Continuation of bifurcations of periodic solutions based on the harmonic balance method

Thibaut Detroux and Gaetan Kerschen
Space Structures and Systems Laboratory (S3L), Structural Dynamics Research Group, Department of Aerospace and Mechanical Engineering, University of Liège, Belgium

Summary. This paper proposes to extend the harmonic balance method to perform the continuation of bifurcations of periodic solutions in a codimension-2 parameter space. Based on the framework of bordering techniques, the procedure utilizes Hill's method to augment the system of equations in order to track bifurcations. As an application to validate the procedure, the design of a nonlinear vibration absorber attached to a Duffing oscillator is studied.

Introduction

Because nonlinearities have an influence on the vibrations of most mechanical structures, it is relevant to study their effects on the evolution of the oscillations with respect to a certain parameter, e.g., the frequency of the external excitation or a system parameter. Besides, the study of bifurcations of periodic solutions, for instance, limit point (LP) and Neimark-Sacker (NS) bifurcations, is also of interest because of the key role they play in structural dynamics.
Software for bifurcation tracking such as MATCONT [1], which uses orthogonal collocation, are well-established and versatile but the computational burden for high-dimensional problems can be important in terms of memory requirements. An interesting alternative for computing the periodic solutions of large-scale systems is the harmonic balance (HB) method [2]. However, to the authors' knowledge, this method has not yet been applied for bifurcation tracking, which is the purpose of this study.

An extended harmonic balance method

Approximation of the periodic solutions

The method is applied to general non-autonomous nonlinear dynamical systems with n degrees of freedom (DOFs). The periodic solutions $\mathbf{x}(t)$ of these systems are approximated by Fourier series truncated to the N_H-th harmonic. Substituting this approximation in the equations of motion and using a Galerkin projection yields the following nonlinear equations of amplitude of the system in the frequency domain:

$$\mathbf{h}(\mathbf{z},\omega) \equiv \mathbf{A}(\omega)\mathbf{z} - \mathbf{b}(\mathbf{z},\omega) = \mathbf{0} \tag{1}$$

where $\mathbf{A}$ is the $(2N_H+1)\,n \times (2N_H+1)\,n$ matrix describing the linear dynamics of the system, $\mathbf{z}$ is the vector containing all the Fourier coefficients of the displacements $\mathbf{x}(t)$, and $\mathbf{b}$ represents the vector of the Fourier coefficients of the (external) linear and nonlinear forces $\mathbf{f}(t)$.
For each continuation iteration along the branch of solutions of (1), a matrix $\mathbf{B}$ whose eigenvalues give the Floquet exponents of the solution is computed with a technique derived from Hill's method [2] and the eigenvalue sorting procedure proposed in [3].

Bifurcation tracking

In order to track bifurcations in parameter space, the system (1) should be augmented. In this work, this is achieved through the use of bordering techniques [4], for which one considers an additional equation of the form

$$g = 0 \tag{2}$$

obtained from the resolution of the system

$$\begin{bmatrix} \mathbf{G} & \mathbf{p} \\ \mathbf{q}^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ g \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \tag{3}$$

where * denotes the conjugate transpose, and where $\mathbf{p}$ and $\mathbf{q}$ are chosen so that the matrix is nonsingular. The resolution of (3) ensures that the determinant of $\mathbf{G}$ vanishes simultaneously with g. $\mathbf{G}$ is replaced by the matrix $\mathbf{B}$ for the tracking of LP bifurcations, and by the bialternate matrix product [5] of the matrix $\mathbf{B}$ for the tracking of NS bifurcations:

$$\mathbf{G}_{LP} = \mathbf{B} \quad \text{and} \quad \mathbf{G}_{NS} = \mathbf{B}_{\odot} \tag{4}$$

Validation of the method

This section studies a nonlinear tuned vibration absorber (NLTVA) attached to a harmonically-forced 1DOF host structure, as depicted in Figure 1. Figures 2(a-b) represent the frequency response of the primary structure computed with the HB method, for two forcing amplitudes, together with the detected bifurcations. In addition to the branch of the main frequency response, one observes the presence of an isolated solution for $F = 0.15$. When $F = 0.19$, these two branches merge, which decreases the performance of the absorber. Interestingly, the analysis of the branches tracking the LPs and NSs in Figures 3(a-b) also indicates the presence of the isola and of the merging, through the elimination of two LPs around $F = 0.18$ and two NSs around $F = 0.19$. A comparison with the orthogonal collocation method shows that the HB method provides accurate results for $N_H = 5$.

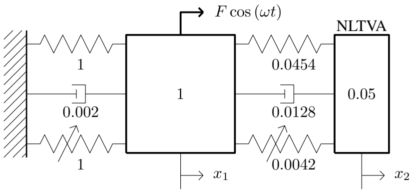

Figure 1: Schematic representation of the 1DOF+NLTVA system. Both nonlinear springs are cubic.

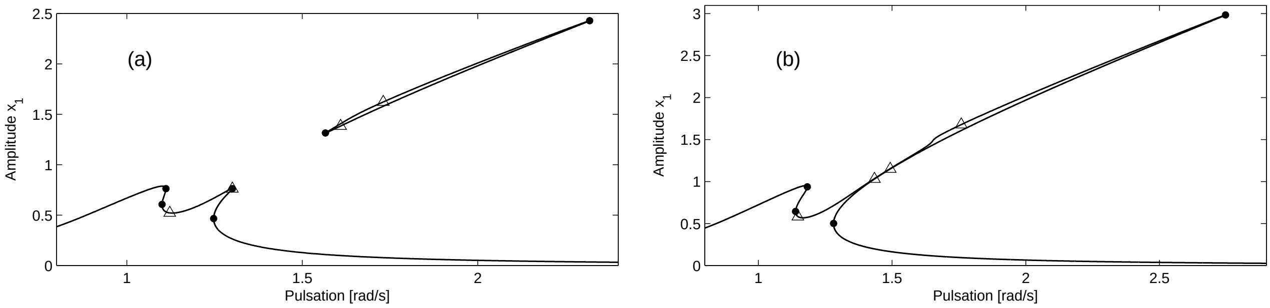

Figure 2: Frequency response of the host structure of the 1DOF+NLTVA system and bifurcation detection for $N_H = 5$. (a) $F = 0.15$, (b) $F = 0.19$. The markers ● and △ highlight LPs and NSs, respectively.

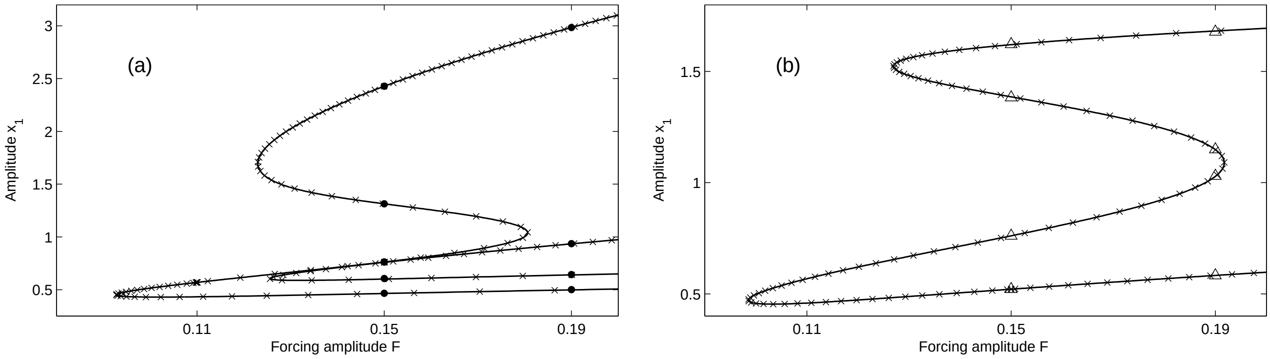

Figure 3: Bifurcation tracking for the host structure of the 1DOF+NLTVA system. (a) Projection of the LP branch on the F–amplitude plane, (b) Projection of the NS branch on the F–amplitude plane. The solid lines are obtained with the HB method $(N_H = 5)$ and the cross markers are obtained with the orthogonal collocation method (4 collocation points and 40 mesh points).

Conclusion

In this paper, the HB method is extended for the continuation of the bifurcations of periodic solutions of nonlinear mechanical systems. The proposed method is capable of accurately tracking LP and NS bifurcations. Future research will extend these results for the study of large-scale, strongly nonlinear systems with a greater number of harmonics.

References

[1] **Dhooge, A., Govaerts, W.** and **Kuznetsov, Y. A.**, *MATCONT: a MATLAB package for numerical bifurcation analysis of ODEs*, ACM Transactions on Mathematical Software (TOMS), Vol. 29, No. 2, pp. 141–164, 2003.

[2] **von Groll, G.** and **Ewins, D. J.**, *The harmonic balance method with arc-length continuation in rotor/stator contact problems*, Journal of Sound and Vibration, Vol. 241, No. 2, pp. 223–233, 2001.

[3] **Lazarus, A.** and **Thomas, O.**, *A harmonic-based method for computing the stability of periodic solutions of dynamical systems*, Comptes Rendus Mécanique, Vol. 338, No. 9, pp. 510–517, 2010.

[4] **Doedel, E. J., Govaerts, W.** and **Kuznetsov, Y. A.**, *Computation of periodic solution bifurcations in ODEs using bordered systems*, SIAM Journal on Numerical Analysis, Vol. 41, No. 2, pp. 401–435, 2003.

[5] **Guckenheimer, J., Myers, M.** and **Sturmfels, B.**, *Computing hopf bifurcations I*, SIAM Journal on Numerical Analysis, Vol. 34, No. 1, pp. 1–21, 1997.