Quadratizations of pseudo-Boolean functions

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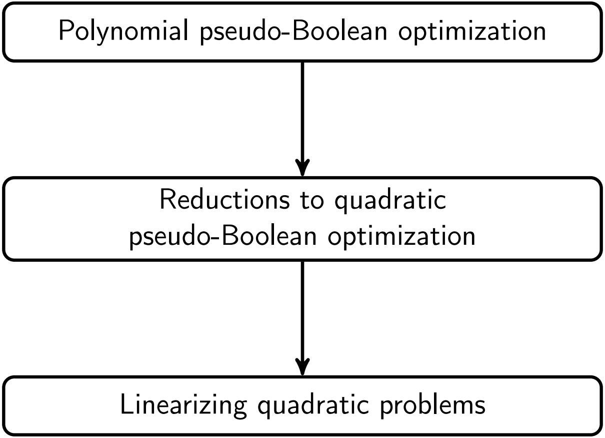
Partially supported by Belspo - IAP Project COMEX

4th December 2014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Reductions to quadratic pseudo-Boolean optimization
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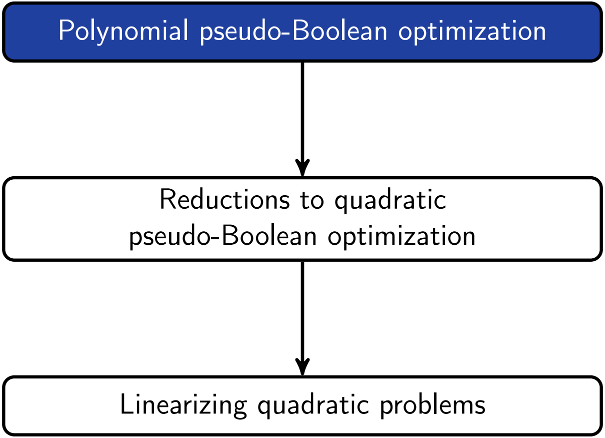
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Definitions

Definition: Pseudo-Boolean functions

A pseudo-Boolean function is a mapping $f : \{0,1\}^n \to \mathbb{R}$.

Multilinear representation

Every pseudo-Boolean function f can be represented uniquely by a multilinear polynomial (Hammer, Rosenberg, Rudeanu [7]).

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Applications: MAX-SAT

MAX-SAT problem

- INPUT: a set of Boolean clauses $C_k = \left(\vee_{i\in A_k}\bar{x}_i\right) \vee \left(\vee_{j\in B_k} x_j\right)$, for $k = 1, \ldots, m$, where $x_i \in \{0,1\}$, and $\bar{x}_i = 1 - x_i$.
- OBJECTIVE: find an assignment of the variables, $x^* \in \{0,1\}^n$ that maximizes the number of satisfied clauses.

Pseudo-Boolean formulation

$$\min \sum_{k=1}^{m} \left(\prod_{i\in A_k} x_i\right)\left(\prod_{j\in B_k} \bar{x}_j\right),$$

C_k takes value 1 *iff* the term $\prod_{i\in A_k} x_i \prod_{j\in B_k} \bar{x}_j$ takes value 0.

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Applications: Computer Vision

Image restoration problems modelled as energy minimization

$$E(l) = \sum_{p \in \mathcal{P}} D_p(l_p) + \sum_{S \subseteq \mathcal{P}, |S| \geq 2} \sum_{p_1,\ldots,p_s \in S} V_{p_1,\ldots,p_s}(l_{p_1}, \ldots, l_{p_s}),$$

where $l_p \in \{0,1\} \;\; \forall p \in \mathcal{P}$.

(Image from "Corel database" with additive Gaussian noise.)

Applications

- Constraint Satisfaction Problem
- Data mining, classification, learning theory...
- Graph theory
- Operations research
- Production management
- ...

Pseudo-Boolean Optimization

Many problems formulated as optimization of a pseudo-Boolean function

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$$\min_{x \in \{0,1\}^n} f(x)$$

- Optimization is $\mathcal{NP}$-hard, even if f is quadratic (MAX-2-SAT, MAX-CUT modelled by quadratic f).

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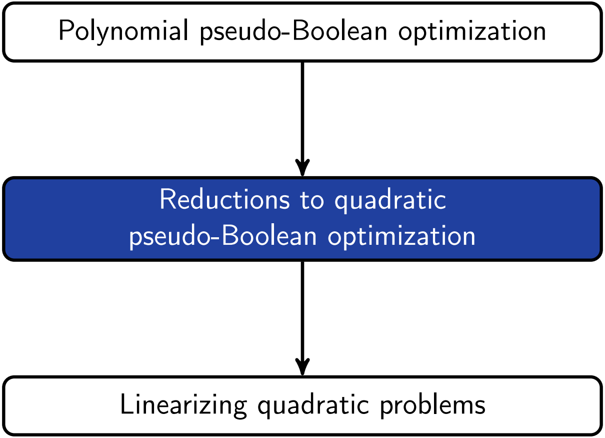
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Definition: Quadratization

Given a pseudo-Boolean function $f(x)$ on $\{0,1\}^n$, we say that $g(x,y)$ is a *quadratization* of f if $g(x,y)$ is a quadratic polynomial depending on x and on m *auxiliary variables* $y_1,\dots,y_m$, such that

$$f(x) = \min\{g(x,y) : y \in \{0,1\}^m\} \quad \forall x \in \{0,1\}^n.$$

Then, $\min\{f(x) : x \in \{0,1\}^n\} = \min\{g(x,y) : x \in \{0,1\}^n, y \in \{0,1\}^m\}$.

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Rosenberg

Rosenberg (1975) [11]: first quadratization method.

1. Take a product $x_i x_j$ from a highest-degree monomial of f and substitute it by a new variable y_{ij}.
2. Add a penalty term $M(x_i x_j - 2x_i y_{ij} - 2x_j y_{ij} + 3y_{ij})$ (M large enough) to the objective function to force $y_{ij} = x_i x_j$ at all optimal solutions.
3. Iterate until obtaining a quadratic function.

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  - Can be applied to any pseudo-Boolean function f.
  - The transformation is polynomial in the size of the input.

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Termwise quadratizations

Multilinear expression of a pseudo-Boolean function:

$$f(x) = \sum_{S \in 2^{[n]}} a_S \prod_{i \in S} x_i$$

Idea: quadratize monomial by monomial, using different sets of auxiliary variables for each monomial.

Termwise quadratizations: negative monomials

Kolmogorov and Zabih [10], Freedman and Drineas [6].

$$a \prod_{i=1}^{n} x_i = \min_{y \in \{0,1\}} ay \left(\sum_{i=1}^{n} x_i - (n-1) \right), a < 0.$$

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Ishikawa [9]

$$a \prod_{i=1}^{d} x_i = a \min_{y_1,\dots y_{n_d} \in \{0,1\}} \sum_{i=1}^{n_d} y_i(c_{i,d}(-S_1 + 2i) - 1) + aS_2,$$

S_1, S_2: elementary linear and quadratic symmetric polynomials in d variables, $\mathbf{n_d} = \lfloor \frac{\mathbf{d-1}}{\mathbf{2}} \rfloor$ and $c_{i,d} = \begin{cases} 1, \text{if } d \text{ is odd and } i = n_d, \\ 2, \text{otherwise.} \end{cases}$

- **Number of variables:** best known bound for positive monomials.
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Number of variables

Using termwise quadratizations:

- One variable per negative monomial and $\lfloor \frac{d-1}{2} \rfloor$ per positive monomial (d: degree of the monomial).
- Best known upper bounds: $O(n^d)$ variables for a polynomial of fixed degree d, $O(n2^n)$ for an arbitrary function.

Can we do better?

Tight upper and lower bounds *independent of the quadratization procedure* by Anthony, Boros, Crama and Gruber [1]

- $\Theta(2^{\frac{n}{2}})$ for a general pseudo-Boolean function.
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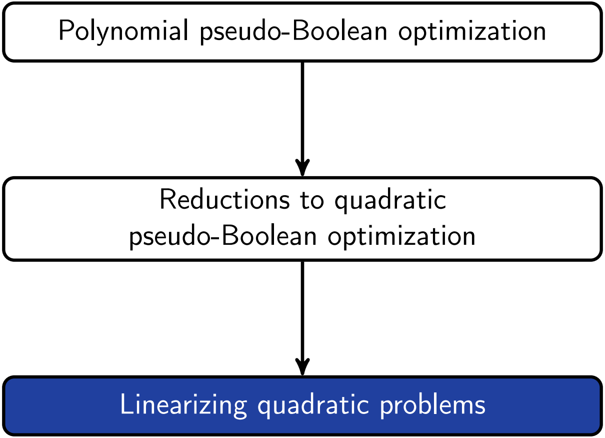
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Standard linearization

Original polynomial problem

$$\min_{x \in \{0,1\}^n} f(x) = \sum_{S \in \mathcal{S}} a_S \prod_{i \in S} x_i$$

$\mathcal{S}$: set of non-constant monomials.

1. Substitute monomials

$$\min_{z_S} \sum_{S \in \mathcal{S}} a_S z_S$$

$$\text{s.t. } z_S = \prod_{i \in S} z_i, \ \forall S \in \mathcal{S}$$

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$$\min_{z_S} \sum_{S\in\mathcal{S}} a_S z_S \qquad \text{(A)}$$

$$\text{s.t. } z_S \leq z_i, \ \forall i \in S, \forall S \in \mathcal{S}$$

$$z_S \geq \sum_{i\in S} z_i - |S| + 1, \ \forall S \in \mathcal{S}$$

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where $\mathcal{Q}$ is the set of non-constant monomials in the original $\{x_1,\ldots,x_n\}$ and the auxiliary $\{x_{n+1},\ldots,x_{n+m}\}$ variables and all $Q\in\mathcal{Q}$ have degree at most 2.

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Comparing relaxations of linearizations

Relaxation of standard linearization (A)

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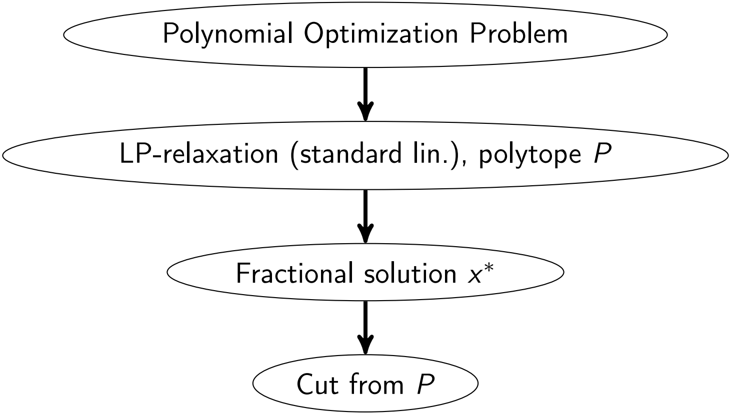
Comparing polytopes at substitution step

Questions:

- Do we have a better knowledge of one of the convex hull of feasible solutions of one of these problems? (i.e., polyhedral description, good separation algorithms...).

Buchheim and Rinaldi's approach



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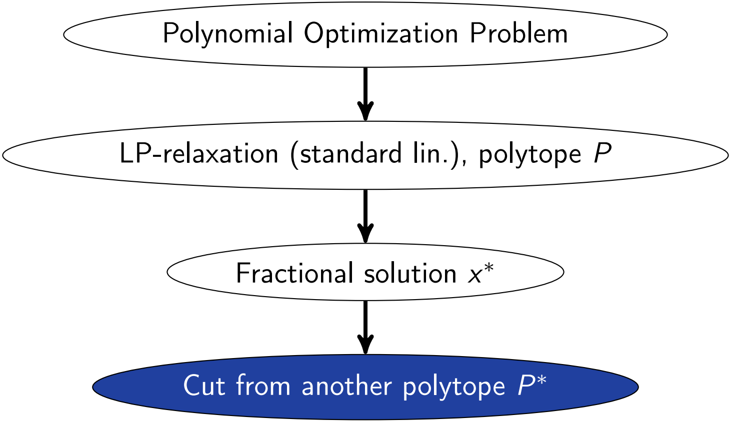

Buchheim and Rinaldi's approach: Polytope P^*?

Presented in [2], [3].

Original polynomial problem

$$\min_{x \in \{0,1\}^n} f(x) = \sum_{S \in \mathcal{S}} a_S \prod_{i \in S} x_i$$

$\mathcal{S}$: set of non-constant monomials

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Buchheim and Rinaldi's formulation over a quadric polytope

Consider the set $\mathcal{S}^* = \{\{S, T\} \mid S, T \in \mathcal{S} \text{ and } S \cup T \in \mathcal{S}\}$.

$$\min_{y_{\{S\}}} \sum_{S \in \mathcal{S}} a_S y_{\{S\}} \tag{C}$$

$$\text{s.t. } y_{\{S,T\}} = y_{\{S\}} y_{\{T\}}, \ \forall \{S, T\} \in \mathcal{S}^*$$

$$y_{\{S,T\}} \in \{0, 1\}, \ \forall \{S, T\} \in \mathcal{S}^*$$

P^*: convex hull of feasible solutions of problem (C)

Observation: P^* is a boolean quadric polytope.

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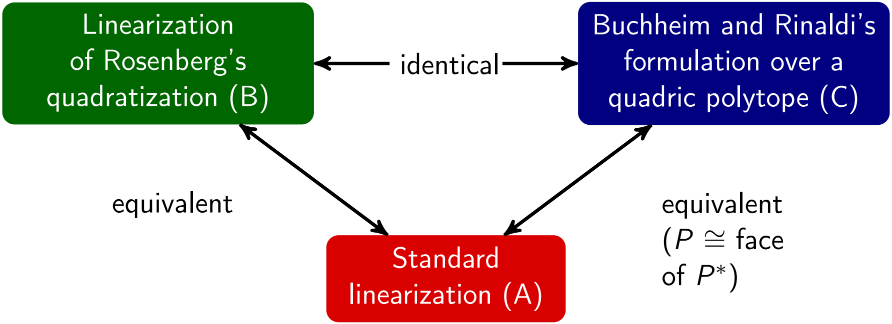
Buchheim and Rinaldi's formulation and Rosenberg's quadratization

Theorem

Buchheim and Rinaldi's formulation over a quadric polytope can be obtained (up to elimination of redundant constraints) by linearizing a variant of Rosenberg's quadratization where:

- the order of substituting variables is induced by the decomposition S_l, S_r of each monomial S, and
- when substituting a product by a variable, we do not impose $y_{ij} = x_i x_j$ with a penalty, but with a constraint.

Assumption: every $S \in \mathcal{S}$ can be written as the union of two *other* monomials $S_l, S_r \in \mathcal{S}$.


Linearization of Rosenberg's quadratization (B)
identical
Buchheim and Rinaldi's formulation over a quadric polytope (C)
equivalent
Standard linearization (A)
equivalent ($P \cong$ face of P^*)

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