0,1) : t_{2k} \neq 0 \text{ for some } k \in \mathbb{N}\}$$

where $(t_k)_{k\in\mathbb{N}}$ is the proper decimal expansion of the real number t belonging to $[0,1)$.

For $n = 1$, we directly have $A_1 = g(B_0) = B_0 \times \{0\}$. In order to construct $B_1 = f(A_1)$, let us take $(x,0) \in A_1$. We have $x_{2k} \neq 0$ for some $k \in \mathbb{N}$ by definition of B_0 and thus

$$f(x,0) = \begin{cases} \sum_{k=1}^{+\infty} \frac{9}{10^{2k-1}} = 0.909090\cdots & \text{if } x = 1 \\ \sum_{k=1}^{+\infty} \frac{x_k}{10^{2k-1}} = 0.x_10x_20x_30\cdots & \text{if } x \neq 1 \end{cases}.$$

We can then write

$$f(x,0) = \sum_{k=1}^{+\infty} \frac{s_k}{10^{2k-1}} = 0.s_10s_20s_30\cdots$$

where $(s_k)_{k\in\mathbb{N}}$ is a sequence satisfying only one of the two following conditions:

(a) $s_k = 9$ for all $k \in \mathbb{N}$,

(b) $(s_k)_{k\in\mathbb{N}}$ is the proper decimal expansion of a real number of $[0,1)$ and $s_{2k} \neq 0$ for some $k \in \mathbb{N}$.

We will denote by Σ the set of sequences which satisfy one of the two previous conditions. We therefore have

$$B_1 = \left\{ t \in [0,1) : t = \sum_{k=1}^{+\infty} \frac{s_k}{10^{2k-1}} \text{ with } (s_k)_{k\in\mathbb{N}} \in \Sigma \right\}.$$

For $n = 2$, the argument is similar. We have $A_2 = g(B_1) = B_1 \times \{0\}$. If $(x,0) \in A_2$, then $x_{2k} = 0$ for all $k \in \mathbb{N}$ and $x_{4k-1} \neq 0$ for some $k \in \mathbb{N}$. Consequently, we have

$$f(x,0) = \sum_{k=1}^{+\infty} \frac{x_{2k-1}}{10^{4k-3}} = 0.x_1 000 x_3 000 x_5 000 \cdots$$

and so

$$B_2 = \left\{ t \in [0,1) : t = \sum_{k=1}^{+\infty} \frac{s_k}{10^{4k-3}} \text{ with } (s_k)_{k\in\mathbb{N}} \in \Sigma \right\}.$$

Going on in this way, we obtain $A_n = B_{n-1} \times \{0\}$ and

$$B_n = \left\{ t \in [0,1) : t = \sum_{k=1}^{+\infty} \frac{s_k}{10^{2^n k - (2^n - 1)}} \text{ with } (s_k)_{k\in\mathbb{N}} \in \Sigma \right\},$$

for all $n \in \mathbb{N}$.

Since $A_0 \neq \emptyset$, $B \setminus \bigcup_{n\in\mathbb{N}_0} B_n \neq \emptyset$ (we have $0 \notin B_n$ for any $n \in \mathbb{N}_0$) and $g^{-1}(x,y) = x$ for $(x,y) \in A \setminus \bigcup_{n\in\mathbb{N}_0} A_n$, we have proved the following proposition thanks to Theorem 2.5.1.

Proposition 2.5.3. *The function f^* defined by*

$$f^*(x,y) := \begin{cases} f(x,y) & \text{if } (x,y) \in \bigcup_{n\in\mathbb{N}_0} A_n \\ x & \text{otherwise} \end{cases}$$

is a bijection from $[0,1]^2$ onto $[0,1]$.

Remark 2.5.4. As expected, we have $f^* = f$ almost everywhere on $[0,1]^2$ (with respect to the Lebesgue measure), since the set $[0,1]^2 \setminus \bigcup_{n\in\mathbb{N}} A_n$ is included in the segment $[0,1] \times \{0\}$, which is a negligible set in $\mathbb{R}^2$. Therefore, we have $f^* = \mathscr{C}$ almost everywhere.

Chapter 3

Continuous Wavelet Transform and Hölder Continuity

The continuous wavelet transform, initially introduced by GROSSMANN and MORLET [48] in the eighties, is a tool to study the Hölder continuity of a function. More precisely, the behaviour of the continuous wavelet transform of a function gives the (pointwise and uniform) Hölder continuity of this function. This description, established twenty years ago, is especially due to JAFFARD and MEYER [59–61, 68, 92] and also HOLSCHNEIDER and TCHAMICHIAN [55].

In this brief chapter, we recall the notions of wavelet and of continuous wavelet transform, firstly in the general setting and secondly in the context of bounded and continuous functions (with a particular wavelet). We then present the tool given by the continuous wavelet transform to characterize Hölder spaces.

3.1 Continuous Wavelet Transform

Let us first recall the notions of wavelet and continuous wavelet transform (see [30, 33, 54, 55, 61, 69, 115]).

3.1.1 General Setting

In the literature, the word "wavelet" is used for several types of functions depending on the context. We take here the following definition.

Definition 3.1.1. The function ψ is a *wavelet* if $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and if ψ satisfies the *admissibility condition*:

$$\xi \mapsto \frac{|\hat{\psi}(\xi)|^2}{|\xi|} \in L^1(\mathbb{R}). \tag{3.1}$$

Using the wavelet ψ, the *continuous wavelet transform* of a function $f \in L^2(\mathbb{R})$ is the function $\mathcal{W}_\psi f$ defined by

$$\mathcal{W}_\psi f(a,b) := \int_{\mathbb{R}} f(x)\,\overline{\psi}_{a,b}(x)\,dx = \langle f, \psi_{a,b}\rangle\,, \quad a \in \mathbb{R}\setminus\{0\},\, b \in \mathbb{R}$$

where

$$\psi_{a,b}(x) := \frac{1}{a}\psi\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R}\,.$$

The admissibility condition plays an important role in the reconstruction of a function from its continuous wavelet transform (see Theorem 3.1.3). Moreover, it implies that $\hat{\psi}(0) = 0$ because

$\psi \in L^1(\mathbb{R})$. Indeed, by contradiction, if we suppose that $|\hat{\psi}(0)| \geq C$ with $C > 0$, there exists $\varepsilon > 0$ such that

$$\frac{|\hat{\psi}(\xi)|^2}{|\xi|} > \frac{C^2}{4|\xi|}$$

for $\xi \in (-\varepsilon, \varepsilon) \setminus \{0\}$ by continuity of $\hat{\psi}$ and we then have a contradiction with Condition (3.1).

Remark 3.1.2. The general setting of the continuous wavelet transform is the space $L^2(\mathbb{R})$. Since a wavelet is an integrable function (in our definition), we can calculate the continuous wavelet transform of a function which belongs to $L^\infty(\mathbb{R})$ (and which is not necessarily in $L^2(\mathbb{R})$). This will just allow to investigate the Hölder continuity of bounded (and continuous) functions from the continuous wavelet transform of these functions.

A square integrable function can be reconstructed from its wavelet continuous transform. This is the object of the following result, which will be proved later in the more general context of nonstationary continuous wavelet transform (see Theorem 6.2.1).

Theorem 3.1.3. *Let ψ be a wavelet such that*

$$\int_{\mathbb{R}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} \, d\xi = 1. \tag{3.2}$$

For all $f, g \in L^2(\mathbb{R})$, we have

$$\iint_{\mathbb{R}^2} \mathcal{W}_\psi \, f(a,b) \, \overline{\mathcal{W}_\psi \, g(a,b)} \, \frac{dadb}{|a|} = \langle f, g \rangle \, .$$

Moreover, for $f \in L^2(\mathbb{R})$, we have

$$\lim_{\substack{\varepsilon \to 0^+ \\ r \to +\infty}} \left\| f(\cdot) - \int_{\{a' \in \mathbb{R} : \varepsilon < |a'| < r\}} \left(\int_{\mathbb{R}} \mathcal{W}_\psi \, f(a,b) \, \psi_{a,b}(\cdot) \, db \right) \frac{da}{|a|} \right\|_{L^2(\mathbb{R})} = 0.$$

There exist some variants of this reconstruction formula. For example, we can recover f from $\mathcal{W}_\psi \, f(a,b)$ with $a > 0$ only and $b \in \mathbb{R}$. In this case, Condition (3.2) is slightly more restrictive.

Theorem 3.1.4. *Let ψ be a wavelet such that*

$$\int_0^{+\infty} \frac{|\hat{\psi}(\xi)|^2}{\xi} \, d\xi = \int_0^{+\infty} \frac{|\hat{\psi}(-\xi)|^2}{\xi} \, d\xi = 1.$$

For all $f, g \in L^2(\mathbb{R})$, we have

$$\iint_{(0,+\infty) \times \mathbb{R}} \mathcal{W}_\psi \, f(a,b) \, \overline{\mathcal{W}_\psi \, g(a,b)} \, \frac{dadb}{a} = \langle f, g \rangle \, .$$

Moreover, for $f \in L^2(\mathbb{R})$, we have

$$\lim_{\substack{\varepsilon \to 0^+ \\ r \to +\infty}} \left\| f(\cdot) - \int_\varepsilon^r \left(\int_{\mathbb{R}} \mathcal{W}_\psi \, f(a,b) \, \psi_{a,b}(\cdot) \, db \right) \frac{da}{a} \right\|_{L^2(\mathbb{R})} = 0.$$

Another possibility consists in the introduction of another wavelet with some specific properties. In the next section, we will come back on this idea in the particular case of a wavelet which belongs to the Hardy space

$$H^2(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \hat{f} = 0 \text{ a.e. on } (-\infty, 0) \right\}.$$

3.1.2 The Particular Setting of Continuous and Bounded Functions

In the next chapter, in order to study the Hölder continuity of generalized Riemann function, we will use a particular wavelet which belongs to $H^2(\mathbb{R})$. The generalized Riemann function is not square integrable, but it is continuous and bounded on $\mathbb{R}$. As announced in the previous subsection (see Remark 3.1.2), its continuous wavelet transform can be investigated. An exact reconstruction formula exists in such a situation: if the wavelet ψ belongs to $H^2(\mathbb{R})$ and if f belongs to a certain class of continuous and bounded functions on $\mathbb{R}$, we can recover f from $\mathcal{W}_\psi f$ using a second wavelet satisfying some additional properties. This result is given below. It is just mentioned in a remark of [55] without a proof of this particular setting. We propose here a proof strongly inspired by Proposition 2.4.2 in [33] and Theorem 2.2 in [55] with some adaptations to our case.

Theorem 3.1.5. *Let ψ be a wavelet which belongs to $H^2(\mathbb{R})$. Let φ be a differentiable wavelet such that $x \mapsto x\varphi(x)$ is integrable on $\mathbb{R}$, such that $D\varphi$ is square integrable on $\mathbb{R}$ and such that*

$$\int_0^{+\infty} \overline{\hat{\psi}}(\xi)\hat{\varphi}(\xi)\,\frac{d\xi}{\xi} = 1. \tag{3.3}$$

If f is a continuous and bounded function on $\mathbb{R}$ and is weakly oscillating around the origin, i.e. such that

$$\lim_{r\to+\infty}\ \sup_{x\in\mathbb{R}} \left| \frac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt \right| = 0,$$

then we have

$$f(x) = \lim_{\substack{\varepsilon\to 0^+\\ r\to+\infty}} 2\int_\varepsilon^r \left(\int_{-\infty}^{+\infty} \mathcal{W}_\psi f(a,b)\,\varphi_{a,b}(x)\,db \right) \frac{da}{a}$$

for all $x \in \mathbb{R}$.

Proof. Let us fix $x \in \mathbb{R}$ and $r > \varepsilon > 0$. We write

$$f_{\varepsilon,r}(x) := \int_\varepsilon^r \left(\int_{-\infty}^{+\infty} \mathcal{W}_\psi f(a,b)\,\varphi_{a,b}(x)\,db \right) \frac{1}{a}\,da.$$

Then, we have

$$f_{\varepsilon,r}(x) = (M_{\varepsilon,r} \star f)(x)$$

by Fubini's theorem, where $M_{\varepsilon,r}$ is defined by

$$M_{\varepsilon,r}(t) := \int_\varepsilon^r \left(\int_{-\infty}^{+\infty} \overline{\psi}\left(-\frac{b}{a}\right) \varphi\left(\frac{t-b}{a}\right)\,db \right) \frac{1}{a^3}\,da, \quad t\in\mathbb{R}\,.$$

Since $M_{\varepsilon,r} \in L^1(\mathbb{R})$ and the support of $\hat{\psi}$ is included in $[0,+\infty)$, we have

$$\hat{M}_{\varepsilon,r}(\xi) = \int_\varepsilon^r \overline{\hat{\psi}}(a\xi)\hat{\varphi}(a\xi)\frac{1}{a}\,da = \begin{cases} 0 & \text{if } \xi \le 0 \\ \displaystyle\int_{\varepsilon\xi}^{r\xi} \overline{\hat{\psi}}(a)\hat{\varphi}(a)\frac{1}{a}\,da & \text{if } \xi > 0 \end{cases}.$$

Moreover, we have

$$\hat{M}_{\varepsilon,r}(\xi) = m(\varepsilon\xi) - m(r\xi) \tag{3.4}$$

for all $\xi \in \mathbb{R}$, where m is defined by

$$m(\xi) := \begin{cases} \displaystyle\int_{\xi}^{+\infty} \overline{\hat{\psi}}(a)\hat{\varphi}(a)\frac{1}{a}\,da & \text{if } \xi \geq 0 \\ \displaystyle\int_{-\xi}^{+\infty} \overline{\hat{\psi}}(-a)\hat{\varphi}(-a)\frac{1}{a}\,da & \text{if } \xi < 0 \end{cases}.$$

It is easy to check that $m(0) = 1$, $m = 0$ on $(-\infty, 0)$ and that m is continuous only on $\mathbb{R}\setminus\{0\}$. Since we have the three following properties: $\hat{\psi}$ is bounded, φ is differentiable and $D\varphi \in L^2(\mathbb{R})$, we obtain

$$|m(\xi)| \leq \left(\int_0^{+\infty} |a\hat{\varphi}(a)|^2 da\right)^{1/2} \left(\int_{\xi}^{+\infty} \frac{|\hat{\psi}(a)|^2}{a^4}\,da\right)^{1/2} \leq \frac{C'}{\xi^{3/2}}$$

for all $\xi > 0$, by Cauchy-Schwarz inequality, where C' is a positive constant. Then, m is bounded and there exists $C > 0$ such that

$$|m(\xi)| \leq \frac{C}{(1 + |\xi|)^{3/2}}$$

for all $\xi \in \mathbb{R}$. So $m \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and we can define M by $M(\xi) := \hat{m}(-\xi)/\pi$ for all $\xi \in \mathbb{R}$. By definition, M is continuous and bounded on $\mathbb{R}$.

Moreover, m is differentiable on $\mathbb{R}\setminus\{0\}$ and

$$Dm(\xi) = \begin{cases} -\overline{\hat{\psi}}(\xi)\hat{\varphi}(\xi)\dfrac{1}{\xi} & \text{if } \xi > 0 \\ 0 & \text{if } \xi < 0 \end{cases}.$$

Since $\hat{\varphi}(0) = 0$ and $x \mapsto x\varphi(x)$ is integrable on $\mathbb{R}$, we have

$$|\hat{\varphi}(\xi)| = \left|\int_{\mathbb{R}} \varphi(x)\left(e^{-ix\xi} - 1\right)dx\right| = \left|\int_{\mathbb{R}} x\varphi(x)\left(\int_0^{\xi} -ie^{-ixt}\,dt\right)dx\right| \leq C''|\xi|$$

for all $\xi \in \mathbb{R}$, where C'' is a positive constant. Consequently, $Dm \in L^2(\mathbb{R})$ because $\psi \in L^2(\mathbb{R})$. So $M \in L^1(\mathbb{R})$ since we can write M as the product of two square integrable functions: for all $x \in \mathbb{R}$, we have

$$M(x) = \frac{1}{\sqrt{1+x^2}}\left(\sqrt{1+x^2}\,M(x)\right),$$

where the second factor is square integrable, because m and Dm are square integrable on $\mathbb{R}$. Moreover, by the Dirichlet condition for Fourier inversion theorem (since m and Dm are piecewise continuous), we have

$$\int_{\mathbb{R}} M(x)\,dx = \hat{M}(0) = m(0^+) + m(0^-) = 1$$

using Equality (3.3) where $m(0^{\pm}) = \lim_{\xi \to 0^{\pm}} m(\xi)$.

By definition of M and by Fourier inversion theorem in Equality (3.4), we have

$$M_{\varepsilon,r}(t) = \frac{1}{2}\left(\frac{1}{\varepsilon}M\left(\frac{t}{\varepsilon}\right) - \frac{1}{r}M\left(\frac{t}{r}\right)\right)$$

for all $t \in \mathbb{R}$ and we then obtain

$$f_{\varepsilon,r}(x) = \frac{1}{2}\left(\int_{\mathbb{R}} \frac{1}{\varepsilon}M\left(\frac{x-t}{\varepsilon}\right)f(t)\,dt - \int_{\mathbb{R}} \frac{1}{r}M\left(\frac{x-t}{r}\right)f(t)\,dt\right).$$

Using the continuity of f, the first integral converges to $f(x)$ as ε tends to 0^+ by Lebesgue's theorem. The second integral converges to 0 as r tends to $+\infty$ because f is bounded and weakly oscillating around on the origin, and $M \in L^1(\mathbb{R})$ is of integral equal to 1 (see Lemma 6.3.3 page 142 in [54] for the proof of this property). The conclusion follows. ■

An example of wavelet which belongs to $H^2(\mathbb{R})$ is the Lusin wavelet ψ_L defined by

$$\psi_L(x) := \frac{1}{\pi(x+i)^2}, \quad x \in \mathbb{R}. \tag{3.5}$$

It is clear that $\psi_L \in H^2(\mathbb{R})$ because

$$\hat{\psi}_L(\xi) = \begin{cases} -2\xi e^{-\xi} & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases}.$$

In the next chapter, we will use the Lusin wavelet in order to study Hölder continuity of generalized Riemann function. We will see that this choice of wavelet will allow to obtain a simple explicit expression of the continuous wavelet transform of the studied function (in comparison with other wavelets as the derivatives of a gaussian function for example).

3.2 Characterization of Hölder Spaces

Thanks to the previous reconstruction formula, the Hölder continuity of a function can be characterized with its continuous wavelet transform (see [55, 61, 69]). This is the object of the following theorem. We will use it in the next chapter (see also [55, 61, 100] for other examples).

Theorem 3.2.1. *Let $\alpha \in (0,1)$, let ψ be a wavelet such that $x \mapsto x^\alpha \psi(x)$ is integrable on $\mathbb{R}$ and let f be a function as in Theorem 3.1.5.*

(a) We have $f \in \Lambda^\alpha(\mathbb{R})$ if and only if there exists $C > 0$ such that

$$|\mathcal{W}_\psi f(a,b)| \leq C\, a^\alpha$$

for all $a > 0$ and $b \in \mathbb{R}$.

(b) Let $x_0 \in \mathbb{R}$. If $f \in \Lambda^\alpha(x_0)$, then there exist $C > 0$ and $\eta > 0$ such that

$$|\mathcal{W}_\psi f(a,b)| \leq C\, a^\alpha \left(1 + \left(\frac{|b - x_0|}{a}\right)^\alpha\right)$$

for all $a \in (0,\eta)$ and $b \in (x_0 - \eta, x_0 + \eta)$. Conversely, if there exist $\alpha' \in (0,\alpha)$, $C > 0$ and $\eta > 0$ such that

$$|\mathcal{W}_\psi f(a,b)| \leq C\, a^\alpha \left(1 + \left(\frac{|b - x_0|}{a}\right)^{\alpha'}\right)$$

for all $a \in (0,\eta)$ and $b \in (x_0 - \eta, x_0 + \eta)$, then $f \in \Lambda^\alpha(x_0)$.

The proof of this theorem (sometimes with some minor variants) can be found in [33, 55, 61, 69, 115]. Hölder spaces with exponent greater than 1 can be also characterized with continuous wavelet transform.

Remark 3.2.2. Let us note that the necessary conditions in Theorem 3.2.1 do not need all the hypotheses on the function f: the continuity and the weak oscillation around the origin of f are not useful for these implications.

Chapter 4

Generalized Riemann Function

In the 19$^{\text{th}}$ century, RIEMANN introduced the function R defined by

$$R(x) := \sum_{n=1}^{+\infty} \frac{\sin(\pi n^2 x)}{n^2}, \quad x \in \mathbb{R},$$

in order to construct a continuous but nowhere differentiable function (see [37] for some historical information). The regularity of this function has been extensively studied by many authors. In 1916, HARDY [49] showed that R is not differentiable at irrational numbers and at some rational numbers. Decades later, GERVER [44] and other people [55, 58, 93, 103, 111] proved that R is only differentiable at the rational numbers $(2p+1)/(2q+1)$ (with $p \in \mathbb{Z}$ and $q \in \mathbb{N}_0$) with a derivative equal to $-1/2$.

Moreover, the Hölder continuity of R was also investigated. Based on a work with LITTLEWOOD [50], HARDY [49] showed that R is not Hölder continuous with exponent $3/4$ at irrational numbers and at some rational numbers. Using the continuous wavelet transform (of R), HOLSCHNEIDER and TCHAMITCHIAN [55] established that R is uniformly Hölder continuous with exponent $1/2$ and gave some results about its Hölder continuity at some particular points. With some similar techniques, JAFFARD and MEYER [61, 68] determined the Hölder exponent of R at each point and proved that R is a multifractal function.

A generalization of R is given by the function $R_{\alpha,\beta}$ defined by

$$R_{\alpha,\beta}(x) := \sum_{n=1}^{+\infty} \frac{\sin(\pi n^\beta x)}{n^\alpha}, \quad x \in \mathbb{R}, \tag{4.1}$$

with $\alpha > 1$ and $\beta > 0$. Other generalizations of R are possible; for example, we can replace the element n^β in the definition of $R_{\alpha,\beta}$ by a polynomial with integer coefficients (see [29, 103]).

The function $R_{\alpha,\beta}$ defined in Expression (4.1) is clearly continuous and bounded on $\mathbb{R}$. If $\beta \in (0, \alpha-1)$, it is easy to check that $R_{\alpha,\beta}$ is continuously differentiable on $\mathbb{R}$ (because the series of derivatives converges uniformly on $\mathbb{R}$). If $\beta \geq \alpha+1$, LUTHER [86] proved that $R_{\alpha,\beta}$ is nowhere differentiable. If $\beta \in [\alpha-1, \alpha+1)$, several partial results about the differentiability of $R_{\alpha,\beta}$ are known (see [86, 103]). Moreover, some results are also known for the cases $\beta = 2$ (see [49, 61]), $\beta = 3$ (see [45]) and $\beta \in \mathbb{N}$ (see [28]). Concerning the Hölder continuity and also the Hölder exponent of $R_{\alpha,\beta}$, several particular cases have been studied (see [21, 28, 61, 68, 73, 118]).

In this chapter, we study the uniform Hölder continuity of $R_{\alpha,\beta}$ with $\beta \geq \alpha - 1$. We apply some obtained results to the more general case of nonharmonic Fourier series. We then present the graphical representation of $R_{2,\beta}$ for some particular values of β. We analyse the particular and amazing behaviour of $R_{\alpha,\beta}$ as β increases. The results presented in this chapter are from [17].

4.1 Hölder Continuity of Generalized Riemann Function

In 2010, Johnsen [73] showed that if $\beta > \alpha - 1$, then $R_{\alpha,\beta}$ is uniformly Hölder continuous with an exponent greater or equal to $(\alpha-1)/\beta$. In order to complete and generalize this result, we use some techniques different from the ones of Johnsen. Our approach is based on the continuous wavelet transform of $R_{\alpha,\beta}$ related to the Lusin wavelet presented in the previous chapter, and follows the approaches used to obtain the Hölder continuity of R in [55, 61, 69]. This method has two advantages: we can consider both cases $\beta = \alpha - 1$ and $\beta > \alpha - 1$ to study the uniform Hölder continuity of $R_{\alpha,\beta}$ and then show the optimality of the obtained exponent. In other words, we calculate the uniform Hölder exponent of $R_{\alpha,\beta}$ for $\beta \geq \alpha - 1$. These results are summarized in the following theorem.

Theorem 4.1.1. *For $\beta \geq \alpha - 1$, we have*

$$H_{R_{\alpha,\beta}}(\mathbb{R}) = \frac{\alpha-1}{\beta}.$$

The generalized Riemann function and the Lusin wavelet satisfy the conditions of Theorem 3.2.1. Indeed, we know that $R_{\alpha,\beta}$ is continuous and bounded and that the Lusin wavelet ψ_L belongs to $H^2(\mathbb{R})$. Moreover, $R_{\alpha,\beta}$ is weakly oscillating around the origin because

$$\left|\frac{1}{2r}\int_{x-r}^{x+r} R_{\alpha,\beta}(t)\,dt\right| \leq \left|\frac{1}{2r}\sum_{n=1}^{+\infty}\frac{\cos((x-r)\pi n^\beta)-\cos((x+r)\pi n^\beta)}{\pi n^{\alpha+\beta}}\right| \leq \frac{\zeta(\alpha+\beta)}{\pi r}$$

for all $x \in \mathbb{R}$ and $r > 0$, where ζ is the well-known Riemann zeta function defined by

$$\zeta(z) := \sum_{n=1}^{+\infty}\frac{1}{n^z}, \quad \Re z > 1.$$

The function $x \mapsto x^\alpha \psi_L(x)$ is clearly integrable for $\alpha \in (0,1)$. Besides, it is easy to find a differentiable wavelet φ such that $x \mapsto x\varphi(x)$ is integrable on $\mathbb{R}$, such that $D\varphi$ is square integrable on $\mathbb{R}$ and such that

$$\int_0^{+\infty}\hat{\varphi}(\xi)e^{-\xi}\,d\xi = -\frac{1}{2}.$$

The function

$$x \mapsto \frac{2i}{\pi(x+i)^3}$$

is a suitable example (of φ).

To prove Theorem 4.1.1, we first need to determine the continuous wavelet transform of $R_{\alpha,\beta}$ related to the Lusin wavelet ψ_L given in Expression (3.5), as in [55, 61, 69] where the case $\alpha = \beta = 2$ is treated.

Proposition 4.1.2. *We have*

$$\mathcal{W}_{\psi_L} R_{\alpha,\beta}(a,b) = ia\pi\sum_{n=1}^{+\infty}\frac{e^{i\pi n^\beta(b+ia)}}{n^{\alpha-\beta}} \tag{4.2}$$

for all $a > 0$ and $b \in \mathbb{R}$.

Proof. We can write

$$R_{\alpha,\beta}(x) = \frac{1}{2}\left(T_{\alpha,\beta}(x) - \widetilde{T}_{\alpha,\beta}(x)\right)$$

for $x \in \mathbb{R}$ with

$$T_{\alpha,\beta}(x) := -i\sum_{n=1}^{+\infty}\frac{e^{i\pi n^\beta x}}{n^\alpha} \qquad \text{and} \qquad \widetilde{T}_{\alpha,\beta}(x) := T_{\alpha,\beta}(-x).$$

In other words, $R_{\alpha,\beta}$ is the odd part of $T_{\alpha,\beta}$.

Let us fix $a > 0$ and $b \in \mathbb{R}$. We have

$$\mathcal{W}_{\psi_L}T_{\alpha,\beta}(a,b) = \int_{\mathbb{R}} T_{\alpha,\beta}(x)\,\frac{1}{a}\overline{\psi}_L\left(\frac{x-b}{a}\right)dx = \frac{a}{\pi}\int_{\mathbb{R}}\frac{T_{\alpha,\beta}(x)}{(x-(b+ia))^2}\,dx.$$

For $\eta > 0$ and $r > 0$, let us denote by $\gamma_{\eta,r}$ the closed path formed by the juxtaposition of the two following ones: the first path describes the segment $[-r+i\eta, r+i\eta]$ and the second one the half-circle of center $i\eta$ and radius r included in $H := \{z \in \mathbb{C} : \Im z > 0\}$. The function $T_{\alpha,\beta}$ is holomorphic on H because the series converges uniformly on every compact set of H. As the point $b + ia$ is situated inside the curve described by $\gamma_{\eta,r}$ for $\eta \in (0,a)$ and $r > a$, we obtain

$$\begin{aligned}
\mathcal{W}_{\psi_L}T_{\alpha,\beta}(a,b) &= \frac{a}{\pi}\lim_{r\to+\infty}\lim_{\eta\to 0^+}\int_{\gamma_{\eta,r}}\frac{T_{\alpha,\beta}(z)}{(z-(b+ia))^2}\,dz\\
&= 2ia\,(DT_{\alpha,\beta})(b+ia)\\
&= 2ia\pi\sum_{n=1}^{+\infty}\frac{e^{i\pi n^\beta(b+ia)}}{n^{\alpha-\beta}},
\end{aligned}$$

thanks to Cauchy's integral formula. Similarly, the continuous wavelet transform of $\widetilde{T}_{\alpha,\beta}$ is given by

$$\mathcal{W}_{\psi_L}\widetilde{T}_{\alpha,\beta}(a,b) = \int_{\mathbb{R}} T_{\alpha,\beta}(-x)\,\frac{1}{a}\overline{\psi}_L\left(\frac{x-b}{a}\right)dx = \frac{a}{\pi}\lim_{r\to+\infty}\lim_{\eta\to 0^+}\int_{\gamma_{\eta,r}}\frac{T_{\alpha,\beta}(z)}{(z-(-b-ia))^2}\,dz = 0$$

by homotopy invariance, because the point $-b-ia$ does not belong to H. We thus have the conclusion. ■

Let us now analyse $\mathcal{W}_{\psi_L}R_{\alpha,\beta}$ in order to study the uniform Hölder continuity of $R_{\alpha,\beta}$ with Theorem 3.2.1. We have

$$|\mathcal{W}_{\psi_L}R_{\alpha,\beta}(a,b)| \le a\pi\sum_{n=1}^{+\infty}\frac{e^{-a\pi n^\beta}}{n^{\alpha-\beta}} = |\mathcal{W}_{\psi_L}R_{\alpha,\beta}(a,0)| \tag{4.3}$$

for $a > 0$ and $b \in \mathbb{R}$. The function $f_{\alpha,\beta} : x \mapsto x^{\beta-\alpha}\,e^{-a\pi x^\beta}$ is differentiable on $(0,+\infty)$ and

$$Df_{\alpha,\beta}(x) = e^{-a\pi x^\beta}\,x^{\beta-\alpha-1}\left((\beta-\alpha) - a\pi\beta x^\beta\right), \quad x > 0.$$

Then, $f_{\alpha,\beta}$ is decreasing on $(0,+\infty)$ if $\beta \in [\alpha-1,\alpha)$ and on $(((\beta-\alpha)/a\pi\beta)^{1/\beta}, +\infty)$ if $\beta \ge \alpha$. The next developments are mainly based on the classical comparison principle between series and integral (when the general term is decreasing).

We note that $f_{\alpha,\beta}$ is integrable on $(0,+\infty)$ only if $\beta > \alpha - 1$. We therefore split the study of the uniform Hölder continuity and the calculus of the uniform Hölder exponent of $R_{\alpha,\beta}$ into two cases: $\beta > \alpha - 1$ and $\beta = \alpha - 1$.

Proposition 4.1.3. *If $\beta > \alpha - 1$, then $R_{\alpha,\beta} \in \Lambda^{\frac{\alpha-1}{\beta}}(\mathbb{R})$ and*

$$H_{R_{\alpha,\beta}}(\mathbb{R}) = \frac{\alpha - 1}{\beta}.$$

Proof. 1. Let us first consider the case $\beta \in (\alpha - 1, \alpha)$. The function $f_{\alpha,\beta}$ is decreasing on $[1, +\infty)$ and we have

$$|\mathcal{W}_{\psi_L} R_{\alpha,\beta}(a, b)| \leq a\pi \left(e^{-a\pi} + \sum_{n=2}^{+\infty} \frac{e^{-a\pi n^\beta}}{n^{\alpha-\beta}} \right) \leq a\pi \left(e^{-a\pi} + \int_1^{+\infty} \frac{e^{-a\pi x^\beta}}{x^{\alpha-\beta}} \, dx \right)$$

for $a > 0$ and $b \in \mathbb{R}$. For the second term of the right hand side of the last inequality, we obtain

$$\int_1^{+\infty} \frac{e^{-a\pi x^\beta}}{x^{\alpha-\beta}} \, dx \leq \int_0^{+\infty} \frac{e^{-a\pi x^\beta}}{x^{\alpha-\beta}} \, dx = \frac{1}{\beta} \pi^{\frac{\alpha-1}{\beta}-1} \, \Gamma\left(\frac{1 + \beta - \alpha}{\beta} \right) a^{\frac{\alpha-1}{\beta}-1}$$

for $a > 0$, where Γ is defined by

$$\Gamma(x) := \int_0^{+\infty} e^{-t} \, t^{x-1} \, dt, \quad x > 0,$$

as usual. For the first term, we note that the function $a \mapsto e^{-a\pi} a^{1-\frac{\alpha-1}{\beta}}$ is bounded on $(0, +\infty)$ because $\alpha - 1 < \beta$. Then, there exists $C_{\alpha,\beta} > 0$ such that

$$|\mathcal{W}_{\psi_L} R_{\alpha,\beta}(a, b)| \leq C_{\alpha,\beta} \, a^{\frac{\alpha-1}{\beta}}$$

for all $a > 0$ and $b \in \mathbb{R}$, which implies $R_{\alpha,\beta} \in \Lambda^{\frac{\alpha-1}{\beta}}(\mathbb{R})$ using Theorem 3.2.1.

Let us show the optimality of this exponent $(\alpha-1)/\beta$ related to the uniform Hölder continuity. Let $C > 0$ and $\eta > 0$; we have

$$|\mathcal{W}_{\psi_L} R_{\alpha,\beta}(a, 0)| = a\pi \sum_{n=1}^{+\infty} \frac{e^{-\pi n^\beta a}}{n^{\alpha-\beta}} \geq a\pi \int_1^{+\infty} \frac{e^{-a\pi x^\beta}}{x^{\alpha-\beta}} \, dx = \frac{1}{\beta} \, (a\pi)^{\frac{\alpha-1}{\beta}} \, \Gamma\left(\frac{\beta - \alpha + 1}{\beta}, a\pi \right)$$

for $a > 0$, where Γ is the incomplete Gamma function defined by

$$\Gamma(x, y) := \int_y^{+\infty} e^{-t} t^{x-1} \, dt, \quad (x, y) \in (0, +\infty) \times [0, +\infty).$$

Since $\Gamma((\beta - \alpha + 1)/\beta, a\pi) \to \Gamma((\beta - \alpha + 1)/\beta)$ and $a^\eta \to 0$ as $a \to 0^+$, there exists $A > 0$ such that, for all $a \in (0, A)$, we have

$$|\mathcal{W}_{\psi_L} R_{\alpha,\beta}(a, 0)| > C \, a^{\frac{\alpha-1}{\beta}+\eta}.$$

Hence the conclusion, using Theorem 3.2.1.

2. Let us now consider the case $\beta \geq \alpha$ and let us write $N_a := \lfloor ((\beta - \alpha)/a\pi\beta)^{1/\beta} \rfloor + 1$. If $a > 1$, then $N_a = 1$ and we can proceed as in the previous case. Let us therefore suppose that

$a \in (0,1]$. We have

$$\begin{aligned}
|\mathcal{W}_{\psi_L} R_{\alpha,\beta}(a,b)| &\leq a\pi \left(\sum_{n=1}^{N_a} \frac{e^{-a\pi n^\beta}}{n^{\alpha-\beta}} + \sum_{n=N_a+1}^{+\infty} \frac{e^{-a\pi n^\beta}}{n^{\alpha-\beta}} \right) \\
&\leq a\pi \left(N_a\, N_a^{\beta-\alpha} + \int_{N_a}^{+\infty} \frac{e^{-a\pi x^\beta}}{x^{\alpha-\beta}}\, dx \right) \\
&\leq a\pi \left(\left(\left(\frac{\beta-\alpha}{\pi\beta} \right)^{\frac{1}{\beta}} + a^{\frac{1}{\beta}} \right)^{\beta-\alpha+1} a^{\frac{\alpha-1}{\beta}-1} + \int_0^{+\infty} \frac{e^{-a\pi x^\beta}}{x^{\alpha-\beta}}\, dx \right) \\
&\leq a^{\frac{\alpha-1}{\beta}} \pi \left(\left(\left(\frac{\beta-\alpha}{\pi\beta} \right)^{\frac{1}{\beta}} + 1 \right)^{\beta-\alpha+1} + \frac{1}{\beta} \pi^{\frac{\alpha-1}{\beta}-1} \Gamma\left(\frac{1+\beta-\alpha}{\beta} \right) \right).
\end{aligned}$$

We then have $R_{\alpha,\beta} \in \Lambda^{\frac{\alpha-1}{\beta}}(\mathbb{R})$, using Theorem 3.2.1.

Let us show the optimality of the exponent related to the uniform Hölder continuity. Let $C > 0$ and $\eta > 0$; we have

$$\begin{aligned}
|\,\mathcal{W}_{\psi_L}\, R_{\alpha,\beta}(a,0)| &= a\pi \sum_{n=1}^{+\infty} \frac{e^{-\pi n^\beta a}}{n^{\alpha-\beta}} \\
&\geq a\pi \sum_{n=N_a}^{+\infty} \frac{e^{-\pi n^\beta a}}{n^{\alpha-\beta}} \\
&\geq a\pi \int_{N_a}^{+\infty} \frac{e^{-a\pi x^\beta}}{x^{\alpha-\beta}}\, dx \\
&= \frac{1}{\beta} (a\pi)^{\frac{\alpha-1}{\beta}} \int_{a\pi N_a^\beta}^{+\infty} e^{-u}\, u^{\frac{\beta-\alpha+1}{\beta}-1}\, du \\
&\geq \frac{1}{\beta} (a\pi)^{\frac{\alpha-1}{\beta}}\, \Gamma\left(\frac{\beta-\alpha+1}{\beta}, \left(\left(\frac{\beta-\alpha}{\beta} \right)^{1/\beta} + (a\pi)^{1/\beta} \right)^\beta \right)
\end{aligned}$$

for $a > 0$. As in the case $\beta \in (\alpha-1, \alpha)$, there exists $A > 0$ such that, for all $a \in (0, A)$, we have

$$|\mathcal{W}_{\psi_L} R_{\alpha,\beta}(a,0)| > C\, a^{\frac{\alpha-1}{\beta}+\eta},$$

hence the conclusion, using once again Theorem 3.2.1. ■

Remark 4.1.4. In fact, taking $b = 2k$ with $k \in \mathbb{Z}$, we can show that $R_{\alpha,\beta} \in \Lambda^{\frac{\alpha-1}{\beta}}(2k)$ and that the exponent cannot be improved because $\mathcal{W}_{\psi_L} R_{\alpha,\beta}(a, 2k) = \mathcal{W}_{\psi_L} R_{\alpha,\beta}(a, 0)$ for all $a > 0$. In other words, we have

$$H_{R_{\alpha,\beta}}(2k) = \frac{\alpha-1}{\beta}.$$

Since this quantity is strictly smaller than 1, $R_{\alpha,\beta}$ is consequently not differentiable at $2k$.

Proposition 4.1.5. *We have $R_{\alpha,\alpha-1} \in \Lambda^{1-\delta}(\mathbb{R})$ for all $\delta \in (0,1)$ and $H_{R_{\alpha,\alpha-1}}(\mathbb{R}) = 1$.*

Proof. We have

$$|\mathcal{W}_{\psi_L} R_{\alpha,\alpha-1}(a,b)| \leq a\pi \left(e^{-a\pi} + \int_1^{+\infty} \frac{e^{-a\pi x^{\alpha-1}}}{x}\, dx \right) = a\pi \left(e^{-a\pi} + \frac{1}{\alpha-1} E_1(a\pi) \right)$$

for $a>0$ and $b\in\mathbb{R}$, where E_1 is the exponential integral defined by

$$E_1(x) := \int_1^{+\infty} \frac{e^{-xt}}{t}\, dt, \quad x>0.$$

Since we have

$$\frac{1}{2}\, e^{-x} \ln\left(1+\frac{2}{x}\right) < E_1(x) < e^{-x} \ln\left(1+\frac{1}{x}\right) \tag{4.4}$$

for all $x>0$ (see [1] page 229), we obtain

$$|\mathcal{W}_{\psi_L} R_{\alpha,\alpha-1}(a,b)| \le a\pi\, e^{-a\pi}\left(1+\frac{1}{\alpha-1}\ln\left(1+\frac{1}{a\pi}\right)\right)$$

for $a>0$ and $b\in\mathbb{R}$. Let us fix $\delta\in(0,1)$. There exists $A>0$ such that, for all $a\in(0,A)$, we have

$$\frac{1}{\alpha-1}\,\frac{\ln\left(1+\frac{1}{a\pi}\right)}{\left(1+\frac{1}{a\pi}\right)^{\delta}} < 1$$

and then

$$|\mathcal{W}_{\psi_L} R_{\alpha,\alpha-1}(a,b)| \le a\pi\, e^{-a\pi}\left(1+\left(1+\frac{1}{a\pi}\right)^{\delta}\right) \le a\pi\left(1+2^{\delta}\left(1+\left(\frac{1}{a\pi}\right)^{\delta}\right)\right).$$

There also exists $A'\in(0,A)$ such that, for all $a\in(0,A')$, we have

$$|\mathcal{W}_{\psi_L} R_{\alpha,\alpha-1}(a,b)| \le C'_\delta a^{1-\delta},$$

where C'_δ is a positive constant (depending only on δ). Since the function

$$a\mapsto a^{\delta} e^{-a\pi}\left(1+\frac{1}{\alpha-1}\ln\left(1+\frac{1}{a\pi}\right)\right)$$

is bounded on $[A',+\infty)$, we also have

$$|\mathcal{W}_{\psi_L} R_{\alpha,\alpha-1}(a,b)| \le C''_\delta a^{1-\delta}$$

for $a\in[A',+\infty)$, where C''_δ is another positive constant. We thus obtain

$$|\mathcal{W}_{\psi_L} R_{\alpha,\alpha-1}(a,b)| \le C_\delta\, a^{1-\delta}$$

for all $a>0$ and $b\in\mathbb{R}$ where $C_\delta := \max\{C'_\delta, C''_\delta\}$, which implies $R_{\alpha,\alpha-1}\in\Lambda^{1-\delta}(\mathbb{R})$ using Theorem 3.2.1.

Let us now show that this exponent of uniform Hölder continuity is optimal. Let $C>0$; we have

$$|\mathcal{W}_{\psi_L} R_{\alpha,\alpha-1}(a,0)| \ge a\pi \int_1^{+\infty} \frac{e^{-a\pi x^{\alpha-1}}}{x}\, dx = \frac{a\pi}{\alpha-1} E_1(a\pi) \ge a\,\frac{\pi}{2(\alpha-1)}\, e^{-a\pi}\ln\left(1+\frac{2}{a\pi}\right)$$

for all $a>0$ thanks to Inequality (4.4). There so exists $A>0$ such that, for all $a\in(0,A)$, we have

$$|\mathcal{W}_{\psi_L} R_{\alpha,\alpha-1}(a,0)| > Ca,$$

hence the conclusion using one last time Theorem 3.2.1.

■

4.2 Extension to Nonharmonic Fourier Series

A part of Theorem 4.1.1 can be adapted for particular nonharmonic Fourier series. Let us first recall the notion of nonharmonic Fourier series (see [66, 86, 122]).

Definition 4.2.1. Let $\boldsymbol{a} := (a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers and let $\boldsymbol{\lambda} := (\lambda_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers which converges to infinity. A *nonharmonic Fourier series* (related to the sequences $\boldsymbol{a}$ and $\boldsymbol{\lambda}$) is a function $S_{\boldsymbol{a},\boldsymbol{\lambda}}$ defined by

$$S_{\boldsymbol{a},\boldsymbol{\lambda}}(x) := \sum_{n=1}^{+\infty} a_n\, e^{i\lambda_n x}, \quad x \in \mathbb{R},$$

if the series converges.

If the series $\sum_{n=1}^{+\infty} a_n$ is absolutely convergent, then the above series (related to $S_{\boldsymbol{a},\boldsymbol{\lambda}}$) converges uniformly on $\mathbb{R}$. We will assume that this is the case in what follows. Such a function $S_{\boldsymbol{a},\boldsymbol{\lambda}}$ is then continuous and bounded on $\mathbb{R}$. As for $R_{\alpha,\beta}$, we can calculate the continuous wavelet transform of $S_{\boldsymbol{a},\boldsymbol{\lambda}}$ (related to the Lusin wavelet).

Since $\lambda_n > 0$ for all $n \in \mathbb{N}$, $S_{\boldsymbol{a},\boldsymbol{\lambda}}$ is a holomorphic function on H and we have

$$\mathcal{W}_{\psi_L} S_{\boldsymbol{a},\boldsymbol{\lambda}}(a,b) = -2a \sum_{n=1}^{+\infty} a_n \lambda_n\, e^{i\lambda_n(b+ia)}$$

for $a > 0$ and $b \in \mathbb{R}$, similarly to Equality (4.2). If we assume that there exist $C_1, C_2, C_3 > 0$, $\alpha > 1$ and $\beta > 0$ such that

$$|a_n| \leq \frac{C_1}{n^\alpha} \quad \text{and} \quad C_2 n^\beta \leq \lambda_n \leq C_3 n^\beta$$

for all $n \in \mathbb{N}$, we then obtain

$$|\mathcal{W}_{\psi_L} S_{\boldsymbol{a},\boldsymbol{\lambda}}(a,b)| \leq 2aC_1C_3 \sum_{n=1}^{+\infty} \frac{e^{-C_2 a n^\beta}}{n^{\alpha-\beta}}$$

for $a > 0$ and $b \in \mathbb{R}$, i.e. an expression similar to the one obtained for $|\mathcal{W}_{\psi_L} R_{\alpha,\beta}(a,b)|$ in Expression (4.3). Using the same development as in the study of the uniform Hölder continuity of $R_{\alpha,\beta}$ with $\alpha > 1$ and $\beta \geq \alpha - 1$, we get the following corollary.

Corollary 4.2.2. *With the previous assumptions on $\boldsymbol{a}$ and $\boldsymbol{\lambda}$, we have $S_{\boldsymbol{a},\boldsymbol{\lambda}} \in \Lambda^{\frac{\alpha-1}{\beta}}(\mathbb{R})$ if $\beta > \alpha - 1$ and $S_{\boldsymbol{a},\boldsymbol{\lambda}} \in \Lambda^{1-\delta}(\mathbb{R})$ for all $\delta \in (0,1)$ if $\beta = \alpha - 1$.*

For example, we obtain the uniform Hölder continuity of the function $S_{\boldsymbol{a},\boldsymbol{\lambda}}$ with $\lambda_n = n^3 + n^2$ and $a_n = n^{-\alpha}$ for $n \in \mathbb{N}$ where $\alpha \in (1,4)$ for example. By the previous corollary, we have $S_{\boldsymbol{a},\boldsymbol{\lambda}} \in \Lambda^{\frac{\alpha-1}{3}}(\mathbb{R})$ since $n^3 \leq \lambda_n \leq 3n^3$ for $n \in \mathbb{N}$. In fact, this example is a part of another generalisation of Riemann function (see [29]).

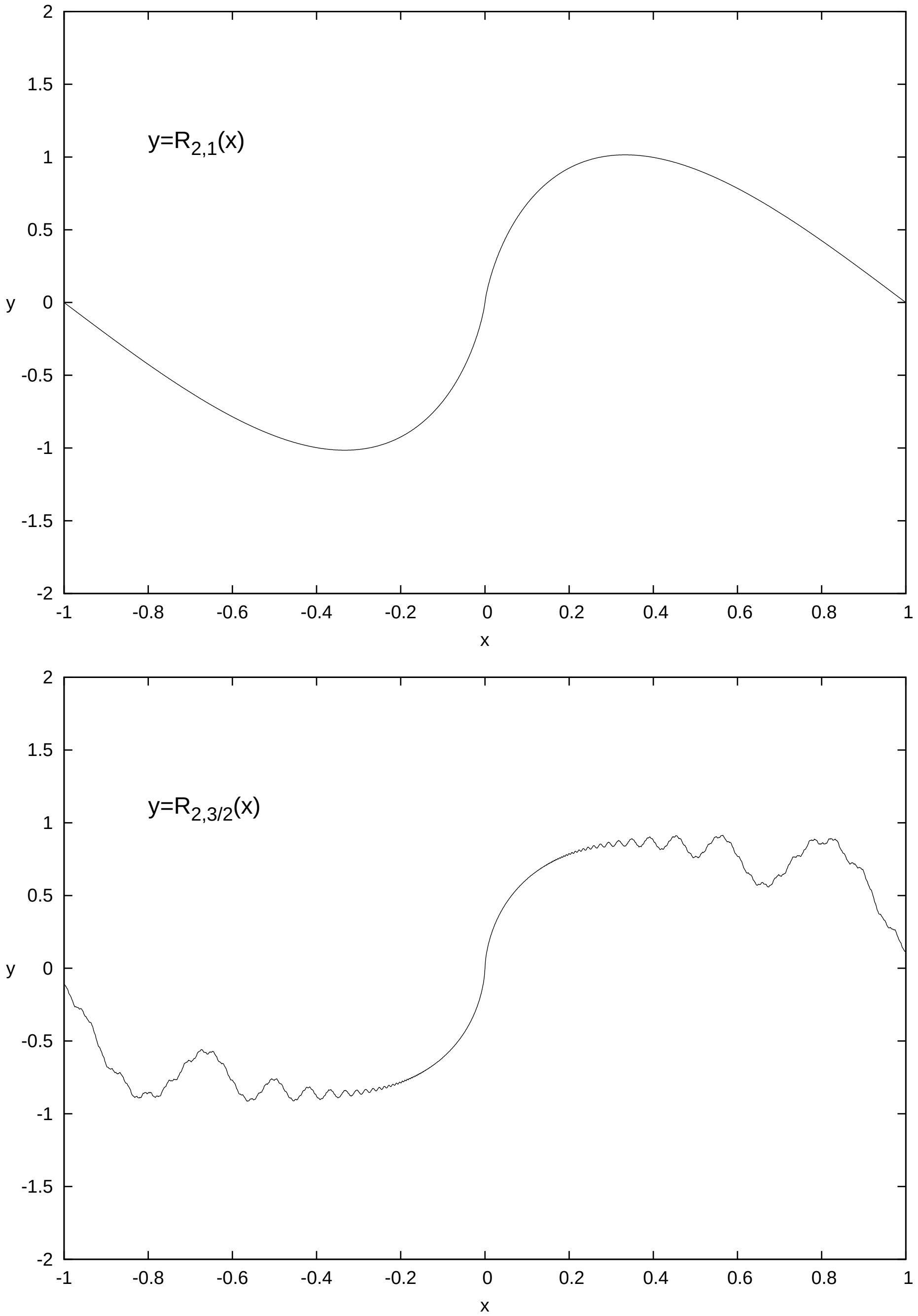

Figure 4.1. Graphical representations of $R_{2,1}$ and $R_{2,3/2}$.

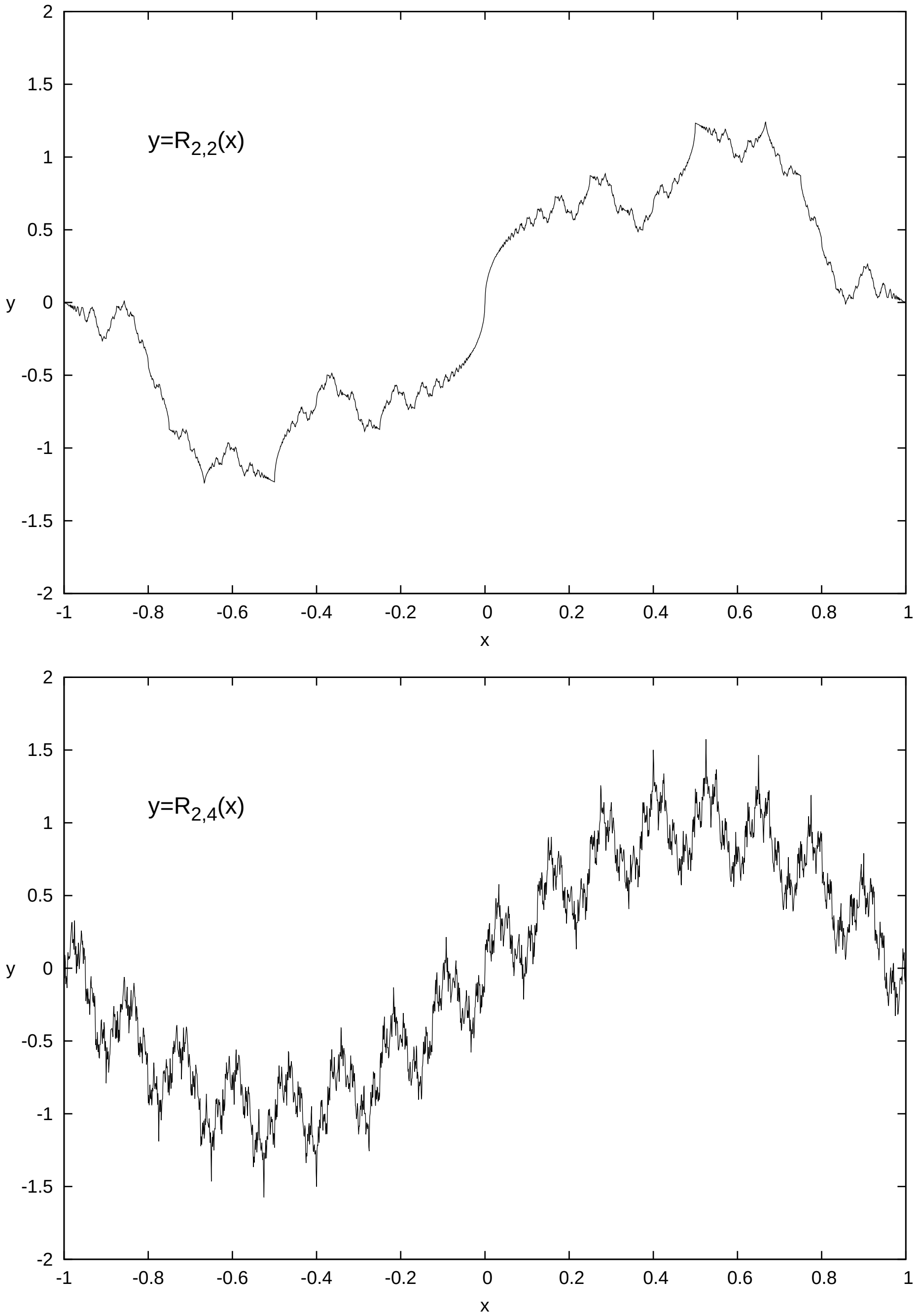

Figure 4.2. Graphical representations of $R_{2,2}$ and $R_{2,4}$.

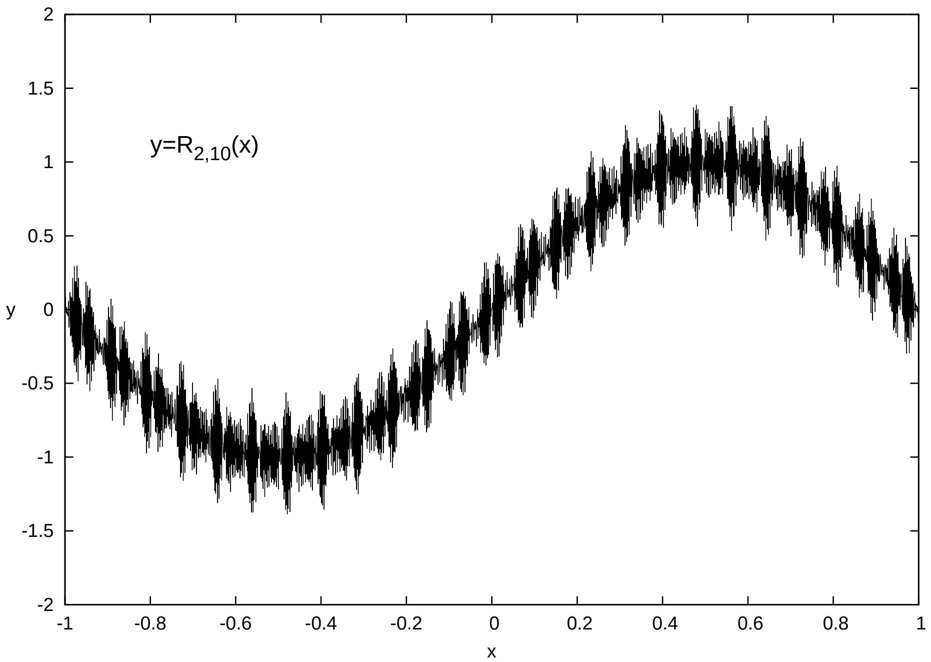

Figure 4.3. Graphical representation of $R_{2,10}$.

4.3 Behaviour of $R_{\alpha,\beta}$ as β Increases

If we fix $\alpha > 1$, we know that the uniform Hölder exponent of $R_{\alpha,\beta}$ decreases as β increases, thanks to Theorem 4.1.1. Moreover, we know that this exponent is exactly the Hölder exponent of $R_{\alpha,\beta}$ at the origin. This phenomenon is clearly illustrated in Figure 4.1, Figure 4.2 and Figure 4.3 in the case $\alpha = 2$.

As β tends to infinity, we note that the graphical representation of $R_{\alpha,\beta}$ looks like to the one of the function $s : x \mapsto \sin(\pi x)$ (in a certain sense to establish), with some noise, fluctuations or oscillations all around. In fact, s is simply the first term of the series defining $R_{\alpha,\beta}$. In the next two propositions, we give a convergence result and show that the fluctuations have a constant amplitude (i.e. independent of β). To do so, let us recall the usual definition of the mean of an integrable function over a bounded interval.

Definition 4.3.1. Let $a, b \in \mathbb{R}$ be such that $a < b$ and let f be an integrable function on (a, b). The *mean of the function* f *over the inverval* (a, b) is defined by

$$m_f^{a,b} := \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proposition 4.3.2. *Let* $\alpha > 1$. *For all* $a, b \in \mathbb{R}$ *such that* $a < b$, *we have*

$$\lim_{\beta\to+\infty} m_{R_{\alpha,\beta}}^{a,b} = m_s^{a,b}.$$

Proof. We have

$$\left|\int_a^b (R_{\alpha,\beta}(x) - \sin(\pi x))\,dx\right| = \left|\sum_{n=2}^{+\infty} \frac{\cos(\pi n^\beta a) - \cos(\pi n^\beta b)}{\pi n^{\alpha+\beta}}\right| \leq \frac{2}{\pi}(\zeta(\alpha+\beta) - 1)$$

and we know that $\zeta(x) \to 1$ as $x \to +\infty$, hence the conclusion. ∎

Proposition 4.3.3. *Let $\alpha > 1$ and let $\beta \in \mathbb{N}$. The function $R_{\alpha,\beta}$ is periodic of period 2 and we have*

$$\int_{-1}^{1} \left(R_{\alpha,\beta}(x) - \sin(\pi x)\right)^2 \, dx = \zeta(2\alpha) - 1.$$

Proof. The periodicity of $R_{\alpha,\beta}$ is easy to check. Let us calculate the integral. By developing $x \mapsto R_{\alpha,\beta}(x) - \sin(\pi x)$ in Fourier series, we have

$$R_{\alpha,\beta}(x) - \sin(\pi x) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} \left(a_m \cos(\pi m x) + b_m \sin(\pi m x)\right)$$

in $L^2([-1,1])$ where $a_0 = a_m = 0$ and

$$\begin{aligned} b_m &= 2\int_0^1 \left(R_{\alpha,\beta}(x) - \sin(\pi x)\right) \sin(\pi m x) \, dx \\ &= \sum_{n=2}^{+\infty} \frac{1}{n^\alpha} \int_0^1 \Big(\cos(x\pi(n^\beta - m)) - \cos(x\pi(n^\beta + m))\Big) \, dx \\ &= \begin{cases} \dfrac{1}{m^{\alpha/\beta}} & \text{if } m = k^\beta \text{ for one } k \in \mathbb{N}\setminus\{1\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for all $m \in \mathbb{N}$. Consequently, by Parseval formula, we obtain

$$\int_{-1}^{1} \left(R_{\alpha,\beta}(x) - \sin(\pi x)\right)^2 \, dx = \sum_{m=1}^{+\infty} b_m^2 = \sum_{k=2}^{+\infty} \frac{1}{k^{2\alpha}} = \zeta(2\alpha) - 1,$$

as expected. ■

The two previous propositions are illustrated in Figure 4.4. Let us end this section with a simple remark about the behaviour of $R_{\alpha,\beta}$ as α tends to infinity.

Remark 4.3.4. Proposition 4.3.2 is also "satisfied" for α: we have

$$\lim_{\alpha \to +\infty} m^{a,b}_{R_{\alpha,\beta}} = m^{a,b}_s$$

for all $\beta > 0$ and all $a, b \in \mathbb{R}$ such that $a < b$. Moreover, by Proposition 4.3.3, we have

$$\lim_{\alpha \to +\infty} \int_{-1}^{1} \left(R_{\alpha,\beta}(x) - \sin(\pi x)\right)^2 \, dx = 0$$

for all $\beta \in \mathbb{N}$. In fact, a stronger result holds: for any fixed $\beta > 0$, $R_{\alpha,\beta}$ converges uniformly on $\mathbb{R}$ to s as α tends to infinity because we have

$$|R_{\alpha,\beta}(x) - \sin(\pi x)| \leq \sum_{n=2}^{+\infty} \frac{1}{n^\alpha} = \zeta(\alpha) - 1$$

for all $x \in \mathbb{R}$.

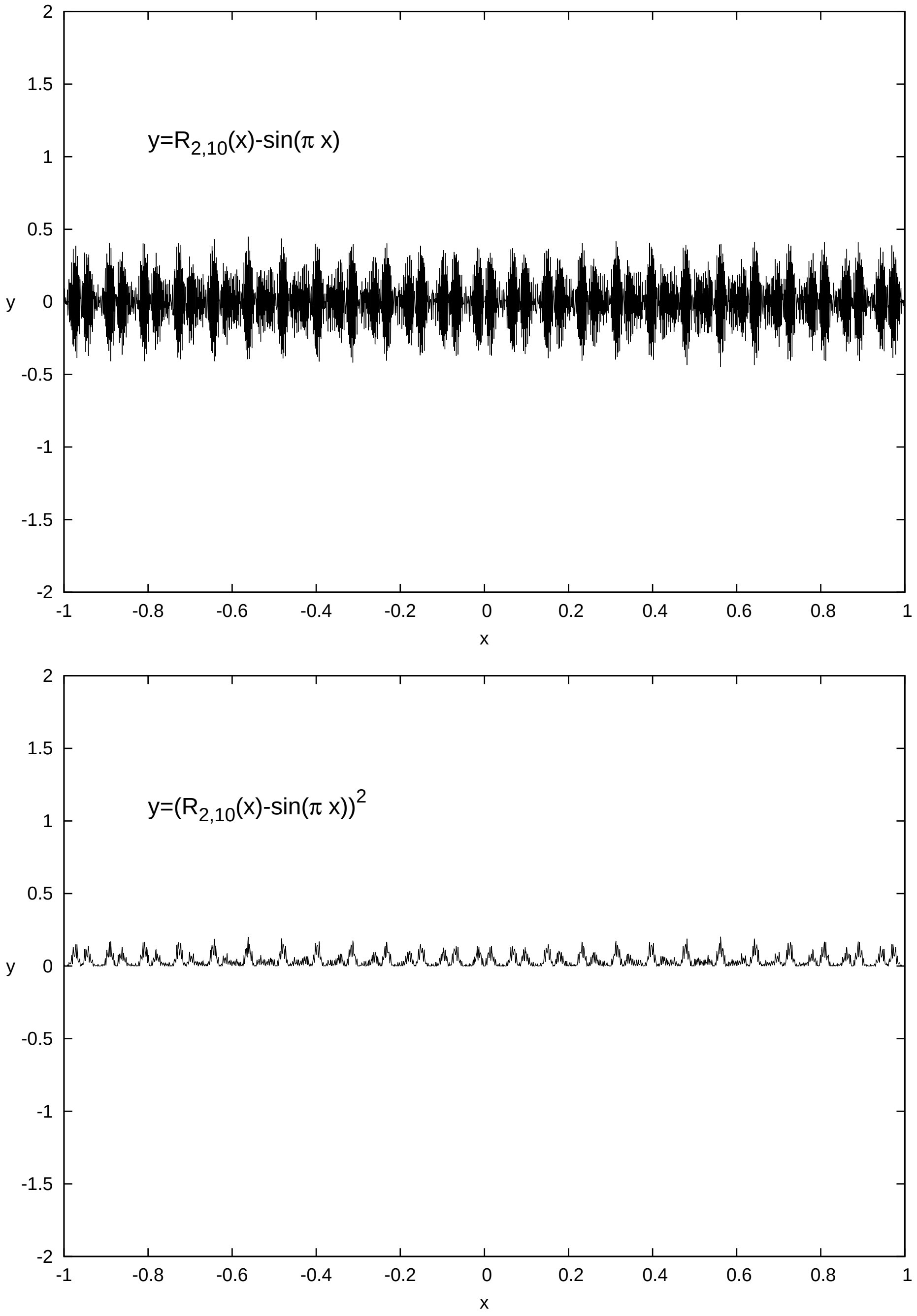

Figure 4.4. Mean value and amplitude of fluctuations of $x \mapsto R_{2,10}(x) - \sin(\pi x)$

Part II

Nonstationary Wavelets

Chapter 5

Nonstationary Orthonormal Basis of Wavelets

The classical theory of wavelets in $L^2(\mathbb{R})$ is now a well known topic and tool in various contexts (functional analysis, signal analysis, multifractal analysis,...). Typically, an orthonormal basis of wavelets of $L^2(\mathbb{R})$ is an orthonormal basis of $L^2(\mathbb{R})$ of type

$$2^{j/2}\psi(2^j\cdot -k),\quad j,k\in\mathbb{Z},$$

where the square integrable function ψ is called the "mother wavelet". Many examples are known and the usual method to obtain such bases consists to use a standard procedure (see [33,87,92]) starting from a multiresolution analysis (or a scaling function). The question arising naturally is whether every orthonormal basis of wavelets can always be obtained from a multiresolution analysis with such a procedure. The answer is negative (see the example given by Journé in [52,87]) and necessary and sufficient conditions have been proposed by several authors in [11, 47,52,81,82].

In several contexts, to answer precise problems which can not be solved in the standard setting, a generalization of the classical definition of multiresolution analysis and orthonormal basis of wavelets have been proposed (see [15,16,35,119]). This new point of view is concerned with the introduction of a "nonstationary" situation, in the sense that the mother wavelet is now admitted to depend on the scale j. The proposed definition (either in the $L^2(\mathbb{R})$ case, see [35], or in the Sobolev case, see [16]) is the following: a nonstationary orthonormal basis of wavelets of $L^2(\mathbb{R})$ is an orthonormal basis of $L^2(\mathbb{R})$ of type

$$2^{j/2}\psi^{(j)}(2^j\cdot -k),\quad j,k\in\mathbb{Z},$$

where the square integrable functions $\psi^{(j)}$, $j\in\mathbb{Z}$, are again called the "mother wavelets". Several explicit examples are known, even in the more general case of biorthogonal wavelets, and all of them have been constructed from a nonstationary multiresolution analysis in a very similar way to the stationary case (see [12,15,16,35,41,119]). More precisely, the papers [12,15,16,35,119] involve Exponential-Splines while the paper [41] is concerned with Splines. On the one hand, the paper [41] shows that it is possible to construct an infinitely differentiable orthonormal basis of wavelets with compact support in a nonstationary setting (it is known that this is not possible in the stationary case). On the other hand, some of the constructions of [15,16,119] lead to functions of the same type, but starting from a different point of view. Firstly, the paper [119] is concerned with $L^2(\mathbb{R})$ and with signal analysis purposes, involving reconstruction of exponential polynomials. Secondly, the papers [15,16] focus on the problem of the construction of regular orthonormal compactly supported basis of wavelets in Sobolev spaces, as a generalization of Daubechies' compactly supported wavelets.

Similarly to the stationary case, a natural question arising in the nonstationary context is whether every nonstationary orthonormal basis of wavelets can always be obtained from a multiresolution analysis, with the introduction of some natural dependence on the scale. The purpose of this chapter is to try to answer this question.

In this chapter, we first give the definition of nonstationary orthonormal basis of wavelets of $L^2(\mathbb{R})$ and a theoretical characterization of such bases. We then consider the construction of such bases from a nonstationary multiresolution analysis of $L^2(\mathbb{R})$ and we present a necessary and sufficient condition about such a building procedure (under some additional asymptotic assumption on the mother wavelets). Finally, we show the non existence of "regular" nonstationary bases of wavelets in the Hardy space $H^2(\mathbb{R})$. The results presented in this chapter are mainly from [**18**].

5.1 Nonstationary Orthonormal Basis of Wavelets

Let us first recall the notion of nonstationary basis of wavelets of $L^2(\mathbb{R})$ (see [**16**, **35**, **98**]).

Definition 5.1.1. Let $\psi^{(j)} \in L^2(\mathbb{R})$ for $j \in \mathbb{Z}$. A *nonstationary orthonormal basis of wavelets of* $L^2(\mathbb{R})$ is an orthonormal basis of $L^2(\mathbb{R})$ of type

$$2^{j/2}\psi^{(j)}(2^j \cdot -k), \quad j,k \in \mathbb{Z}\,.$$

The functions $\psi^{(j)}$, $j \in \mathbb{Z}$, are called the *mother wavelets* of this basis.

Remark 5.1.2. The mother wavelets $\psi^{(j)}$ are not wavelets in the sense of Definition 3.1.1. They are just square integrable functions.

The study of two series involving a sequence of square integrable functions allows to determine whether this sequence leads to a nonstationary orthonormal basis of wavelets. It is the object of the following "theoretical" characterization of nonstationary orthonormal bases of wavelets. This result will be useful in the following, especially for the theorem concerning the construction of a nonstationary orthonormal basis of wavelets starting from a nonstationary multiresolution analysis. The proof is inspired from the stationary case (see [**47**, **52**]) and is presented in Section 5.5.

Theorem 5.1.3. *For* $j \in \mathbb{Z}$, *let* $\psi^{(j)} \in L^2(\mathbb{R})$ *such that* $\|\psi^{(j)}\|_{L^2(\mathbb{R})} = 1$.

(a) If we have

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}^{(-j)}(2^j\xi)|^2 = 1 \tag{5.1}$$

for almost all $\xi \in \mathbb{R}$ *and*

$$t_{p,q}(\xi) := \sum_{j=0}^{+\infty} \hat{\psi}^{(p-j)}(2^j\xi)\,\overline{\hat{\psi}^{(p-j)}(2^j(\xi+2q\pi))} = 0 \tag{5.2}$$

for almost all $\xi \in \mathbb{R}$ *and for all* $p \in \mathbb{Z}$ *and* $q \in 2\,\mathbb{Z}+1$, *then* $\{2^{j/2}\psi^{(j)}(2^j \cdot -k) : j,k \in \mathbb{Z}\}$ *is an orthonormal basis of* $L^2(\mathbb{R})$.

(b) *Conversely, if* $\{2^{j/2}\psi^{(j)}(2^j\cdot -k) : j,k\in\mathbb{Z}\}$ *is an orthonormal basis of* $L^2(\mathbb{R})$ *and if there exist* $\alpha, A > 0$ *such that*

$$\int_{\mathbb{R}} (1+|\xi|)^{\alpha} |\hat{\psi}^{(-j)}(\xi)|^2 \, d\xi \leq A \tag{5.3}$$

for all $j\in\mathbb{N}$, *Equalities* (5.1) *and* (5.2) *are satisfied almost everywhere on* $\mathbb{R}$.

Contrary to the stationary case, some additional dependences on the scale j appear. Moreover, Condition (5.3) has also been added and will be called it *Additional asymptotic condition* in the following of this chapter. We mainly use it to show the integrability of some series on the scale index j of mother wavelets (see Expressions (5.13) and (5.19)). It is inspired from Condition (5.6) (see [**16**]).

Remark 5.1.4. In [**98**], at the same time, independently, the authors got the same result with the following additional condition instead of Condition (5.3): the series

$$\sum_{j=1}^{+\infty} 2^j |\hat{\psi}^{(-j)}(2^j\cdot)|^2 \tag{5.4}$$

converges in $L^1_{\mathrm{loc}}(\mathbb{R}\setminus\{0\})$. In fact, this condition is weaker than Condition (5.3) and is actually also visible in the proof of Theorem 5.1.3 (see the end of the proof of Lemma 5.5.4).

Let us already analyse the convergence of the series appearing in Theorem 5.1.3. The second series (i.e. the series $t_{p,q}$ in Equality (5.2)) converges in $L^1(\mathbb{R})$ thanks to Cauchy-Schwarz's inequality and $\|\psi^{(j)}\|_{L^2(\mathbb{R})} = 1$ for all $j\in\mathbb{Z}$. It then converges almost everywhere on $\mathbb{R}$ by Levi's theorem. By contrast, it is difficult to show that the first series converges almost everywhere because the sum is over all the integers. We wait for Section 5.5 and more precisely for Proposition 5.5.5.

5.2 Nonstationary Multiresolution Analysis

A classical method to construct a nonstationary orthonormal basis of wavelets of $L^2(\mathbb{R})$ consists to start from a nonstationary multiresolution analysis (or from scaling functions) of $L^2(\mathbb{R})$ (see [**16**, **35**, **98**, **119**]).

Definition 5.2.1. A *nonstationary multiresolution analysis of* $L^2(\mathbb{R})$ is an increasing sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R})$ such that

(a) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

(b) for $j\in\mathbb{Z}$, there exists $\varphi^{(j)}\in V_j$ such that $\{2^{j/2}\varphi^{(j)}(2^j\cdot -k) : k\in\mathbb{Z}\}$ is an orthonormal basis of V_j.

The functions $\varphi^{(j)}$, $j\in\mathbb{Z}$, are called *scaling functions*.

In fact, the second point of this definition can be weakened as follows: for $j\in\mathbb{Z}$, there exists $g^{(j)}\in V_j$ such that $\{2^{j/2}g^{(j)}(2^j\cdot -k) : k\in\mathbb{Z}\}$ is a *Riesz basis* of V_j. For fixed $j\in\mathbb{Z}$, it means that

(a) for each $f \in V_j$, there exists a unique sequence $(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$f(\cdot) = \sum_{k\in\mathbb{Z}} c_k 2^{j/2} g^{(j)}(2^j \cdot -k)$$

in $L^2(\mathbb{R})$,

(b) there exist $A_j, B_j > 0$ such that

$$A_j \sum_{k\in\mathbb{Z}} |c_k|^2 \leq \left\| \sum_{k\in\mathbb{Z}} c_k 2^{j/2} g^{(j)}(2^j \cdot -k) \right\|^2_{L^2(\mathbb{R})} \leq B_j \sum_{k\in\mathbb{Z}} |c_k|^2$$

for all sequence $(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

Thanks to Lemma below (see [16]), it then suffices to define $\varphi^{(j)}$ by

$$\hat{\varphi}^{(j)}(\xi) := \frac{\hat{g}^{(j)}(\xi)}{\sqrt{\sum_{k\in\mathbb{Z}} |\hat{g}^{(j)}(\xi + 2k\pi)|^2}}$$

for almost every $\xi \in \mathbb{R}$ and $\{2^{j/2}\varphi^{(j)}(2^j \cdot -k) : k \in \mathbb{Z}\}$ is an orthonormal basis of V_j.

Lemma 5.2.2. *Let $g \in L^2(\mathbb{R})$ and $j \in \mathbb{Z}$. The functions $2^{j/2}g(2^j \cdot -k)$, $k \in \mathbb{Z}$, are orthonormal in $L^2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = 1$$

for almost every $\xi \in \mathbb{R}$.

Without going into the details, let us give some information about the construction of a nonstationary multiresolution analysis of $L^2(\mathbb{R})$ from scaling functions (see [16]).

Proposition 5.2.3. *For $j \in \mathbb{Z}$, let $\varphi^{(j)} \in L^2(\mathbb{R})$. Let us assume that, for each $j \in \mathbb{Z}$, the functions $2^{j/2}\varphi^{(j)}(2^j \cdot -k)$, $k \in \mathbb{Z}$, are orthonormal in $L^2(\mathbb{R})$. Let us set*

$$V_j := \overline{\operatorname{span}\left\{2^{j/2}\varphi^{(j)}(2^j \cdot -k) : k \in \mathbb{Z}\right\}}, \quad j \in \mathbb{Z}\,.$$

(a) We have $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$ if and only if, for all $j \in \mathbb{Z}$, there exists a 2π-periodic and locally square integrable function $m_0^{(j+1)}$ such that

$$\hat{\varphi}^{(j)}(2\xi) = m_0^{(j+1)}(\xi)\hat{\varphi}^{(j+1)}(\xi) \tag{5.5}$$

for almost every $\xi \in \mathbb{R}$.

(b) The union of $(V_j)_{j\in\mathbb{Z}}$ is dense in $L^2(\mathbb{R})$ if and only if

$$\lim_{j\to+\infty} |\hat{\varphi}^{(j)}(2^{-j}\xi)| = 1$$

for almost every $\xi \in \mathbb{R}$.

(c) If there exist $A, \alpha > 0$ such that

$$\int_{\mathbb{R}} (1 + |\xi|)^\alpha \, |\hat{\varphi}^{(j)}(\xi)|^2 \, d\xi \leq A \tag{5.6}$$

for all $j \in -\mathbb{N}$, then the intersection of $(V_j)_{j\in\mathbb{Z}}$ is reduced to $\{0\}$.

The functions $m_0^{(j)}$, $j \in \mathbb{Z}$, are called *filters*. Let us mention that, for all $j \in \mathbb{Z}$, they satisfy the equality

$$|m_0^{(j)}(\xi)|^2 + |m_0^{(j)}(\xi + \pi)|^2 = 1 \tag{5.7}$$

for almost all $\xi \in \mathbb{R}$. Equation (5.5) is often called the *scaling equation*. Condition (5.6) is similar to Additional asymptotic condition (5.3).

The following result allows to construct a nonstationary orthonormal basis of wavelets of $L^2(\mathbb{R})$ from scaling functions (and filters).

Theorem 5.2.4. *For $j \in \mathbb{Z}$, let $\varphi^{(j)} \in L^2(\mathbb{R})$. Let us assume that the spaces*

$$V_j := \overline{\operatorname{span}\left\{2^{j/2}\varphi^{(j)}(2^j \cdot -k) : k \in \mathbb{Z}\right\}}, \quad j \in \mathbb{Z}$$

form a nonstationary multiresolution analysis of $L^2(\mathbb{R})$. For $j \in \mathbb{Z}$, let us define $\psi^{(j)} \in L^2(\mathbb{R})$ by

$$\hat{\psi}^{(j)}(\xi) = e^{-i\xi/2}\, \overline{m_0^{(j+1)}(\xi/2 + \pi)}\, \hat{\varphi}^{(j+1)}(\xi/2)$$

for almost every $\xi \in \mathbb{R}$, where $m_0^{(j+1)}$ is a filter coming from Scaling equation (5.5). Then, $\{2^{j/2}\psi^{(j)}(2^j \cdot -k) : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Under Additional asymptotic condition (5.3) (of the theoretical characterization of nonstationary orthonormal bases of wavelets), the following result gives a necessary and sufficient condition to obtain a nonstationary basis of wavelets from a nonstationary multiresolution analysis. Again, we generalize the proof of [52] to the nonstationary case, which is presented in Section 5.6.

Theorem 5.2.5. *For $j \in \mathbb{Z}$, let $\psi^{(j)} \in L^2(\mathbb{R})$. Let us assume that $\{2^{j/2}\psi^{(j)}(2^j \cdot -k) : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$.*

(a) If the mother wavelets $\psi^{(j)}$, $j \in \mathbb{Z}$, come from a nonstationary multiresolution analysis, then

$$D_j(\xi) := \sum_{n=1}^{+\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}^{(j-n)}(2^n(\xi + 2k\pi))|^2 = 1 \tag{5.8}$$

for almost all $\xi \in \mathbb{R}$ and for all $j \in \mathbb{Z}$.

(b) Conversely, if $D_j = 1$ almost everywhere on $\mathbb{R}$ for all $j \in \mathbb{Z}$ and if we assume that Additional asymptotic condition (5.3) is satisfied, then the mother wavelets $\psi^{(j)}$, $j \in \mathbb{Z}$, come from a nonstationary multiresolution analysis of $L^2(\mathbb{R})$.

For $j \in \mathbb{Z}$, the function D_j is sometimes called the *dimension function* of the mother wavelet $\psi^{(j)}$ (see [22]). For all $j \in \mathbb{Z}$, the double series in Expression (5.8) converges in $L^1([0, 2\pi])$ because $\|\psi^{(j)}\|_{L^2(\mathbb{R})} = 1$ and then almost everywhere on $\mathbb{R}$ by Levi's theorem and by periodicity.

Additional asymptotic condition (5.3) is mentioned because we use the theoretical characterization of wavelets in the second part of the proof of Theorem 5.2.5 (and more precisely in Lemma 5.6.2). In fact, it is not necessary if the nonstationary orthonormal basis of wavelets verifies Equalities (5.1) and (5.2) of Theorem 5.1.3.

In the second part of Theorem 5.2.5, it actually suffices to have $D_j > 0$ almost everywhere on $(0, 2\pi)$ for all $j \in \mathbb{Z}$. This is the purpose of the following proposition.

Proposition 5.2.6. *For $j \in \mathbb{Z}$, let $\psi^{(j)} \in L^2(\mathbb{R})$. Let us assume that $\{2^{j/2}\psi^{(j)}(2^j \cdot - k) : j,k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Let also assume that Additional asymptotic condition (5.3) is satisfied. For all $j \in \mathbb{Z}$, we have $D_j = 1$ almost everywhere on $\mathbb{R}$ if and only if we have $D_j > 0$ almost everywhere on $(0, 2\pi)$.*

We give the proof of this proposition later, since it uses some considerations of the proof of Theorem 5.2.5 (see Section 5.6).

5.3 The Example of Exponential-Splines

In this section, we illustrate the previous results with the example of the Exponential-Splines. The Exponential-Spline of parameter $\lambda \in \mathbb{C}^n$ ($n \in \mathbb{N}$) is the function N_λ defined by

$$\hat{N}_\lambda(\xi) := \prod_{\ell=1}^{n} \frac{e^{\lambda_\ell - i\xi} - 1}{\lambda_\ell - i\xi}$$

for almost every $\xi \in \mathbb{R}$ (see [35, 79]). The classical Spline corresponds to the case $\lambda = 0$. Except this particular case, the usual structure of (stationary) multiresolution analysis cannot be applied to construct a (stationary) orthonormal basis of wavelets of $L^2(\mathbb{R})$ from Exponential-Splines (because N_λ cannot be expressed in terms of its 2-dilates). The nonstationary setting allows it (see [35, 79]).

Let us consider in details the case of the Exponential-Spline $M_\mu := N_{i\mu}$ with $\mu \in \mathbb{R}\setminus\{0\}$. By definition, we clearly have

$$M_\mu(x) = e^{i\mu x}\, \chi_{[0,1]}(x), \quad x \in \mathbb{R}$$

and

$$\hat{M}_\mu(\xi) = \begin{cases} e^{i\frac{\mu-\xi}{2}} \dfrac{\sin\left(\frac{\mu-\xi}{2}\right)}{\frac{\mu-\xi}{2}} & \text{if } \xi \neq \mu \\ 1 & \text{if } \xi = \mu \end{cases}.$$

For all $j \in \mathbb{Z}$, it is easy to check that $\{2^{j/2} M_{2^{-j}\mu}(2^j \cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal family of $L^2(\mathbb{R})$. Let us set

$$V_j := \overline{\operatorname{span}\left\{2^{j/2} M_{2^{-j}\mu}(2^j \cdot - k) : k \in \mathbb{Z}\right\}}$$

for $j \in \mathbb{Z}$ and let us show that $(V_j)_{j\in\mathbb{Z}}$ is a nonstationary multiresolution analysis of $L^2(\mathbb{R})$ with Proposition 5.2.3.

(a) For all $j \in \mathbb{Z}$, we have $V_j \subset V_{j+1}$ because we have the following scaling equation:

$$\hat{M}_{2^{-j}\mu}(2\xi) = e^{\frac{i}{2}(2^{-(j+1)}\mu - \xi)} \cos\left(\frac{2^{-(j+1)}\mu - \xi}{2}\right) \hat{M}_{2^{-(j+1)}\mu}(\xi)$$

for almost every $\xi \in \mathbb{R}$.

(b) For almost every $\xi \in \mathbb{R}$, we directly have

$$\lim_{j\to+\infty} |\hat{M}_{2^{-j}\mu}(2^{-j}\xi)| = \lim_{j\to+\infty} \left| \frac{\sin(2^{-j}\frac{\mu-\xi}{2})}{2^{-j}\frac{\mu-\xi}{2}} \right| = 1$$

and the union of V_j, $j \in \mathbb{Z}$, is therefore dense in $L^2(\mathbb{R})$.

(c) Let $\alpha > 0$; the function $\xi \mapsto (1+|\xi|)^\alpha |\hat{M}_{2^{-j}\mu}(\xi)|^2$ is integrable on $\mathbb{R}$ only for $\alpha \in (0,1)$. We have

$$\int_{\mathbb{R}} (1+|\xi|)^\alpha |\hat{M}_{2^{-j}\mu}(\xi)|^2 = 2\int_{\mathbb{R}} (1+|2^{-j}\mu - 2t|)^\alpha \left|\frac{\sin(t)}{t}\right|^2 dt \tag{5.9}$$

and then, Proposition 5.2.3 does not allow to show that the intersection of V_j, $j \in \mathbb{Z}$, is reduced to $\{0\}$. However, [35] studies the dimension of the intersection of V_j, $j \in \mathbb{Z}$, and proves that it is well reduced to $\{0\}$ since μ is a real parameter.

Consequently, $(V_j)_{j\in\mathbb{Z}}$ is a nonstationary multiresolution analysis of $L^2(\mathbb{R})$. If we define the square integrable function $\psi^{(j)}$ by

$$\hat{\psi}^{(j)}(\xi) := i\, e^{-i\frac{\xi}{2}} \frac{\sin^2\left(\frac{2^{-j}\mu-\xi}{4}\right)}{\frac{2^{-j}\mu-\xi}{4}}$$

for almost every $\xi \in \mathbb{R}$ and for all $j \in \mathbb{Z}$, the family $\{2^{j/2}\psi^{(j)}(2^j\cdot - k) : j,k \in \mathbb{Z}\}$ is a nonstationary orthonormal basis of wavelets of $L^2(\mathbb{R})$ by Theorem 5.2.4.

Remark 5.3.1. The previous example shows that Condition (5.6) is only sufficient, but not necessary to have the triviality of the intersection of V_j, $j \in \mathbb{Z}$, defined in Proposition 5.2.3.

Let us now show that the nonstationary orthonormal basis of wavelets constructed from the scaling functions $M_{2^{-j}\mu}$, $j \in \mathbb{Z}$, satisfies Equalities (5.1) and (5.2) of Theorem 5.1.3. To get that, we use the following equalities.

Lemma 5.3.2. *For all $x \in \mathbb{R}$, we have*

$$\sum_{j=0}^{+\infty} \frac{\sin^4(2^j x)}{2^{2j}} = \sin^2(x) \quad \textit{and} \quad \sum_{j=1}^{+\infty} \frac{\sin^4(2^{-j}x)}{2^{-2j}} = x^2 - \sin^2(x).$$

Proof. The two series are clearly convergent. Let us first remark that

$$\sin^4(y) = \sin^2(y) - \frac{1}{4}\sin^2(2y)$$

for all $y \in \mathbb{R}$. Then, we have

$$\begin{aligned}
\sum_{j=0}^{+\infty} \frac{\sin^4(2^j x)}{2^{2j}} &= \lim_{J\to+\infty} \left(\sum_{j=0}^{J} \left(\frac{\sin(2^j x)}{2^j}\right)^2 - \sum_{j=0}^{J} \left(\frac{\sin(2^{j+1} x)}{2^{j+1}}\right)^2 \right) \\
&= \lim_{J\to+\infty} \sin^2(x) - \left(\frac{\sin(2^{J+1}x)}{2^{J+1}}\right)^2 \\
&= \sin^2(x)
\end{aligned}$$

for all $x \in \mathbb{R}$. Similarly, we also have

$$\sum_{j=1}^{+\infty} \frac{\sin^4(2^{-j}x)}{2^{-2j}} = \lim_{J\to+\infty} \left(\left(\frac{\sin(2^{-J}x)}{2^{-J}}\right)^2 - \sin^2(x) \right) = x^2 - \sin^2(x)$$

for all $x \in \mathbb{R}$. ∎

Firstly, for almost every $\xi \in \mathbb{R}$, we have

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}^{(-j)}(2^j\xi)|^2 = \sum_{j\in\mathbb{Z}} \frac{\sin^4\left(\frac{2^j\mu-2^j\xi}{4}\right)}{\left(\frac{2^j\mu-2^j\xi}{4}\right)^2} = \frac{1}{\theta^2}\sum_{j\in\mathbb{Z}} \frac{\sin^4(2^j\theta)}{2^{2j}} = 1,$$

setting $\theta := (\mu-\xi)/4$ and using Lemma 5.3.2. Secondly, for all $p \in \mathbb{Z}$ and $q \in 2\mathbb{Z}+1$ and for almost every $\xi \in \mathbb{R}$, we have

$$\begin{aligned} t_{p,q}(\xi) &= \sum_{j=0}^{+\infty} e^{i2^jq\pi} \frac{\sin^2\left(\frac{2^{j-p}-2^j\xi}{4}\right)}{\frac{2^{j-p}\mu-2^j\xi}{4}} \frac{\sin^2\left(\frac{2^{j-p}-2^j(\xi+2q\pi)}{4}\right)}{\frac{2^{j-p}\mu-2^j(\xi+2q\pi)}{4}} \\ &= \frac{1}{\theta(\theta-\frac{q\pi}{2})}\left(-\sin^2(\theta)\cos^2(\theta) + \sum_{j=1}^{+\infty}\frac{\sin^4(2^j\theta)}{2^{2j}}\right) \\ &= \frac{-\sin^2(\theta)\cos^2(\theta)+\sin^2(\theta)-\sin^4(\theta)}{\theta(\theta-\frac{q\pi}{2})} \\ &= 0 \end{aligned}$$

setting $\theta := (2^{-p}\mu-\xi)/4$ and using again Lemma 5.3.2. Consequently, thanks to Theorem 5.1.3, we have again proved that $\{2^{j/2}\psi^{(j)}(2^j\cdot -k) : j,k \in \mathbb{Z}\}$ is a nonstationary orthonormal basis of wavelets of $L^2(\mathbb{R})$.

Remark 5.3.3. The functions $\psi^{(j)}$, $j \in \mathbb{Z}$, satisfy the Equalities (5.1) and (5.2) of Theorem 5.1.3 and are the mother wavelets of a nonstationary orthonormal basis of wavelets of $L^2(\mathbb{R})$. However, they do not verify Additional asymptotic condition (5.3) by an argument similar to Expression (5.9). Moreover, they do not verify Condition (5.4) (i.e. the other condition proposed in [**98**]). Indeed, for $J \in \mathbb{N}$ and for $a, b > 0$ such that $a < |\mu| < b$, we have

$$\begin{aligned} \sum_{j=1}^{J} 2^j \int_{\mathbb{R}} |\hat{\psi}^{(-j)}(2^j\xi)|^2 \chi_{[a,b]}(|\xi|)\, d\xi &= \sum_{j=1}^{J}\int_{\mathbb{R}} \chi_{[2^ja,2^jb]}(|t|) \frac{\sin^4\left(\frac{2^j\mu-t}{4}\right)}{\left(\frac{2^j\mu-t}{4}\right)^2}\, dt \\ &= 4\sum_{j=1}^{J}\left(\int_{2^j\frac{\mu-b}{4}}^{2^j\frac{\mu-a}{4}} \frac{\sin^4(y)}{y^2}\, dy + \int_{2^j\frac{a+\mu}{4}}^{2^j\frac{b+\mu}{4}} \frac{\sin^4(y)}{y^2}\, dy\right) \end{aligned}$$

and the general term of this sum does not tend to 0 if $j \to +\infty$ since 0 belongs to one of the two domains of integration of the previous integrals. Consequently, this example shows that both Conditions (5.3) and (5.4) of Theorem 5.1.3 are only sufficient, but not necessary.

Let us end this section with the computation of the dimension D_j of the mother wavelet $\psi^{(j)}$ for all $j \in \mathbb{Z}$. We know that they verify the two equalities of Theorem 5.1.3. If we show that $D_j = 1$ almost everywhere for all $j \in \mathbb{Z}$, then the mother wavelets come from a nonstationary multiresolution analysis of $L^2(\mathbb{R})$ by Theorem 5.2.5, what we already know since we use the scaling functions $M_{2^{-j}\mu}$, $j \in \mathbb{Z}$, to construct $\psi^{(j)}$, $j \in \mathbb{Z}$. For all $j \in \mathbb{Z}$ and for almost all $\xi \in \mathbb{R}$, we have

$$\begin{aligned} D_j(\xi) &= \sum_{n=1}^{+\infty}\sum_{k\in\mathbb{Z}} |\hat{\psi}^{(j-n)}(2^n(\xi+2k\pi))|^2 \\ &= \sum_{n=1}^{+\infty}\sum_{k\in\mathbb{Z}} \frac{\sin^4\left(2^n\frac{2^{-j}\mu-\xi}{4}-2^n\frac{k\pi}{2}\right)}{\left(2^n\left(\frac{2^{-j}\mu-\xi}{4}-\frac{k\pi}{2}\right)\right)^2} \\ &= 4\sum_{n=1}^{+\infty}\frac{\sin^4(2^n\theta)}{2^{2n}}\sum_{k\in\mathbb{Z}}\frac{1}{(2\theta-k\pi)^2}, \end{aligned}$$

setting $\theta := (2^{-j}\mu-\xi)/4$. The first series is equal to $\sin^2(\theta)-\sin^4(\theta) = \sin^2(2\theta)/4$ by Lemma 5.3.2 and the second series to $1/\sin^2(2\theta)$, using the summation by residues. Thus, we obtain $D_j(\xi)=1$.

5.4 Smooth Nonstationary Orthonormal Basis of Wavelets in the Hardy Space $H^2(\mathbb{R})$

We know that there exists no "regular" orthonormal basis of wavelets in the Hardy space $H^2(\mathbb{R})$ (see [**10**, **52**]). Is there such a result in the nonstationary case? The answer is given by the following result.

Theorem 5.4.1. *There is no sequence $(\psi^{(j)})_{j\in\mathbb{Z}}$ of functions which belong to $H^2(\mathbb{R})$ such that*

(a) $|\hat{\psi}^{(j)}|$ is continuous on $\mathbb{R}$ for all $j\in\mathbb{Z}$,

(b) there exist $\alpha, A>0$ such that

$$|\hat{\psi}^{(j)}(\xi)| \leq \frac{A}{(1+\xi)^{\alpha+1/2}}$$

for all $\xi\geq 0$ and $j\in\mathbb{Z}$,

(c) there exist $\beta, B, \eta>0$ such that

$$|\hat{\psi}^{(j)}(\xi)| \leq B\,\xi^{\beta}$$

for all $\xi\in[0,\eta)$ and $j\in\mathbb{N}$,

and such that $\{2^{j/2}\psi^{(j)}(2^j\cdot -k) : j,k\in\mathbb{Z}\}$ forms an orthonormal basis of $H^2(\mathbb{R})$.

The proof of this theorem is given in Section 5.7, because it is based on some results used in the developments of the proofs of Theorem 5.1.3 and Theorem 5.2.5 (see the two following sections). The first steps are similar to the stationary case (see [**52**]).

5.5 Proof of Theorem 5.1.3

Before proving Theorem 5.1.3, let us make some observations, similarly to the stationary case in [**52**].

5.5.1 Auxiliary Results and Notations

The following proposition gives a way to check that the functions $\psi_{j,k}(\cdot) := 2^{j/2}\psi^{(j)}(2^j \cdot -k)$, $j,k \in \mathbb{Z}$, form an orthonormal basis of $L^2(\mathbb{R})$ (see for example [**33**, **52**]).

Proposition 5.5.1. *Let $\{e_j : j \in \mathbb{N}\}$ be a family of elements of a Hilbert space $\mathcal{H}$ such that $\|e_j\|_{\mathcal{H}} = 1$ for $j \in \mathbb{N}$. Then, $\{e_j : j \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{H}$ if and only if*

$$\sum_{j=1}^{+\infty} |\langle f, e_j\rangle|^2 = \|f\|^2_{\mathcal{H}} \tag{5.10}$$

for all $f \in \mathcal{H}$. Moreover, if Equality (5.10) is verified for all $f \in \mathcal{D}$ where $\mathcal{D}$ is a dense subset of $\mathcal{H}$, Equality (5.10) holds for all $f \in \mathcal{H}$.

In our case, since we assume that $\|\psi^{(j)}\|_{L^2(\mathbb{R})} = 1$ for all $j \in \mathbb{Z}$, the family $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ if and only if

$$\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 = \|f\|^2_{L^2(\mathbb{R})}$$

for all f in the dense subspace

$$\mathcal{D} := \left\{ f \in L^2(\mathbb{R}) : \hat{f} \in L^\infty(\mathbb{R}) \ \text{ and } \ \text{supp}(\hat{f}) \text{ is a compact subset of } \mathbb{R}\setminus\{0\} \right\}$$

of $L^2(\mathbb{R})$. The fact that the support of $\hat{f}$ is a compact of $\mathbb{R}\setminus\{0\}$ is used to have the convergence of some series (see Expression (5.14) in the proof of Lemma 5.5.4 where $a > 0$). The following lemma returns to the density of $\mathcal{D}$ in $L^2(\mathbb{R})$.

Lemma 5.5.2. *The set $\mathcal{D}$ is dense in $L^2(\mathbb{R})$.*

Proof. Let $f \in L^2(\mathbb{R})$ and $\varepsilon > 0$. Since $\hat{f} \in L^2(\mathbb{R})$, there exists $\rho \in \mathcal{D}(\mathbb{R})$ such that

$$\|\hat{f} - \rho\|_{L^2(\mathbb{R})} \leq \sqrt{\frac{\pi}{2}}\,\varepsilon.$$

Let us set $\rho_m := \frac{1}{2\pi}(\rho - \rho\chi_{[-\frac{1}{m},\frac{1}{m}]})$ for $m \in \mathbb{N}$. There exists $M \in \mathbb{N}$ such that

$$\|\rho\chi_{[-\frac{1}{m},\frac{1}{m}]}\|_{L^2(\mathbb{R})} \leq \sqrt{\frac{\pi}{2}}\,\varepsilon$$

for $m \geq M$. Consequently, for $m \geq M$, we obtain

$$\|f - \check{\rho}_m\|_{L^2(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}}(\|\hat{f} - \rho\|_{L^2(\mathbb{R})} + \|\rho\chi_{[-\frac{1}{m},\frac{1}{m}]}\|_{L^2(\mathbb{R})}) \leq \varepsilon.$$

Since $\check{\rho}_M \in \mathcal{D}$ by construction, we have the conclusion. ∎

Let us then calculate the quantity

$$I := \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2, \quad f \in \mathcal{D}.$$

For $f \in \mathcal{D}$, we have

$$\begin{aligned} I &= \frac{1}{(2\pi)^2} \sum_{j,k\in\mathbb{Z}} \left| \left\langle \hat{f}, \hat{\psi}_{j,k} \right\rangle \right|^2 \\ &= \frac{1}{(2\pi)^2} \sum_{j,k\in\mathbb{Z}} \left| \int_{\mathbb{R}} 2^{-j/2}\, \hat{f}(\xi)\, e^{i2^{-j}\xi k}\, \overline{\hat{\psi}^{(j)}(2^{-j}\xi)}\, d\xi \right|^2 \\ &= \frac{1}{(2\pi)^2} \sum_{j,k\in\mathbb{Z}} 2^j \left| \int_{\mathbb{R}} \hat{f}(2^j\xi)\, \overline{\hat{\psi}^{(j)}(\xi)}\, e^{ik\xi}\, d\xi \right|^2. \end{aligned}$$

For $j \in \mathbb{Z}$, let us set $F_j(\xi) := \hat{f}(2^j\xi)\overline{\hat{\psi}^{(j)}(\xi)}$ for almost every $\xi \in \mathbb{R}$. By construction, $F_j \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\operatorname{supp}(F_j)$ is a compact subset of $\mathbb{R}\setminus\{0\}$ for $j \in \mathbb{Z}$. We use the following lemma for F_j (see [30, 52] for example).

Lemma 5.5.3. *Let $F \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\operatorname{supp}(F)$ is a compact of $\mathbb{R}\setminus\{0\}$. Then, the series*

$$\sum_{k\in\mathbb{Z}} F(\cdot + 2k\pi)$$

converges almost everywhere on $\mathbb{R}$ to a $2\pi-$periodic and square integrable function Φ and we have

$$\int_{\mathbb{R}} \Phi(\xi)\, \overline{F(\xi)}\, d\xi = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} |\hat{F}(k)|^2.$$

For $j \in \mathbb{Z}$, we set $\Phi_j(\xi) := \sum_{k\in\mathbb{Z}} F_j(\xi + 2k\pi)$ for almost every $\xi \in \mathbb{R}$ as in the previous lemma. We then have

$$\begin{aligned} I &= \frac{1}{(2\pi)^2} \sum_{j,k\in\mathbb{Z}} 2^j \left| \int_{\mathbb{R}} F_j(\xi)\, e^{ik\xi}\, d\xi \right|^2 \\ &= \frac{1}{(2\pi)^2} \sum_{j,k\in\mathbb{Z}} 2^j |\hat{F}_j(k)|^2 \\ &= \frac{1}{2\pi} \sum_{j\in\mathbb{Z}} 2^j \int_{\mathbb{R}} \Phi_j(\xi)\, \overline{F_j(\xi)}\, d\xi \\ &= \frac{1}{2\pi} \sum_{j\in\mathbb{Z}} 2^j \int_{\mathbb{R}} \overline{\hat{f}(2^j\xi)}\, \hat{\psi}^{(j)}(\xi) \sum_{k\in\mathbb{Z}} \hat{f}(2^j(\xi + 2k\pi))\, \overline{\hat{\psi}^{(j)}(\xi + 2k\pi)}\, d\xi. \end{aligned}$$

Taking care of the convergence of series (see later), we get

$$I = I_0 + I_1$$

with

$$I_0 := \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \sum_{j\in\mathbb{Z}} |\hat{\psi}^{(j)}(2^{-j}\xi)|^2\, d\xi$$

and

$$I_1 := \frac{1}{2\pi} \sum_{j\in\mathbb{Z}} 2^j \int_{\mathbb{R}} \overline{\hat{f}(2^j\xi)}\, \hat{\psi}^{(j)}(\xi) \sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{f}(2^j(\xi + 2k\pi))\, \overline{\hat{\psi}^{(j)}(\xi + 2k\pi)}\, d\xi. \tag{5.11}$$

Let us look at the convergence of the series I_0 and I_1. To do that, we use the following lemma.

Lemma 5.5.4. *Under Additional asymptotic condition (5.3), the series*

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}\setminus\{0\}} 2^j\,|\hat{f}(2^j\,\cdot)|\,|\hat{f}(2^j(\cdot+2k\pi))|\,|\hat{\psi}^{(j)}(\cdot)|^2 \tag{5.12}$$

converges almost everywhere on $\mathbb{R}$ and defines an integrable function on $\mathbb{R}$ for all $f\in\mathcal{D}$.

Proof. Since $f\in\mathcal{D}$, we can assume that $\mathrm{supp}(\hat{f})\subset\{\xi\in\mathbb{R}\setminus\{0\}: a<|\xi|<b\}$ for $b>a>0$. We write $\delta:=\mathrm{diam}(\mathrm{supp}(\hat{f}))$.

(a) If $2^j\,2\pi>\delta$, then at most one of the points $2^j\xi$ or $2^j(\xi+2k\pi)$ belongs to $\mathrm{supp}(\hat{f})$ for $\xi\in\mathbb{R}\setminus\{0\}$ and $k\in\mathbb{Z}\setminus\{0\}$. Hence, in the sum on j in Expression (5.12), we only consider $j\le j_0$ where j_0 is the largest integer number which verifies $2^{j_0}2\pi\le\delta$.

(b) We have $\hat{f}(2^j(\xi+2k\pi))\neq 0$ for at most $1+\delta/2^j2\pi$ integer number k. Using the definition of j_0 and the fact that $f\in\mathcal{D}$, we have

$$\begin{aligned}2^j\sum_{k\in\mathbb{Z}\setminus\{0\}}|\hat{f}(2^j(\xi+2k\pi))| &\le 2^j\left(1+\frac{\delta}{2^j2\pi}\right)\|\hat{f}\|_{L^\infty(\mathbb{R})}\le\left(2^{j_0}+\frac{\delta}{2\pi}\right)\|\hat{f}\|_{L^\infty(\mathbb{R})}\\ &\le \frac{\delta}{\pi}\|\hat{f}\|_{L^\infty(\mathbb{R})}\end{aligned}$$

for all $j\le j_0$ and almost all $\xi\in\mathbb{R}\setminus\{0\}$.

(c) If $\hat{f}(2^j\xi)\neq 0$, then we have $2^{-j}a\le|\xi|\le 2^{-j}b$.

Hence, for almost all $\xi\in\mathbb{R}\setminus\{0\}$, we have

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}\setminus\{0\}}2^j\,|\hat{f}(2^j\xi)|\,|\hat{f}(2^j(\xi+2k\pi))|\,|\hat{\psi}^{(j)}(\xi)|^2\le\frac{\delta}{\pi}\|\hat{f}\|^2_{L^\infty(\mathbb{R})}\sum_{j=-\infty}^{j_0}\chi_{[2^{-j}a,2^{-j}b]}(|\xi|)\,|\hat{\psi}^{(j)}(\xi)|^2.$$

It only remains to show that the series

$$\sum_{j=-\infty}^{j_0}\chi_{[2^{-j}a,2^{-j}b]}(|\cdot|)\,|\hat{\psi}^{(j)}(\cdot)|^2 \tag{5.13}$$

is integrable on $\mathbb{R}$. Indeed, the sequence $(g_J)_{J\in\mathbb{N}}$ of integrable functions on $\mathbb{R}$ defined by

$$g_J(\xi):=\sum_{j=-J}^{j_0}\chi_{[2^{-j}a,2^{-j}b]}(|\xi|)\,|\hat{\psi}^{(j)}(\xi)|^2$$

for almost every $\xi\in\mathbb{R}$ is clearly increasing. Moreover, using Condition (5.3), we have

$$\begin{aligned}\int_{\mathbb{R}}g_J(\xi)\,d\xi &= \sum_{j=-J}^{j_0}\int_{\mathbb{R}}\frac{\chi_{[2^{-j}a,2^{-j}b]}(|\xi|)}{(1+|\xi|)^\alpha}(1+|\xi|)^\alpha|\hat{\psi}^{(j)}(\xi)|^2\,d\xi\\ &\le \sum_{j=-J}^{j_0}\frac{1}{(1+2^{-j}a)^\alpha}\int_{\mathbb{R}}(1+|\xi|)^\alpha|\hat{\psi}^{(j)}(\xi)|^2\,d\xi\\ &\le A\sum_{j=-\infty}^{j_0}\frac{1}{(1+2^{-j}a)^\alpha}.\end{aligned} \tag{5.14}$$

Thus, by Levi's theorem, the series in Expression (5.13) and then in Expression (5.12) are integrable on $\mathbb{R}$ and converge almost everywhere on $\mathbb{R}$. ■

Proposition 5.5.5. *Under Additional asymptotic condition (5.3),*

(a) the series I_1 is convergent,

(b) the series I converges if and only if

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}^{(j)}(2^{-j}\,\cdot)|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}\setminus\{0\}). \tag{5.15}$$

Proof. (a) Since

$$2|\hat{\psi}^{(j)}(\xi)||\hat{\psi}^{(j)}(\xi+2k\pi)| \leq |\hat{\psi}^{(j)}(\xi)|^2 + |\hat{\psi}^{(j)}(\xi+2k\pi)|^2 \tag{5.16}$$

for all $k\in\mathbb{Z}\setminus\{0\}$ and for almost all $\xi\in\mathbb{R}$, I_1 is convergent thanks to Lemma 5.5.4.

(b) With the previous item, I converges if and only if I_0 converges. We then have to show that I_0 converges if and only if the series in Expression (5.15) is locally integrable on $\mathbb{R}\setminus\{0\}$. If we suppose that I_0 is convergent for all $f\in\mathcal{D}$, let K be a compact of $\mathbb{R}\setminus\{0\}$. Taking f such that $\hat{f} := \chi_K$, we have

$$I_0 = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \sum_{j\in\mathbb{Z}} |\hat{\psi}^{(j)}(2^{-j}\xi)|^2\, d\xi = \frac{1}{2\pi}\int_K \sum_{j\in\mathbb{Z}} |\hat{\psi}^{(j)}(2^{-j}\xi)|^2\, d\xi$$

and then the series in Expression (5.15) is locally integrable on $\mathbb{R}\setminus\{0\}$. Reciprocally, if we suppose that the series in Expression (5.15) is locally integrable on $\mathbb{R}\setminus\{0\}$, we have

$$I_0 = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \sum_{j\in\mathbb{Z}} |\hat{\psi}^{(j)}(2^{-j}\xi)|^2\, d\xi \leq \frac{1}{2\pi}\|\hat{f}\|^2_{L^\infty(\mathbb{R})} \int_{\mathrm{supp}(\hat{f})} \sum_{j\in\mathbb{Z}} |\hat{\psi}^{(j)}(2^{-j}\xi)|^2\, d\xi$$

since $f\in\mathcal{D}$ and we have the conclusion. ∎

The series of Equality (5.1) of Theorem 5.1.3 directly appears in the definition of I_0. The series of Equality (5.2) also appears in I_1 when we write I_1 as follows:

$$I_1 = \frac{1}{2\pi}\int_{\mathbb{R}} \sum_{p\in\mathbb{Z}} \sum_{q\in 2\mathbb{Z}+1} 2^p\, \overline{\hat{f}(2^p\xi)}\, \hat{f}(2^p(\xi+2q\pi))\, t_{p,q}(\xi)\, d\xi.$$

Let us get this. For every $k\in\mathbb{Z}\setminus\{0\}$, there exist unique $\ell\in\mathbb{N}_0$ and $q\in 2\mathbb{Z}+1$ such that $k=2^\ell q$. Then, from Expression (5.11), since I_1 is convergent, we have

$$\begin{aligned}
2\pi I_1 &= \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} \overline{\hat{f}(\xi)}\, \hat{\psi}^{(-j)}(2^j\xi) \sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{f}(\xi+2^{-j}2k\pi)\, \overline{\hat{\psi}^{(-j)}(2^j\xi+2k\pi)}\, d\xi \\
&= \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} \overline{\hat{f}(\xi)}\, \hat{\psi}^{(-j)}(2^j\xi) \sum_{q\in 2\mathbb{Z}+1} \sum_{\ell=0}^{+\infty} \hat{f}(\xi+2^{-j}\,2\,2^\ell q\pi)\, \overline{\hat{\psi}^{(-j)}(2^j\xi+2\,2^\ell q\pi)}\, d\xi \\
&= \int_{\mathbb{R}} \overline{\hat{f}(\xi)} \sum_{q\in 2\mathbb{Z}+1} \sum_{\ell=0}^{+\infty} \sum_{p\in\mathbb{Z}} \hat{\psi}^{(p-\ell)}(2^{\ell-p}\xi)\, \overline{\hat{\psi}^{(p-\ell)}(2^\ell(2^{-p}\xi+2q\pi))}\, \hat{f}(\xi+2^p\,2q\pi)\, d\xi \\
&= \int_{\mathbb{R}} \sum_{p\in\mathbb{Z}} \sum_{q\in 2\mathbb{Z}+1} 2^p\, t_{p,q}(\xi)\, \overline{\hat{f}(2^p\xi)}\, \hat{f}(2^p(\xi+2q\pi))\, d\xi.
\end{aligned}$$

Before proving Theorem 5.1.3, let us recall some elements about the notion of Lebesgue point (see [106]).

Definition 5.5.6. Let F be a measurable and locally integrable function on $\mathbb{R}$. The real x_0 is a Lebesgue point for F if

$$\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} |F(x) - F(x_0)|\, dx = 0.$$

Proposition 5.5.7. *If F is a measurable and locally integrable function on $\mathbb{R}$, then almost every real number is a Lebesgue point for F.*

This previous proposition will be useful to prove the necessary condition of Theorem 5.1.3.

5.5.2 Proof of the Sufficient Condition of Theorem 5.1.3

We are now armed to prove Theorem 5.1.3. We proceed as in [52], with some adaptations to the nonstationary case. The sufficient condition is relatively simple.

Using Equalities (5.2) and (5.1) and the previous considerations on I, I_0 and I_1, we successively obtain

$$\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 = I = I_0 = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \sum_{j\in\mathbb{Z}} |\hat{\psi}^{(j)}(2^{-j}\xi)|^2\, d\xi = \|f\|^2_{L^2(\mathbb{R})}$$

for all $f \in \mathcal{D}$. Hence the conclusion by Proposition 5.5.1.

5.5.3 Proof of the Necessary Condition of Theorem 5.1.3

Let us now show the necessary condition of Theorem 5.1.3. Let us assume that $\{2^{j/2}\psi^{(j)}(2^j \cdot -k) : j,k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Equality (5.1)

Let us begin with Equality (5.1). Because the series I converges by hypothesis (and Proposition 5.5.1), the function

$$S(\cdot) := \sum_{j\in\mathbb{Z}} |\hat{\psi}^{(-j)}(2^j \cdot)|^2$$

is locally integrable on $\mathbb{R}\setminus\{0\}$ thanks to Proposition 5.5.5. With Proposition 5.5.7, it suffices to show that $S(\xi_0) = 1$ for some Lebesgue point $\xi_0 \neq 0$ of S. Let $\delta > 0$ such that $[\xi_0 - \delta, \xi_0 + \delta] \subset \mathbb{R}\setminus\{0\}$. We denote by $I^{(\delta)}$, $I_0^{(\delta)}$ and $I_1^{(\delta)}$ respectively the quantities I, I_0 and I_1 when we take $f = f_\delta$ where

$$\hat{f}_\delta(\xi) := \frac{1}{\sqrt{2\delta}} \chi_{[\xi_0-\delta,\xi_0+\delta]}(\xi)$$

for almost every $\xi \in \mathbb{R}$. By construction, $f_\delta \in \mathcal{D}$. On the one hand, we have

$$I^{(\delta)} = \sum_{j,k\in\mathbb{Z}} |\langle f_\delta, \psi_{j,k}\rangle|^2 = \|f_\delta\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|\hat{f}_\delta\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{\xi_0-\delta}^{\xi_0+\delta} \frac{1}{2\delta}\, d\xi = \frac{1}{2\pi}$$

and on the other hand, we have

$$I^{(\delta)} = I_0^{(\delta)} + I_1^{(\delta)} = \frac{1}{2\pi}\int_{\xi_0-\delta}^{\xi_0+\delta} \frac{1}{2\delta} \sum_{j\in\mathbb{Z}} |\hat{\psi}^{(-j)}(2^j\xi)|^2\, d\xi + I_1^{(\delta)} = \frac{1}{2\pi}\frac{1}{2\delta}\int_{\xi_0-\delta}^{\xi_0+\delta} S(\xi)\, d\xi + I_1^{(\delta)}.$$

Consequently, taking the limit as $\delta \to 0^+$, we obtain

$$1 = S(\xi_0) + 2\pi \lim_{\delta\to 0^+} I_1^{(\delta)}$$

since ξ_0 is a Lebesgue point of S. It only remains to prove that $\lim_{\delta\to 0^+} I_1^{(\delta)} = 0$. We adapt the proof of Lemma 5.5.4 as follows.

Let us consider $\xi_0 > 0$ (the case $\xi_0 < 0$ is similar). Using Inequality (5.16), we have

$$2\pi\, |I_1^{(\delta)}| \le \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}\setminus\{0\}} 2^j\, |\hat{f}_\delta(2^j\xi)|\, |\hat{f}_\delta(2^j(\xi+2k\pi))|\, |\hat{\psi}^{(j)}(\xi)|^2\, d\xi.$$

Let j_0 be the largest integer number which verifies $2^{j_0}\pi \le \delta$. Since $\xi_0 - \delta > 0$, that $\|\hat{f}_\delta\|_{L^\infty(\mathbb{R})} = 1/\sqrt{2\delta}$ and that

$$2^j\xi \in \operatorname{supp}(\hat{f}_\delta) \quad \Rightarrow \quad \xi \ge 2^{-j_0}(\xi_0 - \delta)$$

with $j \le j_0$, we obtain

$$2\pi\, |I_1^{(\delta)}| \le \frac{1}{\pi} \int_{\frac{\pi}{\delta}(\xi_0-\delta)}^{+\infty} \sum_{j=-\infty}^{j_0} \chi_{[2^{-j}(\xi_0-\delta),2^{-j}(\xi_0+\delta)]}(\xi)\, |\hat{\psi}^{(j)}(\xi)|^2\, d\xi,$$

as in the proof of Lemma 5.5.4. For fixed δ, we also know that the series

$$\sum_{j=-\infty}^{j_0} \chi_{[2^{-j}(\xi_0-\delta),2^{-j}(\xi_0+\delta)]}(\cdot)\, |\hat{\psi}^{(j)}(\cdot)|^2$$

is integrable on $\mathbb{R}$. As $[2^{-j}(\xi_0-\delta), 2^{-j}(\xi_0+\delta)] \subset [2^{-j}(\xi_0-\delta'), 2^{-j}(\xi_0+\delta')]$ for $\delta < \delta'$ with $\delta' \in (0,\xi_0)$, we have

$$2\pi|I_1^{(\delta)}| \le \int_{\frac{\pi}{\delta}(\xi_0-\delta)}^{+\infty} \sum_{j=-\infty}^{j_0} \chi_{[2^{-j}(\xi_0-\delta'),2^{-j}(\xi_0+\delta')]}(\xi)\, |\hat{\psi}^{(j)}(\xi)|^2\, d\xi \to 0$$

if $\delta \to 0^+$ by Lebesgue's theorem. Thus, $\lim_{\delta\to 0^+} I_1^{(\delta)} = 0$ and $S(\xi_0) = 1$.

Equality (5.2)

Let us now prove Equality (5.2). Let $p_0 \in \mathbb{Z}$ and $q_0 \in 2\,\mathbb{Z}+1$. With Proposition 5.5.7 again, it suffices to show that $t_{p_0,q_0}(\xi_0) = 0$ for some Lebesgue point ξ_0 of the integrable function t_{p_0,q_0}. First, from $\|f\|^2_{L^2(\mathbb{R})} = I = I_0 + I_1$ (by Proposition 5.5.1) and $I_0 = \|f\|^2_{L^2(\mathbb{R})}$ (by Equality (5.1), now acquired with the previous paragraph) for $f \in \mathcal{D}$, we get $I_1 = 0$. Then, we have

$$\int_{\mathbb{R}} \sum_{p\in\mathbb{Z}} \sum_{q\in 2\,\mathbb{Z}+1} 2^p\, \overline{\hat{f}(2^p\xi)}\, \hat{g}(2^p(\xi+2q\pi))\, t_{p,q}(\xi)\, d\xi = 0 \tag{5.17}$$

for all $f, g \in \mathcal{D}$ thanks to the polarization identity because the application

$$(f,g) \in \mathcal{D}\times\mathcal{D} \mapsto \int_{\mathbb{R}} \sum_{p\in\mathbb{Z}} \sum_{q\in 2\,\mathbb{Z}+1} 2^p\, \overline{\hat{f}(2^p\xi)}\, \hat{g}(2^p(\xi+2q\pi))\, t_{p,q}(\xi)\, d\xi \in \mathbb{C}$$

is a sesquilinear form.

Let us assume that $\xi_0 \neq 0$ and $\xi_0 + 2q_0\pi \neq 0$. Let $\delta > 0$ be such that $0 \notin [\xi_0 - \delta, \xi_0 + \delta]$ and $0 \notin [\xi_0 + 2q_0\pi - \delta, \xi_0 + 2q_0\pi + \delta]$. Let us define the functions f_δ and g_δ by

$$\hat{f}_\delta(\xi) := \frac{1}{\sqrt{2\delta}} \chi_{[2^{p_0}(\xi_0-\delta), 2^{p_0}(\xi_0+\delta)]}(\xi) \quad \text{and} \quad \hat{g}_\delta(\xi) := \hat{f}_\delta(\xi - 2^{p_0} 2q_0\pi)$$

for almost every $\xi \in \mathbb{R}$. By construction, we have $f_\delta, g_\delta \in \mathcal{D}$ and $\hat{f}_\delta(\xi)\hat{g}_\delta(\xi + 2^{p_0}2q_0\pi) = (1/2\delta)\chi_{[2^{p_0}(\xi_0-\delta), 2^{p_0}(\xi_0+\delta)]}(\xi)$ for almost all $\xi \in \mathbb{R}$. With Equality (5.17) for $f = f_\delta$ and $g = g_\delta$, we then obtain

$$0 = 2^{p_0} \int_{\mathbb{R}} \overline{\hat{f}_\delta(2^{p_0}\xi)}\, \hat{g}_\delta(2^{p_0}(\xi + 2q_0\pi))\, t_{p_0,q_0}(\xi)\, d\xi + J_\delta = \frac{2^{p_0}}{2\delta} \int_{\xi_0-\delta}^{\xi_0+\delta} t_{p_0,q_0}(\xi)\, d\xi + J_\delta$$

where

$$J_\delta := \int_{\mathbb{R}} \sum_{\substack{p\in\mathbb{Z} \\ (p,q)\neq(p_0,q_0)}} \sum_{q\in 2\mathbb{Z}+1} 2^p\, \overline{\hat{f}_\delta(2^p\xi)}\, \hat{g}_\delta(2^p(\xi + 2q\pi))\, t_{p,q}(\xi)\, d\xi.$$

Since ξ_0 is a Lebesgue point of t_{p_0,q_0}, we have

$$0 = 2^{p_0} t_{p_0,q_0}(\xi_0) + \lim_{\delta\to 0^+} J_\delta$$

and it only remains to prove that $\lim_{\delta\to 0^+} J_\delta = 0$ to have the conclusion.

Let us suppose that $\xi_0 > 0$ (the case $\xi_0 < 0$ is similar) and $\delta < \pi$. Let us fix $\xi \in \mathbb{R}$ and $q \in 2\mathbb{Z}+1$. If $\overline{\hat{f}_\delta(2^p\xi)}\hat{g}_\delta(2^p(\xi + 2q\pi)) \neq 0$, we must have $|2^p\xi - 2^{p_0}\xi_0| \leq 2^{p_0}\delta$ and $|2^p(\xi + 2q\pi) - 2^{p_0}(\xi_0 + 2q_0\pi)| \leq 2^{p_0}\delta$. Consequently, we have

$$|2^p q - 2^{p_0} q_0| \leq \frac{1}{2\pi} \left(|(2^p 2q\pi - 2^{p_0} 2q_0\pi - (2^p\xi - 2^{p_0}\xi_0)| + |2^p\xi - 2^{p_0}\xi_0|\right) \leq 2^{p_0} \frac{\delta}{\pi} < 2^{p_0}. \quad (5.18)$$

If $p \geq p_0$, we can easily show that $|2^{p_0}q_0 - 2^p q|$ is greater than 2^{p_0} because q and q_0 are odd, which is in contradiction with Inequality (5.18). If $p < p_0$, we have $|2^{p_0}q_0 - 2^p q| \geq 2^p$. Let j_0 be the largest integer number such that $2^{j_0} \leq 2^{p_0}\delta/\pi$. We so obtain

$$J_\delta = \int_{\mathbb{R}} \sum_{p=-\infty}^{j_0} \sum_{q\in 2\mathbb{Z}+1} 2^p\, \overline{\hat{f}_\delta(2^p\xi)}\, \hat{g}_\delta(2^p(\xi + 2q\pi))\, t_{p,q}(\xi)\, d\xi.$$

Using a similar argument as in Inequality (5.16), we can write

$$|J_\delta| \leq J_{\delta,1} + J_{\delta,2}$$

where we set

$$J_{\delta,1} := \int_{\mathbb{R}} \sum_{p=-\infty}^{j_0} \sum_{q\in 2\mathbb{Z}+1} 2^p |\hat{f}_\delta(2^p\xi)|\, |\hat{g}_\delta(2^p(\xi + 2q\pi))|\, \frac{1}{2}\, \tau_p(\xi)\, d\xi$$

and

$$J_{\delta,2} := \int_{\mathbb{R}} \sum_{p=-\infty}^{j_0} \sum_{q\in 2\mathbb{Z}+1} 2^p |\hat{f}_\delta(2^p(\xi + 2q\pi))|\, |\hat{g}_\delta(2^p\xi)|\, \frac{1}{2}\, \tau_p(\xi)\, d\xi$$

with

$$\tau_p(\xi) := \sum_{\ell=0}^{+\infty} |\hat{\psi}^{(p-\ell)}(2^\ell\xi)|^2$$

for almost all $\xi \in \mathbb{R}$. Since $\|\psi^{(j)}\|_{L^2(\mathbb{R})} = 1$ for $j \in \mathbb{Z}$, this last series converges in $L^1(\mathbb{R})$ by Levi's theorem.

Similarly to the proof of Lemma 5.5.4 (and the proof of the previous paragraph), we obtain

$$J_{\delta,1} \leq \frac{2^{p_0}}{\pi} \int_{\frac{\pi}{\delta}(\xi_0-\delta)}^{+\infty} \sum_{p=-\infty}^{j_0} \chi_{[2^{p_0-p}(\xi_0-\delta),2^{p_0-p}(\xi_0+\delta)]}(\xi)\, \tau_p(\xi)\, d\xi$$

and the series

$$\sum_{p=-\infty}^{j_0} \chi_{[2^{p_0-p}(\xi_0-\delta),2^{p_0-p}(\xi_0+\delta)]}(\cdot)\, \tau_p(\cdot) \tag{5.19}$$

is integrable on $\mathbb{R}$ by Levi's theorem. Indeed, as in the proof of Lemma 5.5.4, the sequence $(h_J)_{J\in\mathbb{N}}$ of integrable functions on $\mathbb{R}$ defined by

$$h_J(\xi) := \sum_{p=-J}^{j_0} \chi_{[2^{p_0-p}(\xi_0-\delta),2^{p_0-p}(\xi_0+\delta)]}(\xi)\, \tau_p(\xi)$$

for almost every $\xi \in \mathbb{R}$ is increasing because $\tau_p \in L^1(\mathbb{R})$ is positive. Moreover, using Additional asymptotic condition (5.3), we have

$$\begin{aligned}
\int_{\mathbb{R}} h_J(\xi)\, d\xi &= \sum_{p=-J}^{j_0} \sum_{\ell=0}^{+\infty} \int_{\mathbb{R}} \frac{\chi_{[2^{p_0-p}(\xi_0-\delta),2^{p_0-p}(\xi_0+\delta)]}(\xi)}{(1+|2^\ell \xi|)^\alpha} (1+|2^\ell\xi|)^\alpha\, |\hat{\psi}^{(p-\ell)}(2^\ell\xi)|^2\, d\xi \\
&\leq \sum_{p=-J}^{j_0} \sum_{\ell=0}^{+\infty} \frac{1}{(1+2^{\ell+p_0-p}(\xi_0-\delta))^\alpha} \int_{\mathbb{R}} (1+|2^\ell\xi|)^\alpha\, |\hat{\psi}^{(p-\ell)}(2^\ell\xi)|^2\, d\xi \\
&= \sum_{p=-J}^{j_0} \sum_{\ell=0}^{+\infty} \frac{2^{-\ell}}{(1+2^{\ell+p_0-p}(\xi_0-\delta))^\alpha} \int_{\mathbb{R}} (1+|\xi|)^\alpha\, |\hat{\psi}^{(p-\ell)}(\xi)|^2\, d\xi \\
&\leq A \sum_{p=-J}^{j_0} \sum_{\ell=0}^{+\infty} \frac{1}{2^\ell 2^{\alpha(\ell+p_0-p)}(\xi_0-\delta)^\alpha} \\
&= \frac{2^{-\alpha p_0}\, A}{(\xi_0-\delta)^\alpha} \sum_{p=-J}^{j_0} 2^{p\alpha} \sum_{\ell=0}^{+\infty} \left(\frac{1}{2^{\alpha+1}}\right)^\ell \\
&\leq \frac{2^{-\alpha p_0}\, A}{(\xi_0-\delta)^\alpha} \frac{1}{1-2^{-(\alpha+1)}} \sum_{p=-\infty}^{j_0} 2^{p\alpha}.
\end{aligned}$$

As $[2^{p_0-p}(\xi_0-\delta), 2^{p_0-p}(\xi_0+\delta)] \subset [2^{p_0-p}(\xi_0-\delta'), 2^{p_0-p}(\xi_0+\delta')]$ for $\delta < \delta'$ with $\delta' \in (0,\xi_0)$, we have

$$J_{\delta,1} \leq \frac{2^{p_0}}{\pi} \int_{\frac{\pi}{\delta}(\xi_0-\delta)}^{+\infty} \sum_{p=-\infty}^{j_0} \chi_{[2^{p_0-p}(\xi_0-\delta'),2^{p_0-p}(\xi_0+\delta')]}(\xi)\, \tau_p(\xi)\, d\xi \to 0$$

as $\delta \to 0^+$ by Lebesgue's theorem. Thus, $\lim_{\delta\to 0^+} J_{\delta,1} = 0$ and $\lim_{\delta\to 0^+} J_{\delta,2} = 0$ by a similar reasoning. Finally, we have $\lim_{\delta\to 0^+} J_\delta = 0$ and $t_{p_0,q_0}(\xi_0) = 0$.

5.6 Proofs of Theorem 5.2.5 and Proposition 5.2.6

Let us now prove Theorem 5.2.5 and let us begin with the necessary condition. We proceed as in the stationary case (see [52]).

5.6.1 Proof of the Necessary Condition of Theorem 5.2.5

By hypothesis, the mother wavelets $\psi^{(j)}$, $j \in \mathbb{Z}$, come from a nonstationary multiresolution analysis of $L^2(\mathbb{R})$ and there exist thus scaling functions $\varphi^{(j)}$ (and filters $m_0^{(j)}$), $j \in \mathbb{Z}$, leading to their construction. The following proposition shows how to get $|\hat{\varphi}^{(j)}|^2$ from $|\hat{\psi}^{(m)}|^2$, $m \in \mathbb{Z}$ such that $m < j$. The proof follows the stationary case with some easy adaptations (see [52]).

Proposition 5.6.1. *For all $j \in \mathbb{Z}$ and for almost all $\xi \in \mathbb{R}$, we have*

$$|\hat{\varphi}^{(j)}(\xi)|^2 = \sum_{n=1}^{+\infty} |\hat{\psi}^{(j-n)}(2^n\xi)|^2.$$

Proof. Let $j \in \mathbb{Z}$. Using Equality (5.7), Equality (5.5) of Proposition 5.2.3 and Theorem 5.2.4, we have

$$|\hat{\varphi}^{(j)}(\xi)|^2 = |\hat{\varphi}^{(j)}(\xi)|^2 \left(|m_0^{(j)}(\xi)|^2 + |m_0^{(j)}(\xi+\pi)|^2\right) = |\hat{\varphi}^{(j-1)}(2\xi)|^2 + |\hat{\psi}^{(j-1)}(2\xi)|^2$$

and then

$$|\hat{\varphi}^{(j)}(\xi)|^2 = |\hat{\varphi}^{(j-N)}(2^N\xi)|^2 + \sum_{n=1}^{N} |\hat{\psi}^{(j-n)}(2^n\xi)|^2$$

for almost all $\xi \in \mathbb{R}$ and for all $N \in \mathbb{N}$. Since $\|\psi^{(m)}\|_{L^2(\mathbb{R})} = 1$ for all $m \in \mathbb{Z}$, the series

$$\sum_{n=1}^{+\infty} |\hat{\psi}^{(j-n)}(2^n \cdot)|^2$$

converges in $L^1(\mathbb{R})$ and then almost everywhere on $\mathbb{R}$ by Levi's theorem. Consequently, the sequence $(|\hat{\varphi}^{(j-N)}(2^N \cdot)|)_{N\in\mathbb{N}}$ converges almost everywhere on $\mathbb{R}$. Moreover,

$$\int_{\mathbb{R}} |\hat{\varphi}^{(j-N)}(2^N\xi)|^2 \, d\xi = 2\pi 2^{-N} \to 0$$

as $N \to +\infty$. Hence

$$\lim_{N\to+\infty} |\hat{\varphi}^{(j-N)}(2^N\xi)|^2 = 0$$

for almost every $\xi \in \mathbb{R}$, which leads to the conclusion. ■

For all $j \in \mathbb{Z}$, using Lemma 5.2.2 and Proposition 5.6.1, we have

$$1 = \sum_{k\in\mathbb{Z}} |\hat{\varphi}^{(j)}(\xi + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \sum_{n=1}^{+\infty} |\hat{\psi}^{(j-n)}(2^n(\xi + 2k\pi))|^2 = D_j(\xi)$$

for almost all $\xi \in \mathbb{R}$, since $\{2^{j/2}\varphi^{(j)}(2^j \cdot -k) : k \in \mathbb{Z}\}$ is an orthonormal family of $L^2(\mathbb{R})$.

5.6.2 Proof of the Sufficient Condition of Theorem 5.2.5

Let us now consider the sufficient condition. Let us assume that $D_j(\xi) = 1$ for all $j \in \mathbb{Z}$ and for almost all $\xi \in \mathbb{R}$ and let us construct scaling functions. Basically, for all $j \in \mathbb{Z}$ and for almost all $\xi \in \mathbb{R}$, since $D_j(\xi) = 1$ by hypothesis, we choose the smallest $n \in \mathbb{N}$ such that

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}^{(j-n)}(2^n(\xi + 2k\pi))|^2 \neq 0$$

and then we define $\varphi^{(j)}$ by

$$\hat{\varphi}^{(j)}(\xi) := \frac{\hat{\psi}^{(j-n)}(2^n\xi)}{\sqrt{\displaystyle\sum_{k\in\mathbb{Z}} |\hat{\psi}^{(j-n)}(2^n(\xi+2k\pi))|^2}}$$

for almost all $\xi \in \mathbb{R}$.

Let us look more precisely at the construction. Let us fix $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ and let us define the infinite vector

$$\Psi_{j,n}(\xi) := \left(\hat{\psi}^{(j-n)}(2^n(\xi+2k\pi))\right)_{k\in\mathbb{Z}}$$

of $\ell^2(\mathbb{Z})$ for almost all $\xi \in \mathbb{R}$. The following lemma will be useful later. The proof uses the theoretical characterization of nonstationary orthonormal bases of wavelets (see Theorem 5.1.3) and thus Additional asymptotic condition (5.3).

Lemma 5.6.2. *For all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, for almost every $\xi \in \mathbb{R}$, we have*

$$\hat{\psi}^{(j-n)}(2^n\xi) = \sum_{r=1}^{+\infty}\sum_{k\in\mathbb{Z}} \hat{\psi}^{(j-n)}(2^n(\xi+2k\pi))\,\overline{\hat{\psi}^{(j-r)}(2^r(\xi+2k\pi))}\,\hat{\psi}^{(j-r)}(2^r\xi). \tag{5.20}$$

Proof. Let us give the idea of the proof. The double series converges almost everywhere for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ thanks to Cauchy-Schwarz's inequality, the convergence of D_j for $j \in \mathbb{Z}$ and Equality (5.1) of Theorem 5.1.3. For the equality, if we denote $G_{j,n}(\xi)$ the second member of Expression (5.20), we have $G_{j,n}(\xi) = G_{j-1,n-1}(2\xi)$ for all $j \in \mathbb{Z}$, all $n \in \mathbb{N}\backslash\{1\}$ and almost all $\xi \in \mathbb{R}$ by Theorem 5.1.3 and Proposition 5.6.3 below. In consequence, for all $j \in \mathbb{Z}$, all $n \in \mathbb{N}$ and almost all $\xi \in \mathbb{R}$, we have $G_{j,n}(\xi) = G_{j-(n-1),1}(2^{n-1}\xi)$ by recursion and thus the conclusion because $G_{j-(n-1),1}(\xi) = \hat{\psi}^{(j-n)}(2\xi)$. ■

Proposition 5.6.3. *The family $\{2^{j/2}\psi^{(j)}(2^j\ \cdot -k) : j,k \in \mathbb{Z}\}$ is orthonormal in $L^2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}^{(j)}(\cdot+2k\pi)|^2 = 1$$

almost everywhere for all $j \in \mathbb{Z}$ and

$$\sum_{k\in\mathbb{Z}} \hat{\psi}^{(j-p)}(2^p(\cdot+2k\pi))\,\overline{\hat{\psi}^{(j)}(\cdot+2k\pi)} = 0$$

almost everywhere for all $j \in \mathbb{Z}$ and $p \in \mathbb{N}$.

Proof. It suffices to adapt the proof of the stationary case (see [**52**]) to the nonstationary case. Let us note that the first equality is similar to the one of Lemma 5.2.2 (see [**16**]). ■

Let us come back to the sufficient condition. Thanks to Lemma 5.6.2, we can write

$$\Psi_{j,n}(\xi) = \sum_{r=1}^{+\infty} \langle \Psi_{j,n}(\xi), \Psi_{j,r}(\xi)\rangle\ \Psi_{j,r}(\xi) \tag{5.21}$$

for almost all $\xi \in \mathbb{R}$. Moreover, for almost all $\xi \in \mathbb{R}$, we can see that

$$\sum_{n=1}^{+\infty} \|\Psi_{j,n}(\xi)\|^2_{\ell^2(\mathbb{Z})} = \sum_{n=1}^{+\infty}\sum_{k\in\mathbb{Z}} |\hat{\psi}^{(j-n)}(2^n(\xi+2k\pi))|^2 = D_j(\xi) = 1. \tag{5.22}$$

For all $j \in \mathbb{Z}$, we define

$$\mathbb{F}_j(\xi) := \overline{\operatorname{span}\{\Psi_{j,n}(\xi) : n \in \mathbb{N}\}}$$

for almost all $\xi \in \mathbb{R}$. It is a subspace of $\ell^2(\mathbb{Z})$ of dimension 1 by the following proposition (see [52]) thanks to Equalities (5.21) and (5.22).

Proposition 5.6.4. *Let $\{v_n : n \in \mathbb{N}\}$ be a family of vectors in a Hilbert space $\mathcal{H}$ such that*

$$\sum_{n=1}^{+\infty} \|v_n\|^2 = C \qquad \text{and} \qquad v_m = \sum_{r=1}^{+\infty} \langle v_m, v_r \rangle \, v_r$$

for all $m \in \mathbb{N}$. Then, the dimension of the subspace $\overline{\operatorname{span}\{v_n : n \in \mathbb{N}\}}$ of $\mathcal{H}$ is equal to C.

In consequence, $\mathbb{F}_j(\xi)$ is generated by only one unit vector $U_j(\xi)$. To construct it, we first make a partition of $[0, 2\pi]$:

$$E_{j,n} := \{\xi \in [0, 2\pi] : \Psi_{j,n}(\xi) \neq 0 \text{ and } \Psi_{j,m}(\xi) = 0 \text{ for } m < n\}, \quad n \in \mathbb{N}$$

and the null set $E_{j,0} := \{\xi \in [0, 2\pi] : D_j(\xi) = 0\}$. We can then define U_j almost everywhere on $[0, 2\pi]$ by

$$U_j(\xi) := \frac{\Psi_{j,n}(\xi)}{\|\Psi_{j,n}(\xi)\|_{\ell^2(\mathbb{Z})}} \quad \text{if } \xi \in E_{j,n}.$$

Let us write $U_j(\xi) := (u_k^{(j)}(\xi))_{k \in \mathbb{Z}}$ and define $\varphi^{(j)}$ almost everywhere on $\mathbb{R}$ by

$$\hat{\varphi}^{(j)}(\xi) := u_k^{(j)}(\xi - 2k\pi) \quad \text{if } \xi \in [0, 2\pi] + 2k\pi \ (k \in \mathbb{Z}).$$

As in the stationary case (see [52]), these $\varphi^{(j)}$, $j \in \mathbb{Z}$, are the sought scaling functions.

5.6.3 Proof of Proposition 5.2.6

Let us now prove Proposition 5.2.6. In fact, it suffices to show that, for all $j \in \mathbb{Z}$, $D_j > 0$ almost everywhere on $(0, 2\pi)$ implies $D_j = 1$ almost everywhere on $\mathbb{R}$.

Let us fix $j \in \mathbb{Z}$. By definition, D_j is 2π-periodic. With the notations of Subsection 5.6.2, we know that $D_j(\xi)$ is the dimension of $\mathbb{F}_j(\xi)$ for almost all $\xi \in \mathbb{R}$ (see Proposition 5.6.4). Consequently, $D_j(\xi) \in \mathbb{N}$ for almost all $\xi \in \mathbb{R}$ because $D_j > 0$ almost everywhere on $(0, 2\pi)$. Moreover, we have

$$\int_0^{2\pi} D_j(\xi)\, d\xi = \sum_{n=1}^{+\infty} \sum_{k \in \mathbb{Z}} \int_{2k\pi}^{2(k+1)\pi} |\hat{\psi}^{(j-n)}(2^n \xi)|^2 \, d\xi = \sum_{n=1}^{+\infty} 2^n \|\hat{\psi}^{(j-n)}\|_{L^2(\mathbb{R})} = 2\pi$$

because $\|\psi^{(m)}\|_{L^2(\mathbb{R})} = 1$ for all $m \in \mathbb{Z}$. We so have $D_j(\xi) = 1$ for almost all $\xi \in \mathbb{R}$.

5.7 Proof of Theorem 5.4.1

Let us now prove Theorem 5.4.1. By contradiction, let us assume that we have an orthonormal basis $\{2^{j/2}\psi^{(j)}(2^j \cdot - k) : j, k \in \mathbb{Z}\}$ of $H^2(\mathbb{R})$ satisfying the given regularity conditions.

Using the two first conditions of regularity, for $j \in \mathbb{Z}$, the series

$$s_j(\cdot) := \sum_{n=1}^{+\infty} |\hat{\psi}^{(j-n)}(2^n \cdot)|^2$$

converges uniformly on compact subsets of $\mathbb{R}\setminus\{0\}$ and then represents a continuous function on $\mathbb{R}\setminus\{0\}$. Moreover, there exists $C > 0$ such that

$$s_j(\xi) \leq \frac{C}{\xi^{2\alpha+1}}$$

for all $\xi > 0$ and $j \in \mathbb{Z}$ because

$$s_j(\xi) \leq \sum_{n=1}^{+\infty} \frac{A^2}{(1+2^n\xi)^{2\alpha+1}} \leq \frac{A^2}{\xi^{2\alpha+1}} \sum_{n=1}^{+\infty} \frac{1}{2^{n(2\alpha+1)}} \leq \frac{C}{\xi^{2\alpha+1}}.$$

By definition, for $j \in \mathbb{Z}$, we can see that

$$D_j(\cdot) = \sum_{k\in\mathbb{Z}} s_j(\cdot + 2k\pi).$$

This series converges uniformly on compact subsets of $[-\pi, 0) \cup (0, \pi]$ and then represents a continuous function on this set. Since $\|\psi^{(m)}\|_{L^2(\mathbb{R})} = 1$ for all $m \in \mathbb{Z}$, $D_j = 1$ almost everywhere on $\mathbb{R}$ by a similar reasoning as in the proof of Proposition 5.2.6 (see Subsection 5.6.3), adapted to the case $H^2(\mathbb{R})$ (Additional asymptotic condition (5.3) is satisfied thanks to the second hypothesis).

Let us fix $j \in \mathbb{Z}$. For all $k \in \mathbb{Z}$, the function $s_j(\cdot - 2k\pi)$ is continuous on $\mathbb{R}\setminus\{2k\pi\}$. The series

$$t_j(\cdot) := \sum_{k\in\mathbb{Z}\setminus\{0\}} s_j(\cdot + 2k\pi)$$

converges uniformly on $[-\pi, \pi]$ and then represents a continuous function on $[-\pi, \pi]$. By construction, we have

$$D_j(\xi) = s_j(\xi) + t_j(\xi)$$

for all $\xi \in [-\pi, 0) \cup (0, \pi]$. By continuity of each term, we obtain

$$1 = \lim_{\xi\to 0^-} D_j(\xi) = \lim_{\xi\to 0^-} (s_j(\xi) + t_j(\xi)) = 0 + t_j(0)$$

because $\psi^{(j)} \in H^2(\mathbb{R})$ and

$$1 = \lim_{\xi\to 0^+} D_j(\xi) = \lim_{\xi\to 0^+} (s_j(\xi) + t_j(\xi)) = \lim_{\xi\to 0^+} s_j(\xi) + t_j(0).$$

Hence, for all $j \in \mathbb{Z}$, we get

$$\lim_{\xi\to 0^+} s_j(\xi) = 0. \tag{5.23}$$

Let us consider $s_j(2^{-j}\xi)$ for $j \in \mathbb{N}$ and $\xi \in (0, \eta)$. On the one hand, there exists $\delta \in (0, \eta)$ such that

$$0 \leq s_j(2^{-j}\xi) \leq \frac{1}{2} \tag{5.24}$$

for all $\xi \in (0,\delta)$ and all $j \in \mathbb{N}$. Indeed, using the third hypothesis, we have

$$\begin{aligned} s_j(2^{-j}\xi) &= \sum_{\ell=1-j}^{+\infty} |\hat{\psi}^{(-\ell)}(2^\ell \xi)|^2 = s_0(\xi) + \sum_{\ell=0}^{j-1} |\hat{\psi}^{(\ell)}(2^{-\ell}\xi)|^2 \\ &\leq s_0(\xi) + B^2\,\xi^{2\beta} \sum_{\ell=0}^{j-1} 2^{-2\ell\beta} = s_0(\xi) + B^2\,\xi^{2\beta}\,\frac{1-4^{-\beta j}}{1-4^{-\beta}} \\ &\leq s_0(\xi) + B_\beta\,\xi^{2\beta} \end{aligned}$$

for all $j \in \mathbb{N}$ and for all $\xi \in (0,\eta)$ where B_β is a constant depending only on β. It follows that, using Equality (5.23) with s_0, we have Inequality (5.24). On the other hand, using Equality (5.1) (of Theorem 5.2.5 in the present setting), we have

$$s_j(2^{-j}\xi) = \sum_{\ell=1-j}^{+\infty} |\hat{\psi}^{(-\ell)}(2^\ell \xi)|^2 \to 1$$

as $j \to +\infty$ for almost every $\xi \geq 0$. Hence we get a contradiction with Inequality (5.24), taking $\xi_0 \in (0,\delta)$ such that $s_j(2^{-j}\xi_0) \to 1$ as $j \to +\infty$.

Chapter 6

Nonstationary Continuous Wavelet Transform

In the previous chapter, we have investigated nonstationary orthonormal bases of wavelets of $L^2(\mathbb{R})$. Initially, this nonstationarity was introduced in various situations: the construction of bases of wavelets in Sobolev spaces (see [15, 16]), the construction of infinitely differentiable compactly supported bases of wavelets in $L^2(\mathbb{R})$ (see [41]),...

Up to now, the nonstationarity has been only considered in the context of orthonormal bases of wavelets. What about the continuous wavelet transform? In [95] (see pages 80-81), the idea of a nonstationary continuous wavelet transform is put forward. Apparently, it could be useful in the study of particular singularities, called oscillating singularities (see [88] for example), of a function.

Let us already mention that the case of the continuous wavelet transform in Sobolev spaces is studied in [105]. In comparison with the case of orthonormal basis of wavelets, it appears that only one wavelet (not a family of wavelets) is sufficient to define the continuous wavelet transform of a distribution which belongs to a Sobolev space and to consider the reconstruction of this distribution from its continuous wavelet transform.

The purpose of this chapter is to present a nonstationary version of the continuous wavelet transform, which does not seem to have been investigated before. In this chapter, we first define the notions of nonstationary family of wavelets and of nonstationary continuous wavelet transform in $L^2(\mathbb{R})$. We then give some examples and we study the reconstruction of a square integrable function from its nonstationary continuous wavelet transform.

6.1 Nonstationary Continuous Wavelet Transform

Let us begin with the introduction of the notions of nonstationary family of wavelets and nonstationary continuous wavelet transform.

Definition 6.1.1. The set $\Psi := \{\psi^{(a)} : a \in \mathbb{R}\setminus\{0\}\}$ is a *nonstationary family of wavelets* if $\psi^{(a)} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for all $a \in \mathbb{R}\setminus\{0\}$ and if Ψ satisfies the *nonstationary admissibility condition*: the function

$$a \mapsto \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{|a|}$$

is integrable on $\mathbb{R}$ for all $\xi \in \mathbb{R}$ and the integral

$$\int_{\mathbb{R}} \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{|a|}\, da$$

is independent of ξ for almost all $\xi \in \mathbb{R}$.

Using the nonstationary family of wavelets Ψ, the *nonstationary continuous wavelet transform* of a function $f \in L^2(\mathbb{R})$ is the function $\mathcal{W}_\Psi f$ defined by

$$\mathcal{W}_\Psi f(a,b) := \int_{\mathbb{R}} f(x)\overline{\psi}_{a,b}(x)\,dx = \langle f, \psi_{a,b}\rangle, \quad a \in \mathbb{R}\setminus\{0\},\ b \in \mathbb{R}$$

where

$$\psi_{a,b}(x) := \frac{1}{a}\psi^{(a)}\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R}.$$

Let us consider some examples of nonstationary family of wavelets.

(a) If ψ is a wavelet, then $\{\psi\}$ is clearly a nonstationary family of wavelets. Indeed, we directly have

$$\int_{\mathbb{R}} \frac{|\hat{\psi}(a\xi)|^2}{|a|}\,da = \int_{\mathbb{R}} \frac{|\hat{\psi}(t)|^2}{|t|}\,dt,$$

which is independent of ξ for all almost $\xi \in \mathbb{R}$. The stationary case is thus a particular case of the nonstationary case.

(b) Let ψ be an even or odd wavelet and let $p \in \mathbb{R}\setminus\{-1\}$. For $a \in \mathbb{R}\setminus\{0\}$, let us set

$$\psi^{(a)}(x) := \frac{1}{|a|^p}\psi\left(\frac{x}{|a|^p}\right), \quad x \in \mathbb{R}.$$

Then, $\Psi := \{\psi^{(a)} : a \in \mathbb{R}\setminus\{0\}\}$ is a nonstationary family of wavelets. Indeed, for almost all $\xi \in \mathbb{R}$, we have $\hat{\psi}^{(a)}(\xi) = \hat{\psi}(|a|^p\xi)$ and

$$\int_{\mathbb{R}} \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{|a|}\,da = \int_{\mathbb{R}} \frac{|\hat{\psi}(|a|^p a\xi)|^2}{|a|}\,da = 2\int_0^{+\infty} \frac{|\hat{\psi}(a^{p+1}\xi)|^2}{a}\,da$$

because $|\hat{\psi}|$ is an even function. We then have

$$\int_{\mathbb{R}} \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{|a|}\,da = \frac{2}{|p+1|}\int_0^{+\infty} \frac{|\hat{\psi}(t)|^2}{t}\,dt = \frac{1}{|p+1|}\int_{\mathbb{R}} \frac{|\hat{\psi}(t)|^2}{|t|}\,dt,$$

which is independent of ξ for almost all $\xi \in \mathbb{R}$. For example,

$$\left\{\frac{1}{|a|^p}\left(-\chi_{[-|a|^p,0)} + \chi_{[0,|a|^p)}\right) : a \in \mathbb{R}\setminus\{0\}\right\}$$

and

$$\left\{x \mapsto -\frac{2x}{|a|^{2p}}e^{-x^2/|a|^{2p}} : a \in \mathbb{R}\setminus\{0\}\right\}$$

are such nonstationary families of wavelets. In this case, the nonstationary continuous wavelet transform of $f \in L^2(\mathbb{R})$ related to Ψ is

$$\mathcal{W}_\Psi f(a,b) = \mathcal{W}_\psi f(a|a|^p, b)$$

for all $a \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$.

(c) If ψ is a wavelet and if ρ is a function defined on $\mathbb{R}\setminus\{0\}$ such that $|\rho| = 1$ on $\mathbb{R}\setminus\{0\}$, then

$$\Psi := \{\rho(a)\psi(\cdot) : a \in \mathbb{R}\setminus\{0\}\}$$

is clearly a nonstationary family of wavelets (thanks to the same argument as Item (a)). In this case, the nonstationary continuous wavelet transform of $f \in L^2(\mathbb{R})$ related to Ψ is

$$\mathcal{W}_\Psi\, f(a,b) = \overline{\rho(a)}\ \mathcal{W}_\psi\, f(a,b)$$

for all $a \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$.

(d) Let p and q be the functions defined on $\mathbb{R}\setminus\{0\}$ by

$$p(a) := \frac{\log(|a|+1)}{|a|} \quad \text{and} \quad q(a) := \sqrt{\frac{|a|}{|a|+1}}\frac{1}{\sqrt{\log(|a|+1)}}.$$

For $a \in \mathbb{R}\setminus\{0\}$, let us set

$$\psi^{(a)}(x) := q(a)\,(xD_x + 1)\,\frac{p(a)}{\pi(p^2(a)+x^2)}, \quad x \in \mathbb{R}$$

and let us note that we have

$$\psi^{(a)}(x) = \frac{q(a)}{p(a)}\,\psi_P\left(\frac{x}{p(a)}\right) = \sqrt{\left(\frac{|a|}{\log(|a|+1)}\right)^3}\frac{1}{\sqrt{|a|+1}}\,\psi_P\left(\frac{|a|x}{\log(|a|+1)}\right)$$

for all $x \in \mathbb{R}$, where ψ_P is the Poisson wavelet:

$$\psi_P(x) := \frac{1}{\pi}\frac{1-x^2}{(1+x^2)^2}, \quad x \in \mathbb{R}\,.$$

Then, $\Psi := \{\psi^{(a)} : a \in \mathbb{R}\setminus\{0\}\}$ is a nonstationary family of wavelets. Indeed, for almost all $\xi \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{|a|}\,da = 2|\xi|^2\int_0^{+\infty}\frac{\log(a+1)}{a+1}\,e^{-2|\xi|\log(a+1)}\,da = 2\int_0^{+\infty} t\,e^{-2t}\,dt = \frac{1}{2}.$$

In this case, the nonstationary continuous wavelet transform of $f \in L^2(\mathbb{R})$ related to Ψ is

$$\mathcal{W}_\Psi\, f(a,b) = q(a)\ \mathcal{W}_{\psi_P}\, f(ap(a),b) = \sqrt{\frac{|a|}{|a|+1}}\frac{1}{\sqrt{\log(|a|+1)}}\ \mathcal{W}_{\psi_P}\, f\left(\frac{a\log(|a|+1)}{|a|},b\right)$$

for all $a \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$.

Let us note that all the previous examples of nonstationary families of wavelets are actually constructed from one wavelet. All the presented nonstationary continuous wavelet transforms can be then reduced to a classical continuous wavelet transform to a multiplicative factor. It is certainly possible to find a nonstationary family of wavelets where such a situation does not occur.

6.2 Reconstruction Formula

If Ψ is a nonstationary family of wavelets, it is possible to reconstruct a square integrable function f from $\mathcal{W}_\Psi f(a,b)$ with $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}$. This is the object of the following result (which is the nonstationary version of Theorem 3.1.3). The proof is very similar to the stationary case (see [33]) and it allows to understand the choice and the use of the nonstationary admissibility condition.

Theorem 6.2.1. *Let $\Psi := \{\psi^{(a)} : a \in \mathbb{R}\backslash\{0\}\}$ be a nonstationary family of wavelets such that*

$$\int_{\mathbb{R}} \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{|a|}\, da = 1 \tag{6.1}$$

for almost all $\xi \in \mathbb{R}$. For all $f, g \in L^2(\mathbb{R})$, we have

$$\iint_{\mathbb{R}^2} \mathcal{W}_\Psi f(a,b)\, \overline{\mathcal{W}_\Psi g(a,b)}\, \frac{dadb}{|a|} = \langle f, g\rangle . \tag{6.2}$$

Moreover, for $f \in L^2(\mathbb{R})$, we have

$$\lim_{\substack{\varepsilon\to 0^+\\ r\to+\infty}} \left\| f(\cdot) - \int_{\{a'\in\mathbb{R}:\varepsilon<|a'|<r\}} \left(\int_{\mathbb{R}} \mathcal{W}_\Psi f(a,b)\, \psi_{a,b}(\cdot)\, db\right) \frac{da}{|a|} \right\|_{L^2(\mathbb{R})} = 0.$$

Proof. 1. Let us first show that

$$(a,b) \mapsto \frac{\mathcal{W}_\Psi h(a,b)}{\sqrt{|a|}}$$

is square integrable on $\mathbb{R}^2$ for all $h \in L^2(\mathbb{R})$. We first have

$$\mathcal{W}_\Psi h(a,b) = \frac{1}{a}\left(h \star \overline{\psi^{(a)}}\left(-\frac{\cdot}{a}\right)\right)(b) = \frac{1}{2\pi}\mathcal{F}^+_{\xi\to b}\left(\hat{h}(\xi)\overline{\hat{\psi}^{(a)}}(a\xi)\right) \tag{6.3}$$

for almost all $a, b \in \mathbb{R}$, where we notice that $\xi \mapsto \hat{h}(\xi)\overline{\hat{\psi}^{(a)}}(a\xi) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ because $\hat{h} \in L^2(\mathbb{R})$ and $\xi \mapsto \overline{\hat{\psi}^{(a)}}(a\xi) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by hypothesis. For almost all fixed $a \in \mathbb{R}$, the function

$$b \mapsto \frac{|\mathcal{W}_\Psi h(a,b)|^2}{|a|} = \frac{1}{4\pi^2|a|}\left|\mathcal{F}^+_{\xi\to b}\left(\hat{h}(\xi)\overline{\hat{\psi}^{(a)}}(a\xi)\right)\right|^2$$

is then integrable on $\mathbb{R}$. Moreover, we have

$$\int_{\mathbb{R}} \frac{|\mathcal{W}_\Psi h(a,b)|^2}{|a|}\, db = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{h}(b)|^2 \frac{|\hat{\psi}^{(a)}(ab)|^2}{|a|}\, db$$

and this function of a is integrable on $\mathbb{R}$ by Fubini's theorem because the function

$$(a,b) \mapsto |\hat{h}(b)|^2 \frac{|\hat{\psi}^{(a)}(ab)|^2}{|a|}$$

is integrable on $\mathbb{R}^2$ by Tonelli's theorem. Indeed, for almost all fixed $b \in \mathbb{R}$, the function $a \mapsto |\hat{\psi}^{(a)}(ab)|^2/|a|$ is integrable on $\mathbb{R}$ because Ψ satisfies the nonstationary admissibility condition. Using Equality (6.1), we have

$$\int_0^{+\infty} |\hat{f}(b)|^2 \frac{|\hat{\psi}^{(a)}(ab)|^2}{|a|}\, da = |\hat{h}(b)|^2$$

and $b \mapsto |\hat{h}(b)|^2$ is integrable on $\mathbb{R}$. By Tonelli's theorem again, we then have the integrability of $(a,b) \mapsto |\mathcal{W}_\Psi\, h(a,b)|^2/|a|$ on $\mathbb{R}^2$.

2. Let us now show Equality (6.2). Using Equality (6.3), we successively have

$$\begin{aligned}
&\iint_{\mathbb{R}^2} \mathcal{W}_\Psi\, f(a,b)\, \overline{\mathcal{W}_\Psi\, g(a,b)}\, \frac{dadb}{|a|} \\
&= \frac{1}{4\pi^2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{F}^+_{\xi \to b} \left(\hat{f}(\xi) \overline{\hat{\psi}^{(a)}}(a\xi) \right) \overline{\mathcal{F}^+_{\xi \to b} \left(\hat{g}(\xi) \overline{\hat{\psi}^{(a)}}(a\xi) \right)} db \right) \frac{da}{|a|} \\
&= \frac{1}{4\pi^2} \int_{\mathbb{R}} \left\langle \hat{F}_a, \hat{G}_a \right\rangle \frac{da}{|a|} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \langle F_a, G_a \rangle \frac{da}{|a|}
\end{aligned}$$

where, for all $a \in \mathbb{R} \backslash \{0\}$, we have setted $F_a(\xi) := \hat{f}(\xi)\overline{\hat{\psi}^{(a)}(a\xi)}$ and $G_a(\xi) := \hat{g}(\xi)\overline{\hat{\psi}^{(a)}(a\xi)}$ for almost all $\xi \in \mathbb{R}$. Using Equality (6.1), we then have

$$\begin{aligned}
\iint_{(0,+\infty)\times\mathbb{R}} \mathcal{W}_\Psi\, f(a,b)\, \overline{\mathcal{W}_\Psi\, g(a,b)}\, \frac{dadb}{a} &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)} \left(\int_{\mathbb{R}} \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{|a|}\, da \right) d\xi \\
&= \frac{1}{2\pi} \left\langle \hat{f}, \hat{g} \right\rangle \\
&= \langle f, g \rangle\,.
\end{aligned}$$

3. Let us finish with the convergence in $L^2(\mathbb{R})$ and let us set

$$I_{\varepsilon,r}(\cdot) := \int_{\{a' \in \mathbb{R} : \varepsilon < |a'| < r\}} \left(\int_{\mathbb{R}} \mathcal{W}_\Psi\, f(a,b)\, \psi_{a,b}(\cdot)\, db \right) \frac{da}{|a|}$$

for $r > \varepsilon > 0$. With Equality (6.2), we directly have

$$\|f - I_{\varepsilon,r}\|_{L^2(\mathbb{R})} = \sup_{\|g\|_{L^2(\mathbb{R})}=1} |\langle f - I_{\varepsilon,r}, g\rangle| = \sup_{\|g\|_{L^2(\mathbb{R})}=1} \left| \iint_X \mathcal{W}_\Psi\, f(a,b)\, \overline{\mathcal{W}_\Psi\, g(a,b)}\, \frac{dadb}{|a|} \right|$$

where $X := (\mathbb{R} \backslash ((-r,-\varepsilon) \cup (\varepsilon, r))) \times \mathbb{R}$. By Cauchy-Schwarz's inequality, we obtain

$$\|f - I_{\varepsilon,r}\|_{L^2(\mathbb{R})} \leq \sup_{\|g\|_{L^2(\mathbb{R})}=1} \sqrt{\iint_X \frac{|\mathcal{W}_\Psi\, f(a,b)|^2}{|a|}\, dadb} \sqrt{\iint_X \frac{|\mathcal{W}_\Psi\, g(a,b)|^2}{|a|}\, dadb}.$$

However, with Equality (6.2), we have

$$\iint_X \frac{|\mathcal{W}_\Psi\, g(a,b)|^2}{|a|}\, dadb \leq \iint_{\mathbb{R}^2} \frac{|\mathcal{W}_\Psi\, g(a,b)|^2}{|a|}\, dadb = \|g\|^2_{L^2(\mathbb{R})}.$$

Consequently, we have

$$\|f - I_{\varepsilon,r}\|_{L^2(\mathbb{R})} \leq \sqrt{\iint_X \frac{|\mathcal{W}_\Psi\, f(a,b)|^2}{|a|}\, dadb} \to 0$$

if $\varepsilon \to 0^+$ and $r \to +\infty$ by Lebesgue's theorem since $(a,b) \mapsto |\mathcal{W}_\Psi\, f(a,b)|^2/|a|$ is integrable on $\mathbb{R}^2$. Hence the conclusion. ∎

As in the stationary case, it is possible to recover a square integrable function f from $\mathcal{W}_\Psi f(a,b)$ with $a>0$ only and $b\in\mathbb{R}$, where Ψ a nonstationary family of wavelets. In this context, we slightly adapt the nonstationary admissibility condition and then also the notion of nonstationary family of wavelets. The set $\Psi := \{\psi^{(a)} : a>0\}$ is a nonstationary family of wavelets if $\psi^{(a)}\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ for all $a>0$, if the function

$$a \mapsto \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{a}$$

is integrable on $(0,+\infty)$ for all $\xi\in\mathbb{R}$ and if the integral

$$\int_0^{+\infty} \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{a}\, da$$

is independent of $\xi\in\mathbb{R}$ for almost all $\xi\in\mathbb{R}$. We have the following reconstruction formula. The proof is similar to the one of the previous theorem.

Theorem 6.2.2. *Let $\Psi := \{\psi^{(a)} : a>0\}$ be a nonstationary family of wavelets such that*

$$\int_0^{+\infty} \frac{|\hat{\psi}^{(a)}(a\xi)|^2}{a}\, da = 1$$

for almost all $\xi\in\mathbb{R}$. For all $f,g\in L^2(\mathbb{R})$, we have

$$\iint_{(0,+\infty)\times\mathbb{R}} \mathcal{W}_\Psi f(a,b)\,\overline{\mathcal{W}_\Psi g(a,b)}\,\frac{dadb}{a} = \langle f,g\rangle\,.$$

Moreover, for $f\in L^2(\mathbb{R})$, we have

$$\lim_{\substack{\varepsilon\to 0^+\\ r\to+\infty}} \left\| f(\cdot) - \int_\varepsilon^r \left(\int_{\mathbb{R}} \mathcal{W}_\Psi f(a,b)\,\psi_{a,b}(\cdot)\,db\right)\frac{da}{a}\right\|_{L^2(\mathbb{R})} = 0.$$

Part III

$\mathcal{S}^\nu$ Spaces Revisited with Wavelet Leaders

Chapter 7

From $\mathcal{S}^\nu$ Spaces to $\mathcal{L}^\nu$ Spaces

The study of the Hölder continuity of a function by means of its wavelet coefficients, i.e. its coefficients in an orthonormal basis of wavelets, is a widely used tool (see [3, 59, 92]). We have already considered this kind of study in Chapter 3 with the continuous wavelet transform of a function. In order to investigate the regularity of a function with the sequence made up of its wavelet coefficients, $\mathcal{S}^\nu$ spaces first (see [64]) and then more recently $\mathcal{L}^\nu$ spaces (see [13]) have been introduced.

Up to now, in Chapter 1, we have presented the notions of Hölder continuity and Hölder exponent to study the regularity of a function. If a function is very irregular, in the sense that its Hölder exponent changes at each point, these notions are not more really relevant. In this case, the spectrum of singularities of the function gives a more appropriate information (see [65] for example). For each possible value h taken by the Hölder exponent of a function, this quantity actually measures the "size" of the set of real numbers where the Hölder exponent of the function is equal to h. In general, it is impossible to calculate the spectrum of singularities of a function because of the determination of several intricate limits which are in its definition. Therefore, one tries to estimate this spectrum from some quantities which are numerically computable (see [65, 67]). It is just the purpose of the methods developed with $\mathcal{S}^\nu$ spaces and $\mathcal{L}^\nu$ spaces. From this point of view, the method based on $\mathcal{L}^\nu$ spaces allows to obtain theoretically better approximations of the spectrum of singularities than the one based on $\mathcal{S}^\nu$ spaces (see [13]), which still improved the one based on Besov spaces (see [64]) given by the Frisch-Parisi conjecture (see [63, 99]).

At first sight, $\mathcal{S}^\nu$ spaces and $\mathcal{L}^\nu$ spaces are spaces of functions. They are both defined from a certain quantity, called wavelet profile in the case of $\mathcal{S}^\nu$ spaces and leader profile in the case of $\mathcal{L}^\nu$ spaces, which depends on the wavelet coefficients of functions. It has been proved that these two profiles and these two types of spaces are actually independent of the chosen orthonormal basis of wavelets to represent the functions (see [13, 64]). Therefore, $\mathcal{S}^\nu$ spaces and $\mathcal{L}^\nu$ spaces can be considered as sequence spaces (and no more as function spaces). Likewise, the two profiles can be directly associated to a sequence (and no more to a function). This will be the point of view that we will adopt in all of this part, except only for some particular remarks or comments.

This chapter is a presentation of $\mathcal{L}^\nu$ spaces and a preparation to the next chapter. After some preliminaries about wavelet coefficients and wavelet leaders in the context of sequences, we recall the notions of wavelet profile and space $\mathcal{S}^\nu$ in a first time and the notions of leader profile and space $\mathcal{L}^\nu$ in a second time. Then, we give some examples and we compare the spaces $\mathcal{S}^\nu$ and $\mathcal{L}^\nu$.

7.1 Wavelet Coefficients and Wavelet Leaders

Initially, $\mathcal{S}^\nu$ and $\mathcal{L}^\nu$ spaces have been introduced to study the regularity of functions from its wavelet coefficients. Since we are interested in local properties of functions, we can assume (as in [9,13,64]) that the functions that we consider are 1-periodic. To represent such functions, we can use an orthonormal basis of wavelets of the space of the 1-periodic functions of $L^2([0,1])$. For that, we take a mother wavelet $\psi \in \mathcal{S}(\mathbb{R})$ (as done in [83]) and we write

$$\psi_{j,k}(\cdot) := \sum_{l\in\mathbb{Z}} \psi(2^j(\cdot - l) - k), \quad j \in \mathbb{N}_0,\, k \in \{0,\ldots,2^j-1\}.$$

We know that the 1-periodic functions $2^{j/2}\psi_{j,k}$, $j \in \mathbb{N}_0$, $k \in \{0,\ldots,2^j-1\}$, together with the constant function 1 form an orthonormal basis of the space of the 1-periodic functions of $L^2([0,1])$ (see [33,88,92] for more details). If f is such a function, we have

$$f = c + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}$$

in $L^2([0,1])$ where $c := \int_0^1 f(x)\,dx$ and

$$c_{j,k} := 2^j \int_0^1 f(x)\psi_{j,k}(x)\,dx, \quad j \in \mathbb{N}_0,\, k \in \{0,\ldots,2^j-1\}.$$

In comparison with Chapter 5, the index k does not vary in $\mathbb{Z}$, but in $\{0,\ldots,2^j-1\}$ for each fixed scale $j \in \mathbb{N}_0$. We are then interested in sequences with a couple of indices (j,k) where $j \in \mathbb{N}_0$ and $k \in \{0,\ldots,2^j-1\}$.

Let us denote

$$\Lambda := \bigcup_{j\in\mathbb{N}_0} \{(j,k) : k \in \{0,\ldots,2^j-1\}\}$$

and $\Omega := \mathbb{C}^\Lambda$. The elements of a sequence $\vec{c} \in \Omega$ are still called *wavelet coefficients* (of $\vec{c}$), even if we are no more in the context of functions. As mentioned in the introduction of this chapter, $\mathcal{S}^\nu$ or $\mathcal{L}^\nu$ can be seen as function or sequence spaces and thus, there is no problem with this abuse of language.

For $j \in \mathbb{N}_0$ and $k \in \{0,\ldots,2^j-1\}$, we use the notation $\lambda(j,k)$, or simply λ if there is no ambiguity, to refer to the dyadic interval

$$\lambda(j,k) := \{x \in \mathbb{R} : 2^j x - k \in [0,1)\} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right).$$

For $j \in \mathbb{N}_0$, Λ_j represents the set of all dyadic intervals of $[0,1)$ of length 2^{-j}. In the following, we will use the two equivalent notations $c_{j,k}$ and c_λ for $(j,k) \in \Lambda$ to denote the elements of $\vec{c} \in \Omega$ (indeed, for any $(j,k) \in \Lambda$ corresponds a unique dyadic interval of $[0,1)$ and reciprocally).

Definition 7.1.1. The *wavelet leaders* of $\vec{c} \in \Omega$ are the quantities

$$d_\lambda := \sup_{\lambda'\subset\lambda} |c_{\lambda'}|, \quad \lambda \in \Lambda_j,\, j \in \mathbb{N}_0.$$

With this definition, it may happen that $d_\lambda = +\infty$. However, in Section 7.3, we will see that all the wavelet leaders of a sequence of $\mathcal{L}^\nu$ are finite. For the wavelet leaders of $\vec{c}$, we will also use the two equivalent notations d_λ and $d_{j,k}$ for $(j,k) \in \Lambda$.

7.2 Wavelet Profile and Space $\mathcal{S}^\nu$

7.2.1 Definitions

Let us recall the notions of wavelet profile and space $\mathcal{S}^\nu$ (see [8, 9, 42, 64]).

Definition 7.2.1. The *wavelet profile* of a sequence $\vec{c} \in \Omega$ is the function $\nu_{\vec{c}}$ defined by

$$\nu_{\vec{c}}(\alpha) := \lim_{\varepsilon\to 0^+}\left(\limsup_{j\to+\infty}\left(\frac{\log(\#E_j(1,\alpha+\varepsilon)(\vec{c}))}{\log(2^j)}\right)\right), \quad \alpha\in\mathbb{R},$$

where

$$E_j(C,\alpha)(\vec{c}) := \left\{k\in\{0,\dots,2^j-1\} : |c_{j,k}| \geq C2^{-\alpha j}\right\}$$

for $j\in\mathbb{N}_0$, $C>0$ and $\alpha\in\mathbb{R}$.

This definition formalizes the idea that at large scales j, there are about $2^{\nu_{\vec{c}}(\alpha)j}$ wavelet coefficients larger in modulus than $2^{-\alpha j}$ (with the convention $2^{-\infty} := 0$). By construction, for $\vec{c}\in\Omega$, $\nu_{\vec{c}}$ is non-decreasing, right-continuous and with values in $\{-\infty\}\cup[0,1]$.

Before giving the definition of space $\mathcal{S}^\nu$, we need the notion of admissible profile.

Definition 7.2.2. An *admissible profile* is a non-decreasing and right-continuous function ν with values in $\{-\infty\}\cup[0,1]$ such that

$$\alpha_{\min} := \inf\{\alpha\in\mathbb{R} : \nu(\alpha)\geq 0\}\in\mathbb{R}\,.$$

Definition 7.2.3. Given an admissible profile ν, a sequence $\vec{c}\in\Omega$ belongs to $\mathcal{S}^\nu$ if

$$\nu_{\vec{c}}(\alpha)\leq\nu(\alpha)$$

for all $\alpha\in\mathbb{R}$.

Equivalently, $\vec{c}$ belongs to $\mathcal{S}^\nu$ if and only if for every $\alpha\in\mathbb{R}$, $\varepsilon>0$ and $C>0$, there exists $J\in\mathbb{N}_0$ such that

$$\#E_j(C,\alpha)(\vec{c})\leq 2^{(\nu(\alpha)+\varepsilon)j}$$

for all $j\geq J$. When $\nu(\alpha)=-\infty$, we use the convention $2^{-\infty j} := 0$ for all $j\in\mathbb{N}_0$. Heuristically, a sequence $\vec{c}$ of Ω belongs to $\mathcal{S}^\nu$ if at each large scale j, the number of k such that $|c_{j,k}|\geq 2^{-\alpha j}$ is of order smaller than $2^{\nu(\alpha)j}$. This space is a vector space (see Section 2 in [8]).

Some examples of $\mathcal{S}^\nu$ spaces for particular admissible profile ν are given in [42].

7.2.2 Basic Results

In this subsection, we summarize the topological properties of $\mathcal{S}^\nu$ established in [8]. This will permit to compare them with the ones of $\mathcal{L}^\nu$ studied in the next chapter.

Theorem 7.2.4. *There exists a unique metrizable topology that is stronger than the topology of the pointwise convergence and that makes $\mathcal{S}^\nu$ a complete topological vector space. This topology is separable.*

More precisely, in order to define a complete metrizable topology on $\mathcal{S}^\nu$, auxiliary spaces were introduced. For any $\alpha \in \mathbb{R}$ and any $\beta \in \{-\infty\} \cup [0, +\infty)$, the space $A(\alpha, \beta)$ is defined by

$$A(\alpha,\beta) := \left\{\vec{c} \in \Omega \ : \ \exists C, C' \geq 0 \text{ such that } \#E_j(C,\alpha)(\vec{c}) \leq C' 2^{\beta j} \quad \forall j \in \mathbb{N}_0\right\}.$$

This space is endowed with the distance

$$\delta_{\alpha,\beta}(\vec{c}, \vec{c}\,') := \inf\left\{C + C' : C, C' \geq 0 \text{ and } \#E_j(C,\alpha)(\vec{c} - \vec{c}\,') \leq C' 2^{\beta j} \ \forall j \in \mathbb{N}_0\right\}$$

for $\vec{c}, \vec{c}\,' \in A(\alpha, \beta)$. Let us remark that if $\beta = -\infty$, then $A(\alpha, -\infty)$ is the space c^α, i.e. the space of sequences $\vec{c} \in \Omega$ such that the sequence $(2^{\alpha j} c_{j,k})_{(j,k)\in\Lambda}$ is bounded. Let us note that $c^0 = \ell^\infty(\Lambda)$. Moreover, $(A(\alpha, -\infty), \delta_{\alpha,-\infty})$ is the topological normed space $(c^\alpha, \|\cdot\|_{c^\alpha})$ where

$$\|\vec{c}\|_{c^\alpha} := \sup_{(j,k)\in\Lambda} 2^{\alpha j} |c_{j,k}|, \quad \vec{c} \in c^\alpha.$$

If $\beta \geq 1$, then $A(\alpha, \beta) = \Omega$. Moreover, if $\beta > 1$, the topology defined by the distance $\delta_{\alpha,\beta}$ is equivalent to the topology of the pointwise convergence.

Proposition 7.2.5. *For any sequence $(\alpha_n)_{n\in\mathbb{N}}$ dense in $\mathbb{R}$ and any sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ of strictly positive numbers decreasing to 0, we have*

$$\mathcal{S}^\nu = \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} A(\alpha_n, \nu(\alpha_n) + \varepsilon_m).$$

The topology of $\mathcal{S}^\nu$ is defined as the projective limit topology, i.e. the coarsest topology that makes each inclusion $\mathcal{S}^\nu \subset A(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$ continuous. This topology is equivalent to the topology given by the distance

$$\delta := \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} 2^{-(m+n)} \frac{\delta_{\alpha_n, \nu(\alpha_n)+\varepsilon_m}}{1 + \delta_{\alpha_n, \nu(\alpha_n)+\varepsilon_m}}$$

(see Section 5 in [8]).

Let us recall the characterization of the compact sets of $\mathcal{S}^\nu$ (see Section 6 in [8]). For $m, n \in \mathbb{N}$, let $C_{m,n}$ and $C'_{m,n}$ be positive or null constants and let us define

$$K_{m,n} := \left\{\vec{c} \in \Omega : \#\{k \in \{0, \ldots, 2^j - 1\} : |c_{j,k}| > C_{m,n}\, 2^{-\alpha_n j}\} \leq C'_{m,n}\, 2^{(\nu(\alpha_n)+\varepsilon_m)j} \ \forall j \in \mathbb{N}_0\right\}$$

(taking the usual sequences of Proposition 7.2.5). We write

$$K := \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} K_{m,n}.$$

Proposition 7.2.6. *A set is a compact subset of $(\mathcal{S}^\nu, \delta)$ if and only if it is closed in $(\mathcal{S}^\nu, \delta)$ and included in some K.*

7.3 Leader Profile and Space $\mathcal{L}^\nu$

7.3.1 Definitions

Let us now define the notions of leader profile of a sequence and space $\mathcal{L}^\nu$ (see firstly [14] and secondly [13] which gives the definitions of leader profile and space $\mathcal{L}^\nu$ in a more general context). In fact, there are just the notions of wavelet profile and space $\mathcal{S}^\nu$ where wavelet coefficients are replaced by wavelet leaders.

Definition 7.3.1. The *leader profile* of $\vec{c} \in \Omega$ is the function $\widetilde{\nu}_{\vec{c}}$ defined by

$$\widetilde{\nu}_{\vec{c}}(\alpha) := \lim_{\varepsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\log(\# \widetilde{E_j}(1, \alpha + \varepsilon)(\vec{c}))}{\log(2^j)} \right) \right), \quad \alpha \in \mathbb{R},$$

where

$$\widetilde{E_j}(C, \alpha)(\vec{c}) := \left\{ k \in \{0, \dots, 2^j - 1\} : d_{j,k} \geq C 2^{-\alpha j} \right\}$$

for $j \in \mathbb{N}_0$, $C > 0$ and $\alpha \in \mathbb{R}$.

This definition formalizes the idea that at large scales j, there are about $2^{\widetilde{\nu}_{\vec{c}}(\alpha) j}$ wavelet leaders larger than $2^{-\alpha j}$.

Definition 7.3.2. Given an admissible profile ν, $\mathcal{L}^\nu$ is the space of sequences $\vec{c} \in \Omega$ such that

$$\widetilde{\nu}_{\vec{c}}(\alpha) \leq \nu(\alpha)$$

for all $\alpha \in \mathbb{R}$.

Just as in the case of $\mathcal{S}^\nu$ spaces, we get the following description of $\mathcal{L}^\nu$ (the proof is a simple adaptation of the proof of Lemma 2.3 in [8]).

Proposition 7.3.3. *Let ν be an admissible profile. A sequence $\vec{c} \in \Omega$ belongs to $\mathcal{L}^\nu$ if and only if for every $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$, there exists $J \in \mathbb{N}_0$ such that*

$$\# \widetilde{E_j}(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha) + \varepsilon) j} \tag{7.1}$$

for all $j \geq J$.

Proof. Let $\vec{c} \in \mathcal{L}^\nu$ and let $\alpha \in \mathbb{R}$, $\eta > 0$ and $C > 0$. By definition of $\widetilde{\nu}_{\vec{c}}$, there exists $\varepsilon > 0$ such that

$$\inf_{J \in \mathbb{N}_0} \sup_{j \geq J} \frac{\log(\# \widetilde{E_j}(1, \alpha + \varepsilon)(\vec{c}))}{\log(2^j)} \leq \nu(\alpha)$$

and then, there exists $J \in \mathbb{N}_0$ such that

$$\frac{\log(\# \widetilde{E_j}(1, \alpha + \varepsilon)(\vec{c}))}{\log(2^j)} \leq \nu(\alpha) + \eta.$$

and that $2^{-\varepsilon j} \leq C$ for all $j \geq J$. Thus, for $j \geq J$, we have

$$\# \widetilde{E_j}(C, \alpha)(\vec{c}) \leq \# \widetilde{E_j}(1, \alpha + \varepsilon)(\vec{c}) \leq 2^{(\nu(\alpha) + \eta) j}.$$

Reciprocally, let $\vec{c} \in \Omega$ be such that $\vec{c}$ satisfies Inequality (7.1). Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. By hypothesis, there exists $J \in \mathbb{N}_0$ such that

$$\# \widetilde{E_j}(1, \alpha + \varepsilon)(\vec{c}) \leq 2^{(\nu(\alpha + \varepsilon) + \varepsilon) j}$$

for all $j \geq J$. Then, we directly obtain

$$\sup_{j \geq J} \frac{\log(\# \widetilde{E_j}(1, \alpha + \varepsilon)(\vec{c}))}{\log(2^j)} \leq \nu(\alpha + \varepsilon) + \varepsilon.$$

Taking the infimum on $J \in \mathbb{N}_0$ and then the limit as $\varepsilon \to 0^+$, we have the conclusion thanks to the right-continuity of ν. ■

7.3.2 First Properties

Let us begin by showing that $\mathcal{L}^\nu$ is a vector space. The proof is similar to the one for $\mathcal{S}^\nu$ (see [64]).

Proposition 7.3.4. *Given an admissible profile ν, $\mathcal{L}^\nu$ is a vector space.*

Proof. It is evident that $\vec{0} \in \mathcal{L}^\nu$. Let $\vec{c}, \vec{c}\,' \in \mathcal{L}^\nu$ and $\theta \in \mathbb{C} \setminus \{0\}$. To have the conclusion, let us show that $\vec{c} + \vec{c}\,' \in \mathcal{L}^\nu$ and $\theta\vec{c} \in \mathcal{L}^\nu$. Let us fix $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$.

On the one hand, by hypothesis and by Proposition 7.3.3, there exists $J \in \mathbb{N}_0$ such that

$$\#\left\{k \in \{0, \ldots, 2^j - 1\} : d_{j,k} \leq \frac{C}{|\theta|}\, 2^{-\alpha j}\right\} \leq 2^{(\nu(\alpha)+\varepsilon)j}$$

and then

$$\#\left\{k \in \{0, \ldots, 2^j - 1\} : \sup_{\lambda' \subset \lambda(j,k)} |\theta c_{\lambda'}| \leq C\, 2^{-\alpha j}\right\} \leq 2^{(\nu(\alpha)+\varepsilon)j}$$

for all $j \geq J$. Thus $\theta\vec{c} \in \mathcal{L}^\nu$.

On the other hand, by hypothesis and by Proposition 7.3.3 again, there exists $J \in \mathbb{N}_0$ such that $\varepsilon j/2 \geq 1$,

$$\#\left\{k \in \{0, \ldots, 2^j - 1\} : d_{j,k} \leq \frac{C}{2}\, 2^{-\alpha j}\right\} \leq 2^{(\nu(\alpha)+\frac{\varepsilon}{2})j}$$

and

$$\#\left\{k \in \{0, \ldots, 2^j - 1\} : d'_{j,k} \leq \frac{C}{2}\, 2^{-\alpha j}\right\} \leq 2^{(\nu(\alpha)+\frac{\varepsilon}{2})j}$$

for all $j \geq J$. Since

$$\sup_{\lambda' \subset \lambda} |c_{\lambda'} + c'_{\lambda'}| \geq C 2^{-\alpha j} \quad \Rightarrow \quad \left[\sup_{\lambda' \subset \lambda} |c_{\lambda'}| \geq \frac{C}{2}\, 2^{-\alpha j} \text{ or } \sup_{\lambda' \subset \lambda} |c'_{\lambda'}| \geq \frac{C}{2}\, 2^{-\alpha j}\right],$$

we have

$$\begin{aligned}
&\#\left\{k \in \{0, \ldots, 2^j - 1\} : \sup_{\lambda' \subset \lambda(j,k)} |c_{\lambda'} + c'_{\lambda'}| \geq C\, 2^{-\alpha j}\right\} \\
&\leq \quad \#\widetilde{E_j}\left(\frac{C}{2}, \alpha\right)(\vec{c}) + \#\widetilde{E_j}\left(\frac{C}{2}, \alpha\right)(\vec{c}\,') \\
&\leq \quad 2\,.\,2^{(\nu(\alpha)+\frac{\varepsilon}{2})j} \\
&\leq \quad 2^{(\nu(\alpha)+\varepsilon)j}
\end{aligned}$$

for all $j \geq J$. Thus, $\vec{c} + \vec{c}\,' \in \mathcal{L}^\nu$. ■

Contrary to the space $\mathcal{S}^\nu$, a sequence of $\mathcal{L}^\nu$ is automatically bounded. This is the object of the following result. Consequently, if a sequence belongs to $\mathcal{L}^\nu$, its wavelet leaders are finite.

Proposition 7.3.5. *Given an admissible profile ν, we have $\mathcal{L}^\nu \subset c^0$.*

Proof. Let $\vec{c} \in \mathcal{L}^\nu$ and let $\alpha < \alpha_{\min}$. By definition of $\alpha_{\min}$ and by Proposition 7.3.3, there exists $J \in \mathbb{N}_0$ such that $d_{j,k} < 2^{-\alpha j}$ for all $j \geq J$ and $k \in \{0, \ldots, 2^j - 1\}$. Moreover, there exists $C' > 0$ such that $2^{\alpha j} d_{j,k} \leq C'$ for all $j \in \{0, \ldots, J-1\}$ and $k \in \{0, \ldots, 2^j - 1\}$. Setting $C := \max\{C', 1\}$, we obtain $d_{j,k} \leq C 2^{-\alpha j}$ for all $(j,k) \in \Lambda$. In particular,

$$d_{0,0} = \sup_{(j,k)\in\Lambda} |c_{j,k}| \leq C$$

and thus, $\vec{c} \in c^0$. ■

Remark 7.3.6. In fact, we can assume that $\alpha_{\min} \geq 0$ in the definition of admissible profile (see Definition 7.2.2) to consider $\mathcal{L}^\nu$ spaces. Let us assume that $\alpha_{\min} < 0$ and let us define the admissible profile $\nu^\dagger$ as follows:

$$\nu^\dagger(\alpha) := \begin{cases} \nu(\alpha) & \text{if } \alpha \geq 0 \\ -\infty & \text{if } \alpha < 0 \end{cases} .$$

We directly have $\mathcal{L}^{\nu^\dagger} \subset \mathcal{L}^\nu$ because $\nu^\dagger \leq \nu$ on $\mathbb{R}$. For the other inclusion, let $\vec{c} \in \mathcal{L}^\nu$. By construction, we have $\nu^\dagger = \nu$ on $(-\infty, \alpha_{\min}) \cup [0, +\infty)$. Let $\alpha \in [\alpha_{\min}, 0)$, $\varepsilon > 0$ and $C > 0$. By Proposition 7.3.3, there exists $J \in \mathbb{N}_0$ such that

$$\#\widetilde{E_j}(C,\alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j}$$

for all $j \geq J$. Since $\vec{c} \in c^0$ by Proposition 7.3.5, there exists $C' > 0$ such that $d_{j,k} \leq C'$ for all $(j,k) \in \Lambda$. Moreover, because $\alpha < 0$, there exists $J' \geq J$ such that $2^{-\alpha j} C > C'$ for all $j \geq J'$. Consequently, $\#\widetilde{E_j}(C,\alpha)(\vec{c}) = 0$ for all $j \geq J'$ and $\vec{c} \in \mathcal{L}^{\nu^\dagger}$. Hence $\mathcal{L}^{\nu^\dagger} = \mathcal{L}^\nu$.

Therefore, from now on, we will always assume that ν is an admissible profile with $\alpha_{\min} \geq 0$.

7.3.3 Examples and Comparison of Spaces $\mathcal{L}^\nu$ and $\mathcal{S}^\nu$

Let us now compare the spaces $\mathcal{L}^\nu$ and $\mathcal{S}^\nu$ and let us give some examples for particular admissible profile ν. From the definition of the wavelet leaders, it is direct to see that $\nu_{\vec{c}} \leq \widetilde{\nu_{\vec{c}}}$ for any sequence $\vec{c} \in \Omega$ since $|c_{j,k}| \leq d_{j,k}$ for every $(j,k) \in \Lambda$. Therefore, given an admissible profile ν, we have

$$\mathcal{L}^\nu \subset \mathcal{S}^\nu . \tag{7.2}$$

Here is an easy example where the inclusion is strict. Let us consider the admissible profile ν defined by

$$\nu(\alpha) := \begin{cases} 1 & \text{if } \alpha \geq 0 \\ -\infty & \text{if } \alpha < 0 \end{cases} \tag{7.3}$$

and let us show that $\mathcal{L}^\nu = c^0$. We know that $\mathcal{L}^\nu \subset c^0$ (see Proposition 7.3.5). For the other inclusion, let $\vec{c} \in c^0$ and let $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$. If $\alpha \geq 0$, we directly have

$$\#\widetilde{E_j}(C,\alpha)(\vec{c}) \leq 2^j \leq 2^{j(\nu(\alpha)+\varepsilon)}$$

for all $j \in \mathbb{N}_0$. Let us now assume that $\alpha < 0$. By hypothesis, there exists $C' > 0$ such that $d_{j,k} \leq C'$ for all $(j,k) \in \Lambda$. Moreover, there exists $J \in \mathbb{N}_0$ such that $C 2^{-\alpha j} > C'$ and then $\#\{k \in \{0, \ldots, 2^j - 1\} : d_{j,k} \geq C 2^{-\alpha j}\} = 0$ for all $j \geq J$. By Proposition 7.3.3, $\vec{c} \in \mathcal{L}^\nu$.

We know that $\mathcal{S}^\nu = \bigcap_{\varepsilon>0} c^{-\varepsilon}$ (see [42]). In this case, $\mathcal{S}^\nu$ is not included in $\mathcal{L}^\nu$. Indeed, on the one hand, the sequence $\vec{c} \in \Omega$ defined by

$$c_{j,k} := \begin{cases} j & \text{if } k = 0 \\ 0 & \text{if } k \in \{1, \ldots, 2^j - 1\} \end{cases} \tag{7.4}$$

for all scale $j \in \mathbb{N}_0$, is not bounded and does not belong to $\mathcal{L}^\nu$. On the other hand, it belongs to $c^{-\varepsilon}$ for all $\varepsilon > 0$ because $j2^{-\varepsilon j}$ tends to 0 for $j \to +\infty$.

In the previous example, the admissible profile is such that $\alpha_{\min} = 0$. In fact, in this case, the inclusion $\mathcal{L}^\nu \subset \mathcal{S}^\nu$ is always strict, as shown in the next proposition.

Proposition 7.3.7. *If ν is an admissible profile such that $\alpha_{\min} = 0$, then $\mathcal{L}^\nu$ is strictly included in $\mathcal{S}^\nu$.*

Proof. Since $\mathcal{L}^\nu$ is included in c^0, it suffices to find an element of $\mathcal{S}^\nu$ which does not belong to c^0. Such an example is given by the sequence $\vec{c} \in \Omega$ defined in Expression (7.4). We know that $\vec{c} \notin c^0$ and let us show that $\vec{c} \in \mathcal{S}^\nu$. Let $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$. If $\alpha < 0$, there exists $J \in \mathbb{N}_0$ such that $j < C2^{-\alpha j}$ and then $\#E_j(C, \alpha)(\vec{c}\,) = 0$ for all $j \geq J$. If $\alpha \geq 0$, we have $\#E_j(C, \alpha)(\vec{c}\,) \leq 1 \leq 2^{(\nu(\alpha)+\varepsilon)j}$ for all $j \in \mathbb{N}_0$. Hence the conclusion. ■

Let us study what happens in the case $\alpha_{\min} > 0$. Let us begin with an example. Let us consider the admissible profile ν defined by

$$\nu(\alpha) := \begin{cases} 1 & \text{if } \alpha \geq a \\ -\infty & \text{if } \alpha < a \end{cases}$$

where $a > 0$. We know that $\mathcal{S}^\nu = \bigcap_{\varepsilon>0} c^{a-\varepsilon}$ (see [42]) and let us show that $\mathcal{L}^\nu = \mathcal{S}^\nu$. Using Inclusion (7.2), it suffices to prove that $\bigcap_{\varepsilon>0} c^{a-\varepsilon} \subset \mathcal{L}^\nu$. Let $\vec{c} \in c^{a-\varepsilon}$ for all $\varepsilon > 0$ and let $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$. If $\alpha \geq a$, we directly have

$$\#\widetilde{E_j}(C, \alpha)(\vec{c}\,) \leq 2^j \leq 2^{j(\nu(\alpha)+\varepsilon)}$$

for all $j \in \mathbb{N}_0$. Let us now assume that $\alpha < a$. There exists $\delta > 0$ such that $a - \delta > 0$ and that $\alpha - a + \delta < 0$. Since $\vec{c} \in c^{a-\delta}$, there exists $C' > 0$ such that $2^{(a-\delta)j}|c_{j,k}| \leq C'$ for all $(j,k) \in \Lambda$. Then, for $j' \geq j$ and $k' \in \{0, \ldots, 2^{j'} - 1\}$, we have

$$|c_{j',k'}| \leq C'\, 2^{-(a-\delta)j'} \leq C'\, 2^{(\alpha-a+\delta)j}\, 2^{-\alpha j}.$$

Since there exists $J \in \mathbb{N}_0$ such that $C'2^{(\alpha-a+\delta)j} \leq C/2$ for all $j \geq J$, we so obtain

$$d_{j,k} < C\, 2^{-\alpha j}$$

for all $j \geq J$ and $k \in \{0, \ldots, 2^j - 1\}$. Thus, $\#\widetilde{E_j}(C, \alpha)(\vec{c}\,) = 0$ for $j \geq J$. Consequently, $\vec{c} \in \mathcal{L}^\nu$.

The next result gives a necessary and sufficient condition on the admissible profile ν to have the equality of the spaces $\mathcal{L}^\nu$ and $\mathcal{S}^\nu$ (see [14]).

Theorem 7.3.8. *Let ν be an admissible profile such that $\alpha_{\min} > 0$. Then, $\mathcal{L}^\nu = \mathcal{S}^\nu$ if and only if*

$$\nu(\alpha) = \alpha \sup_{\alpha' \in (0,\alpha]} \frac{\nu(\alpha')}{\alpha'} \tag{7.5}$$

for all $\alpha \in [\alpha_{\min}, \inf_{\alpha' \geq \alpha_{\min}} \frac{\alpha'}{\nu(\alpha')}]$.

In fact, Condition (7.5) means that the admissible profile ν is with increasing-visibility on the given interval (see [89]). It is indeed the case in the previous example.

Without going into the details, let us give a last example. If ν is an admissible profile which is concave, $\mathcal{L}^\nu$ can be described as a countable intersection of oscillation spaces (see [14]).

In the next chapter, we will endow $\mathcal{L}^\nu$ spaces with a natural topology, in a similar way as $\mathcal{S}^\nu$ spaces (see [8]). We will also study some classical topological properties like separability or compact subsets.

To finish this chapter, let us mention that, if we consider $\mathcal{L}^\nu$ as a function space (see the beginning of this chapter), the topology that we will define on $\mathcal{L}^\nu$ is a "good" topology, in the sense that it is also independent of the chosen orthonormal basis of wavelets (see [14]). This will allow to consider the space $\mathcal{L}^\nu$ as either a topological function space or a topological sequence space.

Chapter 8

Topology on $\mathcal{L}^\nu$ Spaces

In [8], $\mathcal{S}^\nu$ spaces are endowed with a natural topology. Some topological properties have been also studied (see also [5–7] for more information). The main elements have been recalled in Section 7.2.

In this chapter, we adapt most of results of [8] in the case of $\mathcal{L}^\nu$ spaces. More precisely, we first define a topology on $\mathcal{L}^\nu$ spaces. To do so, we introduce auxiliary spaces. We then study the compact subsets and the separability of $\mathcal{L}^\nu$. We finish by the comparison of the topologies of the spaces $\mathcal{S}^\nu$ and $\mathcal{L}^\nu$. The results presented in this chapter are from [14].

8.1 Auxiliary Spaces

As for the case of $\mathcal{S}^\nu$ spaces, a useful description can also be obtained by the introduction of auxiliary spaces. These new spaces will be used to define a topology on $\mathcal{L}^\nu$.

Definition 8.1.1. Let $\alpha \in \mathbb{R}$ and $\beta \in \{-\infty\} \cup [0, +\infty)$. A sequence $\vec{c} \in \Omega$ belongs to the auxiliary space $\widetilde{A}(\alpha, \beta)$ if there exist $C, C' \geq 0$ such that

$$\#\widetilde{E_j}(C, \alpha)(\vec{c}\,) \leq C' 2^{\beta j}$$

for all $j \in \mathbb{N}_0$.

Let us first note that the auxiliary spaces are vector spaces. To prove it, it suffices to adapt the proof of Proposition 7.3.4. For some particular β, we can identify the space $\widetilde{A}(\alpha, \beta)$. This is the object of the following remark.

Remark 8.1.2. (a) If $\beta = -\infty$, then $\widetilde{A}(\alpha, \beta)$ is the set of the sequences $\vec{c} \in \Omega$ such that $(2^{\alpha j} d_{j,k})_{(j,k)\in\Lambda}$ is bounded. In fact, we even have

$$\widetilde{A}(\alpha, -\infty) = \begin{cases} c^\alpha & \text{if } \alpha > 0 \\ c^0 & \text{if } \alpha \leq 0 \end{cases} .$$

Indeed, on the one hand, if $\alpha > 0$, it is clear that $\widetilde{A}(\alpha, -\infty) \subset c^\alpha$ because $|c_{j,k}| \leq d_{j,k}$ for all $(j, k) \in \Lambda$ and all $\vec{c} \in \Omega$. Moreover, if there exists $C > 0$ such that $2^{\alpha j}|c_{j,k}| \leq C$ for all $(j, k) \in \Lambda$, we have

$$|c_{j',k'}| \leq C 2^{-\alpha j'} \leq C 2^{-\alpha j}$$

for all $j' \geq j$ and $k' \in \{0, \ldots, 2^{j'} - 1\}$ and then $2^{\alpha j} d_{j,k} \leq C$ for all $(j, k) \in \Lambda$. So, $c^\alpha \subset \widetilde{A}(\alpha, -\infty)$. On the other hand, if $\alpha \leq 0$, we have $\widetilde{A}(\alpha, -\infty) \subset c^0$ because $|c_{j,k}| \leq 2^0 d_{0,0}$ for all $(j, k) \in \Lambda$ and $\vec{c} \in \Omega$. Moreover, we have the other inclusion because $2^{\alpha j} d_{j,k} \leq d_{0,0}$ for all $(j, k) \in \Lambda$ and $\vec{c} \in \Omega$.

(b) If $\beta \geq 1$, then $\widetilde{A}(\alpha,\beta) = \Omega$ since, for all $\vec{c} \in \Omega$ and all $j \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$ and $C > 0$, we have

$$\#\widetilde{E}_j(C,\alpha)(\vec{c}\,) \leq 2^j \leq 2^{\beta j}.$$

As for $\mathcal{S}^\nu$ spaces, we have the following result which allows to describe $\mathcal{L}^\nu$ spaces as a countable intersection (of auxiliary spaces). This proof is a simple adaptation of the proof of Theorem 5.4 in [8].

Proposition 8.1.3. *For any dense sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and any sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ of strictly positive numbers which converges to 0, we have*

$$\mathcal{L}^\nu = \bigcap_{\varepsilon>0}\bigcap_{\alpha\in\mathbb{R}} \widetilde{A}(\alpha,\nu(\alpha)+\varepsilon) = \bigcap_{m\in\mathbb{N}}\bigcap_{n\in\mathbb{N}} \widetilde{A}(\alpha_n,\nu(\alpha_n)+\varepsilon_m).$$

Proof. Let us show the following inclusions:

$$\mathcal{L}^\nu \subset \bigcap_{\varepsilon>0}\bigcap_{\alpha\in\mathbb{R}} \widetilde{A}(\alpha,\nu(\alpha)+\varepsilon) \subset \bigcap_{m\in\mathbb{N}}\bigcap_{n\in\mathbb{N}} \widetilde{A}(\alpha_n,\nu(\alpha_n)+\varepsilon_m) \subset \mathcal{L}^\nu\,.$$

1. For the first inclusion, let $\vec{c} \in \mathcal{L}^\nu$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. By Proposition 7.3.3, there exists $J \in \mathbb{N}_0$ such that

$$\#\widetilde{E}_j(1,\alpha)(\vec{c}\,) \leq 2^{(\nu(\alpha)+\varepsilon)j}$$

for all $j \geq J$. Moreover, there exists $C' > 0$ such that $2^{\alpha j} d_{j,k} < C'$ for all $j \in \{0,\dots,J-1\}$ and $k \in \{0,\dots,2^j-1\}$. Then,

$$\#\widetilde{E}_j(C',\alpha)(\vec{c}\,) = 0 \leq 2^{(\nu(\alpha)+\varepsilon)j}$$

for all $j \in \{0,\dots,J-1\}$. Consequently, setting $C := \max\{C',1\}$, we have

$$\#\widetilde{E}_j(C,\alpha)(\vec{c}\,) \leq 2^{(\nu(\alpha)+\varepsilon)j}$$

for all $j \in \mathbb{N}_0$ and thus, $\vec{c} \in \widetilde{A}(\alpha,\nu(\alpha)+\varepsilon)$. We so have the first inclusion.

2. The second inclusion is evident.

3. For the third inclusion, let $\vec{c} \in \widetilde{A}(\alpha_n,\nu(\alpha_n)+\varepsilon_m)$ for all $m,n \in \mathbb{N}$. Let us fix $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$. Let us consider the two following cases.

(a) If $\nu(\alpha) = -\infty$, then there exists $n \in \mathbb{N}$ such that $\nu(\alpha_n) = -\infty$ and that $\alpha_n > \alpha$ by hypothesis. Then, $\vec{c} \in \widetilde{A}(\alpha_n,-\infty)$ and there exists $C' > 0$ such that $d_{j,k} \leq C'2^{-\alpha_n j}$ for all $(j,k) \in \Lambda$. Moreover, there exists $J \in \mathbb{N}_0$ such that $C'2^{(\alpha-\alpha_n)j} < C$ for all $j \geq J$, and so $d_{j,k} < C2^{-\alpha j}$ for all $j \geq J$ and $k \in \{0,\dots,2^j-1\}$. Thus, for $j \geq J$, we have

$$\#\widetilde{E}_j(C,\alpha)(\vec{c}\,) = 0 \leq 2^{(\nu(\alpha)+\varepsilon)j}.$$

(b) If $\nu(\alpha) \in [0,1]$, there exist $m,n \in \mathbb{N}$ such that

$$\alpha_n > \alpha, \quad 3\varepsilon_m \leq \varepsilon \quad \text{and} \quad \nu(\alpha) \leq \nu(\alpha_n) \leq \nu(\alpha)+\varepsilon_m$$

by hypothesis. Since $\vec{c} \in \widetilde{A}(\alpha_n,\nu(\alpha_n)+\varepsilon_m)$, there exist $C_0, C_0' \geq 0$ such that

$$\#\widetilde{E}_j(C_0,\alpha_n) \leq C_0' 2^{(\nu(\alpha_n)+\varepsilon_m)j}$$

for all $j \in \mathbb{N}_0$. Moreover, there exists $J \in \mathbb{N}_0$ such that $C_0 2^{-\alpha_n j} \leq C2^{-\alpha j}$ and that $C_0' \leq 2^{j\varepsilon/3}$ for all $j \geq J$. Consequently, for $j \geq J$, we have

$$\#\widetilde{E}_j(C,\alpha)(\vec{c}\,) \leq \#\widetilde{E}_j(C_0,\alpha_n)(\vec{c}\,) \leq C_0' 2^{(\nu(\alpha_n)+\varepsilon_m)j} \leq 2^{j\frac{\varepsilon}{3}}\, 2^{j(\nu(\alpha)+\frac{2\varepsilon}{3})} \leq 2^{j(\nu(\alpha)+\varepsilon)}.$$

Thus, $\vec{c} \in \mathcal{L}^\nu$ by Proposition 7.3.3. Hence the conclusion.

■

Let us now define a distance on these auxiliary spaces. The proof is adapted from the proof of Lemma 3.3 in [8] to the case of wavelet leaders.

Definition 8.1.4. Let $\alpha \in \mathbb{R}$ and $\beta \in \{-\infty\} \cup [0, +\infty)$. For $\vec{c}, \vec{c}\,' \in \widetilde{A}(\alpha, \beta)$, we write

$$\widetilde{\delta}_{\alpha,\beta}(\vec{c}, \vec{c}\,') := \inf \left\{ C + C' : C, C' \geq 0 \text{ and } \# \widetilde{E_j}(C, \alpha)(\vec{c} - \vec{c}\,') \leq C' 2^{\beta j} \ \forall j \in \mathbb{N}_0 \right\}.$$

Lemma 8.1.5. *For $\alpha \in \mathbb{R}$ and $\beta \in \{-\infty\} \cup [0, +\infty)$, $\widetilde{\delta}_{\alpha,\beta}$ is a distance on $\widetilde{A}(\alpha, \beta)$ which is invariant by translation and which satisfies*

$$\widetilde{\delta}_{\alpha,\beta}(\theta \vec{c}, \vec{0}) \leq \max\{1, |\theta|\} \, \widetilde{\delta}_{\alpha,\beta}(\vec{c}, \vec{0}) \tag{8.1}$$

for all $\vec{c} \in \widetilde{A}(\alpha, \beta)$ and $\theta \in \mathbb{C}$.

Proof. 1. By definition, it is clear that $\widetilde{\delta}_{\alpha,\beta}$ is positive, symmetric and invariant by translation.

2. Let us show that if $\widetilde{\delta}_{\alpha,\beta}(\vec{c}, \vec{c}\,') = 0$ for $\vec{c}, \vec{c}\,' \in \widetilde{A}(\alpha, \beta)$, then $\vec{c} = \vec{c}\,'$. Thanks to the translation invariance, it suffices to prove it for $\vec{c}\,' = \vec{0}$. Let $\vec{c} \in \widetilde{A}(\alpha, \beta)$ be such that $\widetilde{\delta}_{\alpha,\beta}(\vec{c}, \vec{0}) = 0$. By hypothesis, for all $\eta > 0$, there exist $C, C' \geq 0$ such that $C + C' \leq \eta$ and that

$$\#\{k \in \{0, \dots, 2^j - 1\} : d_{j,k} \geq C 2^{-\alpha j}\} \leq C' 2^{\beta j}$$

for all $j \in \mathbb{N}_0$. Let us take $j_0 \in \mathbb{N}_0$, $\varepsilon \in (0, 1)$ and $\eta := \min\{\varepsilon 2^{-\beta j_0}, \varepsilon 2^{\alpha j_0}\}$. Then, we have

$$\#\{k \in \{0, \dots, 2^j - 1\} : d_{j_0,k} \geq C 2^{-\alpha j_0}\} \leq C' 2^{\beta j_0} \leq \varepsilon < 1$$

and then $d_{j_0,k} < C 2^{-\alpha j_0} \leq \varepsilon$ for all $k \in \{0, \dots, 2^{j_0} - 1\}$. As ε and j_0 are chosen arbitrarily, we obtain $d_{j_0,k} = 0$ for all $(j_0, k) \in \Lambda$. Hence $\vec{c} = \vec{0}$.

3. Let us prove the triangle inequality. With the translation invariance, it suffices to show that

$$\widetilde{\delta}_{\alpha,\beta}(\vec{c} - \vec{c}\,', \vec{0}) \leq \widetilde{\delta}_{\alpha,\beta}(\vec{c}, \vec{0}) + \widetilde{\delta}_{\alpha,\beta}(\vec{c}\,', \vec{0})$$

for all $\vec{c}, \vec{c}\,' \in \widetilde{A}(\alpha, \beta)$. By definition of $\widetilde{\delta}_{\alpha,\beta}$, for all $\eta > 0$, there exist $C_1, C_1', C_2, C_2' \geq 0$ such that $C_1 + C_1' \leq \eta/2 + \widetilde{\delta}_{\alpha,\beta}(\vec{c}, \vec{0})$, $C_2 + C_2' \leq \eta/2 + \widetilde{\delta}_{\alpha,\beta}(\vec{c}\,', \vec{0})$,

$$\# \widetilde{E_j}(C_1, \alpha)(\vec{c}\,) \leq C_1' 2^{\beta j} \quad \text{and} \quad \# \widetilde{E_j}(C_2, \alpha)(\vec{c}\,') \leq C_2' 2^{\beta j}$$

for all $j \in \mathbb{N}_0$.

Let us fix $j \in \mathbb{N}_0$. If $k \notin \widetilde{E_j}(C_1, \alpha)(\vec{c}\,) \cup \widetilde{E_j}(C_2, \alpha)(\vec{c}\,')$, we have

$$\sup_{\lambda' \subset \lambda(j,k)} |c_{\lambda'} - c'_{\lambda'}| \leq d_{j,k} + d'_{j,k} < (C_1 + C_2) 2^{-\alpha j},$$

that means that $k \notin \widetilde{E_j}(C_1 + C_2, \alpha)(\vec{c} - \vec{c}\,')$. We so obtain

$$\widetilde{E_j}(C_1 + C_2, \alpha)(\vec{c} - \vec{c}\,') \subset \left(\widetilde{E_j}(C_1, \alpha)(\vec{c}\,) \cup \widetilde{E_j}(C_2, \alpha)(\vec{c}\,') \right).$$

Then, we have

$$\# \widetilde{E_j}(C_1 + C_2, \alpha)(\vec{c} - \vec{c}\,') \leq \# \widetilde{E_j}(C_1, \alpha)(\vec{c}\,) + \# \widetilde{E_j}(C_2, \alpha)(\vec{c}\,') \leq (C_1' + C_2') 2^{\beta j}.$$

Consequently, we successively have

$$\widetilde{\delta}_{\alpha,\beta}(\vec{c}-\vec{c}\,',\vec{0}) \le (C_1+C_2)+(C_1'+C_2') \le \eta+\widetilde{\delta}_{\alpha,\beta}(\vec{c},\vec{0})+\widetilde{\delta}_{\alpha,\beta}(\vec{c}\,',\vec{0})$$

and the conclusion follows since η is chosen arbitrarily.

With these three points, we can conclude that $\widetilde{\delta}_{\alpha,\beta}$ is a distance on $\widetilde{A}(\alpha,\beta)$.

4. To finish, let us show Inequality (8.1). Let $\vec{c}\in\widetilde{A}(\alpha,\beta)$ and $\theta\in\mathbb{C}$. If $|\theta|\le 1$, we directly have $\widetilde{\delta}_{\alpha,\beta}(\theta\vec{c},\vec{0})\le\widetilde{\delta}_{\alpha,\beta}(\vec{c},\vec{0})$ because

$$\#\left\{k\in\{0,\dots,2^j-1\}:\sup_{\lambda'\subset\lambda(j,k)}|\theta c_{\lambda'}|\ge C2^{-\alpha j}\right\}\le\#\left\{k\in\{0,\dots,2^j-1\}:d_{j,k}\ge C2^{-\alpha j}\right\}$$

for all $j\in\mathbb{N}_0$ and all $C>0$. If $|\theta|>1$, we have $\widetilde{\delta}_{\alpha,\beta}(\theta\vec{c},\vec{0})\le|\theta|\widetilde{\delta}_{\alpha,\beta}(\vec{c},\vec{0})$ because

$$\left\{k\in\{0,\dots,2^j-1\}:d_{j,k}\ge C2^{-\alpha j}\right\}=\left\{k\in\{0,\dots,2^j-1\}:\sup_{\lambda'\subset\lambda(j,k)}|\theta c_{\lambda'}|\ge C|\theta|2^{-\alpha j}\right\}$$

for all $j\in\mathbb{N}_0$ and $C>0$. ■

If $\beta=-\infty$, then $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$ is the topological normed space $(c^\alpha,\|\cdot\|_{c^\alpha})$ if $\alpha>0$ and $(c^0,\|\cdot\|_{c^0})$ if $\alpha\le 0$. Moreover, if $\beta\ge 1$, we have $\widetilde{\delta}_{\alpha,\beta}\le 1$. In the following proposition, we also get more information about the topology in the case $\beta>1$. The proofs of some points are similar to the ones of Proposition 3.5 in [8].

For auxiliary spaces of $\mathcal{S}^\nu$, it is known that the topology defined by $\delta_{\alpha,\beta}$ is stronger than the pointwise topology; these topologies are equivalent when $\beta>1$. In the $\mathcal{L}^\nu$ case, the topology defined by $\widetilde{\delta}_{\alpha,\beta}$ is also stronger than the pointwise topology. In fact, it is even stronger than the *uniform topology*, i.e. the topology defined by the norm of c^0. The equivalence with uniform topology happens if $\beta>1$.

Proposition 8.1.6. *Let $\alpha\in\mathbb{R}$ and $\beta\in\{-\infty\}\cup[0,+\infty[$.*

(a) The addition is continuous on $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$.

(b) The space $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$ has a stronger topology than the uniform topology. Moreover, every Cauchy sequence in $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$ is also a uniform Cauchy sequence.

(c) If $\beta>1$, the topology defined by the distance $\widetilde{\delta}_{\alpha,\beta}$ is equivalent to the uniform topology.

(d) (i) If B is a bounded set of $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$, then there exists $r>0$ such that

$$\begin{aligned}B &\subset \left\{\vec{c}\in\Omega:\#\{k\in\{0,\dots,2^j-1\}:d_{j,k}\ge r\,2^{-\alpha j}\}\le r\,2^{\beta j}\ \forall j\in\mathbb{N}_0\right\}\\ &\subset \left\{\vec{c}\in\Omega:\#\{k\in\{0,\dots,2^j-1\}:d_{j,k}> r\,2^{-\alpha j}\}\le r\,2^{\beta j}\ \forall j\in\mathbb{N}_0\right\}.\end{aligned}$$

(ii) Let $r,r'\ge 0$, $\alpha'\ge\alpha$ and $\beta'\le\beta$. The set

$$B:=\left\{\vec{c}\in\Omega:\#\{k\in\{0,\dots,2^j-1\}:d_{j,k}> r\,2^{-\alpha' j}\}\le r'\,2^{\beta' j}\ \forall j\in\mathbb{N}_0\right\}$$

is a bounded set of $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$. Moreover, B is closed for the uniform convergence.

(e) The space $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$ is a complete metric space.

Proof. (a) The first point is obvious using the triangle inequality with the distance $\widetilde{\delta}_{\alpha,\beta}$.

(b) Let $(\vec{c}^{\,(m)})_{m\in\mathbb{N}}$ be a sequence of elements of $\widetilde{A}(\alpha,\beta)$ which converges to $\vec{c}$ in $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$. If $\beta=-\infty$, it suffices to observe that we have

$$\sup_{(j,k)\in\Lambda}|c^{(m)}_{j,k}-c_{j,k}| = 2^{\alpha 0}\sup_{\lambda'\subset\lambda(0,0)}|c^{(m)}_{\lambda'}-c_{\lambda'}| \leq \sup_{(j,k)\in\Lambda} 2^{\alpha j}\sup_{\lambda'\subset\lambda(j,k)}|c^{(m)}_{\lambda'}-c_{\lambda'}| = \widetilde{\delta}_{\alpha,-\infty}(\vec{c}^{\,(m)},\vec{c})$$

for every $m\in\mathbb{N}$. Let us consider now the case $\beta\geq 0$. Let $\varepsilon>0$ and $\eta:=\min\{\frac{1}{2},\varepsilon\}$. By hypothesis, there exists $M\in\mathbb{N}$ such that

$$\#\left\{k\in\{0,\dots,2^j-1\} : \sup_{\lambda'\subset\lambda(j,k)}|c^{(m)}_{\lambda'}-c_{\lambda'}|\geq \eta 2^{-\alpha j}\right\}\leq \eta 2^{\beta j}$$

for all $j\in\mathbb{N}_0$ and $m\geq M$. Consequently, taking $j=0$, we obtain for all $m\geq M$,

$$\sup_{(j_0,k_0)\in\Lambda}|c^{(m)}_{j_0,k_0}-c_{j_0,k_0}| = \sup_{\lambda'\subset\lambda(0,0)}|c^{(m)}_{\lambda'}-c_{\lambda'}|<\eta\leq\varepsilon.$$

The proof is similar for Cauchy sequences.

(c) With the previous point, it only remains to show that the uniform topology is stronger than the topology defined by the distance $\widetilde{\delta}_{\alpha,\beta}$ (in the case $\beta>1$). Let $(\vec{c}^{\,(m)})_{m\in\mathbb{N}}$ be a sequence of $\widetilde{A}(\alpha,\beta)=\Omega$ which converges uniformly to $\vec{c}$ and let $\varepsilon>0$. There exists $J\in\mathbb{N}_0$ such that $2^j\leq\varepsilon 2^{\beta j}$ for every $j\geq J$ because $\beta>1$ and then we have

$$\#\left\{k\in\{0,\dots,2^j-1\} : \sup_{\lambda'\subset\lambda(j,k)}|c^{(m)}_{\lambda'}-c_{\lambda'}|\geq \varepsilon 2^{-\alpha j}\right\}\leq 2^j\leq \varepsilon 2^{\beta j}$$

for every $j\geq J$ and $m\in\mathbb{N}$. Let us now fix $j\in\{0,\dots,J-1\}$. Using the uniform convergence, there exists $M\in\mathbb{N}$ (which only depends on ε) such that

$$\sup_{\lambda'\subset\lambda(j,k)}|c^{(m)}_{\lambda'}-c_{\lambda'}|<\varepsilon 2^{-\alpha j}$$

for every $k\in\{0,\dots,2^j-1\}$ and $m\geq M$. So, for every $m\geq M$, we have

$$\#\left\{k\in\{0,\dots,2^j-1\} : \sup_{\lambda'\subset\lambda(j,k)}|c^{(m)}_{\lambda'}-c_{\lambda'}|\geq \varepsilon 2^{-\alpha j}\right\}=0\leq \varepsilon 2^{\beta j}.$$

Consequently, we have $\widetilde{\delta}_{\alpha,\beta}(\vec{c}^{\,(m)},\vec{c})\leq 2\varepsilon$ for all $m\geq M$ and thus $(\vec{c}^{\,(m)})_{m\in\mathbb{N}}$ converges to $\vec{c}$ in $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$.

(d)(i) The second inclusion is clear. Let us prove the first inclusion. Since B is a bounded set in the metric space $(\widetilde{A}(\alpha,\beta),\widetilde{\delta}_{\alpha,\beta})$, there exists $C>0$ such that $\widetilde{\delta}_{\alpha,\beta}(\vec{x},\vec{y})<C$ for all $\vec{x},\vec{y}\in B$. Let $\vec{x}\in B$ be such that $\widetilde{\delta}_{\alpha,\beta}(\vec{x},\vec{0})\leq C$. By the triangle inequality, we then have

$$\widetilde{\delta}_{\alpha,\beta}(\vec{c},\vec{0})\leq\widetilde{\delta}_{\alpha,\beta}(\vec{c},\vec{x})+\widetilde{\delta}_{\alpha,\beta}(\vec{x},\vec{0})<2C$$

for all $\vec{c}\in B$. Consequently, we obtain

$$B\subset\left\{\vec{c}\in\Omega : \#\{k\in\{0,\dots,2^j-1\} : d_{j,k}\geq C\,2^{-\alpha j}\}\leq C\,2^{\beta j}\ \forall j\in\mathbb{N}_0\right\}.$$

(d)(ii) By definition and by hypothesis, it is clear that $B \subset \widetilde{A}(\alpha, \beta)$. By the triangle inequality again, we have

$$\widetilde{\delta}_{\alpha,\beta}(\vec{x}, \vec{y}) \leq \widetilde{\delta}_{\alpha,\beta}(\vec{x}, \vec{0}) + \widetilde{\delta}_{\alpha,\beta}(\vec{y}, \vec{0}) \leq 2(r + r')$$

for all $\vec{x}, \vec{y} \in B$ and then, B is bounded in $(\widetilde{A}(\alpha, \beta), \widetilde{\delta}_{\alpha,\beta})$. Let us now show that B is closed for the uniform convergence. Let $(\vec{c}^{\,(m)})_{m\in\mathbb{N}}$ be a sequence of B which converges uniformly to $\vec{c}$ and let $\varepsilon > 0$. Then, there exists $M \in \mathbb{N}$ such that

$$\sup_{\lambda' \subset \lambda(0,0)} |c_{\lambda'}^{(m)} - c_{\lambda'}| < \varepsilon$$

for all $m \geq M$. For $(j, k) \in \Lambda$, we have

$$d_{j,k} > r2^{-\alpha' j} \quad \Rightarrow \quad d_{j,k}^{(M)} > r2^{-\alpha' j}.$$

Otherwise, $d_{j,k}^{(M)} \leq r2^{-\alpha' j}$ and then, taking ε smaller if needed, we have

$$r2^{-\alpha' j} < d_{j,k} - \varepsilon \leq \sup_{\lambda' \subset \lambda(j,k)} |c_{\lambda'}^{(M)} - c_{\lambda'}| + d_{j,k}^{(M)} - \varepsilon \leq r2^{-\alpha' j},$$

which is absurd. So $\vec{c} \in B$ because

$$\#\{k \in \{0, \ldots, 2^j - 1\} : d_{j,k} > r2^{-\alpha' j}\} \leq \#\{k \in \{0, \ldots, 2^j - 1\} : d_{j,k}^{(M)} > r2^{-\alpha' j}\} \leq r'2^{\beta' j}$$

for all $j \in \mathbb{N}_0$.

(e) Since $(\widetilde{A}(\alpha, \beta), \widetilde{\delta}_{\alpha,\beta})$ is a metric space, it only remains to show that if $(\vec{c}^{\,(m)})_{m\in\mathbb{N}}$ is a Cauchy sequence in $(\widetilde{A}(\alpha, \beta), \widetilde{\delta}_{\alpha,\beta})$, it converges in $(\widetilde{A}(\alpha, \beta), \widetilde{\delta}_{\alpha,\beta})$. From Item (b) of this proposition, $(\vec{c}^{\,(m)})_{m\in\mathbb{N}}$ is also a uniform Cauchy sequence and then it converges uniformly to $\vec{c}$. By hypothesis, if $\eta > 0$, there exists $M \in \mathbb{N}$ such that

$$\#\left\{k \in \{0, \ldots, 2^j - 1\} : \sup_{\lambda' \subset \lambda(j,k)} |c_{\lambda'}^{(p)} - c_{\lambda'}^{(q)}| > \eta 2^{-\alpha j}\right\} \leq \eta 2^{\beta j}$$

for all $j \in \mathbb{N}_0$ and for all $p, q \geq M$. Then, $\vec{c}^{\,(q)}$ belongs to the set

$$\left\{\vec{a} \in \Omega : \#\left\{k \in \{0, \ldots, 2^j - 1\} : \sup_{\lambda' \subset \lambda(j,k)} |c_{\lambda'}^{(p)} - a_{\lambda'}| > \eta 2^{-\alpha j}\right\} \leq \eta 2^{\beta j}\ \forall j \in \mathbb{N}_0\right\}$$

for all $p, q \geq M$. As the previous set is closed for the uniform convergence (it is similar to the last part of the proof of Item (d) of this proposition), $\vec{c}$ also belongs to

$$\left\{\vec{a} \in \Omega : \#\left\{k \in \{0, \ldots, 2^j - 1\} : \sup_{\lambda' \subset \lambda(j,k)} |c_{\lambda'}^{(p)} - a_{\lambda'}| > \eta 2^{-\alpha j}\right\} \leq \eta 2^{\beta j}\ \forall j \in \mathbb{N}_0\right\}$$

for all $p \geq M$. Thus, $\vec{c} \in \widetilde{A}(\alpha, \beta)$ and $\widetilde{\delta}_{\alpha,\beta}(\vec{c}^{\,(p)}, \vec{c}) \leq 2\eta$ for all $p \geq M$. Hence the conclusion. ■

Remark 8.1.7. If $\beta \in [0, 1]$ and $\alpha > 0$, the scalar multiplication

$$(\theta, \vec{c}) \in \mathbb{C} \times \widetilde{A}(\alpha, \beta) \mapsto \theta\vec{c} \in \widetilde{A}(\alpha, \beta)$$

is not continuous and consequently, the space $(\widetilde{A}(\alpha, \beta), \widetilde{\delta}_{\alpha,\beta})$ is not a topological vector space.

Indeed, let $\vec{c}$ be the sequence defined by

$$c_{j,k} := \begin{cases} j2^{-\alpha j} & \text{if } k \in \{0, \dots, \lfloor 2^{\beta j} \rfloor - 1\} \\ 0 & \text{if } k \in \{\lfloor 2^{\beta j} \rfloor, \dots, 2^j - 1\} \end{cases}$$

for $j \in \mathbb{N}_0$. From some scale, this sequence is strictly decreasing. Moreover, for large scale j, we have $\lfloor 2^{\beta(j+1)} \rfloor / 2 \leq \lfloor 2^{\beta j} \rfloor$, which implies that we do not have non-zero coefficients in a dyadic interval $\lambda(j,k)$ with $k \in \{0, \dots, 2^j - 1\}$ where $c_{j,k} = 0$. In other words, there exists $J \in \mathbb{N}_0$ such that $d_{j,k} = c_{j,k}$ for all $j \geq J$ and $k \in \{0, \dots, 2^j - 1\}$ and so,

$$\# \widetilde{E_j}(C, \alpha)(\vec{c}\,) \leq \lfloor 2^{\beta j} \rfloor \leq 2^{\beta j}$$

for all $j \geq J$. For $j \in \{0, \dots, J-1\}$, we have

$$\# \widetilde{E_j}(C, \alpha)(\vec{c}\,) \leq 2^j \leq 2^{\beta j}\, 2^{(1-\beta)J}.$$

Thus, setting $C' := 2^{(1-\beta)J} \geq 1$, we have

$$\# \widetilde{E_j}(C, \alpha)(\vec{c}\,) \leq C'\, 2^{\beta j}$$

for all $j \in \mathbb{N}_0$ and $\vec{c} \in \widetilde{A}(\alpha, \beta)$.

Let us now prove that the sequence $(\vec{c}/m)_{m \in \mathbb{N}}$ does not converge to $\vec{0}$ in $(\widetilde{A}(\alpha, \beta), \widetilde{\delta}_{\alpha,\beta})$, following the idea of Proposition 3.5 in [8]. By contradiction, let us assume that we have the convergence. Then, there exists $M \geq J$ such that

$$\# \left\{ k \in \{0, \dots, 2^j - 1\} : \frac{1}{m}\, d_{j,k} \geq \frac{1}{2}\, 2^{-\alpha j} \right\} \leq \frac{1}{2}\, 2^{\beta j}$$

for all $m \geq M$ and $j \in \mathbb{N}_0$. Taking $j = m$, we have

$$\# \left\{ k \in \{0, \dots, 2^m - 1\} : \frac{1}{m}\, c_{m,k} \geq \frac{1}{2}\, 2^{-\alpha m} \right\} \leq \frac{1}{2}\, 2^{\beta m}$$

and then

$$\lfloor 2^{\beta m} \rfloor \leq \frac{1}{2}\, 2^{\beta m}$$

for all $m \geq M$. Hence a contradiction. If $\beta = 0$, it is clear. If $\beta \in (0, 1]$, we actually have $m \leq 1/\beta$ and we have the contradiction if we assume that M is also strictly greater than $1/\beta$.

This counterexample also shows that the topology defined by $\widetilde{\delta}_{\alpha,\beta}$ and the uniform topology are not equivalent for such β and α.

Let us end this section with some relations between auxiliary spaces. The second part is useful to obtain the continuity of the scalar multiplication in $\mathcal{L}^\nu$.

Lemma 8.1.8. *(a) If $\alpha \geq \alpha'$ and $\beta \leq \beta'$, then*

$$\widetilde{A}(\alpha, \beta) \subset \widetilde{A}(\alpha', \beta') \quad \textit{and} \quad \widetilde{\delta}_{\alpha',\beta'} \leq \widetilde{\delta}_{\alpha,\beta}.$$

(b) Let $\alpha' > \alpha$ and $\beta' < \beta$. If the sequence $(\theta_m)_{m \in \mathbb{N}}$ converges to θ in $\mathbb{C}$ and if the sequence $(\vec{c}^{\,(m)})_{m \in \mathbb{N}}$ of c^0 converges to $\vec{c}$ in $(\widetilde{A}(\alpha, \beta), \widetilde{\delta}_{\alpha,\beta})$ with $\vec{c} \in \widetilde{A}(\alpha', \beta')$, then the sequence $(\theta_m \vec{c}^{\,(m)})_{m \in \mathbb{N}}$ converges to $\theta \vec{c}$ in $(\widetilde{A}(\alpha, \beta), \widetilde{\delta}_{\alpha,\beta})$.

Proof. The first item is obvious. The second one is similar to the one given for the $\mathcal{S}^\nu$ case (see Lemma 4.2 in [8]). Since the sequence $(\theta_m)_{m\in\mathbb{N}}$ converges to θ in $\mathbb{C}$, there exists $D>0$ such that $|\theta_m-\theta|\leq D$ for all $m\in\mathbb{N}$. We have

$$\theta_m\vec{c}^{\,(m)}-\theta\vec{c}=(\theta_m-\theta)(\vec{c}^{\,(m)}-\vec{c})-\theta(\vec{c}^{\,(m)}-\vec{c})+(\theta_m-\theta)\vec{c}$$

and then

$$\widetilde{\delta}_{\alpha,\beta}(\theta_m\vec{c}^{\,(m)},\theta\vec{c})\leq\max\{1,D\}\,\widetilde{\delta}_{\alpha,\beta}(\vec{c}^{\,(m)},\vec{c})+\max\{1,|\theta|\}\,\widetilde{\delta}_{\alpha,\beta}(\vec{c}^{\,(m)},\vec{c})+\widetilde{\delta}_{\alpha,\beta}((\theta_m-\theta)\vec{c},\vec{0})$$

thanks to Lemma 8.1.5. The two first terms converge to 0, using hypotheses and the first point of this lemma. Let us now consider the convergence of the third term. Since $\vec{c}\in\widetilde{A}(\alpha',\beta')$, there exist $C,C'\geq 0$ such that

$$\#\widetilde{E_j}(C,\alpha')(\vec{c})\leq C'2^{\beta' j}$$

for all $j\in\mathbb{N}_0$. Let $\eta>0$. Then, there exists $J\in\mathbb{N}_0$ such that $DC2^{-j(\alpha'-\alpha)}\leq\eta$ and $C'2^{-j(\beta-\beta')}\leq\eta$ for all $j\geq J$. Consequently, we have, for all $j\geq J$ and $m\in\mathbb{N}$,

$$\#\left\{k\in\{0,\dots,2^j-1\}:|\theta_m-\theta|\,d_{j,k}\geq\eta 2^{-\alpha j}\right\}\leq\eta 2^{\beta j}$$

because $|\theta_m-\theta|\leq D$ for all $m\in\mathbb{N}$. Since the sequence $(\theta_m)_{m\in\mathbb{N}}$ converges to θ and that $\vec{c}\in c^0$, there exists $M\in\mathbb{N}$ such that

$$|\theta_m-\theta|d_{j,k}<\eta 2^{-\alpha j}$$

for all $m\geq M$, $j\in\{0,\dots,J-1\}$ and $k\in\{0,\dots,2^j-1\}$. Hence $\widetilde{\delta}_{\alpha,\beta}((\theta_m-\theta)\vec{c},\vec{0})\leq 2\eta$ for all $m\geq M$ and we get the conclusion. ■

Remark 8.1.9. (a) The assumption that the sequences belong to c^0 will not be restrictive because we know that $\mathcal{L}^\nu\subset c^0$ (see Proposition 7.3.5).

(b) If $\beta=\beta'=-\infty$, this lemma remains true.

8.2 Topology on $\mathcal{L}^\nu$

By Proposition 8.1.3, we know that $\mathcal{L}^\nu$ is a countable intersection of auxiliary spaces. As in the case of $\mathcal{S}^\nu$ spaces, this description allows to obtain a structure of complete metric space on $\mathcal{L}^\nu$. Indeed, the idea is to use the following classical result of functional analysis (see for example [72]) to define a topology on $\mathcal{L}^\nu$.

Proposition 8.2.1. *For $m\in\mathbb{N}$, let E_m be a space endowed with the topology defined by the distance d_m. Let us set $E:=\bigcap_{m\in\mathbb{N}}E_m$. On E, let us consider the topology τ defined as follows: for every $e\in E$, a basis of neighbourhoods of e is given by the family of sets*

$$\bigcap_{(m)}\{f\in E:d_m(e,f)\leq r_m\}$$

where $r_m>0$ for every $m\in\mathbb{N}$ and (m) means that it is an intersection on a finite number of values of m. Then, this topology satisfies the following properties.

(a) For every $m\in\mathbb{N}$, the identity $i:(E,\tau)\to(E_m,d_m)$ is continuous and τ is the weakest topology on E which verifies this property.

(b) *The topology τ is equivalent to the topology defined on E by the distance d given by*

$$d(e,f) := \sum_{m=1}^{+\infty} 2^{-m} \frac{d_m(e,f)}{1+d_m(e,f)}, \quad e,f \in E.$$

(c) *A sequence is a Cauchy sequence in (E,τ) if and only if it is a Cauchy sequence in (E_m, d_m) for every $m \in \mathbb{N}$.*

(d) *A sequence converges to e in (E,τ) if and only if it converges to e in (E_m, d_m) for every $m \in \mathbb{N}$.*

Using some properties of the auxiliary spaces $(\widetilde{A}(\alpha,\beta), \widetilde{\delta}_{\alpha,\beta})$ and Proposition 8.2.1, we can define a distance on the spaces $\mathcal{L}^\nu$ and obtain some additional information on these spaces. The reasoning is an adaptation of Section 5 in [8].

Definition 8.2.2. Let $\boldsymbol{\alpha} := (\alpha_n)_{n\in\mathbb{N}}$ be a dense sequence in $\mathbb{R}$ and $\boldsymbol{\varepsilon} := (\varepsilon_m)_{m\in\mathbb{N}}$ be a sequence of $(0,+\infty)$ which converges to 0. We denote

$$\widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}} := \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} 2^{-(m+n)} \frac{\widetilde{\delta}_{\alpha_n,\nu(\alpha_n)+\varepsilon_m}}{1+\widetilde{\delta}_{\alpha_n,\nu(\alpha_n)+\varepsilon_m}}.$$

Proposition 8.2.3. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ be sequences chosen as above.*

(a) *The application $\widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}}$ is a distance on $\mathcal{L}^\nu$.*

(b) *The topology defined by $\widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}}$ on $\mathcal{L}^\nu$ is the weakest topology such that, for every $m,n \in \mathbb{N}$, the identity $i : \mathcal{L}^\nu \to \widetilde{A}(\alpha_n, \nu(\alpha_n)+\varepsilon_m)$ is continuous.*

(c) *A sequence in $\mathcal{L}^\nu$ is a Cauchy sequence in $(\mathcal{L}^\nu, \widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}})$ if and only if, for every $m,n \in \mathbb{N}$, it is a Cauchy sequence in $(\widetilde{A}(\alpha_n,\nu(\alpha_n)+\varepsilon_m), \widetilde{\delta}_{\alpha_n,\nu(\alpha_n)+\varepsilon_m})$.*

(d) *A sequence in $\mathcal{L}^\nu$ converges in $(\mathcal{L}^\nu, \widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}})$ if and only if, for every $m,n \in \mathbb{N}$, it converges in $(\widetilde{A}(\alpha_n,\nu(\alpha_n)+\varepsilon_m), \widetilde{\delta}_{\alpha_n,\nu(\alpha_n)+\varepsilon_m})$.*

(e) *The space $(\mathcal{L}^\nu, \widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}})$ is a complete topological metric space.*

Proof. The four first items are simply consequences of Proposition 8.2.1 and of some results concerning auxiliary spaces (see Section 8.1). Let us prove the last item.

It is clear that the addition is continuous in $(\mathcal{L}^\nu, \widetilde{\delta})$ thanks to Item (a) of Proposition 8.1.6 and the second item of this proposition. Let us show that the scalar multiplication $(\theta, \vec{c}) \in \mathbb{C}\times\mathcal{L}^\nu \mapsto \theta\vec{c} \in \mathcal{L}^\nu$ is also continuous in $(\mathcal{L}^\nu, \widetilde{\delta})$. Let $(\theta_l)_{l\in\mathbb{N}}$ be a sequence of $\mathbb{C}$ which converges to θ and let $(\vec{c}^{(l)})_{l\in\mathbb{N}}$ be a sequence of $\mathcal{L}^\nu$ which converges to $\vec{c}$ in $(\mathcal{L}^\nu, \widetilde{\delta})$. If $(\theta_l\vec{c}^{(l)})_{l\in\mathbb{N}}$ converges to $\theta\vec{c}$ in $(\widetilde{A}(\alpha,\nu(\alpha)+\varepsilon), \widetilde{\delta}_{\alpha,\nu(\alpha)+\varepsilon})$ for all $\alpha\in\mathbb{R}$ and all $\varepsilon>0$, we have the conclusion thanks to Item (d) of this proposition. Let us fix $\alpha\in\mathbb{R}$ and $\varepsilon>0$. Then, there exist $m,n\in\mathbb{N}$ such that

$$\varepsilon_m < \varepsilon, \quad \alpha_n > \alpha \quad \text{and} \quad \nu(\alpha_n)+\varepsilon_m < \nu(\alpha)+\varepsilon.$$

Using Item (d) of this proposition, the sequence $(\vec{c}^{(l)})_{l\in\mathbb{N}}$ converges to $\vec{c}$ in $(\widetilde{A}(\alpha_n,\nu(\alpha_n)+\varepsilon_m), \widetilde{\delta}_{\alpha_n,\nu(\alpha_n)+\varepsilon_m})$. By Proposition 8.1.3, $\vec{c}\in\widetilde{A}(\alpha,\nu(\alpha)+\varepsilon)$. Consequently, $(\theta_l\vec{c}^{(l)})_{l\in\mathbb{N}}$ converges to $\theta\vec{c}$ in $(\widetilde{A}(\alpha,\nu(\alpha)+\varepsilon), \widetilde{\delta}_{\alpha,\nu(\alpha)+\varepsilon})$ by Lemma 8.1.8. Thus, $(\mathcal{L}^\nu, \widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}})$ is a topological metric space.

Moreover, $(\mathcal{L}^\nu, \widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}})$ is complete thanks to Items (d) and (c) of this proposition and Item (e) of Proposition 8.1.6. ■

In fact, all the distances $\widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}}$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ are sequences as in Definition 8.2.2 define the same topology on $\mathcal{L}^\nu$. We even have the following more general result.

Proposition 8.2.4. *If $\widetilde{\delta}_1$ and $\widetilde{\delta}_2$ define complete topologies on $\mathcal{L}^\nu$ which are stronger than the pointwise topology, then these topologies are equivalent.*

Proof. It is a direct consequence of the closed graph theorem. ∎

With the two previous propositions, the choice of sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ of Definition 8.2.2 has thus no importance for the topology defined on $\mathcal{L}^\nu$ from the distance $\widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}}$. Therefore, in the following, we write $\widetilde{\delta}$ this distance on $\mathcal{L}^\nu$, independently of these $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$.

Remark 8.2.5. Combining Proposition 8.2.3 (Item (d)) and Proposition 8.1.6 (Item (b)), the space $(\mathcal{L}^\nu, \widetilde{\delta})$ has a stronger topology than the uniform topology. Moreover, the inclusion $\mathcal{L}^\nu \subset c^0$ is continuous.

8.3 Compact Subsets of $\mathcal{L}^\nu$

Let us continue with the characterization of compact subsets of $(\mathcal{L}^\nu, \widetilde{\delta})$. This characterization will only holds if $\alpha_{\min} > 0$. It is particularly useful to prove the convergence of sequences in $\mathcal{L}^\nu$. For $m, n \in \mathbb{N}$, let $C_{m,n}$ and $C'_{m,n}$ be positive or null constants and let us define

$$\widetilde{K}_{m,n} := \left\{ \vec{c} \in \Omega : \#\{k \in \{0, \dots, 2^j - 1\} : d_{j,k} > C_{m,n}\, 2^{-\alpha_n j}\} \le C'_{m,n}\, 2^{(\nu(\alpha_n)+\varepsilon_m)j}\ \forall j \in \mathbb{N}_0 \right\}$$

(by taking the usual sequences of Proposition 8.1.3 and Definition 8.2.2). We write

$$\widetilde{K} := \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \widetilde{K}_{m,n}. \tag{8.2}$$

Let us note that $\widetilde{K}_{m,n}$ is a bounded set of $(\widetilde{A}(\alpha_n, \nu(\alpha_n)+\varepsilon_m), \widetilde{\delta}_{\alpha_n,\nu(\alpha_n)+\varepsilon_m})$ by Proposition 8.1.6 (Item (d)) and that $\widetilde{K} \subset \mathcal{L}^\nu$ by Proposition 8.1.3.

Here are some useful observations to obtain the characterization of compact subsets of $(\mathcal{L}^\nu, \widetilde{\delta})$.

Lemma 8.3.1. *(a) From any sequence of $\widetilde{K}$, we can extract a subsequence which converges pointwise.*

(b) Let $\alpha > 0$ and let B be a bounded set of $(c^\alpha, \|\cdot\|_{c^\alpha})$. If $(\vec{c}^{(l)})_{l\in\mathbb{N}}$ is a sequence of B which converges pointwise to $\vec{c}$, then it converges uniformly to $\vec{c}$.

(c) Let $\alpha_0 \in \mathbb{R}$ and $\beta_0 \ge 0$ and let B be a bounded set of $(\widetilde{A}(\alpha_0, \beta_0), \widetilde{\delta}_{\alpha_0,\beta_0})$. If $(\vec{c}^{(l)})_{l\in\mathbb{N}}$ is a sequence of B which converges uniformly to $\vec{c}$, then it converges to $\vec{c}$ in $(\widetilde{A}(\alpha,\beta), \widetilde{\delta}_{\alpha,\beta})$ for all α and β such that $\alpha < \alpha_0$ and $\beta > \beta_0$.

(d) Let $\alpha_0 \ge 0$ and let B be a bounded set of $(c^{\alpha_0}, \|\cdot\|_{c^{\alpha_0}})$. If $(\vec{c}^{(l)})_{l\in\mathbb{N}}$ is a sequence of B which converges uniformly to $\vec{c}$, then it converges to $\vec{c}$ in $(c^\alpha, \|\cdot\|_{c^\alpha})$ for all $\alpha < \alpha_0$.

Proof. (a) Let $(\vec{c}^{(l)})_{l\in\mathbb{N}}$ be a sequence of $\widetilde{K}$. There exists $n \in \mathbb{N}$ such that $\alpha_n < \alpha_{\min}$ and then we have

$$|c_{j,k}^{(l)}| \le 2^{-\alpha_n} C_{m,n}$$

for all $l \in \mathbb{N}$ and $(j,k) \in \Lambda$. This means that the sequence $(\vec{c}^{\,(l)})_{l\in\mathbb{N}}$ is pointwise bounded in $\mathbb{C}$ and we can thus extract a pointwise convergent subsequence.

(b) Since B is a bounded set of $(c^\alpha, \|\cdot\|_{c^\alpha})$, there exists $r > 0$ such that

$$B \subset B' := \left\{ \vec{a} \in \Omega : 2^{\alpha j} \sup_{\lambda' \subset \lambda(j,k)} |a_{\lambda'}| \leq r \ \forall (j,k) \in \Lambda \right\}$$

and B' is closed for the uniform and then the pointwise convergence by Proposition 8.1.6 (Item (d)). Moreover, B' is a bounded set of $(c^\alpha, \|\cdot\|_{c^\alpha})$. So, $\vec{c} \in B' \subset c^\alpha$ and $(\vec{c}^{\,(l)} - \vec{c})_{l\in\mathbb{N}}$ is bounded in $(c^\alpha, \|\cdot\|_{c^\alpha})$. Consequently, using again Proposition 8.1.6 (Item (d)), there exists $R > 0$ such that $|c^{(l)}_{j,k} - c_{j,k}| \leq R2^{-\alpha j}$ for all $(j,k) \in \Lambda$ and all $l \in \mathbb{N}$. Let $\eta > 0$. On the one hand, since $\alpha > 0$, there exists $J \in \mathbb{N}_0$ such that $R2^{-\alpha j} < \eta$ for every $j \geq J$ and then

$$|c^{(l)}_{j,k} - c_{j,k}| < \eta$$

for all $l \in \mathbb{N}$, $j \geq J$ and $k \in \{0, \ldots, 2^j - 1\}$. On the other hand, thanks to the pointwise convergence, there exists $L \in \mathbb{N}$ (which only depends on η) such that

$$|c^{(l)}_{j,k} - c_{j,k}| < \eta$$

for all $l \geq L$, $j \in \{0, \ldots, J-1\}$ and $k \in \{0, \ldots, 2^j - 1\}$. Thus, for all $l \geq L$, we obtain

$$\sup_{(j,k)\in\Lambda} |c^{(l)}_{j,k} - c_{j,k}| < \eta.$$

(c) Since the sequence $(\vec{c}^{\,(l)} - \vec{c})_{l\in\mathbb{N}}$ is bounded in $(\widetilde{A}(\alpha_0, \beta_0), \widetilde{\delta}_{\alpha_0,\beta_0})$ (by the same argument as in the previous item of this proposition), there exist $R, R' \geq 0$ such that

$$\# \left\{ k \in \{0, \ldots, 2^j - 1\} : \sup_{\lambda' \subset \lambda(j,k)} |c^{(l)}_{\lambda'} - c_{\lambda'}| > R2^{-\alpha_0 j} \right\} \leq R' 2^{\beta_0 j}$$

for all $j \in \mathbb{N}_0$ and $l \in \mathbb{N}$, using Proposition 8.1.6 (Item (d)). Let $\eta > 0$. Since $\alpha < \alpha_0$ and $\beta > \beta_0$, there exists $J \in \mathbb{N}_0$ such that $R2^{-\alpha_0 j} < \eta 2^{-\alpha j}$ and $R' 2^{\beta_0 j} < \eta 2^{\beta j}$ for every $j \geq J$ and then

$$\# \left\{ k \in \{0, \ldots, 2^j - 1\} : \sup_{\lambda' \subset \lambda(j,k)} |c^{(l)}_{\lambda'} - c_{\lambda'}| \geq \eta 2^{-\alpha j} \right\} \leq \eta 2^{\beta j}$$

for all $l \in \mathbb{N}$ and $j \geq J$. Moreover, thanks to the uniform convergence, there exists $L \in \mathbb{N}$ (which only depends on η) such that

$$\sup_{\lambda' \subset \lambda(j,k)} |c^{(l)}_{\lambda'} - c_{\lambda'}| < \eta 2^{-\alpha j}$$

for all $l \geq L$, $j \in \{0, \ldots, J-1\}$ and $k \in \{0, \ldots, 2^j - 1\}$, and then

$$\# \left\{ k \in \{0, \ldots, 2^j - 1\} : \sup_{\lambda' \subset \lambda(j,k)} |c^{(l)}_{\lambda'} - c_{\lambda'}| \geq \eta 2^{-\alpha j} \right\} = 0 \leq \eta 2^{\beta j}$$

for all $l \geq L$ and $j \in \{0, \ldots, J-1\}$. Thus, we have $\widetilde{\delta}_{\alpha,\beta}(\vec{c}^{\,(l)}, \vec{c}) \leq 2\eta$ for every $l \geq L$.

(d) The proof of this item is similar to the two previous ones. ∎

Proposition 8.3.2. *Let us assume that $\alpha_{\min} > 0$. A set is a compact subset of $(\mathcal{L}^\nu, \widetilde{\delta})$ if and only if it is closed in $(\mathcal{L}^\nu, \widetilde{\delta})$ and included in some $\widetilde{K}$.*

Proof. Since any compact set of a metric space is closed and bounded, the condition is obviously necessary.

To prove that the condition is also sufficient, it suffices to show that $\widetilde{K}$ is compact. Let $(\vec{c}^{\,(l)})_{l\in\mathbb{N}}$ be a sequence of $\widetilde{K}$. By Lemma 8.3.1 (Item (a)), we can extract a subsequence which converges pointwise. Let us note $(\vec{c}^{\,(p(l))})_{l\in\mathbb{N}}$ this subsequence and $\vec{c}$ its pointwise limit. Let us show that $(\vec{c}^{\,(p(l))})_{l\in\mathbb{N}}$ converges to $\vec{c}$ in $(\mathcal{L}^\nu, \widetilde{\delta})$.

As $\alpha_{\min} > 0$, there exists $n_0 \in \mathbb{N}$ such that $0 < \alpha_{n_0} < \alpha_{\min}$. By construction, $\vec{c}^{\,(p(l))} \in \widetilde{K}_{m,n_0}$ for all $l \in \mathbb{N}$ and $m \in \mathbb{N}$ and we know that $\widetilde{K}_{m,n_0}$ is bounded in $(c^{\alpha_{n_0}}, \|\cdot\|_{c^{\alpha_{n_0}}})$ by Proposition 8.1.6 (Item (d)). Using Lemma 8.3.1 (Item (b)), we get that $(\vec{c}^{\,(p(l))})_{l\in\mathbb{N}}$ converges uniformly to $\vec{c}$.

Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. If $\nu(\alpha) \in \mathbb{R}$, there exist $n, m \in \mathbb{N}$ such that

$$\varepsilon_m < \varepsilon, \quad \alpha_n > \alpha \quad \text{and} \quad \nu(\alpha_n) + \varepsilon_m < \nu(\alpha) + \varepsilon.$$

Lemma 8.3.1 (Item (c)) implies that $(\vec{c}^{\,(p(l))})_{l\in\mathbb{N}}$ converges to $\vec{c}$ in $(\widetilde{A}(\alpha, \nu(\alpha)+\varepsilon), \widetilde{\delta}_{\alpha,\nu(\alpha)+\varepsilon})$. If $\nu(\alpha) = -\infty$, there exists $n \in \mathbb{N}$ such that $\alpha_n > \alpha$ and $\nu(\alpha_n) = -\infty$. By Lemma 8.3.1 (Item (d)), $(\vec{c}^{\,(p(l))})_{l\in\mathbb{N}}$ converges to $\vec{c}$ in $(\widetilde{A}(\alpha, \nu(\alpha)+\varepsilon), \widetilde{\delta}_{\alpha,\nu(\alpha)+\varepsilon})$. Proposition 8.2.3 gives the conclusion. ∎

In fact, we also have obtained within this last proof the following result.

Corollary 8.3.3. *Every sequence of $\widetilde{K}$ which converges pointwise converges also in $(\mathcal{L}^\nu, \widetilde{\delta})$ to an element of $\widetilde{K}$.*

Remark 8.3.4. The characterization is not longer valid in the case $\alpha_{\min} = 0$. Indeed, let ν be the admissible profile defined by

$$\nu(\alpha) := \begin{cases} -\infty & \text{if} \quad \alpha < 0 \\ 1 & \text{if} \quad \alpha \geq 0 \end{cases}$$

as in Expression (7.3). In this case, we know that $\mathcal{L}^\nu = c^0$ (see Subsection 7.3.3 in the previous chapter). If we assume that we have this characterization of subset compacts of $\mathcal{L}^\nu$, then the (closed) unit ball of c^0 would be compact (it is easy to show that it is included in some $\widetilde{K}$) and therefore the space would be finite dimensional. This leads to a contradiction.

8.4 Separability

As for the characterization of the compact subsets of $\mathcal{L}^\nu$, we have to consider separately the two following cases: $\alpha_{\min} > 0$ and $\alpha_{\min} = 0$. Let us start with a first difference described in the following lemma.

Lemma 8.4.1. *If $\vec{c} \in \Omega$, let $(\vec{c}^{\,N})_{N\in\mathbb{N}_0}$ be the sequence of Ω defined by*

$$c^N_{j,k} := \begin{cases} c_{j,k} & \text{if } j \leq N \text{ and } k \in \{0, \dots, 2^j - 1\} \\ 0 & \text{if } j > N \text{ and } k \in \{0, \dots, 2^j - 1\} \end{cases} \tag{8.3}$$

for every $N \in \mathbb{N}_0$.

(a) If $\alpha_{\min} > 0$, $(\vec{c}^N)_{N\in\mathbb{N}_0}$ converges to $\vec{c}$ in $(\mathcal{L}^\nu, \widetilde{\delta})$ for all $\vec{c} \in \mathcal{L}^\nu$.

(b) If $\alpha_{\min} = 0$, there exists $\vec{c} \in \mathcal{L}^\nu$ such that $(\vec{c}^N)_{N\in\mathbb{N}_0}$ does not converge to $\vec{c}$ in $(\mathcal{L}^\nu, \widetilde{\delta})$.

Proof. (a) Since the characterization of compacts of $\mathcal{L}^\nu$ (when $\alpha_{\min} > 0$) is similar to the one in $\mathcal{S}^\nu$ case, the proof of this first item only needs some adaptations of Lemma 6.3 in [8] with wavelet leaders.

Since $\vec{c} \in \mathcal{L}^\nu$, $\vec{c} \in \widetilde{A}(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$ for all $m, n \in \mathbb{N}$ by Proposition 8.1.3. Then, for all $m, n \in \mathbb{N}$, there exist $C_{m,n}, C'_{m,n} \geq 0$ such that

$$\#\left\{k \in \{0, \dots, 2^j - 1\} : d_{j,k} > C_{m,n} 2^{-\alpha_n j}\right\} \leq C'_{m,n} 2^{(\nu(\alpha_n)+\varepsilon_m)j}$$

for all $j \in \mathbb{N}_0$ and so, $\vec{c} \in \widetilde{K}$ where $\widetilde{K}$ is defined as in Expression (8.2). For all $N \in \mathbb{N}_0$, we also have $\vec{c}^N \in \widetilde{K}$ because $d^N_{j,k} \leq d_{j,k}$ for all $(j,k) \in \Lambda$ by definition of $\vec{c}^N$. Moreover, $(\vec{c}^N)_{N\in\mathbb{N}_0}$ converges pointwise to $\vec{c}$. Corollary 8.3.3 gives the conclusion.

(b) Let us now suppose that $\alpha_{\min} = 0$ and let us consider the sequence $\vec{c}$ defined by

$$c_{j,k} := \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \in \{1, \dots, 2^j - 1\} \end{cases}.$$

for each scale $j \in \mathbb{N}_0$. We have $d_{j,k} = c_{j,k}$ for all $(j,k) \in \Lambda$. Using the assumption $\alpha_{\min} = 0$, it is easy to check that $\vec{c}$ belongs to $\mathcal{L}^\nu$. By contradiction, let us assume that $(\vec{c}^N)_{N\in\mathbb{N}_0}$ converges to $\vec{c}$ in $(\mathcal{L}^\nu, \widetilde{\delta})$. We know that the space $(\mathcal{L}^\nu, \widetilde{\delta})$ has a stronger topology than the uniform topology (see Remark 8.2.5). However, for $N \in \mathbb{N}_0$, we have

$$\sup_{(j,k)\in\Lambda} |c_{j,k} - c^N_{j,k}| = 1,$$

hence a contradiction. ■

Let us begin by studying the separability of $\mathcal{L}^\nu$ with $\alpha_{\min} > 0$.

Lemma 8.4.2. *Let B be a pointwise bounded set of sequences and let us assume that there exists $N \in \mathbb{N}_0$ such that*

$$\forall \vec{c} \in B,\ \forall j > N,\ \forall k \in \{0, \dots, 2^j - 1\},\ c_{j,k} = 0.$$

If $\alpha_{\min} > 0$, then B is included in a compact subset of $\mathcal{L}^\nu$.

Proof. Since B is a pointwise bounded set, there exists a constant $C > 0$ such that

$$\sup_{j\in\{0,\dots,N\}} \sup_{k\in\{0,\dots,2^j-1\}} |c_{j,k}| \leq C$$

for all $\vec{c} \in B$. Let $\vec{c} \in B$. Then, $c_{j,k} = 0$ and therefore $d_{j,k} = 0$ for all $j > N$ and $k \in \{0, \dots, 2^j - 1\}$. Moreover, for all $j \in \{0, \dots, N\}$, $k \in \{0, \dots, 2^j - 1\}$ and $n \in \mathbb{N}$, we have

$$2^{\alpha_n j} d_{j,k} \leq 2^{\alpha_n j} \sup_{j'\in\{0,\dots,N\}} \sup_{k'\in\{0,\dots,2^{j'}-1\}} |c_{j',k'}| \leq C 2^{\alpha_n j} \leq C \sup_{j'\in\{0,\dots,N\}} \sup_{k'\in\{0,\dots,2^{j'}-1\}} 2^{\alpha_n j'}$$

Setting $C_{m,n} := C \sup_{j'\in\{0,\dots,N\}} \sup_{k'\in\{0,\dots,2^{j'}-1\}} 2^{\alpha_n j'}$ for $m, n \in \mathbb{N}$, we so obtain

$$\#\left\{k \in \{0, \dots, 2^j - 1\} : d_{j,k} > 2^{-\alpha_n j} C_{m,n}\right\} = 0 \leq C'_{m,n} 2^{(\nu(\alpha_n)+\varepsilon_m)j}$$

for all $j \in \mathbb{N}_0$ and all constant $C'_{m,n} \geq 0$. Consequently, $\vec{c} \in \widetilde{K}$ where $\widetilde{K}$ is defined as in Expression (8.2). ■

Proposition 8.4.3. *If $\alpha_{\min} > 0$, the metric space $(\mathcal{L}^\nu, \widetilde{\delta})$ is separable.*

Proof. Let us prove that the set

$$U := \left\{ \vec{c} \in \Omega \ : \ c_{j,k} \in \mathbb{Q} + i\,\mathbb{Q} \ \text{ and } \exists N \in \mathbb{N}_0 \ \text{ such that } c_{j,k} = 0 \ \forall j > N, \ k \in \{0, \dots, 2^j - 1\} \right\}$$

is dense in $(\mathcal{L}^\nu, \widetilde{\delta})$. Let $\vec{c} \in \mathcal{L}^\nu$; by Lemma 8.4.1, the sequence $(\vec{c}^{\,N})_{N \in \mathbb{N}_0}$ defined in Expression (8.3) converges to $\vec{c}$ in $(\mathcal{L}^\nu, \widetilde{\delta})$. Using the density of $\mathbb{Q} + i\,\mathbb{Q}$ in $\mathbb{C}$, we can find for all $N \in \mathbb{N}_0$, a sequence $(\vec{q}_N^{\,(l)})_{l \in \mathbb{N}}$ of U which converges pointwise to $\vec{c}^{\,N}$. By Lemma 8.4.2 and Corollary 8.3.3, the convergence also holds in $(\mathcal{L}^\nu, \widetilde{\delta})$, hence the conclusion. ∎

Let us consider now the case where the admissible profile ν is such that $\alpha_{\min} = 0$. The previous result is no longer valid. Indeed, with the admissible profile considered in Remark 8.3.4, the space $\mathcal{L}^\nu$ is c^0 which is not separable. More generally, we have the following property.

Proposition 8.4.4. *If $\alpha_{\min} = 0$, the metric space $(\mathcal{L}^\nu, \widetilde{\delta})$ is not separable.*

Proof. Let us consider the uncountable set A of sequences $\vec{c}$ of Ω such that for each scale $j \in \mathbb{N}_0$, $c_{j,0} \in \{0, 1\}$ and the other coefficients are equal to 0. Using the hypothesis $\alpha_{\min} = 0$, we easily prove that A is a subset of $\mathcal{L}^\nu$. Indeed, let $\vec{c} \in A$ and let $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$. If $\alpha < 0$, there exists $J \in \mathbb{N}_0$ such that $C2^{-\alpha j} > 1$ for all $j \geq J$ and we then have

$$d_{j,k} \leq 1 < C2^{-\alpha j}$$

for all $j \geq J$ and $k \in \{0, \dots, 2^j - 1\}$. If $\alpha \geq 0$, we have

$$\# \widetilde{E_j}(C, \alpha)(\vec{c}\,) \leq 1 \leq 2^{(\nu(\alpha) + \varepsilon) j}$$

for all $j \in \mathbb{N}_0$. Thus, $\vec{c} \in \mathcal{L}^\nu$. Moreover, we clearly have $\|\vec{c} - \vec{c}\,'\|_{c^0} = 1$ for all distinct elements $\vec{c}$ and $\vec{c}\,'$ of A.

Let D be a dense subset of $(\mathcal{L}^\nu, \widetilde{\delta})$. For every $\vec{c} \in A$, there exists a sequence $(\vec{c}^{\,(m)})_{m \in \mathbb{N}}$ of elements of D which converges in $(\mathcal{L}^\nu, \widetilde{\delta})$ to $\vec{c} \in \mathcal{L}^\nu$. Moreover, the convergence also holds in c^0 by Remark 8.2.5. Consequently, there exists $M \in \mathbb{N}$ such that

$$\|\vec{c} - \vec{c}^{\,(m)}\|_{c^0} < \frac{1}{2}$$

for all $m \geq M$. In particular, there exists $\vec{a} \in D$ such that

$$\|\vec{c} - \vec{a}\|_{c^0} < \frac{1}{2}.$$

Since $\|\vec{c} - \vec{c}\,'\|_{c^0} = 1$ for two distinct elements $\vec{c}$ and $\vec{c}\,'$ of A , D must contain at least as many elements as A and cannot be countable. ∎

8.5 Comparison with the Topology of $\mathcal{S}^\nu$

In the end of the previous chapter, we have studied the inclusions between $\mathcal{L}^\nu$ and $\mathcal{S}^\nu$. Let us recall that $\mathcal{L}^\nu \subset \mathcal{S}^\nu$ for all admissible profile ν. Let us now compare the topologies of $\mathcal{L}^\nu$ and $\mathcal{S}^\nu$. We have the following proposition; its proof is straightforward.

Proposition 8.5.1. *(a) If $\alpha \in \mathbb{R}$ and $\beta \in \{-\infty\} \cup [0, +\infty[$, then we have*

$$\widetilde{A}(\alpha,\beta) \subset A(\alpha,\beta) \quad \text{and} \quad \delta_{\alpha,\beta} \leq \widetilde{\delta}_{\alpha,\beta}.$$

(b) If a sequence converges in $(\widetilde{A}(\alpha,\beta), \widetilde{\delta}_{\alpha,\beta})$, it converges in $(A(\alpha,\beta), \delta_{\alpha,\beta})$ to the same limit. If a sequence is a Cauchy sequence in $(\widetilde{A}(\alpha,\beta), \widetilde{\delta}_{\alpha,\beta})$, it is a Cauchy sequence in $(A(\alpha,\beta), \delta_{\alpha,\beta})$.

(c) The space $(\widetilde{A}(\alpha,\beta), \widetilde{\delta}_{\alpha,\beta})$ has a stronger topology than the topology induced by the distance $\delta_{\alpha,\beta}$.

(d) The space $(\mathcal{L}^\nu, \widetilde{\delta})$ has a stronger topology than the topology induced by the distance δ.

Proof. Let us prove the first item. Let $\vec{c} \in \widetilde{A}(\alpha,\beta)$. By definition, there exist $C, C' \geq 0$ such that

$$\#\{k \in \{0,\ldots,2^j-1\} : d_{j,k} \geq C2^{-\alpha j}\} \leq C'2^{\beta j}$$

for all $j \in \mathbb{N}_0$. Since $|c_{j,k}| \leq d_{j,k}$ for all $(j,k) \in \Lambda$, we directly have $\vec{c} \in A(\alpha,\beta)$. The same argument shows that $\delta_{\alpha,\beta}(\vec{c},\vec{0}) \leq \widetilde{\delta}_{\alpha,\beta}(\vec{c},\vec{0})$.

The other items result from the first item of this proposition. ■

The topology induced by δ on $\mathcal{L}^\nu$ is not equivalent to the one induced by $\widetilde{\delta}$. It is the object of this last result.

Proposition 8.5.2. *If $\mathcal{L}^\nu$ is strictly included in $\mathcal{S}^\nu$, then $\mathcal{L}^\nu$ is not closed in $\mathcal{S}^\nu$.*

Proof. Let $\vec{c} \in \mathcal{S}^\nu \setminus \mathcal{L}^\nu$ and let $(\vec{c}^{\,N})_{N\in\mathbb{N}_0}$ be the sequence defined from $\vec{c}$ as in Expression (8.3). For all $N \in \mathbb{N}_0$, $\vec{c}^{\,N}$ belongs to $\mathcal{L}^\nu$ and then to $\mathcal{S}^\nu$ because it has only a finite number of non zero coefficients. The sequence $(\vec{c}^{\,N})_{N\in\mathbb{N}_0}$ converges to $\vec{c}$ for the topology of $\mathcal{S}^\nu$ (see Lemma 6.3 in [8]). Since $\vec{c} \notin \mathcal{L}^\nu$, we have the conclusion. ■

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List of Symbols (by Section)

1. W, T

1.1. $\Lambda^\alpha(x_0)$, $\Lambda^\alpha(\mathbb{R})$, $\underline{\alpha}$

1.2. $h_f(x_0)$, R, $H_f(x_0)$, $H_f(\mathbb{R})$

1.3. D

2.

2.1. E, D, I, $\mathcal{N}$, d, $\boldsymbol{a}$, $\boldsymbol{a} \wedge \boldsymbol{b}$, d', $[a_1, \ldots, a_n]$, $p_j(\boldsymbol{a})$, $q_j(\boldsymbol{a})$, $[a_1, \ldots]$, $[\boldsymbol{a}]$, $I_n(x)$

2.2. $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}$, Q

2.3. σ

2.4. κ_k, τ, $R_n(\boldsymbol{a})$, $S_n(\boldsymbol{a})$

2.5. $\mathscr{C}$, Σ

3.

3.1. ψ, $\mathcal{W}_\psi\, f$, $\psi_{a,b}$, $H^2(\mathbb{R})$, ψ_L

3.2.

4. R, $R_{\alpha,\beta}$

4.1. ζ, H, Γ, E_1

4.2. $S_{\boldsymbol{a},\boldsymbol{\lambda}}$

4.3. s, $m_f^{a,b}$

5.

5.1. $\psi^{(j)}$, $t_{p,q}$

5.2. V_j, $\varphi^{(j)}$, $m_0^{(j)}$, D_j

5.3. N_λ, M_μ

5.4.

5.5. $\psi_{j,k}$, $\mathcal{H}$, $\mathcal{D}$, I, I_0, I_1, S

5.6. $\Psi_{j,n}$, $\mathbb{F}_j(\xi)$

5.7.

6.

6.1. Ψ, $\psi^{(a)}$, $\mathcal{W}_\Psi\, f$, $\psi_{a,b}$, ψ_P

6.2.

7.

7.1. $\psi_{j,k}$, $c_{j,k}$, Λ, Ω, $\lambda(j,k)$, λ, Λ_j, c_λ, d_λ, $d_{j,k}$

7.2. $\nu_{\vec{c}}$, $E_j(C,\alpha)(\vec{c})$, ν, $\alpha_{\min}$, $\mathcal{S}^\nu$, $A(\alpha,\beta)$, $\delta_{\alpha,\beta}$, c^α, δ, $K_{m,n}$, K

7.3. $\widetilde{\nu}_{\vec{c}}$, $\widetilde{E_j}(C,\alpha)(\vec{c})$, $\mathcal{L}^\nu$

8.

8.1. $\widetilde{A}(\alpha,\beta)$, $\widetilde{\delta}_{\alpha,\beta}$

8.2. $\boldsymbol{\alpha}$, $\boldsymbol{\varepsilon}$, $\widetilde{\delta}_{\boldsymbol{\alpha},\boldsymbol{\varepsilon}}$, $\widetilde{\delta}$

8.3. $\widetilde{K}_{m,n}$, $\widetilde{K}$

8.4. $\vec{c}^{\,N}$

8.5.

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