Dual approach using a variant perimeter constraint and efficient sub-iteration scheme for topology optimization

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Abstract

To prevent the checkerboard and intermediate grey regions of material distributions in compliance-volume topology optimization, an efficient sub-iteration scheme is established for the implementation of the perimeter constraint in dual approach. Considering the high nonlinearity of the perimeter function, a new variant quadratic formulation of the perimeter control is proposed. In each explicit subproblem, the satisfaction of the variant is achieved by sequential diagonal quadratic approximations. It is shown that the implementation of the sub-iteration scheme is very efficient and reliable without needs of move-limits and any artificial control parameters. Numerical results show that this variant perimeter constraint is effective to regularize the topology solution. The relaxation of the perimeter bound tends to generate a checkerboard free and satisfactory optimum topology solution.

Keywords: Topology optimization; Perimeter constraint; Dual approach

1. Introduction

Nowadays, topology optimization is a subject which has being extensively studied in the engineering community. Based on the fundamental work of Bendsoe and Kikuchi [1], this approach has gained great success in automobile, aeronautical, micro-systems, mechanism and other industrial applications. It is generally recognized that this is a very efficient method for material layout at the preliminary design stage. Compared to other design problems, topology optimization has some basic characteristics. A large number of design variables defined by the concerned element densities are often involved but few constraints exist. Numerically, this is very beneficial to utilize the dual approach because the original minimization problem will be indirectly solved by maximizing a quasi-non constraint dual problem of few variables. Secondly, due to the poor finite element formulation of the 0-1 discrete topology optimization problem, it was found by Diaz and Sigmund [2] that topology solution involves the common trouble of checkerboards, small holes in members and local minima. These phenomena often depend upon the finite element mesh, starting point and numerical optimization algorithms, e.g., checkerboards increase along with the refinement of finite element mesh.

To obtain the desired distribution pattern of materials regarding the manufacturability, a variety of measures are proposed to regularize the solution. Based on the theoretical proof of Ambrosio and Buttazzo [3] on the existence of optimal solution when the perimeter constraint is added in the problem formulation, Haber et al. [4,5] realize the first numerical implementation of the perimeter constraint to avoid the altering solid-void checkerboard solution. Thereafter, Sigmund [6] presented an alternative filtering method to avoid the checkerboards based on the image processing scheme. The

latter proceeds by smoothing the first-order derivatives of structural compliance with respect to density variables. Furthermore, Petersson and Sigmund [7] proposed a sort of local density slope control method. Recently, a variant scheme is proposed by Zhou *et al.* [8], in which the whole set of linear constraints associated with local slope control are approximately replaced with simple side constraints to design variables. An up-to-date overview is given by Rozvany [9] about the history, theoretical background and numerical methods of topology optimization. In this work, the perimeter constraint scheme is studied. Following earlier results of Duysinx [10, 11] and Beckers [12], the severe nonlinearity of the perimeter constraint usually results in poor approximations so that the constraint violation, the fluctuation of design variables and the instability of iteration procedure are inevitable. To remedy this, move-limits or artificial convexifications are often utilized to limit the variation of design variables. Here, an alternative perimeter control using quadratic function is proposed here. In our formulation, we adopt the SIMP (Solid Isotropic Micro-structure with Penalization), i.e., the power law to describe the dependence of element stiffness upon the density variable. Details are presented below.

2. Finite element formulation of topology optimization problem

In general, the discrete topology optimization problem is stated as follows

$$Min \quad C(X) = F^T U(X)$$

$$V(X) = \sum_{i=1}^{n} x_i v_i \le \overline{V}$$

$$P(X) = \sum_{k=1}^{M} l_k \left(\sqrt{(x_i - x_j)^2 + \varepsilon^2} - \varepsilon \right) \le \overline{P} \tag{1}$$

$$0 \le \delta \le x_i \le 1 \quad i = 1, n$$

The minimization of mean compliance $C(X)$ aims at finding element density variables to achieve the maximum rigidity of the structure while the total material volume $V(X)$ over the design domain is limited by its upper bound $\overline{V}$. $\overline{V}$ is often defined by the volume fraction of the entire domain for $x_i = 1$ with i=1, n. The symbol δ refers to a mall value (e.g. $\delta = 10^{-5}$) used to avoid the singularity of the element stiffness matrix. The perimeter $P(X)$ is a global constraint used to prevent the growth of small holes in member and checkerboards over the design domain. Geometrically, it means that the total jump of density variables between all adjacent elements has to be limited when the material volume is given. In the expression, the symbol ε is an artificial smoothing parameter used to ensure the differentiability of the perimeter function when two adjacent variables have equal values, l_k is the interface length between adjacent elements i and j. M is the total number of interface between to adjacent elements.

To solve the above problem, sensitivity analysis is a basic computing task if gradient-based methods are used. By definition, the SIMP law means that the following relation is adopted for the stiffness matrix K_i of element i,

$$K_i = x_i^p \overline{K}_i \tag{2}$$

Thus, the global stiffness matrix is assembled by

$$K = \sum_{i=1}^{n} L_i^T K_i L_i = \sum_{i=1}^{n} L_i^T x_i^p \overline{K}_i L_i \tag{3}$$

In the above expression, the exponent is often chosen as p=3 or 4. Suppose that the external load vector F is independent upon the density variable, then by differentiating the finite element system equation

$$KU = F \tag{4}$$

the sensitivity of compliance function can be calculated as follows

$$\frac{\partial C}{\partial x_i} = -F^T K^{-1} \frac{\partial K}{\partial x_i} U = -(F^T K^{-1} L_i^T) p x_i^{p-1} \overline{K}_i (L_i U) = -(L_i K^{-1} F)^T p x_i^{p-1} \overline{K}_i (L_i U) \tag{5}$$

To make easy the computation, the above expression can be rewritten as

$$\frac{\partial C}{\partial x_i} = -U_i^T p \frac{K_i}{x_i} U_i = -\frac{p}{x_i} (U_i^T K_i U_i) = -\frac{p}{x_i} (F_i^T U_i) \tag{6}$$

in which F_i and U_i designate the internal node force vector and displacement vector related to element I, respectively. This relation shows that the sensitivity of compliance function is always negative and can be evaluated just as a simple scaling of the potential deformation energy of related element. This can be very easily performed after the fulfilment of finite element analysis. Therefore, this scheme is not only efficient in time but also very advantageous to implement the topology optmization capability into any available finite element system without the source code. In our work, a topology optimization system is established on the basis of the SAMCEF finite element software. As for the volume constraint, the gradient simply corresponds to the set of coefficients due to its linearity.

In contrast, it can be easily proved that the perimeter is a non monotonous function because the first-order partial derivative of the perimeter function may be either positive or negative depending upon relative values of design variables. Therefore, a main problem is how to establish an appropriate approximation scheme. As observed by Duysinx [10], monotonous approximations are unsuitable for the perimeter constraint. They are likely to cause the divergence and numerical oscillations even quadratic approximations are used.

3. Dual solution strategy using quadratic formulation of the perimeter and sub-iteration scheme

Instead of the above perimeter definition, a variant quadratic form is proposed here to restrict the element density variation over the whole domain. The expression is written as,

$$P(X) = \sum_{k=1}^{M} (x_i - x_j)^2 \leq \overline{P} \tag{7}$$

M is still the total number of quadratic terms, equals the number of interfaces between two adjacent elements i and j. Clearly, this is a convex function since each constitutive quadratic term has a positive semi-definite Hessian matrix. Compared to the original one, this simplified formulation excludes the smoothing parameter and becomes differentiable.

To solve the topology optimization problem, the dual approach is an efficient method due to the large number of design variables and few constraints. It proceeds by solving a sequence of explicit subproblems. To ensure the convexity of each subproblem, the compliance function is approximated by the reciprocal approximation. As for the perimeter function (7), a quadratic approximation of diagonal Hessian matrix is applied to ensure the approximation quality. As a result, the obtained subproblem has a separable form in terms of x_i and it can be written in the following compact form if the approximation is made at point X^0.

$$Min \quad c^0 + \sum_i \frac{\overline{c_i}}{x_i}$$

$$v^0 + \sum_i v_i x_i \leq 0$$

$$p^0 + \sum_i p_i x_i + \sum_i d_i x_i^2 \leq 0 \tag{8}$$

$$\delta \leq x_i \leq 1 \quad i = 1, n$$

Where the coefficients c^0, c_i, v^0, v_i p^0, p_i and d_i are all constants depending upon the function values and first-order derivatives of the compliance and volume at the developing design point X^0;

$$c^0 = C(X^0) + \sum_{-} \frac{\partial C(X^0)}{\partial x_i} x_i^0 \; ; \; c_i = -\frac{\partial C(X^0)}{\partial x_i}(x_i^0)^2 > 0$$

$$v^0 = V(X^0) - \overline{V}$$

$$p^0 = P(X^0) - \overline{P} - \sum_{i=1}^{n} \frac{\partial P(X^0)}{\partial x_i} x_i^0 + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 P(X^0)}{\partial x_i^2}(x_i^0)^2 \tag{9}$$

$$p_i = \frac{\partial P(X^0)}{\partial x_i} - \frac{\partial^2 P(X^0)}{\partial x_i^2} x_i^0 \quad ; \quad d_i = \frac{1}{2}\frac{\partial^2 P(X^0)}{\partial x_i^2}$$

From the standpoint of duality, the minimization problem will be transformed into a maximization problem in the space of dual variables, i.e., Lagrangian multipliers. The min-max problem corresponds to

$$\underset{\lambda_j \geq 0}{Max} \; \underset{\delta \leq x_i \leq 1}{Min} \; L(X, \lambda) \tag{10}$$

where the Lagrangian function is expressed as

$$L(X, \lambda) = c^0 + \sum_i \frac{\overline{c_i}}{x_i} + \lambda_1(v^0 + \sum_i v_i x_i) + \lambda_2(p^0 + \sum_i p_i x_i + \sum_i d_i x_i^2) \tag{11}$$

The separability of primal variables x_i makes it possible to group related terms of x_i as,

$$L_i(x_i, \lambda) = \frac{c_i}{x_i} + (\lambda_1 v_i + \lambda_2 p_i)x_i + \lambda_2 d_i x_i^2 \tag{12}$$

Hence, the minimization in (10) is equivalent to find the solution of

$$\underset{\delta \leq x_i \leq 1}{Min} L_i(x_i, \lambda) = \underset{\delta \leq x_i \leq 1}{Min}\left[\frac{c_i}{x_i} + (\lambda_1 v_i + \lambda_2 p_i)x_i + \lambda_2 d_i x_i^2\right] \tag{13}$$

It can be observed that the nonlinearity of this one-dimensional function makes it impossible to derive analytically the optimum x_i in function of dual variables λ_1 and λ_2. For this reason, the Newton-Raphson iteration scheme is firstly applied to solve (13) before performing the maximization in the dual space. Details about this computation can be found in a similar work of authors [13].

After solving (8), the intermediate solution obtained can guarantee the absolute satisfaction of the volume constraint instead the perimeter in (7) because the former is exactly approximated without error. To a great extent, the violation of perimeter constraint may accumulate and break the iteration process for the next subproblem. The sub-iteration scheme is a good approach to bypass this difficulty. It was proposed by authors [13] in the hydrodynamic shape optimization of a swatch hull. In that case, the geometrical condition associated with the hull volume has to be kept unchanged to ensure the body's stability during optimization and the sub-iteration scheme is used to update the nonlinear equality constraint defined by this condition. In topology optimization, Duysinx [11] used the same

idea for the perimeter constraint defined in (1). The sub-iteration scheme is to solve repetitively the subproblem (8) for which approximations of the compliance and volume at X^0 remain unchanged whereas the perimeter constraint is gradually updated in an internal cycle until (7) is satisfied. The idea is based on the fact that the perimeter constraint (7) is a sort of geometrical condition unlike the compliance. Thus, function and derivative evaluations are very simple and fast without the need of finite element analysis. Let X^k be an intermediate solution point of (8), then the updated subproblem corresponds to

$$Min \quad c^0 + \sum_i \frac{\overline{c_i}}{x_i}$$

$$v^0 + \sum_i v_i x_i \leq 0$$

$$p^k + \sum_i p_i x_i + \sum_i d_i x_i^2 \leq 0 \qquad (14)$$

$$\delta \leq x_i \leq 1 \quad i = 1, n$$

with coefficients to be

$$p^k = P(X^k) - \overline{P} - \sum_{i=1}^{n} \frac{\partial P(X^0)}{\partial x_i} x_i^0 + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 P(X^0)}{\partial x_i^2} (x_i^0)^2$$

$$p_i = \frac{\partial P(X^k)}{\partial x_i} - \frac{\partial^2 P(X^k)}{\partial x_i^2} x_i^k \quad ; \quad d_i = \frac{1}{2} \frac{\partial^2 P(X^k)}{\partial x_i^2} \qquad (15)$$

In this way, the sub-iteration ensures the consideration of coupling term effects of (7) in the subproblem definition.

4. Numerical examples

In this section, two numerical tests are solved to validate the proposed method. Investigations are made to the formulation, approximation quality and effects of the perimeter constraint upon checkerboards. A 50% volume fraction is used as the upper bound of the volume constraint. The exponent $p = 4$ is chosen for the SIMP law in all cases.

4.1. Topology optimization of a 2D cantilever beam subject to a concentrated load

The problem is shown in Fig. 1. It is a clamped rectangular design domain loaded by a concentrated force F. Initial data are given below.

| Domain length: L=32 in, Width: H=20 in, Thickness: t= 1 in, Volume: V= 640 in³ |
| Load: F=100 lb, Physical properties: E =21×104 psi, ν =0.3 |

Two load cases are considered. Suppose that the force is applied at the middle point A and the bottom point B of the right side, respectively. A 48×30 finite element mesh is used for the domain discretization. To solve the topology optimization problem, initial values are set to be $x_i = 0.1$ for all density variables. In Fig.2, a set of optimum solutions are obtained by varying the upper bound of the perimeter whose values are increased from 100 to 300. It can be seen that along with the relaxation of perimeter bound, each member in the design domain is gradually outlined, internal small holes tend to decrease and grey regions associated with intermediate material density values between (0, 1) begin to disappear. Finally, the iteration gives rise to a quasi 0/1 discrete solution for which the perimeter constraint becomes inactive. In this example, the minimum value of compliance decreases along with the relaxation of the perimeter bound. However, if the optimum solution is directly obtained without the use of the perimeter constraint, the solution is clearly less interesting. As shown in Fig. 2, due to the existence of local optima, grey elements and small holes involve inside the members. The

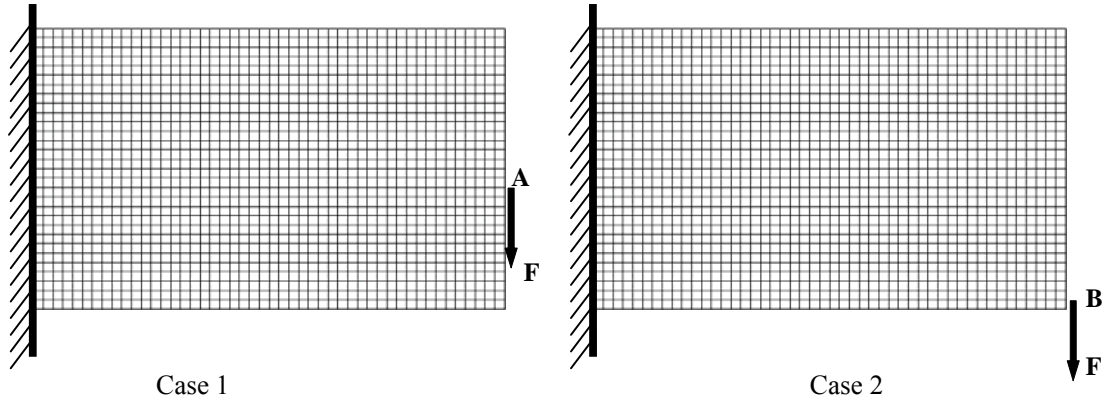compliance value C=2.0506 is even larger than that when $\overline{P} = 300$.

Case 1   Case 2

Fig. 1. 2D topology optimisation problem

$\overline{P}$ =100, C=2.01212

$\overline{P}$ =150, C=1.88721

$\overline{P}$ =200, C=1.836337

$\overline{P}$ =250, C=1.806325

P*=266, $\overline{P}$ =300, C=1.80359

Without perimeter constraint, C=2.050554
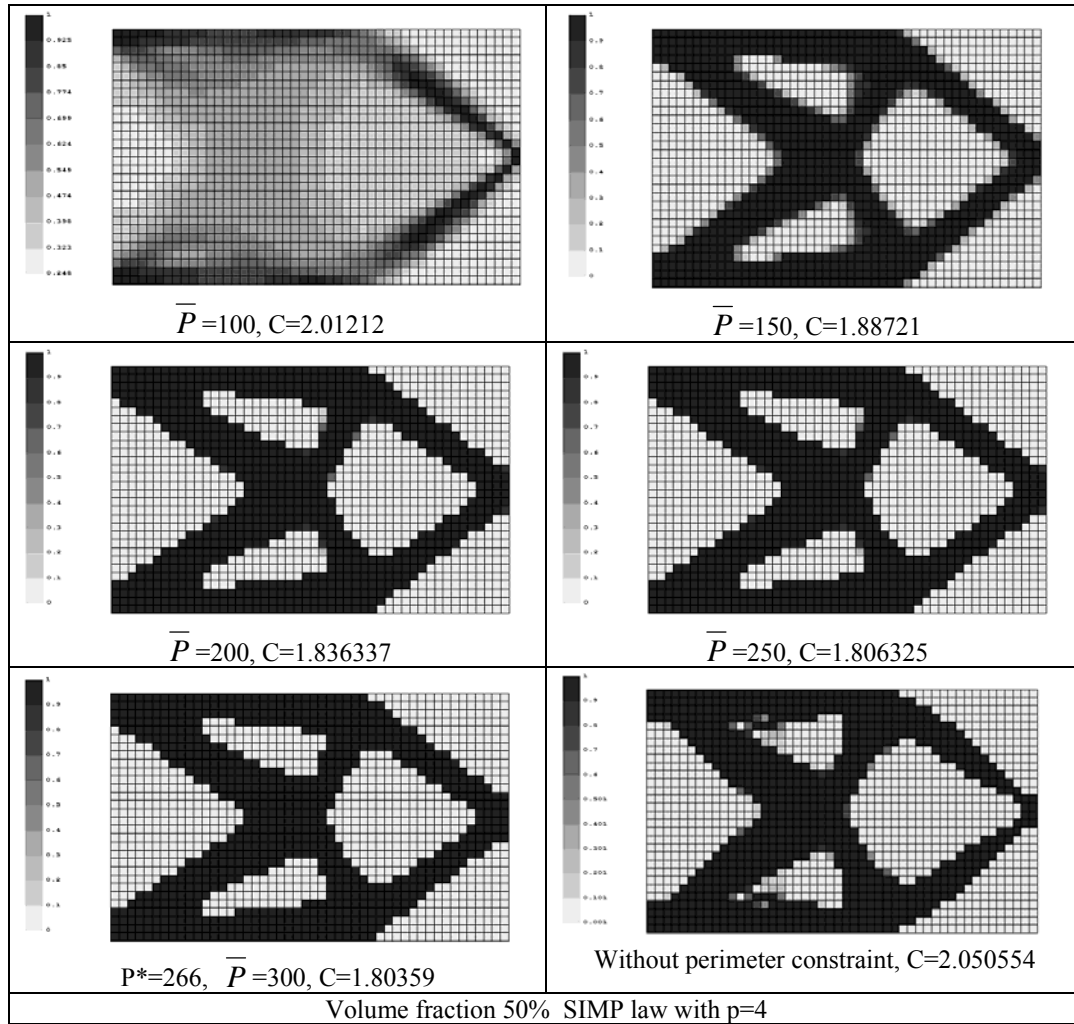
Volume fraction 50% SIMP law with p=4

Fig. 2. Topology optimum solutions in the first case

In the second case, the same conclusion can be drawn from numerical results shown in Fig. 3. The effect of progressive relaxation of perimeter constraint is significant and effective to achieve a satisfactory material distribution pattern. Additionally, the numerical iteration process is stable in both cases. Based on the diagonal quadratic approximation, the proposed perimeter constraint is found to be exactly satisfied after the sub-iteration process.
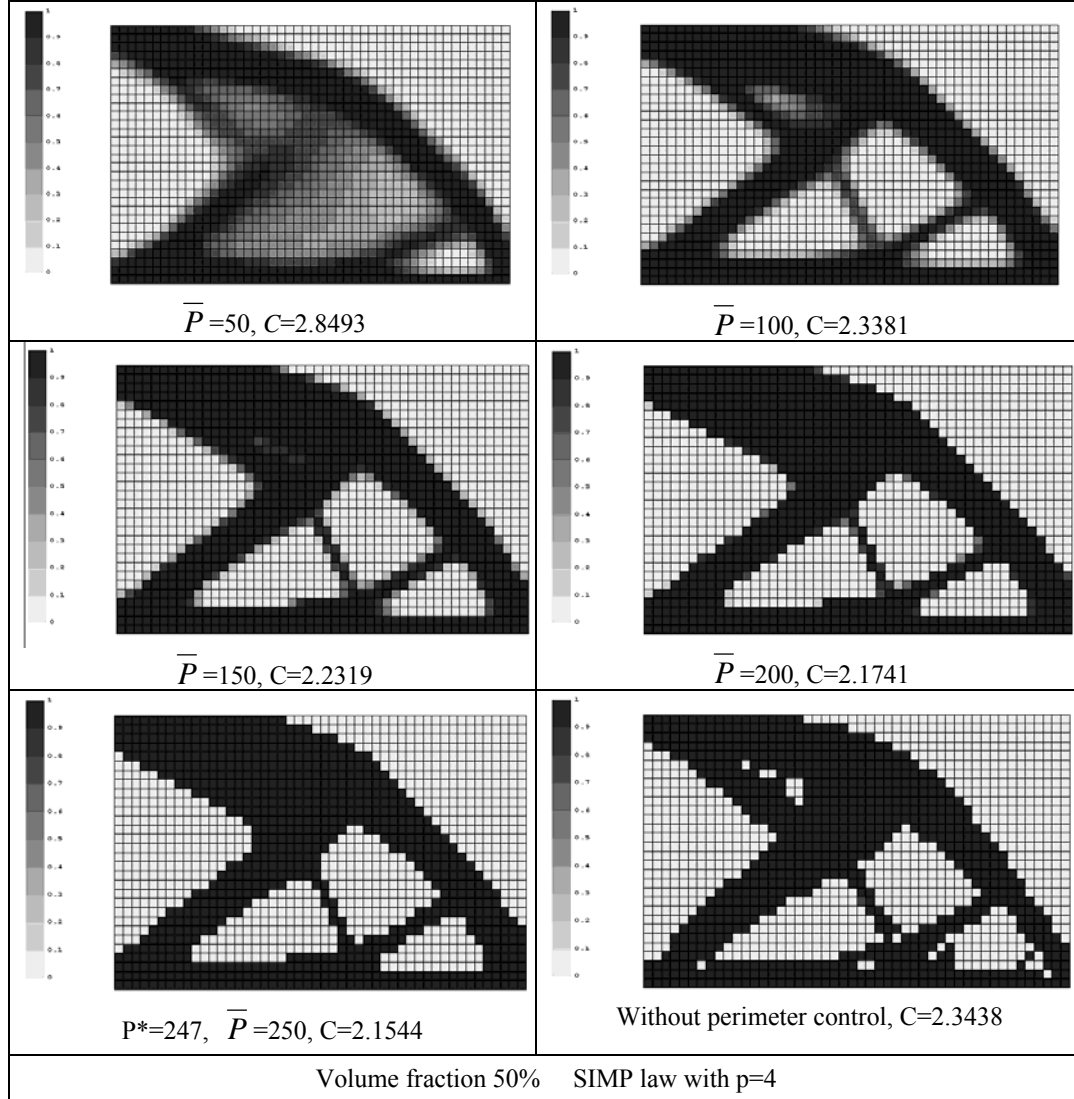

$\overline{P}$ =50, C=2.8493	$\overline{P}$ =100, C=2.3381
$\overline{P}$ =150, C=2.2319	$\overline{P}$ =200, C=2.1741
P*=247, $\overline{P}$ =250, C=2.1544	Without perimeter control, C=2.3438
Volume fraction 50% SIMP law with p=4	

Fig. 3. Topology optimum solutions in the first case

4.2. Topology optimization of a 3D cantilever beam subject to a concentrated load

To test the efficiency of the developed topology optimization procedure, the above problem is now extended to a 3D one by increasing the plate thickness to 4 in. Suppose that the load has the same magnitude and is applied at the middle point A. In this case, a 48×30×4 mesh of 8-node finite element is used for the domain discretization. In the same way, by increasing the perimeter bound, the optimum solution nearly tends to a discrete 0/1 design pattern. To some extent, this solution may be comparable to that obtained directly by the discrete optimization algorithm. As shown in Fig. 4, two internal grey regions in the member ultimately vanish when the perimeter constraint becomes inactive

with $\overline{P}$ =1500. Similarly, the last optimum pattern in Fig. 4 indicates that small holes with grey elements are involves if the perimeter control is not used.

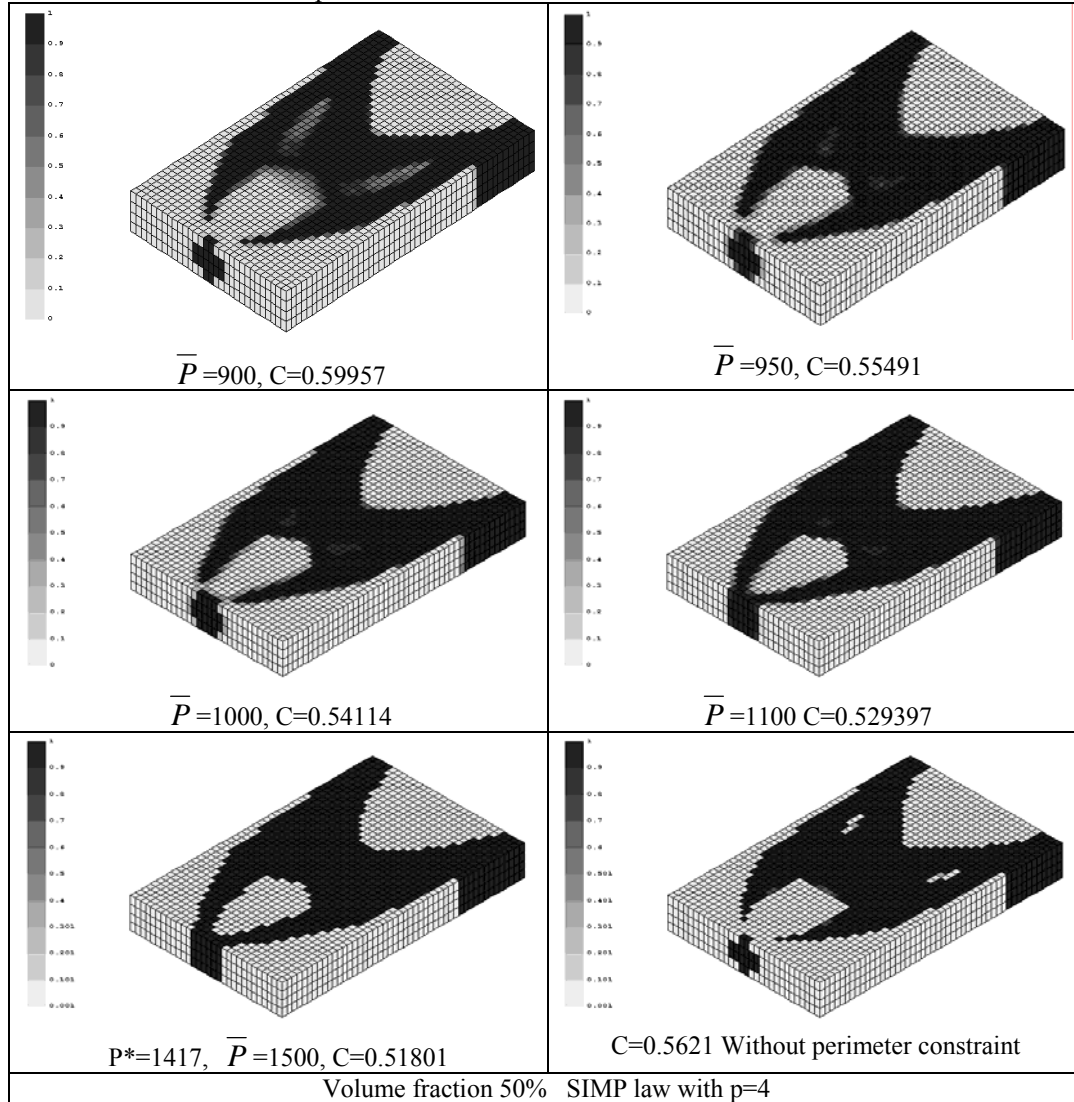| $\overline{P}$ =900, C=0.59957 | $\overline{P}$ =950, C=0.55491 |
$\overline{P}$ =1000, C=0.54114	$\overline{P}$ =1100 C=0.529397
P*=1417, $\overline{P}$ =1500, C=0.51801	C=0.5621 Without perimeter constraint
Volume fraction 50% SIMP law with p=4	

Fig. 4. Topology optimum solution with gradual relaxation of perimeter constraint

5. Conclusions

An interactive topology optimization procedure is developed and integrated with the SAMCEF commercial finite element system. The integration relies basically on the scaling of element deformation energy for sensitivity analysis. Based on the perimeter control method, a variant quadratic formulation of perimeter function is proposed to prevent checkerboards and grey element appearances. This variant is proved to be very suitable to the diagonal approximation and easily implemented in the dual approach. An efficient sub-iteration scheme is established to update the approximation of the perimeter. Numerical results show that the optimization procedure works well. The perimeter constraint can be satisfied at the final solution. The gradual relaxation of the proposed perimeter control has the effect of reducing small holes inside the structural members and leads to a checkerboard free material pattern.

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