Université de Liège

Graded-commutative nonassociative algebras: higher octonions and Krichever-Novikov superalgebras; their structures, combinatorics and non-trivial cocycles.

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Abstract

This dissertation consists of two parts. The first one is the study of a series of real (resp. complex) noncommutative and nonassociative algebras $\mathbb{O}_{p,q}$ (resp. $\mathbb{O}_n$) generalizing the algebra of octonion numbers $\mathbb{O}$. This generalization is similar to the one of the algebra of quaternion numbers in Clifford algebras. Introduced by Morier-Genoud and Ovsienko, these algebras have a natural $\mathbb{Z}_2^n$-grading ($p + q = n$), and they are characterized by a cubic form over the field $\mathbb{Z}_2$. We establish all the possible isomorphisms between the algebras $\mathbb{O}_{p,q}$ preserving the structure of $\mathbb{Z}_2^n$-graded algebra. The classification table of $\mathbb{O}_{p,q}$ is quite similar to that of the real Clifford algebras $\mathcal{C}l_{p,q}$, the main difference is that the algebras $\mathbb{O}_{n,0}$ and $\mathbb{O}_{0,n}$ are exceptional. We also provide a periodicity for the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ analogous to the periodicity for the Clifford algebras $\mathcal{C}l_n$ and $\mathcal{C}l_{p,q}$.

In the second part we consider superalgebras of Krichever-Novikov (K-N) type. Krichever and Novikov introduced a family of Lie algebras with two marked points generalizing the Witt algebra and its central extension called the Virasoro algebra. The K-N Lie (super)algebras for more than two marked points were studied by Schlichenmaier. In particular, he extended the explicit formula of 2-cocycles due to Krichever and Novikov to multiple-point situation. We give an explicit construction of central extensions of Lie superalgebras of K-N type and we establish a 1-cocycle with values in its dual space. In the case of Jordan superalgebras related to superalgebras of K-N type, we calculate a 1-cocycle with coefficients in the dual space.

Key words: octonion, Clifford algebra, binary cubic form, twisted group algebra, nonassociative and noncommutative algebra, graded algebra, Krichever-Novikov Lie superalgebra, Jordan superalgebra, Lie antialgebra, non-trivial cocycle.

Résumé

Cette dissertation est divisée en deux parties. La première partie consiste en l'étude d'une série d'algèbres non commutatives et non associatives réelles $\mathbb{O}_{p,q}$ (respectivement complexes $\mathbb{O}_n$ où $p + q = n$). Cette généralisation est similaire à celle de l'algèbre des quaternions en algèbres de Clifford. Introduites par Morier-Genoud et Ovsienko, ces algèbres ont une $\mathbb{Z}_2^n$-graduation naturelle et sont caractérisées par une forme cubique sur le corps commutatif $\mathbb{Z}_2$. Nous établissons tous les isomorphismes possibles entre les algèbres $\mathbb{O}_{p,q}$ préservant la graduation. La table de classification des algèbres $\mathbb{O}_{p,q}$ est similaire à celle des algèbres de Clifford $\mathcal{C}l_{p,q}$, la différence majeure étant que les algèbres $\mathbb{O}_{0,n}$ et $\mathbb{O}_{n,0}$ sont exceptionnelles. Nous donnons aussi une périodicité des algèbres $\mathbb{O}_n$ et $\mathbb{O}_{p,q}$ analogue à la périodicité sur les algèbres de Clifford $\mathcal{C}l_n$ et $\mathcal{C}l_{p,q}$.

Dans la seconde partie, nous étudions les 1- et 2-cocycles des superalgèbres de type Krichever-Novikov (K-N). Krichever et Novikov ont introduit une famille d'algèbres de Lie avec deux points marqués généralisant l'algèbre de Witt ainsi que son extension centrale appelée l'algèbre de Virasoro. Par la suite, Schlichenmaier a étudié les (super)algèbres de Lie de type K-N ayant plus de deux points marqués. En particulier, il a étendu la formule explicite des 2-cocycles de Krichever et Novikov au cas des multiples points marqués. Nous donnons une construction explicite d'extensions centrales des superalgèbres de Lie de type K-N et nous déterminons un 1-cocycle à valeur dans l'espace dual. Dans le cas des superalgèbres de Jordan de type K-N reliées aux superalgèbres de Lie de type K-N, nous calculons un 1-cocycle à valeur dans l'espace dual.

Mots clés: octonion, algèbre de Clifford, forme cubique binaire, algèbre de groupe tordue, algèbre non commutative et non associative, algèbre graduée, superalgèbre de Lie de type Krichever-Novikov, superalgèbre de Jordan, antialgèbre de Lie, cocycle non-trivial.

Le doctorat est un moment unique et intense d'apprentissage de la Recherche par la Recherche. Un combat de tous les jours, bien souvent solitaire, parsemé d'incertitudes et de questionnements mais aussi d'émerveillements et de rencontres insolites. Il aura fallu un brin de folie pour se lancer dans cette aventure, de la résistance pour y faire face et beaucoup de ténacité pour en sortir.

Mes remerciements visent en premier lieu mes deux promoteurs de thèse, Pierre Lecomte et Valentin Ovsienko. Le premier m'a donné l'opportunité de réaliser ce doctorat à Liège. Depuis 2010, il m'accompagne ainsi, au jour le jour, tant dans mes travaux de recherche que dans le cadre de mon assistanat. Le second, malgré la distance, a su témoigner d'une disponibilité permanente et d'un soutien inconditionnel, notamment tout au long de la rédaction de ce document. Durant ce doctorat, j'ai particulièrement apprécié la manière avec laquelle ils partagent leur passion de la recherche et de l'enseignement.

Je souhaite également remercier Sophie Morier-Genoud, avec laquelle j'ai eu le plaisir de collaborer, ainsi que Martin Schlichenmaier, Pierre Mathonet et Georges Hansoul, qui ont accepté de faire partie de mon jury de thèse.

Au fur et à mesure de mes recherches, de nombreux échanges de qualité avec mes collègues et amis doctorants ont contribué à façonner ma vision des mathématiques et du monde. Mais plus encore, le doctorat m'a permis de rencontrer une multitude de personnes dans les différents environnements de travail que j'ai fréquentés. À Liège et à Lyon principalement, mais également lors de conférences dans divers endroits d'Europe et aux Etats-Unis, ces nombreuses expériences de vie m'ont enrichie sur les plans professionnel et personnel

Un doctorat est une tranche de vie qui ne se vit qu'une fois. Une expérience humaine que j'ai partagée avec les gens que j'aime. Mes pensées vont à mes parents, à mes deux sœurs, Sarah et Fanny, ainsi qu'à Magali, qui ont pu m'épauler à chaque instant.

À Emilie,
Là-bas.

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Introduction

Algebras generalizing the Octonions

The algebra of octonion numbers $\mathbb{O}$ is the normed division algebra over the real number which has the biggest dimension. The algebra $\mathbb{O}$ is noncommutative and nonassociative. Due to the Cayley-Dickson construction [Cay45] and [Dic19], the algebra $\mathbb{O}$ can be viewed as an extension of the algebra of quaternion numbers $\mathbb{H}$. Elduque in [Eld98], understood the algebra $\mathbb{O}$ as a graded algebra, where in particular a $\mathbb{Z}_2^3$-grading was considered. Albuquerque and Majid in [AM99] considered $\mathbb{O}$ as twisted group algebra over $\mathbb{Z}_2^3$. They likewise proved that Clifford algebras are obtained by twisting of group algebras of $\mathbb{Z}_2^n$ by a cocycle, see [AM02]. The Clifford algebras are associative algebras but noncommutative generalizing $\mathbb{H}$ differently from the Cayley-Dickson process.

Four years ago, Morier-Genoud and Ovsienko [MGO11] introduced a series of noncommutative and nonassociative real algebras $\mathbb{O}_{p,q}$ and their complexification $\mathbb{O}_n$. They generalize the octonion numbers (and the split-octonions) in the same way as the Clifford algebras generalize the quaternion numbers. They are obtained by twisting of group algebras of $\mathbb{Z}_2^n$ by a cochain.

The first part of this thesis is dedicated to the study of properties of the algebras $\mathbb{O}_{p,q}$ and $\mathbb{O}_n$. The real Clifford algebras $\mathcal{C}l_{p,q}$ and the complex $\mathcal{C}l_n$ are completely classified and there exist a lot of periodicities, one of the best known is the *Bott Periodicity*. We give here similar results on the algebras $\mathbb{O}_{p,q}$ and $\mathbb{O}_n$ about classification and the periodicities and compare it with the known cases on the Clifford algebras. These results come from the papers [KMG15] and [Kre15] of the author.

Superalgebras of Krichever-Novikov type

The Lie algebras of Krichever-Novikov (K-N) type are a family of Lie algebras generalizing the Virasoro algebra. Introduced by Krichever and Novikov in [KN87b], [KN87a] and [KN89], these algebras are Lie algebras of meromorphic vector fields on a Riemman surface of arbitrary genus g with at first two marked points. Later, Schlichenmaier studied the K-N Lie algebras for more than two marked points, [Sch90b], [Sch90c], [Sch90a]. In particular in [Sch03], he showed the existence of local 2-cocycles and central extensions for multiple-point K-N Lie algebras, extending the explicit formula of 2-cocycles given by Krichever and Novikov in [KN87b].

The notion of Lie antialgebra was introduced by Ovsienko, [Ovs11]. Lie antialgebras are particular cases of Jordan superalgebras and are related to Lie superalgebras. One of the main examples of Lie antialgebra is $\mathcal{AK}(1)$ related to the Virasoro algebra and the Neveu-Schwarz Lie superalgebra $\mathcal{K}(1)$. Leidwanger and Morier-Genoud in [LMG12a] studied the Lie superalgebras of K-N type generalizing $\mathcal{K}(1)$, $\mathcal{L}_{g,N}$, and the relation with Jordan superalgebras of K-N type generalizing $\mathcal{AK}(1)$, $\mathcal{J}_{g,N}$.

Bryant in [Bry90] studied cocycles of the Lie superalgebra $\mathcal{L}_{g,N}$, N=2. Recently and independently of this work, Schlichenmaier studied the case of Lie superalgebras $\mathcal{L}_{g,N}$ in [Sch13] and [Sch14].

The second part of this thesis is devoted to the study of extensions of superalgebras of K-N type: $\mathcal{L}_{g,N}$ and $\mathcal{J}_{g,N}$. These results come from the paper of the author [Kre13]. We give an explicit formula for a local non-trivial 2-cocycle on $\mathcal{L}_{g,N}$, where only integration over a separating cycle is considered. Then we give a formula for a local 1-cocycle on $\mathcal{J}_{g,N}$ with coefficients in the dual space. Interesting explicit examples of superalgebras arise in the case of the Riemman sphere with three marked points, we have: $\mathcal{L}_{0,3} \supset \mathcal{K}(1) \supset \mathrm{osp}(1|2)$ and $\mathcal{J}_{0,3} \supset \mathcal{AK}(1) \supset \mathcal{K}_3$, where $\mathcal{K}_3$ is the Kaplansky Jordan superalgebra which is related to osp(1|2), see [MG09]. We calculate explicitly the 2-cocycle on the Lie superalgebra $\mathcal{L}_{0,3}$ that is unique up to isomorphism and vanishes on the Lie superalgebra osp(1|2). Finally, we also give an explicit formula for a 1-cocycle on the Lie antialgebra $\mathcal{J}_{0,3}$ with values in its dual space that vanishes on $\mathcal{K}_3$.

This document

The text is divided in five chapters. The three first chapters are dedicated to the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$. It is based on the following two papers

[Kre15] M. Kreusch. Bott type periodicity for higher octonions, 2015. *J. Noncommut. Geom, to appear*;

[KMG15] M. Kreusch and S. Morier-Genoud. Classification of the algebras $\mathbb{O}_{p,q}$, 2015. *Comm. Alg, to appear.*

In **Chapter** 1, we define the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ as twisted group algebras over $\mathbb{Z}_2^n$. Then we characterize them as graded algebras and view them as non associative extension of Clifford algebras. These algebras are determined by a generating cubic form and they give explicit solutions to the the Hurwitz problem of "Sum of squares". In **Chapter** 2, we answer the question [KMG15]: Given two algebras $\mathbb{O}_{p,q}$ and $\mathbb{O}_{p',q'}$ with $p+q=p'+q'$, what are the conditions on p, q, p' and q' to have $\mathbb{O}_{p,q} \simeq \mathbb{O}_{p',q'}$? In **Chapter** 3, we answer the question [Kre15]: How do the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ $(p+q=n)$ depend on the parameter n?

The last two chapters are devoted to the superalgebras of Krichever-Novikov type and their cocyles. It is based on the following paper

[Kre13] M. Kreusch. Extensions of superalgebras of Krichever-Novikov type. *Lett. Math. Phys.,* 103(11):1171-1189, 2013.

In **Chapter** 4, we introduce the notions of K-N (super)algebras. We define the Lie superalgebras $\mathcal{L}_{g,N}$ with significant examples: $\mathcal{K}(1)$ and $\mathcal{L}_{0,3}$. We define Lie antialgebras which are particular class of Jordan superalgebras, explain the geometrical origin, present examples and detail their related Lie superalgebras. Finally, we introduce Jordan superalgebras of K-N type and the significant examples: $\mathcal{AK}(1)$ and $\mathcal{J}_{0,3}$. In **Chapter** 5, we present a local non-trivial 2-cocycle on $\mathcal{L}_{g,N}$ together with an explicit formula in the particular case $\mathcal{L}_{0,3}$. We also provide a local 1-cocycle on $\mathcal{J}_{g,N}$ with value in the dual space and an explicit formula in the particular case $\mathcal{J}_{0,3}$.

Chapter 1

The Algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ Generalizing the Octonions

There are exactly four normed division algebras over $\mathbb{R}$: the algebra of real numbers $\mathbb{R}$, the algebra of complex numbers $\mathbb{C}$, the algebra of quaternion numbers $\mathbb{H}$ and the algebra of octonion numbers $\mathbb{O}$. This classical theorem was proved by Hurwitz in 1898 [Hur98]. These algebras are the first ones of the sequence of *Cayley-Dickson algebras*, see Figure 1.1.

$$\mathbb{R} \longrightarrow \mathbb{C} \longrightarrow \mathbb{H} \longrightarrow \mathbb{O} \longrightarrow \cdots \qquad \boxed{\text{Cayley-Dickson algebras}}$$

Figure 1.1: Illustration of the first Cayley-Dickson algebras.

In 1843, Hamilton discovered the algebra of quaternion numbers. A bit later, in 1844, Graves found out the algebra of octonion numbers. While the algebras of quaternion numbers is noncommutative but still associative, the algebra of octonion numbers is neither commutative nor associative.

From the sequence of Cayley-Dickson algebras, a series of other algebras were developed. For example, in 1876, Clifford found associative algebras generalizing the algebra of quaternion numbers (differently from the Cayley-Dickson process). The Clifford algebras $\mathcal{C}l_n$ over $\mathbb{C}$ or $\mathcal{C}l_{p,q}$ over $\mathbb{R}$ $(p+q=n)$ are intimately connected to the theory of quadratic forms. Note that the algebra of real numbers appears as $\mathcal{C}l_{0,0}$, the algebra of complex numbers as $\mathcal{C}l_{0,1}$ and the algebra of quaternion numbers as $\mathcal{C}l_{0,2}$.

A series of noncommutative and nonassociative algebras $\mathbb{O}_{p,q}$ over $\mathbb{R}$ and their complexification $\mathbb{O}_n$ $(n = p+q)$ was introduced in [MGO11]. The algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ generalize the classical algebras of octonion numbers and split-octonions in the same way as the Clifford algebras generalize the algebra of quaternion numbers. Note that the algebra of octonion numbers appears in the series as $\mathbb{O}_{0,3}$, whereas the algebra of split-octonions is isomorphic to $\mathbb{O}_{3,0}$, $\mathbb{O}_{1,2}$ and $\mathbb{O}_{2,1}$. The properties of the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ are very different from those of the classical Cayley-Dickson algebras. The series of algebras $\mathbb{O}_{p,q}$ can be illustrated by Figure 1.2.

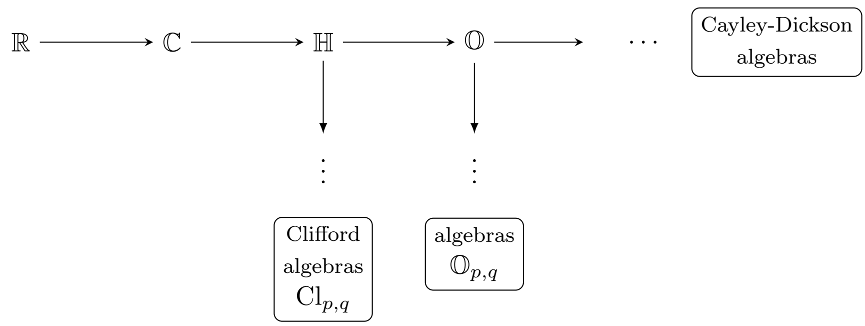

Figure 1.2: Families of $\mathbb{Z}_2^n$-graded algebras.

The idea to understand the classical algebra of the octonion numbers $\mathbb{O}$ as a graded algebra was suggested by Elduque [Eld98] in which a $\mathbb{Z}_2^3$-grading was particularly considered. In [AM99], Albuquerque and Majid, understood the algebra $\mathbb{O}$ as a twisted group algebra over $\mathbb{Z}_2^3$ which has a graded commutative and a graded associative structure. Later they showed that Clifford algebras are obtained by twisting of group algebras of $\mathbb{Z}_2^n$ by a cocycle [AM02].

The complex algebras $\mathbb{O}_n$ and the real algebras $\mathbb{O}_{p,q}$ are graded algebras over the abelian group $\mathbb{Z}_2^n$ and were introduced by Morier-Genoud and Ovsienko in [MGO11]. They studied their properties like simplicity, uniqueness and characterization. Indeed, these algebras are determined by a cubic form

$$\alpha : \mathbb{Z}_2^n \to \mathbb{Z}_2,$$

where $\mathbb{Z}_2^n$ is understood as a vector space of dimension n over the field $\mathbb{Z}_2$ of two elements, see [MGO11]. This is the main property of the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ that distinguishes them from other series of algebras generalizing the octonions, such as the Cayley-Dickson algebras.

The complex algebras $\mathbb{O}_n$ and especially the real algebras $\mathbb{O}_{0,n}$ have applications to the classical Hurwitz problem of sum of square identities and related problems; see [LMGO11, MGO13]. The algebras $\mathbb{O}_{0,n}$ and $\mathbb{O}_n$ give rise to explicit formulas of the famous square identity

$$(a_1^2 + \cdots a_r^2)(b_1^2 + \cdots b_s^2) = c_1^2 + \cdots c_N^2$$

where c_l, $(l = 1, \ldots, N)$ are bilinear functions of a_i $(i = 1, \ldots, r)$ and of b_j $(j = 1, \ldots, s)$. Note that an application of $\mathbb{O}_n$ to additive combinatorics was suggested in [MGO14].

The first problem of classification of the real algebras $\mathbb{O}_{p,q}$ with *fixed* $n = p+q$ depending on the signature (p, q), was formulated in [MGO11]. This problem was solved in [KMG15] and is detailed in Chapter 2. The result is as follows. The classification table of $\mathbb{O}_{p,q}$ for p and $q \neq 0$, coincides with the well known table of the real Clifford algebras. The algebras $\mathbb{O}_{0,n}$ and $\mathbb{O}_{n,0}$ are exceptional.

The second problem concerns the periodicity on the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ with $p + q = n$ depending on the parameter n. This problem was solved in [Kre15] and is detailed in Chapter 3. The result is as follows. In the complex case, we link together the algebras $\mathbb{O}_n$ and $\mathbb{O}_{n+4}$, showing a modulo 4 periodicity. In the real case, we establish a modulo 4 periodicity of the algebras $\mathbb{O}_{p,q}$ $(pq > 0)$ which is different from the modulo 4 periodicity of the exceptional algebras $\mathbb{O}_{0,n}$ and $\mathbb{O}_{n,0}$.

In this **chapter** we define the main objects of the first three chapters, the complex algebras $\mathbb{O}_n$ and the real algebras $\mathbb{O}_{p,q}$. First, in **Section** 1.1 they are defined as twisted group algebras over $\mathbb{Z}_2^n$. This definition insures continuity to the previous work done by Albuquerque and Majid in [AM02] on the Clifford algebras viewed as twisted group algebras.

The motivation for introducing the algebras $\mathbb{O}_{p,q}$ is presented in **Section** 1.2 coming from the fundamental paper [MGO11] of Morier-Genoud and Ovsienko and more in the papers [MGO10], [MGO13] and [MGO14] of the same authors. These algebras have applications to the Hurwitz problem "Sum of Squares", but also in combinatorics and appear in the code loop theory.

In **Section** 1.3, we give an other point of view of the algebras $\mathbb{O}_{p,q}$ in term of generators and relations considered as graded algebras. This follows from the work of Elduque in [Eld98] and of Albuquerque and Majid in [AM99] on the different gradings on octonions. The algebras $\mathbb{O}_{p,q}$, as a third perspective, are viewed in term of nonassociative extension of Clifford algebras in **Section** 1.4. These two last standpoints are mainly used in Chapter 2.

The last and important **Section** 1.5 is the characterization of twisted group algebras (up to isomorphism) thanks to a cubic form. This is in particular the case for the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ that differ from the Cayley-Dickson algebras. This cubic form is widely used in Chapter 3 to state results on periodicity.

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1.1 Twisted group algebras over $\mathbb{Z}_2^n$

Throughout the dissertation, we denote by $\mathbb{Z}_2$ the quotient $\mathbb{Z}/2\mathbb{Z}$ understood as abelian group, and also as a field of two elements $\{0,1\}$. The ground field $\mathbb{K}$ is assumed to be $\mathbb{R}$ or $\mathbb{C}$ for the three first chapters.

1.1.1 Twisted group algebras

Let f be an arbitrary function in two arguments

$$f : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2,$$

such that $f(x,0) = 0 = f(0,x)$ for all x in $\mathbb{Z}_2^n$. The *twisted group algebra* $(\mathbb{K}[\mathbb{Z}_2^n], f)$ is defined as the 2^n-dimensional vector space with the basis $\{u_x, x \in \mathbb{Z}_2^n\}$, i.e.

$$\mathbb{K}[\mathbb{Z}_2^n] = \bigoplus_{x\in\mathbb{Z}_2^n} \mathbb{K}u_x,$$

and equipped with the product

$$u_x \cdot u_y = (-1)^{f(x,y)} u_{x+y},$$

for all $x, y \in \mathbb{Z}_2^n$.

The algebra $(\mathbb{K}[\mathbb{Z}_2^n], f)$ is always unital, the unit being u_0. In general $(\mathbb{K}[\mathbb{Z}_2^n], f)$ is neither commutative nor associative. The defect of commutativity and associativity is measured by a symmetric function $\beta : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2$ and a function $\phi : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2$, respectively

$$u_x \cdot u_y = (-1)^{\beta(x,y)}\, u_y \cdot u_x, \tag{1.1}$$

$$u_x \cdot (u_y \cdot u_z) = (-1)^{\phi(x,y,z)}\, (u_x \cdot u_y) \cdot u_z, \tag{1.2}$$

where explicitly

$$\beta(x,y) = f(x,y) + f(y,x), \tag{1.3}$$

$$\phi(x,y,z) = f(y,z) + f(x+y,z) + f(x,y+z) + f(x,y). \tag{1.4}$$

Note that the formula (1.4) reads $\phi = df$ where d is the coboundary operator on the space of cochains $C^*(\mathbb{Z}_2^n, \mathbb{Z}_2)$, see Subsection 1.1.2. The function ϕ is a trivial 3-cocycle or a coboundary on $\mathbb{Z}_2^n$ with coefficients in $\mathbb{Z}_2$ and the algebra $(\mathbb{K}[\mathbb{Z}_2^n], f)$ is associative if and only if f is a 2-cocycle on $\mathbb{Z}_2^n$. Such structures were studied in a more general setting, see e.g. [AM99, AEPI01].

Example 1.1. **(a)** The (real) algebra of quaternion numbers $\mathbb{H}$ ($\simeq \mathcal{C}l_{0,2}$), and more generally every complex or real Clifford algebra with n generators can be realized as twisted group algebras over $\mathbb{Z}_2^n$; see [AM02]. For arbitrary elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{Z}_2^n$, set

$$\begin{aligned} f_{\mathcal{C}l_n}(x,y) &= \sum_{1\le i\le j\le n} x_i y_j, \\ f_{\mathcal{C}l_{p,q}}(x,y) &= f_{\mathcal{C}l_n}(x,y) + \sum_{1\le i\le p} x_i y_i \qquad (n = p+q). \end{aligned} \tag{1.5}$$

Then the twisted group algebras $(\mathbb{R}[\mathbb{Z}_2^n], f_{\mathcal{C}l_{p,q}})$ and $(\mathbb{C}[\mathbb{Z}_2^n], f_{\mathcal{C}l_n})$ are respectively isomorphic to $\mathcal{C}l_{p,q}$ and to $\mathcal{C}l_n$. In particular, the twisting function $f_{\mathbb{H}}(x,y) = x_1y_1 + x_1y_2 + x_2y_2$ corresponds to the algebra of quaternions.

(b) The (real) algebra of octonion numbers $\mathbb{O}$ is a twisted group algebra over $\mathbb{Z}_2^3$; see [AM99]. The twisting function is cubic and given by

$$f_{\mathbb{O}}(x,y) = (x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3) + \sum_{1\le i\le j\le 3} x_i y_j. \tag{1.6}$$

Definition 1.1. ([MGO11]) The *complex algebra* $\mathbb{O}_n$ and the *real algebra* $\mathbb{O}_{p,q}$ with $p + q = n \geq 3$ are the twisted group algebras with the twisting functions

$$\begin{aligned} f_{\mathbb{O}_n}(x,y) &= \sum_{1\le i<j<k\le n} (x_ix_jy_k + x_iy_jx_k + y_ix_jx_k) + \sum_{1\le i\le j\le n} x_iy_j, \\ f_{\mathbb{O}_{p,q}}(x,y) &= f_{\mathbb{O}_n}(x,y) + \sum_{1\le i\le p} x_iy_i, \end{aligned} \tag{1.7}$$

respectively.

Recall that the element $1 := u_0$ is the unit of the algebra. The real algebra $\mathbb{O}_{0,3}$ is nothing but the classical algebra $\mathbb{O}$ of the octonion numbers. For both series of algebras $\mathcal{C}l_{p,q}$ and $\mathbb{O}_{p,q}$ the index (p,q) is called the *signature*, and throughout the dissertation we assume $p+q=n$.

Comparing the definitions of the twisting function (1.6), (1.5) and (1.7) gives a first point of view to understand $\mathbb{O}_{p,q}$ as a generalization of $\mathbb{O}$ in the same way as $\mathcal{C}l_{p,q}$ generalize $\mathbb{H}$.

1.1.2 Cohomology $H^*(\mathbb{Z}_2^n, \mathbb{Z}_2)$

Let us consider the abelian group $\mathbb{Z}_2^n$ and the $\mathbb{Z}_2^n$-module $\mathbb{Z}_2$, that is an abelian group together with the trivial right and left actions.

For $m \in \mathbb{N}$, let $C^m(\mathbb{Z}_2^n, \mathbb{Z}_2)$ be the set of all functions from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2$ in m arguments, called the space of *m-cochains*, and let

$$d : C^m(\mathbb{Z}_2^n, \mathbb{Z}_2) \longrightarrow C^{m+1}(\mathbb{Z}_2^n, \mathbb{Z}_2)$$

be given by

$$\begin{aligned}(dc)(g_1,\ldots,g_{m+1}) &= c(g_1,\ldots,g_m) + c(g_2,\ldots,g_{m+1}) \\ &\quad + \sum_{i=1}^{m} c(g_1,\ldots,g_{i-1},g_i+g_{i+1},g_{i+2},\ldots,g_{m+1})\end{aligned}$$

for all $g_1,\ldots,g_{m+1} \in \mathbb{Z}_2^n$. It is a coboundary operator, that is $d \circ d = 0$. The group of *m-cocycles* is $\ker(d)$ and denoted by $Z^m(\mathbb{Z}_2^n, \mathbb{Z}_2)$. The group of *m-coboundaries* is $\mathrm{Im}(d)$ and denoted by $B^m(\mathbb{Z}_2^n, \mathbb{Z}_2)$ for $m \geq 1$ and we set $B^0(\mathbb{Z}_2^n, \mathbb{Z}_2) = 0$. The *$m^{th}$-space of cohomology*, denoted by $H^m(\mathbb{Z}_2^n, \mathbb{Z}_2)$, is

$$H^m(\mathbb{Z}_2^n, \mathbb{Z}_2) := Z^m(\mathbb{Z}_2^n, \mathbb{Z}_2) \big/ B^m(\mathbb{Z}_2^n, \mathbb{Z}_2).$$

Note that the cohomology ring

$$H^*(\mathbb{Z}_2^n, \mathbb{Z}_2) := \bigoplus_m H^m(\mathbb{Z}_2^n, \mathbb{Z}_2)$$

is entirely determined, see [AM04] for more details.

We are particularly interested in the case where $m = 1, 2$ or 3. For example, a 3-cochain ϕ is a 3-cocycle if $d\phi = 0$, i.e.

$$\phi(x,y,z)+\phi(y,z,t)+\phi(x+y,z,t)+\phi(x,y+z,t)+\phi(x,y,z+t)=0,$$

for all $x,y,z,t\in\mathbb{Z}_2^n$. A 3-cochain ϕ is a 3-coboundary if there exists a function $f:\mathbb{Z}_2^n\times\mathbb{Z}_2^n\longrightarrow\mathbb{Z}_2$ such that $\phi=df$. It is exactly the case for the function ϕ given in (1.4) for a twisted group algebra $(\mathbb{K}[\mathbb{Z}_2^n],f)$.

A 2-cochain β is a 2-cocycle if $d\beta=0$, i.e.

$$\beta(y,z)+\beta(x,y)+\beta(x+y,z)+\beta(x,y+z)=0,$$

for all $x,y,z\in\mathbb{Z}_2^n$. A 2-cochain β is a 2-coboundary if there exists a function $\alpha:\mathbb{Z}_2^n\longrightarrow\mathbb{Z}_2$ such that $\beta=d\alpha$, i.e., $\beta(x,y)=\alpha(y)+\alpha(x)+\alpha(x+y)$ for all $x,y\in\mathbb{Z}_2^n$. We will see in Section 1.5 that such a function α exists for the algebras $\mathbb{O}_{p,q}$ and $\mathbb{O}_n$.

1.2 The algebras give solutions to the Hurwitz problem on square identities

The algebras $\mathbb{O}_n$ in the complex case and the algebras $\mathbb{O}_{0,n}$ in the real case are used to give explicit formulas for solutions of the classical problem on square identities. Recall first the general setting of the problem of "Sum of Squares", formulated by Hurwitz [Hur98].

Let us consider the following equation

$$(a_1^2+\cdots+a_r^2)(b_1^2+\cdots+b_s^2)=c_1^2+\cdots+c_N^2$$

where c_l, $l\in\{1,\ldots,N\}$, are bilinear functions depending on a_i, $i\in\{1,\ldots,r\}$, and b_j, $j\in\{1,\ldots,s\}$, with coefficients in $\mathbb{K}$. We say that these identities are of type $[r,s,N]$ where r, s and N are integers. An identity of type $[r,s,N]$ is called *optimal* if r and s cannot be increased and N cannot be decreased.

Although a number of identities of type $[r,s,N]$ are known, the general problem to determine all of them is still open. Furthermore, the problem of optimality is often difficult. Trivial cases include identities of type $[r,s,rs]$ or $[1,N,N]$. A first non-trivial result was given by Hurwitz.

Theorem 1.1 ([Hur98]). *The identities of type* $[N, N, N]$ *only exist for* N *equals to* $1, 2, 4, 8$.

This corresponds to the only real normed division algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ of dimension $1, 2, 4$ and 8.

The so-called *Hurwitz-Radon problem* consists in finding identities of type $[r, N, N]$. The *Hurwitz-Radon function* $\rho(N)$ is defined as follows. Write N in the form $N = 2^n(2m+1)$ for n and m in $\mathbb{N}$. The function ρ is then given by

$$\rho(N) = \begin{cases} 2n+1 & \text{if } n \equiv 0 \bmod 4, \\ 2n & \text{if } n \equiv 1, 2 \bmod 4, \\ 2n+2 & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

A second result is the following.

Theorem 1.2 ([Hur98], [Rad22]). *The identities of type* $[r, N, N]$ *only exist for* $r \leq \rho(N)$.

Hurwitz proved the theorem[1] in [Hur22] with complex coefficients while independently, Radon solved it in [Rad22] with real coefficients. The constructive proof of Hurwitz, leads to $[r, N, N]$-identities with integers coefficients. Later, it was proved by Gabel in [Gab74] that the bilinear forms c_k can be chosen with coefficients ± 1. Note that the interesting case for optimal identity is $N = 2^n$ which coincide with the dimension of the algebras $\mathbb{O}_n$ and $\mathbb{O}_{0,n}$.

Apart from the Hurwitz-Radon problem the complete list of optimal identities of type $[r, s, N]$ with $10 \leq r, s \leq 16$ is given in [Yiu94]. Other examples of identities of type $[r, s, N]$ are given in the literature, but the optimality is hard to figure out.

The class of $\mathbb{Z}_2^n$-graded nonassociative and noncommutative algebras $\mathbb{O}_n$ and $\mathbb{O}_{0,n}$ are used in [MGO11] (resp. [LMGO11]) to give a method to find explicit formulas for square identities of type $[r, N, N]$ (resp. $[r, s, N]$). The algebras $\mathbb{O}_n$ and $\mathbb{O}_{0,n}$ are not composition algebras. However, they have a natural

[1] The date of publication is posthumous since Hurwitz died in 1919.

euclidean norm

$$\mathcal{N}(a) = \sum_{x \in \mathbb{Z}_2^n} (a_x)^2, \quad \text{for } a = \sum_{x \in \mathbb{Z}_2^n} a_x u_x,$$

where $a_x \in \mathbb{K}$ and where $\{u_x : x \in \mathbb{Z}_2^n\}$ is the basis of $\mathbb{K}[\mathbb{Z}_2^n]$. Whenever we find two subspaces R, T of $\mathbb{O}_n$ (or of $\mathbb{O}_{0,n}$) such that[2] $\mathcal{N}(u)\mathcal{N}(v) = \mathcal{N}(uv)$ for $u \in R$ and $v \in T$, we obtain an identity of the type $[r, s, N]$ on $\mathbb{K}$ where $r = \dim R$, $s = \dim T$ and $N = 2^n$.

Explicit identities of type $[\rho(2^n), 2^n, 2^n]$, for n not a multiple of 4, are given in [MGO11] thanks to the theory on the algebras $\mathbb{O}_n$ and $\mathbb{O}_{0,n}$. Later, Lenzhen, Morier-Genoud and Ovsienko in [LMGO11] showed the existence of specific infinite series of identities of the type $[r, s, N]$. These solutions depend on the residue of n modulo 4 ($N = 2^n(2m+1)$). They extended the method used in [MGO11] to obtain more general identities, but unfortunately without proof of optimality. As an illustration they showed that, given in term of binomial coefficients, there exist identities of type

$$\left[2n, 2\binom{n}{m}, 2\binom{n+1}{m}\right] \quad \text{and} \quad \left[2n+2, 2\binom{n}{m}, 2\binom{n+1}{m}\right]$$

for $n \equiv 1 \bmod 4$ and $n \equiv 3 \bmod 4$ respectively, where $m = (n-1)/2$.

1.3 Graded algebras, generators and relations

By definition, every twisted group algebra $(\mathbb{K}[\mathbb{Z}_2^n], f)$ is a $\mathbb{Z}_2^n$-graded algebra. Consider the following natural basis elements of $\mathbb{Z}_2^n$

$$e_i = (0, \dots, 0, 1, 0, \dots, 0)$$

where 1 stands at i^{th} position. The corresponding homogeneous elements $u_i := u_{e_i}$ with $1 \leq i \leq n$, form a set of generators $\{u_1, \dots, u_n\}$ of the algebra

[2] An explicit condition is given in [MGO11] (prop 8.2) or [LMGO11] (prop 3.5).

$(\mathbb{K}[\mathbb{Z}_2^n], f)$. The *degree* of every generator is an element of $\mathbb{Z}_2^n$ as follows

$$\bar{u}_i := e_i.$$

Let $u = u_{i_1} \cdots u_{i_k}$ be a monomial, define its degree independently from the parenthesizing in the monomial by

$$\bar{u} := \bar{u}_{i_1} + \cdots + \bar{u}_{i_k},$$

which is again an element of $\mathbb{Z}_2^n$. The relations between the generators u_i are entirely determined by the function f.

The twisted group algebra $(\mathbb{K}[\mathbb{Z}_2^n], f)$ is also *graded-commutative* and *graded-associative*, as relation of type (1.1) and (1.2) hold between homogeneous elements. In particular, the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ are graded-commutative and graded-associative. From [MGO11], consider the trilinear (or tri-additive) function $\phi : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2$, such that

$$\phi(e_i, e_j, e_k) = \begin{cases} 1 & \text{if } i, j, k \in \{1, \dots, n\} \text{ are all disctincts,} \\ 0 & \text{otherwise.} \end{cases}$$

This function is obviously *symmetric*, i.e.,

$$\phi(x, y, z) = \phi(x, z, y) = \cdots = \phi(z, y, x), \qquad \forall x, y, z \in \mathbb{Z}_2^n$$

and *alternate*, i.e.

$$\phi(x, x, y) = \phi(x, y, x) = \phi(y, x, x) = 0, \qquad \forall x, y \in \mathbb{Z}_2^n.$$

The following point of view of the algebras $\mathbb{O}_{p,q}$ comes from [MGO11].

The *algebra* $\mathbb{O}_{p,q}$ is the unique real unital algebra, generated by n elements $u_1, \dots, u_n$, where $n = p + q$, subject to the relations

$$\begin{aligned} u_i^2 &= \begin{cases} 1 & \text{if } 1 \leq i \leq p, \\ -1 & \text{if } p+1 \leq i \leq n, \end{cases} \\ u_i \cdot u_j &= -u_j \cdot u_i, \qquad \text{if } 1 \leq i < j \leq n. \end{aligned}$$

together with the graded-associativity

$$u \cdot (v \cdot w) = (-1)^{\phi(\bar{u},\bar{v},\bar{w})} (u \cdot v) \cdot w, \tag{1.8}$$

where u, v, w are monomials.

The *algebra* $\mathbb{O}_n$ is the complexification of $\mathbb{O}_{p,q}$, its generators satisfy the same relations.

Note that in particular, the relation (1.8) implies

$$u_i \cdot (u_j \cdot u_k) = -(u_i \cdot u_j) \cdot u_k,$$

for distinct i, j and k in $\{1, \ldots, n\}$, that is, the generators anti-associate with each other. Note that, the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ are *graded-alternative*, i.e.,

$$u_x \cdot (u_x \cdot u_y) = (u_x)^2 \cdot u_y \qquad \text{and} \qquad (u_y \cdot u_x) \cdot u_x = u_y \cdot (u_x)^2$$

for all homogeneous elements u_x and u_y.

With abuse of language, we will sometime denote (p, q), the *signature* of the set of generators.

1.4 Nonassociative extension of Clifford algebras

Let us describe the third way to view the algebras $\mathbb{O}_{p,q}$, see [KMG15]. Consider the subalgebra of the algebra $\mathbb{O}_{p,q}$ consisting in the elements of even degree. It has the following basis

$$\{u_x : \ x \in \mathbb{Z}_2^n, |x| \equiv 0 \mod 2\}.$$

Proposition 1.1. *The subalgebra of $\mathbb{O}_{p,q}$ of even elements is isomorphic to the Clifford algebra $\mathcal{C}l_{p,q-1}$, if $q > 0$.*

Proof. Consider the following elements

$$v_i := u_{e_i+e_n}, \quad \text{for all } 1 \leq i \leq n-1. \tag{1.9}$$

They generate all the even elements and they satisfy

$$\begin{cases} v_i^2 & = & 1, & \text{if} & 1 \le i \le p, \\ v_i^2 & = & -1, & \text{if} & p+1 \le i < p+q, \\ v_i \cdot v_j & = & -v_j \cdot v_i, & \text{for all} & i \ne j < p+q, \\ v_i \cdot (v_j \cdot v_k) & = & (v_i \cdot v_j) \cdot v_k, & \text{for all} & i, j, k < p+q. \end{cases}$$

Linearity of the function ϕ implies that v_i generate an associative algebra. The above system of generators is therefore a presentation for the real Clifford algebra $\mathcal{C}l_{p,q-1}$. □

Note that if $q = 0$, then the subalgebra of $\mathbb{O}_{n,0}$ of even elements is isomorphic to $\mathcal{C}l_{0,n-1}$. Indeed, the elements v_i given in (1.9) are such that $v_i^2 = -1$ for all i in $\{1, \ldots, n-1\}$.

Example 1.2. Figure 1.3 present the examples of the algebras of quaternion and octonion numbers.

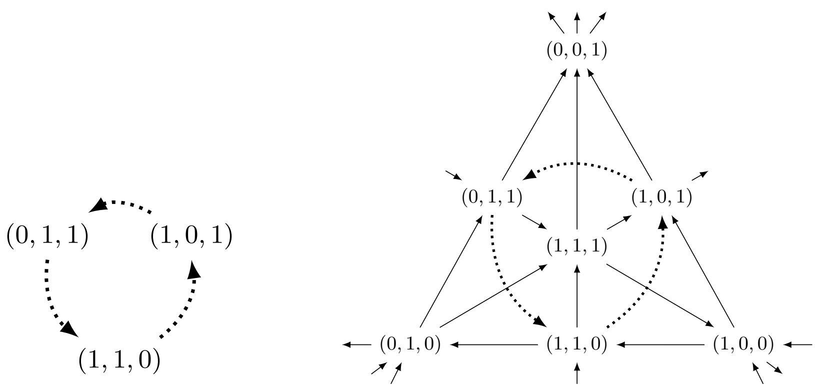

Figure 1.3: Multiplication in $\mathbb{H}$ (left) and $\mathbb{O}$ (right).

In other words, Proposition 1.1 means that the algebra $\mathbb{O}_{p,q}$, for $q \ne 0$, contains the following *Clifford subalgebra*

$$\mathcal{C}l_{p,q-1} \subset \mathbb{O}_{p,q}. \tag{1.10}$$

The algebra $\mathbb{O}_{p,q}$ thus can be viewed as a nonassociative extension of this

Clifford subalgebra by one odd element, u_n, anticommuting and antiassociating with all the generators of the Clifford algebra. As a vector space, $\mathbb{O}_{p,q}$ can be seen as

$$\mathcal{C}l_{p,q-1} \oplus \left(\mathcal{C}l_{p,q-1} \cdot u_n\right).$$

We will sometimes use the notation $\mathbb{O}_{p,q} \simeq \mathcal{C}l_{p,q-1} \star u_n$.

1.5 The algebras are determined by a generating cubic form

We will be needing a theory, developed in [MGO11], about a class of twisted algebras over $\mathbb{Z}_2^n$ that are characterized by a cubic form; this is the case for the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$. The structure of twisted group algebras that can be equipped with a generating function is much simpler than that of arbitrary twisted group algebras. Note that this class contains such interesting algebras as the code loops [Gri86], whereas the Cayley-Dickson algebras higher that the algebra of octonion numbers do not belong to this class. Let us mention that the generating cubic form is a very useful tool for the study of the algebras and is used in Chapter 2 and widely exploited in Chapter 3.

Definition 1.2. Given a twisted group algebra $(\mathbb{K}\left[\mathbb{Z}_2^n\right], f)$, a function $\alpha : \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2$ is called a *generating function* if

$$\begin{array}{llcl}
(i) & f(x,x) & = & \alpha(x), \\
(ii) & \beta(x,y) & = & \alpha(x+y)+\alpha(x)+\alpha(y), \\
(iii) & \phi(x,y,z) & = & \alpha(x+y+z)+\alpha(x+y)+\alpha(x+z)+\alpha(y+z) \\
 & & & +\alpha(x)+\alpha(y)+\alpha(z),
\end{array}$$

where $x, y, z \in \mathbb{Z}_2^n$ and where β and ϕ are as in (1.3) and (1.4).

Note that the identity (i) of Definition 1.2 implies that α vanishes on the zero element $(0, \dots, 0)$ of $\mathbb{Z}_2^n$. We already knew that the unit u_0 of the algebra commutes and associates with all other elements. One of the main results of [MGO11] is the following.

Theorem 1.3. *([MGO11])*

(i) A twisted group algebra $(\mathbb{K}[\mathbb{Z}_2^n], f)$ *has a generating function if and only if the function* ϕ *is symmetric.*

(ii) The generating function α *is a polynomial on* $\mathbb{Z}_2^n$ *of degree* ≤ 3.

(iii) Given any polynomial α *on* $\mathbb{Z}_2^n$ *of degree* ≤ 3*, there exists a unique (up to isomorphism) twisted group algebra* $(\mathbb{K}[\mathbb{Z}_2^n], f)$ *having* α *as a generating function.*

The first result defines a restricted class of twisted algebra and the second one points out that every algebra of this class is characterized by a cubic form on $\mathbb{Z}_2^n$.

Remind that every cubic form $\alpha : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is as follows

$$\alpha(x) = \sum_{1 \leq i \leq j \leq k \leq n} A_{ijk}\, x_i x_j x_k, \tag{1.11}$$

where the coefficients $A_{ijk} = 0$ or 1. Note that over $\mathbb{Z}_2$ one has $x_i^3 = x_i^2 = x_i$, and therefore, every polynomial of degree ≤ 3 is a homogeneous cubic form. The general theory of such cubic forms is not completely developed, the classification seems to be only done for special cubic functions of 8 variables; see [Hou96] and [Hou98].

The last point states that if a twisted algebra has a generating function then it is (completely) characterized by this function. The existence of such function was proven in [MGO11]. Indeed, one can define in a canonical way a twisting function f_α associated with a cubic form α according to the following explicit procedure. To every monomial one associates

$$\begin{array}{rcl} x_i x_j x_k & \longmapsto & x_i x_j y_k + x_i y_j x_k + y_i x_j x_k, \\ x_i x_j & \longmapsto & x_i y_j, \\ x_i & \longmapsto & x_i y_i, \end{array} \tag{1.12}$$

where $1 \leq i < j < k \leq n$. Then one extends the above map to the cubic polynomial α by linearity in monomial. A nice property of f_α is that it is linear in the second argument. The procedure (1.12) is not the unique way to associate the twisting function to a cubic form but any other procedure lead to an isomorphic algebra.

Let us prove the uniqueness of (*iii*) coming from [KMG15]. Assume that the twisted group algebra $(\mathbb{K}[\mathbb{Z}_2^n], f)$ has a generating cubic form α. Let us show that α determines (up to isomorphism) completely the algebra $(\mathbb{K}[\mathbb{Z}_2^n], f)$.

First, notice that α uniquely determines the relations of degree 2 and 3 between the generators u_i. Indeed, $u_i^2 = -1$ if and only if α contains the linear term x_i (otherwise $u_i^2 = 1$); u_i and u_j anticommute if and only if α contains the quadratic term $x_i x_j$ (otherwise, they commute); $u_i \cdot (u_j \cdot u_k) = -(u_i \cdot u_j) \cdot u_k$ if and only if α contains the cubic term $x_i x_j x_k$ (otherwise, the generators associate).

The monomials

$$u'_x = u_{i_1} \cdot (u_{i_2} \cdot (\cdots \; (u_{i_{l-1}} \cdot u_{i_l}) \cdots)),$$

for $x = e_{i_1} + e_{i_2} + \cdots + e_{i_l}$ with $i_1 < i_2 < \cdots < i_l$ form a basis of the algebra. The product $u'_x \cdot u'_y$ of two such monomials is equal to the monomial $\pm u'_{x+y}$. The sign can be determined by using only sequences of commutation and association between the generators, and the squares of the generators. Therefore the structure constants related to this basis are completely determined. A twisting function f' is deduced from

$$u'_x \cdot u'_y =: (-1)^{f'(x,y)} u'_{x+y}.$$

1.5.1 The generating functions of $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$

The algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ have the following generating functions

$$\begin{aligned} \alpha_n(x) &= \sum_{1 \le i < j < k \le n} x_i x_j x_k + \sum_{1 \le i < j \le n} x_i x_j + \sum_{1 \le i \le n} x_i, \\ \alpha_{p,q}(x) &= \alpha_n(x) + \sum_{1 \le i \le p} x_i. \end{aligned}$$

The cubic form α_n of $\mathbb{O}_n$ is invariant under the action of the group of permutations of the coordinates. Therefore, the value $\alpha_n(x)$ depends only on the weight (i.e. the number of nonzero components) of x and can be easily computed. More precisely, for $x = (x_1, \dots, x_n)$ in $\mathbb{Z}_2^n$, we denote the *Hamming weight* of x by

$$|x| = \#\{x_i \neq 0\}.$$

One has

$$\alpha_n(x) = \begin{cases} 0, & \text{if } |x| \equiv 0 \mod 4, \\ 1, & \text{otherwise.} \end{cases}$$

Remark 1.1. The Clifford algebras $\mathcal{C}l_n$ and $\mathcal{C}l_{p,q}$ also have generating functions that are quadratic forms given by

$$\begin{aligned} \alpha_n^{\mathcal{C}l}(x) &= \sum_{1 \leq i < j \leq n} x_i x_j + \sum_{1 \leq i \leq n} x_i, \\ \alpha_{p,q}^{\mathcal{C}l}(x) &= \alpha_n^{\mathcal{C}l}(x) + \sum_{1 \leq i \leq p} x_i. \end{aligned}$$

Moreover, $\alpha_n^{\mathcal{C}l}$ depends also only on the Hamming weight and one has

$$\alpha_n^{\mathcal{C}l}(x) = \begin{cases} 0, & \text{if } \ |x| \equiv 0, 3 \mod 4, \\ 1, & \text{if } \ |x| \equiv 1, 2 \mod 4. \end{cases}$$

The existence of a generating cubic form gives a way to distinguish the algebras $\mathcal{C}l_{p,q}$ ($\mathcal{C}l_n$) and $\mathbb{O}_{p,q}$ ($\mathbb{O}_n$) from other twisted group algebras. For instance, the Cayley-Dickson algebras do not have in general generating cubic forms. Let us mention again that the generating cubic form is a very useful tool for the study of the algebras.

1.5.2 Overview of the algebras in term of cocycles

The identity (ii) of Definition 1.2 means that β is the differential of α, therefore $d\beta = 0$. This is the case for the Clifford algebras and for the algebras $\mathbb{O}_{p,q}$, for the Cayley-Dickson algebras this is false.

The algebra $(\mathbb{K}[\mathbb{Z}_2^n], f)$ is commutative if and only if $\beta = 0$, or equivalently (if α exists) if and only if $d\alpha = 0$ and it is associative if and only if $\phi = 0$ or equivalently if $df = 0$.

The Cayley-Dickson algebras higher that the algebra of octonion numbers are neither commutative nor associative, and there is no function α in one argument that characterizes these algebras. Nevertheless, Albuquerque and Majid in [AM99] showed that the Cayley-Dickson algebras higher that the algebra of octonion numbers are also twisted group algebras for suitable twisting function. Figure 1.4 exposes an overview of the situation.

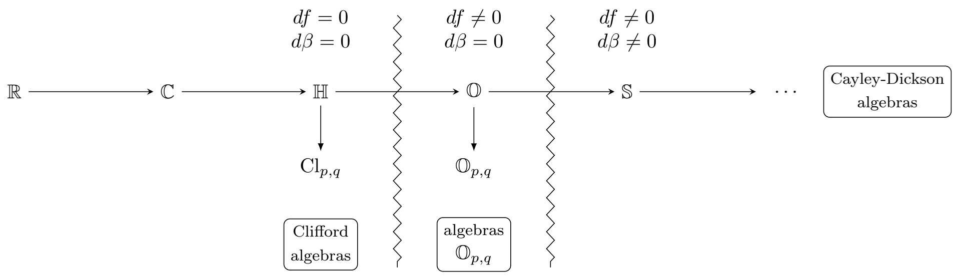

Figure 1.4: Cohomological properties of families of $\mathbb{Z}_2^n$-graded algebras.

1.5.3 The problem of equivalence

Two isomorphic algebras may have different generating functions. For instance the algebra determined by

$$\alpha(x) = \sum_{1 \le i < j \le n-1} x_i x_j x_n + \sum_{1 \le i \le j \le n} x_i x_j + \sum_{1 \le i \le p} x_i$$

is isomorphic to $\mathbb{O}_{p,q}$, when $q > 0$ (by sending u_i to $u_{e_i + e_n}$, $1 \le i \le n-1$, and u_n to u_n). We introduce the following definitions.

Definition 1.3. Two cubic forms α and α' on $\mathbb{Z}_2^n$ are *equivalent* if there exists a linear transformation $G \in \mathrm{GL}_n(\mathbb{Z}_2)$ such that

$$\alpha(x) = \alpha'(Gx).$$

Definition 1.4. An isomorphism from a twisted group algebra $(\mathbb{K}[\mathbb{Z}_2^n], f)$ to another $(\mathbb{K}[\mathbb{Z}_2^n], f')$ *preserves the structure* of $\mathbb{Z}_2^n$-graded algebra if the isomorphism sends homogeneous elements into homogeneous.

The main method that we use to establish isomorphisms between twisted group algebras with generating functions is based on the fact that two equivalent cubic forms give rise to isomorphic algebras. More precisely, one has the following statement which is an obvious corollary of the uniqueness of the generating function.

Proposition 1.2. *Given two twisted group algebras,* $(\mathbb{K}[\mathbb{Z}_2^n], f)$ *and* $(\mathbb{K}[\mathbb{Z}_2^n], f')$ *with equivalent generating functions* α *and* α'*, then these algebras are isomorphic as* $\mathbb{Z}_2^n$*-graded algebras.*

Recall that the general problem of classification of cubic forms on $\mathbb{Z}_2^n$ is an old open problem, see [Hou96]. Specific tools are setting up to detect properties (especially periodicities) on our cubic forms α_n and $\alpha_{p,q}$, see Chapter 3.

Chapter 2

Classification of the Algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$

As mentioned in Chapter 1, there is a strong analogy between the algebras $\mathcal{C}l_{p,q}$ (resp. $\mathcal{C}l_n$) and $\mathbb{O}_{p,q}$ (resp. $\mathbb{O}_n$). In **Chapter** 2, we compare the two series and classify the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$.

In **Section** 2.1, the simplicity of the algebras $\mathbb{O}_{p,q}$ are recalled from [MGO11]. On the complex case, the classification of the algebras $\mathbb{O}_n$ is accomplished with the study of the simplicity, while the real case is more tricky and the criterion of simplicity is just the first step.

In **Section** 2.2, we classify the isomorphisms $\mathbb{O}_{p,q} \simeq \mathbb{O}_{p',q'}$ that preserve the structure of $\mathbb{Z}_2^n$-graded algebra. Different cases according to the residue modulo 4 of the number of generators are settled. This new result come from the paper [KMG15].

The **Section** 2.3 is a comparison between the known classification of the Clifford algebras $\mathcal{C}l_{p,q}$ and the classification of the algebras $\mathbb{O}_{p,q}$.

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2.1 Simplicity of the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$

One of the first properties studied in [MGO11] is the simplicity. An algebra is *simple* if it contains no proper ideal. Note that that in this context we don't speak about *graded-simple* algebra which is an algebra that contains no proper graded-ideal. This latter notion is much weaker and rather a property of the grading. A graded-simple algebra is not necessarily a simple algebra.

2.1.1 The complex case $\mathbb{O}_n$

The classification and simplicity in the complex case is quite easy. Comparison of results on $\mathcal{C}l_n$ from for example [Lou97], and on $\mathbb{O}_n$ from [MGO11] can be expressed as follows.

Clifford algebras	**Algebras $\mathbb{O}_n$**
There is only one Clifford algebra in each dimension	There is only one algebra $\mathbb{O}_n$ in each dimension
$\mathcal{C}l_n$ is simple if and only if $n \not\equiv 1 \mod 2$	$\mathbb{O}_n$ is simple if and only if $n \not\equiv 0 \mod 4$
For all $k \in \mathbb{N}$, one has $\mathcal{C}l_{2k+1} \simeq \mathcal{C}l_{2k} \oplus \mathcal{C}l_{2k}$	For all $k \in \mathbb{N}\backslash\{0\}$, one has $\mathbb{O}_{4k} \simeq \mathbb{O}_{4k-1} \oplus \mathbb{O}_{4k-1}$

Table 2.1: Simplicity in the complex case for $\mathcal{C}l_n$ and $\mathbb{O}_n$.

Observe that a modulo 2 periodicity appears in the Clifford case, while a modulo 4 periodicity appears in the algebras $\mathbb{O}_n$. This can be linked with the periodicity of the generating cubic forms $\alpha_n^{\mathcal{C}l}$ and α_n according to the Hamming weight.

2.1.2 The real case $\mathbb{O}_{p,q}$

The real case is significantly more complicated for the Clifford algebras as for the algebras $\mathbb{O}_{p,q}$. Let us recall the results of [MGO11] on the simplicity of the algebras $\mathbb{O}_{p,q}$ and compare them with the known Clifford case.

Clifford algebras	**Algebras $\mathbb{O}_{p,q}$**
$\mathcal{C}l_{p,q}$ is simple if and only if $p+q \not\equiv 1 \mod 2$ or $(p+q \equiv 1 \mod 2)$ and $(p-q \equiv 3 \mod 4)$	$\mathbb{O}_{p,q}$ is simple if and only if $p+q \not\equiv 0 \mod 4$ or $(p+q \equiv 0 \mod 4)$ and $(p, q$ odd$)$
if $\mathcal{C}l_{p,q}$ is not simple, then $\mathcal{C}l_{p,q} \simeq \mathcal{C}l_{p-1,q} \oplus \mathcal{C}l_{p,q-1}$ and $\mathcal{C}l_{p-1,q} \simeq \mathcal{C}l_{p,q-1}$	if $\mathbb{O}_{p,q}$ is not simple, then $\mathbb{O}_{p,q} \simeq \mathbb{O}_{p-1,q} \oplus \mathbb{O}_{p,q-1}$ and $\mathbb{O}_{p-1,q} \simeq \mathbb{O}_{p,q-1}$
if $\mathcal{C}l_{p,q}$ is simple and $p+q \equiv 1 \mod 2$, then $\mathcal{C}l_{p,q} \simeq \mathcal{C}l_{p,q-1} \otimes \mathbb{C}$ $\simeq \mathcal{C}l_{p-1,q} \otimes \mathbb{C}$	if $\mathbb{O}_{p,q}$ is simple and $p+q \equiv 0 \mod 4$, then $\mathbb{O}_{p,q} \simeq \mathbb{O}_{p,q-1} \otimes \mathbb{C}$ $\simeq \mathbb{O}_{p-1,q} \otimes \mathbb{C}$

Table 2.2: Simplicity in the real case for $\mathcal{C}l_{p,q}$ and $\mathbb{O}_{p,q}$.

Note that in Table 2.2 the signatures (p, q) of the algebras $\mathcal{C}l_{p,q}$ (resp. $\mathbb{O}_{p,q}$) must be such that $pq \geq 0$ (resp. $pq \geq 0$ and $p + q \geq 3$). In the last raw of Table 2.2, the algebras $\mathcal{C}l_{p,q}$ and $\mathbb{O}_{p,q}$ are $\mathbb{C}$-algebra considered over $\mathbb{R}$.

The criterion of simplicity in the real case is not enough to classify the algebras $\mathbb{O}_{p,q}$ on the contrary of the complex case. The next section is dedicated to the classification of the real algebras $\mathbb{O}_{p,q}$ up to isomorphism.

2.2 Classification of the algebras $\mathbb{O}_{p,q}$

In this section, we formulate and prove the main result of classification for the algebras $\mathbb{O}_{p,q}$ from [KMG15]. The problem we consider is to classify the isomorphisms $\mathbb{O}_{p,q} \simeq \mathbb{O}_{p',q'}$ that preserve the structure of $\mathbb{Z}_2^n$-graded algebra (meaning that the isomorphism sends homogeneous elements into homogeneous). We will see that the classification of the algebras $\mathbb{O}_{p,q}$ behave as in the Clifford case. The obtained results are the only possible ones.

The main result of this chapter is as follows.

Theorem 2.1. *If $pq \neq 0$, then there are the following isomorphisms of graded algebras*

(i) $\mathbb{O}_{p,q} \simeq \mathbb{O}_{q,p}$;

(ii) $\mathbb{O}_{p,q+4} \simeq \mathbb{O}_{p+4,q}$;

(iii) Every isomorphism between the algebras $\mathbb{O}_{p,q}$ preserving the structure of $\mathbb{Z}_2^n$-graded algebra is a combination of the above isomorphisms.

(iv) For $n \geq 5$, the algebras $\mathbb{O}_{n,0}$ and $\mathbb{O}_{0,n}$ are not isomorphic, and are not isomorphic to any other algebras $\mathbb{O}_{p,q}$ with $p+q=n$.

Part (i) of the theorem gives a "vertical symmetry" with respect to $p-q=0$ and may be compared to the vertical symmetry with respect to $p-q=1$ in the case of Clifford algebras

$$\mathcal{C}l_{p,q} \simeq \mathcal{C}l_{q+1,p-1}.$$

Part (ii) gives a shift of the signature (modulo 4) that also holds in the Clifford case

$$\mathcal{C}l_{p+4,q} \simeq \mathcal{C}l_{p,q+4}.$$

Another way to formulate Theorem 2.1 is as follows.

Corollary 2.1. *Assume that $p, p', q, q' \neq 0$, then $\mathbb{O}_{p,q} \simeq \mathbb{O}_{p',q'}$ if and only if the corresponding Clifford subalgebras* (1.10) *are isomorphic; the algebras $\mathbb{O}_{n,0}$ and $\mathbb{O}_{0,n}$ are exceptional.*

Note that we conjecture that whenever there is no isomorphism preserving the structure of $\mathbb{Z}_2^n$-graded algebra the algebras are not isomorphic.

2.2.1 Construction of the isomorphisms

In this section, we establish a series of lemmas that provides all possible isomorphisms between the algebras $\mathbb{O}_{p,q}$. Most of them have to be treated according to the residue class of $p+q$ modulo 4.

We start with the algebras of small dimensions where exceptional results happen in comparison with the general case.

Lemma 2.1. *For* $n=3$, *one has*

$$\mathbb{O}_{3,0} \simeq \mathbb{O}_{2,1} \simeq \mathbb{O}_{1,2} \not\simeq \mathbb{O}_{0,3}.$$

Proof. To establish the isomorphisms $\mathbb{O}_{3,0} \simeq \mathbb{O}_{2,1}$ and $\mathbb{O}_{2,1} \simeq \mathbb{O}_{1,2}$, we consider the following coordinate transformations

$$\left\{\begin{array}{lcl} x'_1 &=& x_1, \\ x'_2 &=& x_1+x_3, \\ x'_3 &=& x_1+x_2+x_3, \end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{lcl} x''_1 &=& x_3, \\ x''_2 &=& x_1+x_3, \\ x''_3 &=& x_1+x_2+x_3. \end{array}\right.$$

It is easy to check that $\alpha_{3,0}(x') = \alpha_{2,1}(x)$ and $\alpha_{2,1}(x'') = \alpha_{1,2}(x)$.
We can also use the Clifford subalgebras. Write $\mathbb{O}_{2,1} \simeq \mathcal{C}l_{2,0} \star u_3$ using

$$v_1 = u_{e_1+e_3}, \qquad v_2 = u_{e_2+e_3},$$

as generators of $\mathcal{C}l_{2,0}$. Change these generators according to

$$v'_1 = v_1 \cdot v_2, \qquad v'_2 = v_2.$$

This gives two generators of $\mathcal{C}l_{1,1} \simeq \mathcal{C}l_{2,0}$ inside $\mathbb{O}_{2,1}$ that still anticommute and antiassociate with u_3. Hence, $\mathbb{O}_{2,1} \simeq \mathcal{C}l_{1,1} \star u_3 \simeq \mathbb{O}_{1,2}$.

Let us prove that $\mathbb{O}_{0,3}$ is not isomorphic to the other algebras. In the algebra $\mathbb{O}_{0,3}$ all seven homogeneous basis elements different from the unit square to -1, whereas in $\mathbb{O}_{3,0}, \mathbb{O}_{2,1}$ and $\mathbb{O}_{1,2}$ three elements square to -1 and four square to 1. Hence, there is no graded-isomorphism over $\mathbb{R}$. □

Remark 2.1. Let us also mention that $\mathbb{O}_{0,3}$ is isomorphic to the classical algebra of octonion numbers, whereas $\mathbb{O}_{3,0} \simeq \mathbb{O}_{2,1} \simeq \mathbb{O}_{1,2}$ are isomorphic to the classical algebra of split-octonions, see [MGO11].

Lemma 2.2. *For $n = 4$, one has*

$$\mathbb{O}_{4,0} \simeq \mathbb{O}_{2,2} \qquad \text{and} \qquad \mathbb{O}_{3,1} \simeq \mathbb{O}_{1,3}.$$

Proof. Consider the classical basis $\{u_x : x \in \mathbb{Z}_2^4\}$ of $\mathbb{O}_{4,0}$, and as usual we denote by $u_i = u_{e_i}$. The following set of elements

$$\begin{cases} u'_1 &= u_{e_1+e_4}, \\ u'_2 &= u_{e_2+e_4}, \\ u'_3 &= u_3, \\ u'_4 &= u_4, \end{cases}$$

forms a system of generators, that anticommute and antiassociate such that

$$(u'_1)^2 = (u'_2)^2 = -1.$$

The signature of this set is $(2, 2)$. Hence $\mathbb{O}_{4,0}$ is isomorphic to $\mathbb{O}_{2,2}$.

Similarly, the following change of generators

$$\begin{cases} u'_1 &= u_{e_2+e_3+e_4}, \\ u'_2 &= u_{e_1+e_3+e_4}, \\ u'_3 &= u_{e_1+e_2+e_4}, \\ u'_4 &= u_{e_1+e_2+e_3}, \end{cases}$$

gives the isomorphism $\mathbb{O}_{3,1} \simeq \mathbb{O}_{1,3}$. The new generators anticommute and antiassociate and have signature $(1, 3)$ if the initial ones had signature $(3, 1)$ and vice versa. □

In order to determine the (anti)commutativity and (anti)associativity between elements one can use the formulas (1.3) and (1.4) of Definition 1.1 and evaluate them using the standard form α_n, since linear terms vanish in the formulas (1.3) and (1.4), which depends only on the *Hamming weight* of the elements.

We now turn to the higher dimensional algebras and extend our method.

The following lemma describes all the cases where the algebras $\mathbb{O}_{p,q}$ are not simple.

Lemma 2.3. *If $p+q=4k$ and p,q are even, then*

$$\mathbb{O}_{p,q} \simeq \mathbb{O}_{p,q-1} \oplus \mathbb{O}_{p-1,q}, \qquad \text{if} \quad p \geq 2 \text{ and } q \geq 2,$$

and

$$\mathbb{O}_{4k,0} \simeq \mathbb{O}_{4k-1,0} \oplus \mathbb{O}_{4k-1,0}, \qquad \mathbb{O}_{0,4k} \simeq \mathbb{O}_{0,4k-1} \oplus \mathbb{O}_{0,4k-1}.$$

This statement is proved in [MGO11] in the complex case (see Theorem 3, p. 100). The proof in the real case is identically the same.

The next four lemmas give the list of isomorphisms with respect to the residue class of $p+q$ modulo 4.

Lemma 2.4. *(i) If $p+q=4k$ and p,q are even, then*

$$\mathbb{O}_{p,q} \simeq \mathbb{O}_{p+4,q-4}, \qquad \text{if} \quad p \geq 2 \text{ and } q-4 \geq 2.$$

(ii) If $p+q=4k$ and p,q are odd, then

$$\mathbb{O}_{p,q} \simeq \mathbb{O}_{p+2,q-2}, \qquad \text{if} \quad p \geq 1 \text{ and } q-2 \geq 1.$$

Proof. We define a new set of generators splitted in blocks of four elements

$$\left\{ \begin{array}{lcl} u'_{4i+1} & = & u_{e_{4i+2}+e_{4i+3}+e_{4i+4}}, \\ u'_{4i+2} & = & u_{e_{4i+3}+e_{4i+4}+e_{4i+1}}, \\ u'_{4i+3} & = & u_{e_{4i+4}+e_{4i+1}+e_{4i+2}}, \\ u'_{4i+4} & = & u_{e_{4i+1}+e_{4i+2}+e_{4i+3}}. \end{array} \right. \tag{2.1}$$

for every $i \in \{0, \dots, k-1\}$. Let us illustrate the situation in Figure 2.1 below where k blocks of four generators are considered and in each block we can perform the change of variables (encoded by the sign $\circlearrowleft$) given by the relations (2.1).

These new generators are still anticommuting and antiassociating: elements within each block and elements between different blocks. The signature in

Figure 2.1: Illustration of splitting into blocks of four elements, $n = 4k$.

each block remains unchanged if the initial one is $(4,0)$, $(2,2)$, or $(0,4)$ and changes from $(3,1)$ to $(1,3)$ and vice versa.
Therefore, one can organize the generators in the blocks in order to obtain the desired isomorphisms. If p and q are even then we can take two blocks with the initial signature $(1,3)$ and the other blocks are taken with the initial signature either $(4,0)$, $(2,2)$ or $(0,4)$ according to the initial signature (p,q) of $\mathbb{O}_{p,q}$. In the same way, if p and q are odd then we take one block with the initial signature $(1,3)$ and the other blocks are taken with the initial signature either $(4,0)$, $(2,2)$ or $(0,4)$ according to the initial signature (p,q) of $\mathbb{O}_{p,q}$. Hence the result. □

Lemma 2.5. *If $p+q = 4k+1$, then*

$$\begin{aligned} \mathbb{O}_{p,q} &\simeq \mathbb{O}_{p+4,q-4}, && \text{if } \; p \geq 1 \text{ and } q-4 \geq 1, \\ \mathbb{O}_{p,q} &\simeq \mathbb{O}_{p+1,q-1}, && \text{if } \; p \geq 2 \text{ and } q-1 \geq 2, \quad p \text{ even.} \end{aligned}$$

Moreover, we have $\mathbb{O}_{4,1} \simeq \mathbb{O}_{1,4}$.

Proof. Consider the same change of the first $4k$ generators as in Lemma 2.4, and change also the last generator as follows

$$\left\{ \begin{array}{lcl} u'_{4i+1} &=& u_{e_{4i+2}+e_{4i+3}+e_{4i+4}}, \\ u'_{4i+2} &=& u_{e_{4i+3}+e_{4i+4}+e_{4i+1}}, \\ u'_{4i+3} &=& u_{e_{4i+4}+e_{4i+1}+e_{4i+2}}, \\ u'_{4i+4} &=& u_{e_{4i+1}+e_{4i+2}+e_{4i+3}}, \end{array} \right. \qquad \text{and} \qquad u'_n = u_z, \tag{2.2}$$

where $i \in \{0, \ldots, k-1\}$, and where z denotes the element of maximal weight

$$z = (1, \ldots, 1) \in \mathbb{Z}_2^n.$$

Let us illustrate the situation in Figure 2.2 below where k blocks of four generators and the last generator alone are considered. In each block and on the last generator we can perform the change of variables given by (2.2) and encoded by the sign $\circlearrowright$.

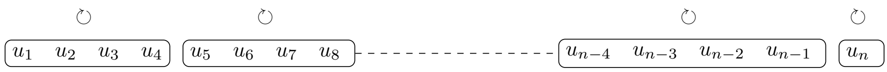

Figure 2.2: Illustration of splitting into blocks of four elements, $n = 4k + 1$.

These new generators are still anticommuting and antiassociating: elements within each block, elements of different blocks and together with the last generators u'_n.

If p is even, then we take one block with the initial signature $(1, 3)$, the last initial generator u_n such that $u_n^2 = 1$ and the other blocks are taken with the initial signature either $(4, 0)$, $(2, 2)$ or $(0, 4)$. Due to this choice, if the initial signature is (p, q) the obtained signature after the change of generators is $(p + 1, q - 1)$. Furthermore, if p is even, then we take also two blocks with the initial signature $(1, 3)$, the last initial generator u_n such that $u_n^2 = -1$ and the other blocks are taken with the initial signature either $(4, 0)$, $(2, 2)$ or $(0, 4)$. Due to this choice, if the initial signature is (p, q) the obtained signature after the change of generators is $(p + 4, q - 4)$.

If p is odd, then we take one block with the initial signature $(1, 3)$, the last initial generator u_n such that $u_n^2 = -1$ and the other blocks are taken with the initial signature either $(4, 0)$, $(2, 2)$ or $(0, 4)$. Due to this choice, if the initial signature is (p, q), the obtained signature after the change of generators is $(p + 3, q - 3)$. In particular, we have $\mathbb{O}_{1,4} \simeq \mathbb{O}_{4,1}$.

The desired isomorphisms are obtained by combining these different cases. $\square$

Lemma 2.6. *(i) If $p + q = 4k + 2$ and p, q are odd, then*

$$\mathbb{O}_{p,q} \simeq \mathbb{O}_{p+4,q-4}, \qquad \text{if} \quad p \geq 1 \text{ and } q - 4 \geq 1.$$

(ii) If $p+q=4k+2$ and p,q are even, then

$$\mathbb{O}_{p,q} \simeq \mathbb{O}_{p+2,q-2}, \qquad \text{if} \quad p \geq 2 \text{ and } q-2 \geq 2.$$

Proof. Consider the following change of generators

$$\begin{cases} u'_i = u_{e_i+w_1}, & \text{if } 1 \leq i \leq 4, \\ u'_i = u_i, & \text{otherwise}, \end{cases} \tag{2.3}$$

where $w_1 = (0,0,0,0,1,\ldots,1)$ is an element in $\mathbb{Z}_2^n$.

If $p \geq 2$ and $q-2 \geq 2$ are even, then consider the situation in Figure 2.3. The first line represents the initial generators together with the signs $+$ or $-$ that indicate the sign of the square of the associated generators. The second line represents the generators after the transformation given by (2.3). Hence, the result for p and q even.

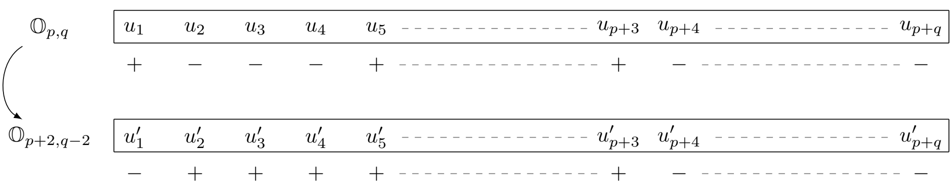

Figure 2.3: Configuration of the square of the generators: p even, $n=4k+2$.

If $p \geq 1$ and $q-4 \geq 1$ are odd, then consider the situation in Figure 2.4 according to the transformation (2.3). Hence, the result for p and q odd.

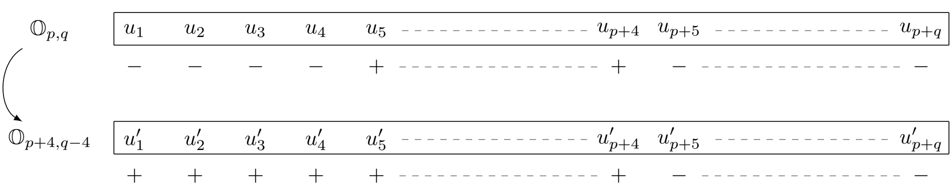

Figure 2.4: Configuration of the square of the generators: p odd, $n=4k+2$.

□

Lemma 2.7. *If* $p+q=4k+3$, *then*

$$\mathbb{O}_{p,q} \simeq \mathbb{O}_{p+4,q-4}, \qquad \text{if } \; p \geq 1 \text{ and } q-4 \geq 1,$$
$$\mathbb{O}_{p,q} \simeq \mathbb{O}_{p-1,q+1}, \qquad \text{if } \; p \geq 1 \text{ and } q \geq 1, \; p \text{ even.}$$

Proof. The isomorphisms are given by considering the following change of generators

$$\begin{cases} u_i' &= u_{e_i+w_2}, \quad \text{if } \; 1 \leq i \leq 4, \\ u_i' &= u_i, \quad \text{if } \; 5 \leq i \leq n-1, \\ u_n' &= u_{w_3}, \end{cases} \tag{2.4}$$

where $w_2 = (0,0,0,0,1,\ldots,1,0)$ and $w_3 = (0,0,0,0,1,\ldots,1)$ are in $\mathbb{Z}_2^n$.

The results are a combination of the transformation given by (2.4) applied to different choices of initial generators. If p is odd and $p \geq 5$, $q \geq 1$ then the following choice of initial generators, see Figure 2.5, induces the following isomorphism $\mathbb{O}_{p,q} \simeq \mathbb{O}_{p-3,q+3}$.

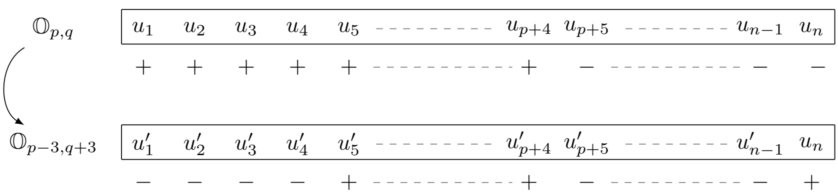

Figure 2.5: Configuration of the square of the generators: p odd, $n=4k+3$.

If $p=1$ and $q \geq 5$ then, the following choice of initial generators, see Figure 2.6, induces the following isomorphism $\mathbb{O}_{1,q} \simeq \mathbb{O}_{6,q-5}$.

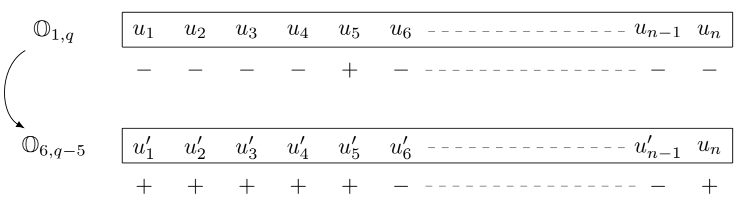

Figure 2.6: Configuration of the square of the generators: $p=1$, $n=4k+3$.

If p is even and $p \geq 4$, $q \geq 1$ then, the following choice of initial generators, see Figure 2.7, induces the following isomorphism $\mathbb{O}_{p,q} \simeq \mathbb{O}_{p-1,q+1}$.

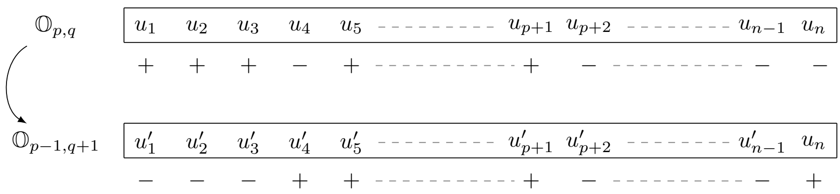

Figure 2.7: Configuration of the square of the generators: p even, $n = 4k+3$.

Taking into account the three configurations, we obtain the result. $\square$

The combination of all the above lemmas implies Theorem 2.1, part (i) and part (ii).

Remark 2.2. The isomorphisms $\mathbb{O}_{p,q} \simeq \mathbb{O}_{p-1,q+1}$, in the case $p+q = 4k+3$, p even, can be established using connection to Clifford algebras. Indeed, consider the Clifford subalgebra $\mathcal{C}l_{p,q-1} \subset \mathbb{O}_{p,q}$, see Section 1.4, with the generators v_i. For $p+q = 4k+3$ and p even, the classical isomorphism is

$$\mathcal{C}l_{p,q-1} \simeq \mathcal{C}l_{p-1,q}.$$

This isomorphism can be given by the change of variables on generators

$$\begin{cases} v'_1 &= v_1 \cdots v_{n-1}, & \\ v'_i &= v_i, & \text{if} \quad 2 \leq i \leq n-1. \end{cases}$$

Add the generator u_n of weight 1 that anticommutes and antiassociates with v'_i, one obtains the algebra $\mathbb{O}_{p-1,q+1}$. Hence the result, since the commutativity and associativity properties are preserved.

The above isomorphism can be illustrated by the following diagram

$$\begin{array}{ccc} \mathcal{C}l_{p,q-1} & & \mathcal{C}l_{p-1,q} \\ \cap & & \cap \\ < v_1, \ldots, v_{n-1} > \star < u_n > & \simeq & < v'_1, \ldots, v'_{n-1} > \star < u_n > \\ \wr| & & |\wr \\ \mathbb{O}_{p,q} & & \mathbb{O}_{p-1,q+1} \end{array}$$

2.2.2 Obstruction to isomorphism

In order to prove Theorem 2.1, Parts (*iii*) and (*iv*), we will define an *invariant* of the algebras: we count how many homogeneous basis elements square to 1. This invariant will be called the *statistics*.

Definition 2.1. Define the *statistics* of the algebra $\mathbb{O}_{p,q}$ by

$$s(p,q) := \# \left\{ x \in \mathbb{Z}_2^n : \alpha_{p,q}(x) = 1 \right\}.$$

Lemma 2.8. *The number $s(p,q)$ is invariant with respect to an isomorphism preserving the structure of $\mathbb{Z}_2^n$-graded algebra.*

Proof. Let us prove by contradiction that a $\mathbb{R}$-isomorphism preserves the sign of u_x^2.

An isomorphism preserving the structure of $\mathbb{Z}_2^n$-graded algebra sends homogeneous basis elements to homogeneous. Suppose having such an isomorphism on $\mathbb{R}$

$$\phi : \mathbb{O}_{p,q} \longrightarrow \mathbb{O}_{p',q'} : u_x \mapsto \kappa_x u_{\phi(x)}$$

such that $s(p,q) \neq s(p',q')$, where $p+q = p'+q' = n$; $\kappa_x \in \{-1, 1\}$ and $x, \phi(x)$ in $\mathbb{Z}_2^n$. Without restriction, we can assume that $s(p,q) > s(p',q')$. Then, there exists an homogeneous element u_y with $y \in \mathbb{Z}_2^n$ such that $(u_y)^2 = -u_0$ and $(u_{\phi(y)})^2 = u_0$. On one hand, we have

$$\phi(u_y \cdot u_y) = \phi(u_y^2) = \phi(-u_0) = -\phi(u_0) = -u_0.$$

On the other hand, we have

$$\phi(u_y \cdot u_y) = \phi(u_y) \cdot \phi(u_y) = \kappa_y^2 u_{\phi(y)} \cdot u_{\phi(y)} = \kappa_y^2 u_0.$$

Hence a contradiction. □

Note that this is not true on the field $\mathbb{C}$ since $(\sqrt{-1})^2 = -1$.

Clearly,

$$0 \leq s(p,q) \leq 2^n,$$

where $n = p + q$. Our goal is to show that for all $p, q \neq 0$, we have

$$s(n,0) < s(p,q) < s(0,n).$$

Lemma 2.9. *Let $n \geq 3$, the algebras $\mathbb{O}_{0,n}$ and $\mathbb{O}_{n,0}$ are not isomorphic.*

Proof. As usually, one denotes by $\binom{n}{j}$ the *Binomial coefficient*. We compute the statistics for these algebras

$$s(n,0) = \sum_{\substack{i=0 \\ 4i+2\leq n}}^{k} \binom{n}{4i+2} \qquad \text{and} \qquad s(0,n) = 2^n - \sum_{\substack{i=0 \\ 4i\leq n}}^{k} \binom{n}{4i},$$

where k is the integer part of $\frac{n}{4}$. Clearly, $s(n,0) \neq s(0,n)$ since

$$\sum_{\substack{i=0 \\ 4i\leq n}}^{k} \binom{n}{4i} + \sum_{\substack{i=0 \\ 4i+2\leq n}}^{k} \binom{n}{4i+2} \quad < \quad \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

□

Let us introduce the (abusive) notation $a \not\simeq b \not\simeq c$ meaning that a, b and c are such that $a \not\simeq b$, $b \not\simeq c$ and $a \not\simeq c$.

Lemma 2.10. *Let $n \geq 5$, the algebras $\mathbb{O}_{n,0}$ and $\mathbb{O}_{0,n}$ are not isomorphic to any algebras $\mathbb{O}_{p,q}$. Furthermore, we have the different cases.*
If $n = 4k$, then

$$\mathbb{O}_{n-1,1} \not\simeq \mathbb{O}_{n-2,2} \not\simeq \mathbb{O}_{n-4,4}.$$

If $n = 4k+1$, then

$$\mathbb{O}_{n-1,1} \not\simeq \mathbb{O}_{n-2,2}.$$

If $n = 4k+2$, then

$$\mathbb{O}_{n-1,1} \not\simeq \mathbb{O}_{n-2,2} \not\simeq \mathbb{O}_{n-3,3}.$$

If $n = 4k+3$, then

$$\mathbb{O}_{n-1,1} \not\simeq \mathbb{O}_{n-3,3}.$$

Proof. Let $n = 4k$, then

$$
\begin{aligned}
s(n-1,1) &= \sum_{i=0}^{k-1} \binom{n-1}{4i} + 2\binom{n-1}{4i+2} + \binom{n-1}{4i+3},\\
s(n-2,2) &= \sum_{i=0}^{k-1} 3\binom{n-2}{4i} + 3\binom{n-2}{4i+2} + \sum_{i=0}^{k-2} 2\binom{n-2}{4i+3},\\
s(n-4,4) &= \sum_{i=0}^{k-1} 14\binom{n-4}{4i} + \sum_{i=0}^{k-2} 4\binom{n-4}{4i+1} + 10\binom{n-4}{4i+2} + 4\binom{n-4}{4i+3}.
\end{aligned}
$$

Let $n = 4k+1$, then

$$
\begin{aligned}
s(n-1,1) &= 1 + \sum_{i=0}^{k-1} \binom{n-1}{4i} + 2\binom{n-1}{4i+2} + \binom{n-1}{4i+3},\\
s(n-2,2) &= \sum_{i=0}^{k-1} 3\binom{n-2}{4i} + 3\binom{n-2}{4i+2} + 2\binom{n-2}{4i+3}.
\end{aligned}
$$

Let $n = 4k+2$, then

$$
\begin{aligned}
s(n-1,1) &= n - 1 + \sum_{i=0}^{k-1} \binom{n-1}{4i} + 2\binom{n-1}{4i+2} + \binom{n-1}{4i+3},\\
s(n-2,2) &= \sum_{i=0}^{k} 3\binom{n-2}{4i} + \sum_{i=0}^{k-1} 3\binom{n-2}{4i+2} + 2\binom{n-2}{4i+3},\\
s(n-3,3) &= \sum_{i=0}^{k-1} 7\binom{n-3}{4i} + \binom{n-3}{4i+1} + 5\binom{n-3}{4i+2} + 3\binom{n-3}{4i+3}.
\end{aligned}
$$

Let $n = 4k+3$, then

$$
\begin{aligned}
s(n-1,1) &= \sum_{i=0}^{k} \binom{n-1}{4i} + 2\binom{n-1}{4i+2} + \sum_{i=0}^{k-1} \binom{n-1}{4i+3},\\
s(n-3,3) &= \sum_{i=0}^{k} 7\binom{n-3}{4i} + \sum_{i=0}^{k-1} \binom{n-3}{4i+1} + 5\binom{n-3}{4i+2} + 3\binom{n-3}{4i+3}.
\end{aligned}
$$

For $n \geq 5$, all the above statistics are distinct and are strictly bounded by $s(n,0)$ from below, and by $s(0,n)$ from above. □

Considering the symmetry and the shift, the above lemmas imply Theorem 2.1, Parts (*iii*) and (*iv*).

Remark 2.3. The function $s(p,q)$ has been first implemented using the program `Mathematica` to detect the real value of the statistics of the algebras $\mathbb{O}_{p,q}$. The code is in Appendix B and more precisely in Section B.2.

2.3 Summary table for algebras $\mathbb{O}_{p,q}$ compared to the Clifford algebras $\mathcal{C}l_{p,q}$

First, we give the table of the known classification of the algebras $\mathcal{C}l_{p,q}$, see Figure 2.8. It is known that the small Clifford algebras are

$$\mathcal{C}l_{0,0} \simeq \mathbb{R}, \qquad \mathcal{C}l_{0,1} \simeq \mathbb{C} \qquad \text{and} \qquad \mathcal{C}l_{0,2} \simeq \mathbb{H}.$$

Every algebras $\mathcal{C}l_{p,q}$ (and $\mathcal{C}l_n$) is isomorphic to a matrix algebra with entries in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ or the direct sum of such matrix algebras.

The two main properties $\mathcal{C}l_{p,q} \simeq \mathcal{C}l_{q+1,p-1}$ and $\mathcal{C}l_{p,q+4} \simeq \mathcal{C}l_{p+4,q}$ are symmetry with respect to the vertical axis ($p - q = 1$) and a shift of each row of the table, respectively. On each row, the Clifford algebras having the same color are isomorphic. Furthermore, represented with two arrows, when $p + q \equiv 1 \mod 2$, the Clifford algebra $\mathcal{C}l_{p,q}$ is not simple if $p - q \equiv 1 \mod 4$.

The original new classification of the algebras $\mathbb{O}_{p,q}$ is given in Figure 2.9. On each row, the algebras $\mathbb{O}_{p,q}$ having the same color are isomorphic. The table, with the corresponding statistics, is useful to understand the symmetry and shift results. Recall that the algebras $\mathbb{O}_{p,q}$ (and $\mathbb{O}_n$) are only defined for $p + q \geq 3$. The first two rows are different than the others as in small dimensions some "degeneracy" occurs. The two main properties $\mathbb{O}_{p,q} \simeq \mathbb{O}_{q,p}$ and $\mathbb{O}_{p,q+4} \simeq \mathbb{O}_{p+4,q}$ are symmetry with respect to the vertical middle axis and a shift on each row of the table, respectively. The algebras, $\mathbb{O}_{n,0}$ and $\mathbb{O}_{0,n}$, are exceptional.

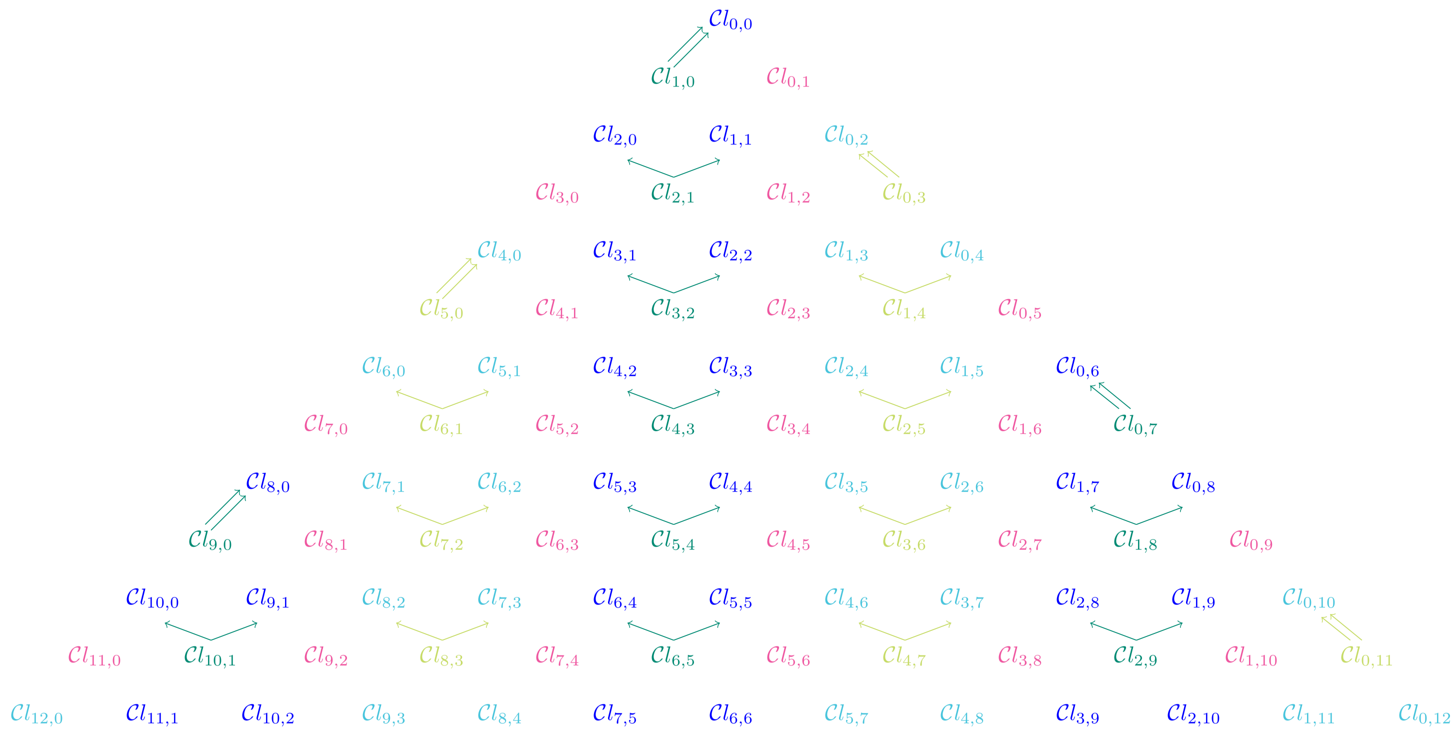

Figure 2.8: Summary table of classification of Clifford algebras $\mathcal{C}l_{p,q}$

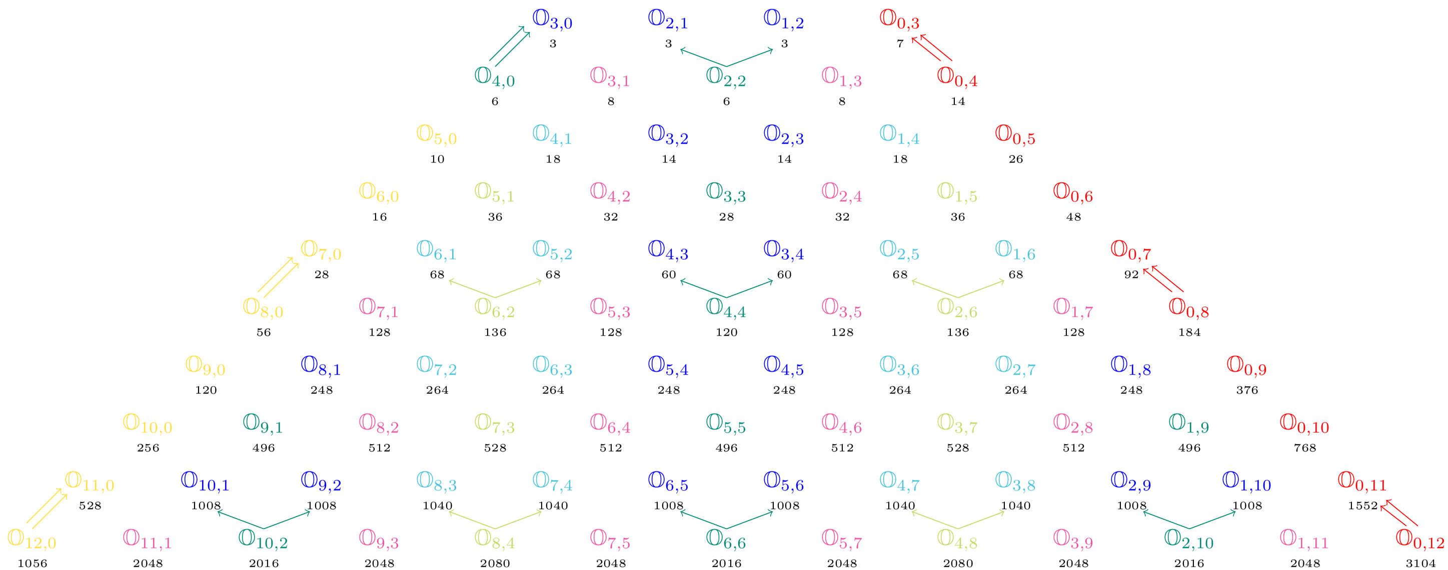

Figure 2.9: Summary table of classification of algebras $\mathbb{O}_{p,q}$

Chapter 3

Periodicity of the Algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$

In **Chapter** 3, we study the following problem: how do the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ with $p+q=n$ depend on the parameter n? Similarity with the Clifford algebras allows one to expect properties of periodicity, in particular it is natural to look for analogs of so-called *Bott periodicity*; see [Bae05].

We consider the problem of periodicity in the complex and in the real cases separately. We establish a periodicity according to the residue class of n modulo 4. In the complex case, we link together the algebras $\mathbb{O}_n$ and $\mathbb{O}_{n+4}$. Note that for the complex Clifford algebras there is a simple periodicity modulo 2. In the real case, we establish a periodicity for the algebras $\mathbb{O}_{p,q}$ according to the residue class of $p+q$ modulo 4 (provided $pq>0$). The situation for the exceptional algebras $\mathbb{O}_{0,n}$ and $\mathbb{O}_{n,0}$ is different. The results are compared to the well-known results for the Clifford algebras $\mathrm{Cl}_{p,q}$.

The main results of periodicity for the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ are given in **Section** 3.1. They come directly from the paper [Kre15]. In order to prove these results, a new tool is set up in **Section** 3.2, namely the triangulated graphs of a cubic form. At the end of this section, the main result is formulate in term of cubic forms. Finally, in **Section** 3.3, we proof with some details the result formulated in term of cubic forms. The Complement A is linked with this section, since it gives more details on some proofs and calculations.

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3.1 Analogue of the Bott periodicity

In this section, we formulate the main results on periodicity in comparison with the classical results about the Clifford algebras. The main difference between the periodicity theorems that we obtain and the classical ones is that all the periodicities for the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ are modulo 4, whereas in the case of Clifford algebras the simplest way to formulate the periodicity properties is modulo 2.

3.1.1 The complex case $\mathbb{O}_n$

Let us recall that for the complex Clifford algebras, one has the following simple statement

$$\mathcal{C}l_{n+2} \simeq \mathcal{C}l_n \otimes \mathcal{C}l_2.$$

Note also that $\mathcal{C}l_2$ is isomorphic to the algebra of complex 2×2-matrices. Our first goal is to establish a similar result for the algebras $\mathbb{O}_n$.

Definition 3.1. The *tensor product* of the algebras $\mathbb{O}_n = (\mathbb{C}[\mathbb{Z}_2^n], f_{\mathbb{O}_n})$ and $\mathbb{O}_5 = (\mathbb{C}[\mathbb{Z}_2^5], f_{\mathbb{O}_5})$, denoted by $\mathbb{O}_n \otimes \mathbb{O}_5$, is the twisted group algebra $(\mathbb{C}[\mathbb{Z}_2^{n+5}], f_{\mathbb{O}_n} + f_{\mathbb{O}_5})$ where the product of two elements of $\mathbb{O}_n \otimes \mathbb{O}_5$ is given by

$$\begin{aligned}(u_x \otimes u_y) \cdot (u_{x'} \otimes u_{y'}) &:= u_x \cdot u_{x'} \otimes u_y \cdot u_{y'} \\ &= (-1)^{f_{\mathbb{O}_n}(x,x') + f_{\mathbb{O}_5}(y,y')} u_{x+x'} \otimes u_{y+y'},\end{aligned}$$

where $x, x' \in \mathbb{Z}_2^n$ and $y, y' \in \mathbb{Z}_2^5$.

The unit is given by $u_0 \otimes u_0$ and the rules of the square of elements, commutativity and associativity are given by

$$(u_x \otimes u_y)^2 = (-1)^{\alpha_n(x,x') + \alpha_5(y,y')} u_0 \otimes u_0,$$

$$(u_x \otimes u_y) \cdot (u_{x'} \otimes u_{y'}) = (-1)^{\beta_n(x,x') + \beta_5(y,y')} (u_{x'} \otimes u_{y'}) \cdot (u_x \otimes u_y),$$

$$\begin{aligned}&(u_x \otimes u_y) \cdot ((u_{x'} \otimes u_{y'}) \cdot (u_{x''} \otimes u_{y''})) \\ = \;& (-1)^{\phi_n(x,x',x'') + \phi_5(y,y',y'')} ((u_x \otimes u_y) \cdot (u_{x'} \otimes u_{y'})) \cdot (u_{x''} \otimes u_{y''}),\end{aligned}$$

where $\alpha(x) = f(x,x)$ and β and ϕ are as in (1.3) and (1.4). Consider the subalgebra of $\mathbb{O}_n \otimes \mathbb{O}_5$, denoted by $\mathcal{P}(\mathbb{O}_n \otimes \mathbb{O}_5)$, consisting in the elements of the form

$$u_{(x_1,x_2,\dots,x_n)} \otimes u_{(x_1,y_2,\dots,y_5)},$$

where $(x_1,x_2,\dots,x_n) \in \mathbb{Z}_2^n$ and $(x_1,y_2,\dots,y_5) \in \mathbb{Z}_2^5$. The dimension of $\mathcal{P}(\mathbb{O}_n \otimes \mathbb{O}_5)$ is 2^{n+4} and some generators are given by

$$u_{e_1+e_i} \otimes u_{e_1}, \qquad u_{e_1} \otimes u_{e_1+e_j}, \qquad u_{e_1} \otimes u_{e_1}$$

where $i \in \{2,\dots,n\}$, $e_i = (0,\dots,0,1,0\dots,0) \in \mathbb{Z}_2^n$ where 1 stands at the i^{th} position and $j \in \{2,\dots,5\}$, $e_j = (0,\dots,0,1,0\dots,0) \in \mathbb{Z}_2^5$ where 1 stands at the j^{th} position. The modulo 4 periodicity on the algebras $\mathbb{O}_n$ involves the subalgebra $\mathcal{P}(\mathbb{O}_n \otimes \mathbb{O}_5)$. Here is the result.

Theorem 3.1. *If $n \geq 3$, there is an isomorphism*

$$\mathbb{O}_{n+4} \simeq \mathcal{P}(\mathbb{O}_n \otimes \mathbb{O}_5).$$

3.1.2 The real case $\mathbb{O}_{p,q}$

In the real case, the periodicity of the algebras $\mathbb{O}_{pq}$, where $p,q>0$ is different than the periodicity of the algebras $\mathbb{O}_{n,0}$ and $\mathbb{O}_{0,n}$. As in the complex case, let us define the tensor product of two algebras.

Definition 3.2. The *tensor product* of the algebra $\mathbb{O}_{p,q} = (\mathbb{R}[\mathbb{Z}_2^n], f_{\mathbb{O}_{p,q}})$ and $\mathbb{O}_{r,s} = (\mathbb{R}[\mathbb{Z}_2^5], f_{\mathbb{O}_{r,s}})$ with $r+s=5$, denoted by $\mathbb{O}_{p,q} \otimes \mathbb{O}_{r,s}$, is the twisted group algebra $(\mathbb{R}[\mathbb{Z}_2^{n+5}], f_{\mathbb{O}_{p,q}} + f_{\mathbb{O}_{r,s}})$ where the product of two elements of $\mathbb{O}_{p,q} \otimes \mathbb{O}_{r,s}$ is given by

$$(u_x \otimes u_y) \cdot (u_{x'} \otimes u_{y'}) := (-1)^{f_{\mathbb{O}_{p,q}}(x,x') + f_{\mathbb{O}_{r,s}}(y,y')} u_{x+x'} \otimes u_{y+y'}.$$

Theorem 3.2. *If $n = p+q \geq 3$ and $pq > 0$ (except for $(p,q) = (1,4)$ and $(p,q) = (4,1)$), then there are the following isomorphisms of graded algebras*

$$\mathbb{O}_{0,n+4} \simeq \mathcal{P}(\mathbb{O}_{0,n} \otimes \mathbb{O}_{5,0}) \simeq \mathcal{P}(\mathbb{O}_{n,0} \otimes \mathbb{O}_{0,5}), \tag{3.1}$$

$$\mathbb{O}_{n+4,0} \simeq \mathcal{P}(\mathbb{O}_{n,0} \otimes \mathbb{O}_{5,0}) \simeq \mathcal{P}(\mathbb{O}_{0,n} \otimes \mathbb{O}_{0,5}), \tag{3.2}$$

$$\mathbb{O}_{p+2,q+2} \simeq \mathcal{P}(\mathbb{O}_{p,q} \otimes \mathbb{O}_{2,3}). \tag{3.3}$$

In order to compare the above theorem with the classical results for the real Clifford algebras, we recall following periodicities

$$\begin{aligned} \mathcal{C}l_{p+2,q} &\simeq \mathcal{C}l_{q,p} \otimes \mathcal{C}l_{2,0}, \\ \mathcal{C}l_{p,q+2} &\simeq \mathcal{C}l_{q,p} \otimes \mathcal{C}l_{0,2}, \\ \mathcal{C}l_{p+1,q+1} &\simeq \mathcal{C}l_{p,q} \otimes \mathcal{C}l_{1,1}. \end{aligned}$$

This in particular implies

$$\mathcal{C}l_{p+8,q} \simeq \mathcal{C}l_{p+4,q+4} \simeq \mathcal{C}l_{p,q+8} \simeq \mathcal{C}l_{p,q} \otimes \mathrm{Mat}_{16}(\mathbb{R}),$$

known as the *Bott periodicity.*

3.1.3 How to use the generating function

In order to illustrate our method and the role of generating functions, let us give two simple proofs of the classical real isomorphisms $\mathrm{Cl}_{p+2,q} \simeq \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{q,p}$ and $\mathrm{Cl}_{p,q+2} \simeq \mathrm{Cl}_{q,p} \otimes \mathrm{Cl}_{0,2}$.

The algebras $\mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{q,p}$ and $\mathrm{Cl}_{q,p} \otimes \mathrm{Cl}_{0,2}$ have respectively the following generating functions

$$\alpha(x) = x_1 x_2 + \sum_{3 \leq i \leq j \leq n+2} x_i x_j + \sum_{p+3 \leq i \leq n+2} x_i,$$

and

$$\alpha'(x) = \sum_{1 \leq i \leq j \leq n} x_i x_j + \sum_{p+1 \leq i \leq n} x_i + x_{n+1} x_{n+2} + x_{n+1} + x_{n+2}.$$

It is easy to check that the coordinate transformations

$$\begin{aligned} x'_1 &= x_1 + x_3 + \cdots + x_{n+2}, & & & x''_i &= x_i, \quad i \leq n, \\ x'_2 &= x_2 + x_3 + \cdots + x_{n+2}, & &\text{and} & x''_{n+1} &= x_1 + \cdots + x_{n+1}, \\ x'_i &= x_i, \quad i > 2, & & & x''_{n+2} &= x_1 + \cdots + x_n + x_{n+2}. \end{aligned} \tag{3.4}$$

send respectively α and α' to the generating quadratic form of $\mathcal{C}l_{p+2,q}$ and of $\mathcal{C}l_{p,q+2}$. The last periodicity statement for the real Clifford algebras, i.e.

$\mathrm{Cl}_{p+1,q+1} \simeq \mathcal{C}l_{1,1} \otimes \mathcal{C}l_{q,p}$ can be proved in a similar way.

Remark 3.1. In the coordinate transformations 3.4, we have

$$x'_1 + x'_2 = x_1 + x_2 \quad \text{and} \quad x''_{n+1} + x''_{n+2} = x_{n+1} + x_{n+2}.$$

where x_1 and x_2 (resp. x_{n+1} and x_{n+2}) are the coordinates that we want to group and isolate from the others. This will have large and nice consequences on the cubic forms associated to the algebras $\mathbb{O}_{p,q}$.

Although coordinate of the cubic forms associated to $\mathbb{O}_{p,q}$ and $\mathbb{O}_n$ usually can NOT be totally grouped and isolated from the others like in the Clifford case, a structure of star with respect to a vantage coordinate[1] will appear.

3.2 A cubic form on $\mathbb{Z}_2^n$ can be interpreted in term of a triangulated graph

In this section, we present a way to interpret a cubic form on $\mathbb{Z}_2^n$ in term of a triangulated graph and reformulate our main results. This will allow us to find the simplest equivalent normal forms for the cubic forms $\alpha_{p,q}$, for which the periodicity statements are very transparent.

3.2.1 The definition

Consider an arbitrary cubic form on $\mathbb{Z}_2^n$

$$\alpha(x) = \sum_{1 \le i < j < k \le n} A_{ijk}\, x_i x_j x_k + \sum_{1 \le i < j \le n} B_{ij}\, x_i x_j + \sum_{1 \le i \le n} C_i\, x_i.$$

Note that this is precisely the form (1.11) by we separate the terms for which some of the indices coincide. We will associate a triangulated graph to every such function. The definition is as follows.

Definition 3.3. Given a cubic form α, the corresponding triangulated graph is as follows.

[1]One vantage coordinate to bind them all.

1. The set of vertices of the graph coincides with the set $\{x_1, x_2, \ldots, x_n\}$. Write $\bullet$ if $C_i = 1$ and $\circ$ if $C_i = 0$.

2. Two distinct vertices, i and j, are joined by an edge —— if $B_{ij} = 1$.

3. Join by a triangle ◁ those (distinct) vertices i, j, k for which $A_{ijk} = 1$.

Note that the defined triangulated graph completely characterizes the cubic form.

Example 3.1. Let us give elementary examples in the 2-dimensional case.

1. The first interesting case is that of the classical algebra of quaternion numbers $\mathbb{H}$. The quadratic form and the corresponding graph are as follows.

$$\alpha^{\mathcal{C}l}_{0,2}(x_1, x_2) = x_1x_2 + x_1 + x_2 \qquad \longleftrightarrow$$

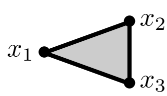
2. The other interesting case is that the Clifford algebra $\mathcal{C}l_{2,0}$. The quadratic form and the corresponding graph are as follows.

$$\alpha^{\mathcal{C}l}_{2,0}(x_1, x_2) = x_1x_2 \qquad \longleftrightarrow$$

Example 3.2. Let us give several examples in the 3-dimensional case.

1. The first interesting case is that of the classical algebra of octonion numbers $\mathbb{O}$. The cubic form and the corresponding graph are as follows.

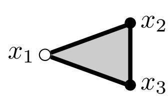
$$\begin{aligned}\alpha_{0,3}(x_1, x_2, x_3) &= x_1x_2x_3 + x_1x_2 + x_1x_3 \qquad \longleftrightarrow \\ &\quad +x_2x_3 + x_1 + x_2 + x_3\end{aligned}$$



Amazingly, the above triangle contains the full information about the cubic form $\alpha_{0,3}$ and therefore about the algebra $\mathbb{O}$.

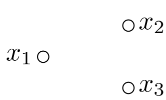
2. The algebra of split-octonions has the following cubic form.

$$\begin{aligned}\alpha_{1,2}(x_1, x_2, x_3) &= x_1x_2x_3 + x_1x_2 + x_1x_3 \qquad \longleftrightarrow \\ &\quad +x_2x_3 + x_2 + x_3\end{aligned}$$



3. The "trivial example".

$$\alpha(x_1, x_2, x_3) \equiv 0 \qquad \longleftrightarrow$$



3.2.2 The forms $\tilde{\alpha}_{0,n}$ and $\tilde{\alpha}_{n,0}$

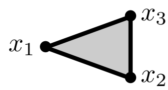
Let us now introduce a series of cubic forms $\tilde{\alpha}_{p,q}$. We will prove in Section 3.3 that they are equivalent to the forms $\alpha_{p,q}$. The advantage of this new way to represent the cubic forms $\alpha_{p,q}$ consists in the fact that the corresponding graphs are very simple. The periodicity properties of the algebras $\mathbb{O}_n$ and $\mathbb{O}_{p,q}$ can be seen directly from the graphs.

Let us start with the case of signature $(0, n)$.

Definition 3.4. The cubic forms $\tilde{\alpha}_{0,n}$ are defined as follows.

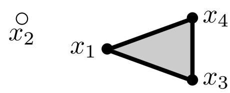
1. $\tilde{\alpha}_{0,3} = \alpha_{0,3}$, i.e. we have

$$\begin{aligned} & \tilde{\alpha}_{0,3}(x_1, x_2, x_3) \\ = \; & x_1x_2x_3 + x_1x_2 + x_1x_3 \\ & + x_2x_3 + x_1 + x_2 + x_3, \end{aligned} \qquad \longleftrightarrow$$

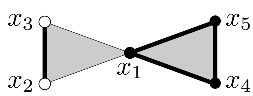


2. The next cases are

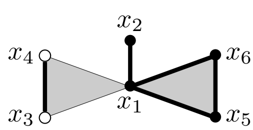
$$\begin{aligned} & \tilde{\alpha}_{0,4}(x_1, \ldots, x_4) \\ = \; & x_1x_3x_4 + x_1x_3 + x_1x_4 \\ & + x_3x_4 + x_1 + x_3 + x_4, \end{aligned} \qquad \longleftrightarrow$$



$$\begin{aligned} & \tilde{\alpha}_{0,5}(x_1, \ldots, x_5) \\ = \; & x_1x_2x_3 + x_1x_4x_5 + x_2x_3 + x_1x_4 \\ & + x_1x_5 + x_4x_5 + x_1 + x_4 + x_5, \end{aligned} \qquad \longleftrightarrow$$



$$\begin{aligned} & \tilde{\alpha}_{0,6}(x_1, \ldots, x_6) \\ = \; & x_1x_3x_4 + x_1x_5x_6 + x_1x_2 \\ & + x_3x_4 + x_1x_5 + x_1x_6 + x_5x_6 \\ & + x_1 + x_2 + x_5 + x_6. \end{aligned} \qquad \longleftrightarrow$$



3. In general, we have the following.

$$\begin{aligned}
\tilde{\alpha}_{0,4k+3}(x_1,\dots,x_{4k+3}) &= \tilde{\alpha}_{0,3}(x_1,x_2,x_3) \\
&\quad + \sum_{i=1}^{k} \left(x_1 + \tilde{\alpha}_{0,5}(x_1,x_{4i},\dots,x_{4i+3})\right), \\
\tilde{\alpha}_{0,4k}(x_1,\dots,x_{4k}) &= \tilde{\alpha}_{0,4}(x_1,\dots,x_4) \\
&\quad + \sum_{i=1}^{k-1} \left(x_1 + \tilde{\alpha}_{0,5}(x_1,x_{4i+1},\dots,x_{4i+4})\right), \\
\tilde{\alpha}_{0,4k+1}(x_1,\dots,x_{4k+1}) &= \tilde{\alpha}_{0,5}(x_1,\dots,x_5) \\
&\quad + \sum_{i=1}^{k-1} \left(x_1 + \tilde{\alpha}_{0,5}(x_1,x_{4i+2},\dots,x_{4i+5})\right), \\
\tilde{\alpha}_{0,4k+2}(x_1,\dots,x_{4k+2}) &= \tilde{\alpha}_{0,6}(x_1,\dots,x_6) \\
&\quad + \sum_{i=1}^{k-1} \left(x_1 + \tilde{\alpha}_{0,5}(x_1,x_{4i+3},\dots,x_{4i+6})\right).
\end{aligned}$$

Table 3.1 gives a series of examples of defined cubic forms. The property of periodicity according to the residue of n modulo 4 is quite obvious. In general, four consecutive variables in $\tilde{\alpha}$ correspond to a butterfly graph where the variable x_1 corresponds to the central vertex.

Definition 3.5. The forms $\tilde{\alpha}_{n,0}$ are defined according the following simple rule

$$\tilde{\alpha}_{n,0}(x_1,\dots,x_n) := \tilde{\alpha}_{0,n}(x_1,\dots,x_n) + x_1.$$

Remark 3.2. The definitions of $\tilde{\alpha}_{0,n}$ and $\tilde{\alpha}_{n,0}$ are such that

$$\begin{aligned}
\tilde{\alpha}_{n+4,0}(x_1,x_2,\dots,x_{n+4}) &= \tilde{\alpha}_{0,n}(x_1,x_2,\dots,x_n) + \tilde{\alpha}_{0,5}(x_1,x_{n+1},\dots,x_{n+4}) \\
&= \tilde{\alpha}_{n,0}(x_1,x_2,\dots,x_n) + \tilde{\alpha}_{5,0}(x_1,x_{n+1},\dots,x_{n+4}), \\
\tilde{\alpha}_{0,n+4}(x_1,x_2,\dots,x_{n+4}) &= \tilde{\alpha}_{n,0}(x_1,x_2,\dots,x_n) + \tilde{\alpha}_{0,5}(x_1,x_{n+1},\dots,x_{n+4}) \\
&= \tilde{\alpha}_{0,n}(x_1,x_2,\dots,x_n) + \tilde{\alpha}_{5,0}(x_1,x_{n+1},\dots,x_{n+4}).
\end{aligned}$$

These periodicities on the cubic forms $\tilde{\alpha}_{0,n}$ and $\tilde{\alpha}_{n,0}$ are exactly the expected periodicities for Theorem 3.2, equations (3.1) and (3.2).

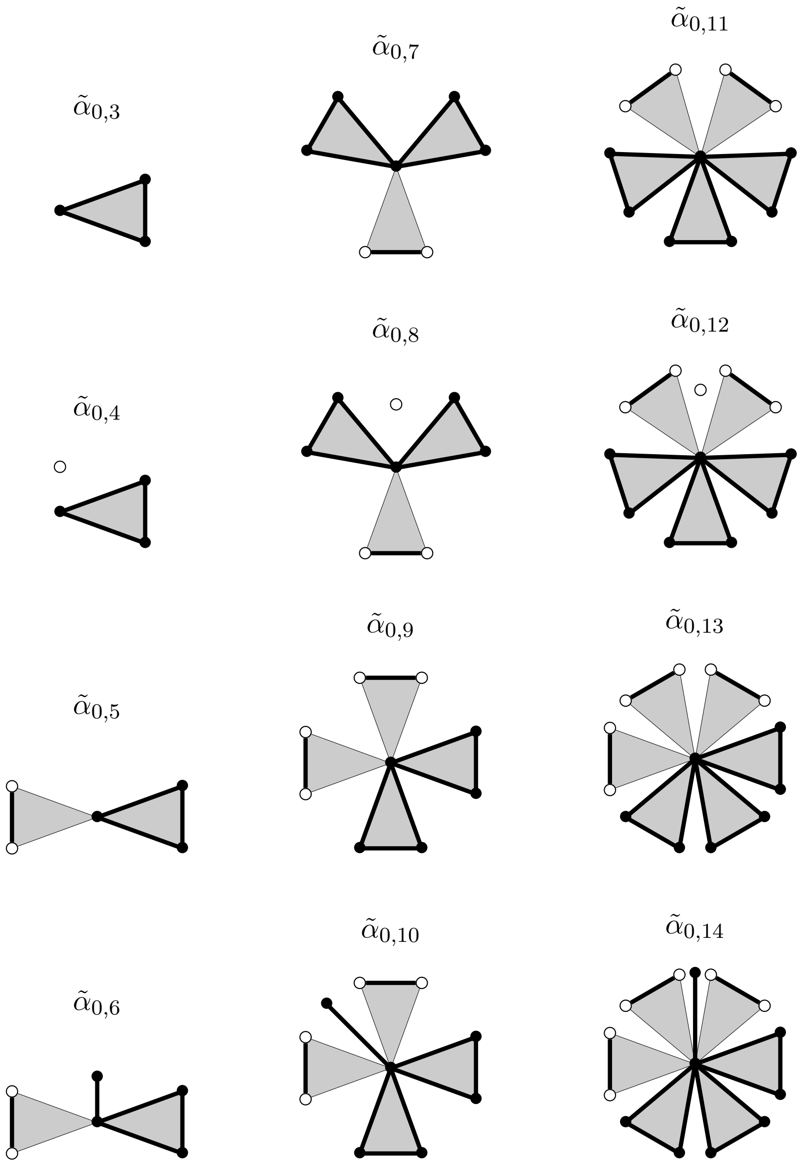
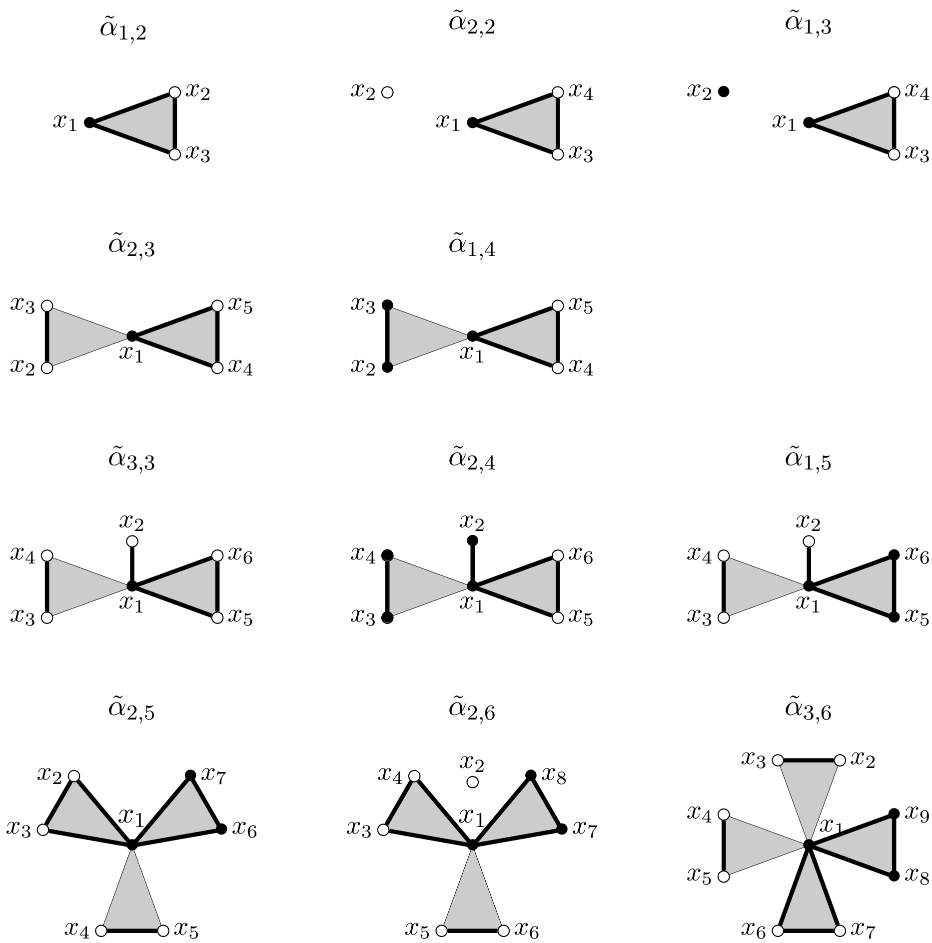


Table 3.1: Example of the cubic form $\tilde{\alpha}_{0,n}$ for $n \in \{3, \ldots, 14\}$.

3.2.3 The forms $\tilde{\alpha}_{p,q}$

The cubic forms $\tilde{\alpha}_{p,q}$ with signature (p,q) such that $p > 0$ and $q > 0$ are defined as follows.

Definition 3.6. 1. The first eleven cases are defined as follows.



The coordinate formulas follow directly from the above graphs.

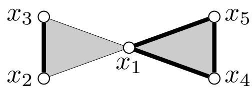
2. We define the forms $\tilde{\alpha}_{p,q}$ with arbitrary $p > 0$ and $q > 0$, except for $(p,q) = (1,4)$ and $(p,q) = (4,1)$ in the last equation, using the following

rules

$$\begin{aligned}
\tilde{\alpha}_{q,p} &:= \tilde{\alpha}_{p,q};\\
\tilde{\alpha}_{p,q+4} &:= \tilde{\alpha}_{p+4,q};\\
\tilde{\alpha}_{p+2,q+2}(x_1,\dots,x_{n+4}) &:= \tilde{\alpha}_{p,q}(x_1,\dots,x_n)\\
&\quad +\tilde{\alpha}_{2,3}(x_1,x_{n+1},x_{n+2},x_{n+3},x_{n+4})+x_1.
\end{aligned}$$

It is easy to check that the first eleven forms suffice to determine the rest. Note that the form $\tilde{\alpha}_{3,6}$ is not defined from $\tilde{\alpha}_{1,4}$ with the rule of induction but is given in the eleven first cases. Moreover, $\tilde{\alpha}_{6,3}$ is defined as the form $\tilde{\alpha}_{3,6}$ and not from $\tilde{\alpha}_{4,1}$ with the rule of induction.

Remark 3.3. The form $\tilde{\alpha}_{2,3}$ is equivalent to



according to the following coordinate transformation

$$\begin{aligned}
x_i' &= x_i, \qquad \text{for } i=1,\dots,4,\\
x_5' &= x_1+x_5.
\end{aligned} \tag{3.5}$$

Hence, the definition of $\tilde{\alpha}_{p,q}$ is such that $\tilde{\alpha}_{p+2,q+2}(x_1,x_2,\dots,x_{n+4})$ is equivalent to

$$\tilde{\alpha}_{p,q}(x_1,x_2,\dots,x_n)+\tilde{\alpha}_{2,3}(x_1,x_{n+1},\dots,x_{n+4}),$$

This periodicity on the cubic forms $\tilde{\alpha}_{p,q}$ is exactly the expected periodicity for Theorem 3.2, equations (3.3).

3.2.4 An equivalent formulation of the main result

Let us give a different way to formulate our main result.

Theorem 3.3. *The cubic forms $\alpha_{p,q}$ and $\tilde{\alpha}_{p,q}$ are equivalent for all p,q.*

Theorems 3.1 and 3.2 will follow from Theorem 3.3 since the forms $\tilde{\alpha}_{p,q}$ $(pq>0)$, $\tilde{\alpha}_{0,n}$ and $\tilde{\alpha}_{n,0}$ have the required periodicity.

3.3 Construction of the periodicity through triangulated graph

In this section, we give explicitly step by step the coordinate transformations that intertwine the cubic forms $\alpha_{p,q}$ and $\tilde{\alpha}_{p,q}$. According to the number of generators congruent modulo 4, different cases appear. The cases where the number of generators is even will be deduced from cases where the number of generators is odd. This is explained in Subsection 3.3.1. In Subsection 3.3.2, we focus on the two cases with odd number of generators. Finally, we finish the proof of Theorem 3.3 in Subsection 3.3.3.

3.3.1 The cases (p, q) with $n = 4k$ and $n = 4k + 2$

The first Lemma 3.1 shows that the case $n = 4k$ can be deduced from the case $n = 4k - 1$. The cubic form $\alpha_{p,q}$ with $p + q = 4k$ is equivalent to a cubic form where the last generator completely disappears or is only present in the linear part.

We introduce the following notation. Consider the projection $\mathbb{Z}_2^n \to \mathbb{Z}_2^{n-1}$ defined by "forgetting" the last coordinate, x_n. The cubic form on $\mathbb{Z}_2^n$ obtained by the pull-back of a cubic form α on $\mathbb{Z}_2^{n-1}$ will be denoted by $\widehat{\alpha}$. In other words,

$$\widehat{\alpha}(x_1, \ldots, x_n) = \alpha(x_1, \ldots, x_{n-1}).$$

Lemma 3.1. *If $n = p + q = 4k$ with $k \in \mathbb{N}\backslash\{0\}$ then, one has the equivalent forms*

$$\alpha_{0,n} \simeq \widehat{\alpha}_{0,n-1} \qquad \textit{and} \qquad \alpha_{n,0} \simeq \widehat{\alpha}_{n-1,0},$$

$$\alpha_{p,q} \simeq \begin{cases} \widehat{\alpha}_{p,q-1} & \textit{if } p, q > 0 \textit{ are even,} \\ \widehat{\alpha}_{p,q-1} + x_n & \textit{if } p, q \geq 1 \textit{ are odd.} \end{cases}$$

Proof. To establish the equivalence, we give all the details in Complement A and more precisely in Section A.1. A change of variable is given and the lemma is first proved in the case of the signature $(0, n)$. Then the other cases of the signatures $(n,0)$ and (p,q) where $p,q \geq 1$ are deduced from the first case. □

The second Lemma 3.2 allows us to reduce the case $n = 4k+2$ to the case $n = 4k+1$. The cubic form $\alpha_{p,q}$ with $p+q = 4k+2$ is equivalent to a cubic form where the last coordinate is only present in the quadratic part and sometimes in the linear part.

Lemma 3.2. *If $n = p+q = 4k+2$ with $k \in \mathbb{N}\backslash\{0\}$ then, one has the equivalent forms*

$$\alpha_{0,n} \simeq \widehat{\alpha}_{0,n-1} + x_n + x_n \sum_{i=1}^{n-1} x_i, \tag{3.6}$$

$$\alpha_{n,0} \simeq \widehat{\alpha}_{n-1,0} + x_n + x_n \sum_{i=1}^{n-1} x_i, \tag{3.7}$$

$$\alpha_{p,q} \simeq \begin{cases} \widehat{\alpha}_{p-1,q} + x_n \sum_{i=1}^{n-1} x_i & \text{if } p,q \geq 1 \text{ are odd,} \\ \widehat{\alpha}_{p-1,q} + x_n + x_n \sum_{i=1}^{n-1} x_i & \text{if } p,q > 0 \text{ are even.} \end{cases} \tag{3.8}$$

Proof. All the details of the proof are given in Complement A and in particular in Section A.2. □

3.3.2 The cases (p,q) with $n = 4k+3$ and $n = 4k+1$

If $n = p+q$ is odd, there exist exactly four distinct algebras $\mathbb{O}_{p,q}$ with $p,q \geq 0$ up to graded isomorphism; see Chapter 2. We will treat each of these four cases independently.

Lemma 3.3. *If $n = 4k+1$, then*

1. *the form $\alpha_{0,n}$ is equivalent to*

$$x_1 + (x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\dots,x_{2k+1}) + \alpha^{\mathcal{C}l}_{0,2k}(x_{2k+2},\dots,x_{4k+1})\right),$$

2. *the form $\alpha_{n,0}$ is equivalent to*

$$(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\dots,x_{2k+1}) + \alpha^{\mathcal{C}l}_{0,2k}(x_{2k+2},\dots,x_{4k+1})\right),$$

3. *the form* $\alpha_{2k,2k+1}$ *is equivalent to*

$$x_1+(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\ldots,x_{2k+1})+\alpha^{\mathcal{C}l}_{0,2k}(x_{2k+2},\ldots,x_{4k+1})\right)+\sum_{i=2k+2}^{4k+1}x_i,$$

4. *the form* $\alpha_{2k-2,2k+3}$ *is equivalent to*

$$x_1+(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\ldots,x_{2k+1})+\alpha^{\mathcal{C}l}_{0,2k}(x_{2k+2},\ldots,x_{4k+1})\right)+\sum_{i=2k+2}^{4k-1}x_i.$$

Proof. The details of the proof are given in Complement A and more precisely in Section A.3. A change of variable is given and the lemma is proved in the case of the signature $(0,n)$. Then the other cases of the signatures $(n,0)$, $(2k,2k+1)$ and $(2k-2,2k+3)$ are deduced from the first case. □

Remark 3.4. Note that the periodicity of the cubic forms $\alpha_{p,q}$ where $p+q=4k+2$ are deduced from the periodicity of the cubic form $\alpha_{p,q}$ where $p+q=4k+1$. The formulas (3.6), (3.7) and (3.8) suggest to have the last coordinate x_n involved in the quadratic part where the periodicity is difficult to catch. As a surprise, if we apply to this quadratic part, i.e. $x_n\sum_{i=1}^{n-1}x_i$, the changes of coordinates used in Lemma 3.3 on the first $n-1$ variables and let x_n unchanged, then this quadratic part appears (after the change of coordinate) only as x_nx_1. The periodicity becomes much easier. For more details about the change of coordinates, look at Complement A, equations (A.19) and (A.24).

Lemma 3.4. *If* $n=4k+3$ *and* k *is odd, then*

1. *the form* $\alpha_{0,n}$ *is equivalent to*

$$x_1+(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2,\ldots,x_{2k+3})+\alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4},\ldots,x_{4k+3})\right),$$

2. *the form* $\alpha_{n,0}$ *is equivalent to*

$$(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2,\ldots,x_{2k+3})+\alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4},\ldots,x_{4k+3})\right),$$

3. *the form* $\alpha_{2k+1,2k+2}$ *is equivalent to*

$$(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2,\ldots,x_{2k+3})+\alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4},\ldots,x_{4k+3})\right)+x_1+\sum_{i=2}^{2k+3}x_i,$$

4. *the form* $\alpha_{2k,2k+3}$ *is equivalent to*

$$(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2,\ldots,x_{2k+3})+\alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4},\ldots,x_{4k+3})\right)$$
$$+x_1+x_{2k+3}+\sum_{i=2}^{2k}x_i.$$

If $n=4k+3$ *and* k *is even, then*

1. *the form* $\alpha_{0,n}$ *is equivalent to*

$$x_1+(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\ldots,x_{2k+1})+\alpha^{\mathcal{C}l}_{0,2k+2}(x_{2k+2},\ldots,x_{4k+3})\right),$$

2. *the form* $\alpha_{n,0}$ *is equivalent to*

$$(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\ldots,x_{2k+1})+\alpha^{\mathcal{C}l}_{0,2k+2}(x_{2k+2},\ldots,x_{4k+3})\right),$$

3. *the form* $\alpha_{2k+1,2k+2}$ *is equivalent to*

$$(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\ldots,x_{2k+1})+\alpha^{\mathcal{C}l}_{0,2k+2}(x_{2k+2},\ldots,x_{4k+3})\right)+x_1+\sum_{i=2k+2}^{4k+3}x_i,$$

4. *the form* $\alpha_{2k,2k+3}$ *is equivalent to*

$$(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\ldots,x_{2k+1})+\alpha^{\mathcal{C}l}_{0,2k+2}(x_{2k+2},\ldots,x_{4k+3})\right)$$
$$+x_1+\sum_{i=2k+3}^{4k+1}x_i+x_{4k+3}.$$

Proof. All the details of the proof are given in Complement A and in particular in Section A.3. □

3.3.3 The end of the proof

Consider some more properties on the quadratic forms of the Clifford algebras. Denote, as above, by $\alpha^{\mathcal{C}l}_{0,2}$ the generating quadratic form of the Clifford algebra $\mathcal{C}l_{0,2}$ and by $\alpha^{\mathcal{C}l}_{2,0}$ the generating quadratic form of the Clifford algebra $\mathcal{C}l_{2,0}$. Denote also

$$(\alpha^{\mathcal{C}l}_{2,0})^l(x_1, \ldots, x_{2l}) := \alpha^{\mathcal{C}l}_{2,0}(x_1, x_2) + \cdots + \alpha^{\mathcal{C}l}_{2,0}(x_{2l-1}, x_{2l}),$$

$$(\alpha^{\mathcal{C}l}_{0,2})^l(x_1, \ldots, x_{2l}) := \alpha^{\mathcal{C}l}_{0,2}(x_1, x_2) + \cdots + \alpha^{\mathcal{C}l}_{0,2}(x_{2l-1}, x_{2l}).$$

The following lemma is useful in the Clifford case.

Lemma 3.5. *If $k > 0$ is even, then*

$$\alpha^{\mathcal{C}l}_{2k,0} \simeq \alpha^{\mathcal{C}l}_{0,2k} \simeq (\alpha^{\mathcal{C}l}_{0,2})^{k/2} + (\alpha^{\mathcal{C}l}_{2,0})^{k/2}.$$

If k is odd, then

$$\alpha^{\mathcal{C}l}_{2k,0} \simeq (\alpha^{\mathcal{C}l}_{2,0})^{\frac{k+1}{2}} + (\alpha^{\mathcal{C}l}_{0,2})^{\frac{k-1}{2}}, \qquad \alpha^{\mathcal{C}l}_{0,2k} \simeq (\alpha^{\mathcal{C}l}_{2,0})^{\frac{k-1}{2}} + (\alpha^{\mathcal{C}l}_{0,2})^{\frac{k+1}{2}}.$$

Proof. Thanks to the coordinate transformation (3.4), we have

$$\alpha^{\mathcal{C}l}_{p,q+2}(x_1, \ldots, x_{n+2}) \simeq \alpha^{\mathcal{C}l}_{q,p}(x_1, \ldots, x_n) + \alpha^{\mathcal{C}l}_{0,2}(x_{n+1}, x_{n+2})$$

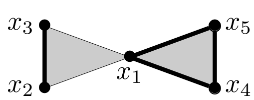
and

$$\alpha^{\mathcal{C}l}_{p+2,q}(x_1, \ldots, x_{n+2}) \simeq \alpha^{\mathcal{C}l}_{2,0}(x_1, x_2) + \alpha^{\mathcal{C}l}_{q,p}(x_3, \ldots, x_{n+2})$$

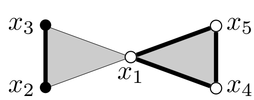
where $n = p + q$. The results are obtained using recursively these formulas in the special cases where $p = 0$ or $q = 0$. □

Lemma 3.5 means that the graph of the quadratic form of a Clifford algebra with even generators, is equivalent to a disconnected graph consisting of components of the type  and .

Lemma 3.6. *The form $\tilde{\alpha}_{2,3}$ and $\tilde{\alpha}_{1,4}$ are equivalent to*



and



respectively. One has

$$\begin{aligned} & \tilde{\alpha}_{0,5}(x_1, x_2, \ldots, x_5) + \tilde{\alpha}_{1,4}(x_1, x_6, \ldots, x_9) \\ = \; & \tilde{\alpha}_{2,3}(x_1, x_2, x_3, x_8, x_9) + \tilde{\alpha}_{2,3}(x_1, x_6, x_7, x_4, x_5) \end{aligned}$$

Proof. For the first part, the first coordinate transformation is given by

$$\begin{aligned} x'_1 &= x_1, \\ x'_2 &= x_3 + x_4 + x_5, \\ x'_3 &= x_2 + x_4 + x_5, \\ x'_4 &= x_2 + x_3 + x_5, \\ x'_5 &= x_2 + x_3 + x_4. \end{aligned}$$

while the second one is given by the coordinate transformation (3.5). The second affirmation is deduced directly from the first one together with the result in Remark 3.3 . □

To finish the proof of Theorem 3.3, we consider four different cases.

1. The case with the signature $(0, n)$. Suppose that $n = 4k + 1$, then according to Lemmas 3.3 and 3.5, $\alpha_{0,n}$ is equivalent to the following form

$$x_1 + (x_1 + 1)\left((\alpha^{\mathcal{C}l}_{2,0})^k(x_2, \ldots, x_{2k+1}) + (\alpha^{\mathcal{C}l}_{0,2})^k(x_{2k+2}, \ldots, x_{4k+1})\right).$$

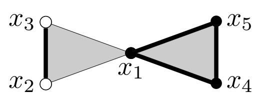
According to Lemmas 3.4 and 3.5, if $n = 4k+3$, then $\alpha_{0,n}$ is equivalent to the following form

$$x_1 + (x_1 + 1)\left((\alpha^{\mathcal{C}l}_{2,0})^k(x_2, \ldots, x_{2k+1}) + (\alpha^{\mathcal{C}l}_{0,2})^{k+1}(x_{2k+2}, \ldots, x_{4k+3})\right).$$

The desired equivalence follows from

$\tilde{\alpha}_{0,5}$

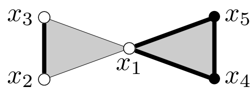
$$x_1 + (x_1 + 1)\left(\alpha^{\mathcal{C}l}_{2,0}(x_2, x_3) + \alpha^{\mathcal{C}l}_{0,2}(x_4, x_5)\right) \quad \longleftrightarrow$$



2. The case where the signature is $(n, 0)$ follows directly from the case of signature $(0, n)$ since we have the following equivalence

$\tilde{\alpha}_{5,0}$

$$(x_1+1)\left(\alpha_{2,0}^{\mathcal{C}l}(x_2,x_3)+\alpha_{0,2}^{\mathcal{C}l}(x_4,x_5)\right) \qquad \longleftrightarrow$$



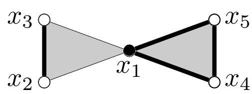
3. When the signature is $(2k, 2k+1)$ with $n = 4k+1$, due to Lemmas 3.3, 3.5 and 3.6, the form $\alpha_{2k,2k+1}$ is equivalent to the following form

$$\begin{aligned}
& (x_1+1)\left((\alpha_{2,0}^{\mathcal{C}l})^k(x_2,\ldots,x_{2k+1})+(\alpha_{0,2}^{\mathcal{C}l})^k(x_{2k+2},\ldots,x_{4k+1})\right) \\
& +x_1+\sum_{i=2k+2}^{4k+1} x_i \\
= \;& (x_1+1)\left((\alpha_{2,0}^{\mathcal{C}l})^{k-1}(x_2,\ldots,x_{2k-1})+(\alpha_{0,2}^{\mathcal{C}l})^{k-1}(x_{2k+2},\ldots,x_{4k-1})\right) \\
& +\sum_{i=2k+2}^{4k-1} x_i+\tilde{\alpha}_{2,3}(x_1,x_{2k},x_{2k+1},x_{4k},x_{4k+1}).
\end{aligned}$$

The desired equivalence follows from

$\tilde{\alpha}_{2,3}$

$$\begin{aligned}
& (x_1+1)\left(\alpha_{2,0}^{\mathcal{C}l}(x_2,x_3)+\alpha_{0,2}^{\mathcal{C}l}(x_4,x_5)\right) \qquad \longleftrightarrow \\
& +x_1+x_4+x_5
\end{aligned}$$



When the signature is $(2k+1, 2k+2)$ with $n = 4k+3$, also due to Lemmas 3.4, 3.5 and 3.6, the form $\alpha_{2k+1,2k+2}$ is equivalent to the following form

$$\begin{aligned}
& (x_1+1)\left((\alpha_{2,0}^{\mathcal{C}l})^k(x_2,\ldots,x_{2k+1})+(\alpha_{0,2}^{\mathcal{C}l})^{k+1}(x_{2k+2},\ldots,x_{4k+3})\right) \\
& +x_1+\sum_{i=2k+2}^{4k+3} x_i \\
= \;& (x_1+1)\left((\alpha_{2,0}^{\mathcal{C}l})^k(x_2,\ldots,x_{2k+1})+(\alpha_{0,2}^{\mathcal{C}l})^k(x_{2k+2},\ldots,x_{4k+1})\right) \\
& +\sum_{i=2k+2}^{4k+1} x_i+\tilde{\alpha}_{1,2}(x_1,x_{4k+2},x_{4k+3}).
\end{aligned}$$

4. When the signature is $(2k, 2k+3)$ with $n = 4k+3$, the form $\alpha_{2k,2k+3}$

is equivalent to the following form

$$\begin{aligned}
& (x_1+1)\left((\alpha^{\mathcal{C}l}_{2,0})^k(x_2,\ldots,x_{2k+1})+(\alpha^{\mathcal{C}l}_{0,2})^{k+1}(x_{2k+2},\ldots,x_{4k+3})\right)\\
& +x_1+\sum_{i=2k+2}^{4k+1} x_i\\
= \; & (x_1+1)\left((\alpha^{\mathcal{C}l}_{2,0})^{k-1}(x_2,\ldots,x_{2k-1})+(\alpha^{\mathcal{C}l}_{0,2})^{k-1}(x_{2k+2},\ldots,x_{4k-1})\right)\\
& +\sum_{i=2k+2}^{4k-1} x_i+\tilde{\alpha}_{2,5}(x_1,x_{2k},x_{2k+1},x_{4k},x_{4k+1},x_{4k+2},x_{4k+3}).
\end{aligned}$$

When the signature is $(2k-1,2k+2)$ with $n=4k+1$, the form $\alpha_{2k-1,2k+2}$ is equivalent to $\alpha_{2k-2,2k+3}$ which is equivalent to the following form

$$\begin{aligned}
& (x_1+1)\left((\alpha^{\mathcal{C}l}_{2,0})^k(x_2,\ldots,x_{2k+1})+(\alpha^{\mathcal{C}l}_{0,2})^{k}(x_{2k+2},\ldots,x_{4k+1})\right)\\
& +x_1+\sum_{i=2k+2}^{4k-1} x_i\\
= \; & (x_1+1)\left((\alpha^{\mathcal{C}l}_{2,0})^{k-2}(x_2,\ldots,x_{2k-3})+(\alpha^{\mathcal{C}l}_{0,2})^{k-2}(x_{2k+2},\ldots,x_{4k-3})\right)\\
& +\tilde{\alpha}_{3,6}(x_1,x_{2k-2},x_{2k-1},x_{2k},x_{2k+1},x_{4k-2},x_{4k-1},x_{4k},x_{4k+1})\\
& +\sum_{i=2k+2}^{4k-3} x_i.
\end{aligned}$$

Theorem 3.3 is proved.

Chapter 4

Algebras of Krichever-Novikov type and Lie Antialgebras

Krichever and Novikov [KN87b], [KN87a] and [KN89] introduced and studied a family of Lie algebras with two marked points generalizing the Virasoro algebra. Krichever-Novikov (K-N) type algebras are algebras of meromorphic objects on compact Riemann surfaces of arbitrary genus g. Schlichenmaier studied the Krichever-Novikov Lie algebras for more than two marked points in [Sch90b], [Sch90c] and [Sch90a]. He showed, in particular, the existence of local 2-cocycles and central extensions for multiple-point K-N algebras [Sch03], extending the explicit formula of 2-cocycles due to Krichever and Novikov. Deformations on theses algebras were studied in [FS03] and [FS05] by Fialowski and Schlichenmaier.

The notion of Lie antialgebra was introduced by Ovsienko in [Ovs11], where the geometric origin was explained. It was then shown in [LMG12b] that these algebras are particular cases of Jordan superalgebras. The most important property of Lie antialgebras is their relationships with Lie superalgebras see [Ovs11], [MG09], [LMG12b] and [LMG12a]; different from the classical Kantor-Koecher-Tits construction for general Jordan superalgebras. One of the main example of [Ovs11] is the conformal Lie antialgebra $\mathcal{AK}(1)$ closely related to the Virasoro algebra and the Neveu-Schwarz Lie superalgebra. In [MG09], Morier-Genoud studied another important finite dimensional Lie antialgebra: $\mathcal{K}_3$, called the Kaplansky Jordan superalgebra which is related to osp(1|2).

Lie superalgebras of K-N type, $\mathcal{L}_{g,N}$, and the relation with Jordan superalgebras of K-N type, $\mathcal{J}_{g,N}$, were studied by Leidwanger and Morier-Genoud in [LMG12a]. They found examples of Lie antialgebras generalizing $\mathcal{AK}(1)$, as the same way that $\mathcal{L}_{g,N}$ generalizes $\mathcal{K}(1)$. In the second part of this thesis, we study central extensions of $\mathcal{L}_{g,N}$ and the corresponding 1-cocycles on $\mathcal{J}_{g,N}$.

The first theorem of this part is an explicit formula for a *local* non-trivial 2-cocycle on $\mathcal{L}_{g,N}$. The notion of locality depends on a structure of *almost-grading* on $\mathcal{L}_{g,N}$. A cocycle is called local if it vanishes whenever the sum of the degrees of the arguments is greater or lower than a certain bound. This formula uses projective connections and is very similar to the formula of Krichever-Novikov and Schlichenmaier. It was first studied in the case of two points in the super setting by Bryant in [Bry90]. We prove that the cohomology class of this 2-cocycle is independent of the choice of the projective connection. Fixing a theta characteristics and a splitting of the set of N points into two non-empty disjoint subsets, one obtains an almost-grading on the Lie superalgebra $\mathcal{L}_{g,N}$. Independently of this work, it was proved by Schlichenmaier [Sch13], that there exists a unique (up to equivalence and rescaling) non-trivial almost-graded central extension on $\mathcal{L}_{g,N}$.

The second theorem is an explicit formula for a local 1-cocycle on $\mathcal{J}_{g,N}$ with coefficient in the dual space. Thanks to conversations with Lecomte and Ovsienko about a cohomology theory for Lie antialgebras, they discovered two non-trivial cohomology classes of the conformal Lie antialgebra $\mathcal{AK}(1)$ analogous to the Gelfand-Fuchs class and to the Godbillon-Vey class. The cocycle on $\mathcal{J}_{g,N}$ studied in this part satisfies similar properties than those found by Lecomte and Ovsienko. It is given by a very simple and geometrically natural formula.

Interesting explicit examples of superalgebras arise in the case of the Riemann sphere with three marked points. These examples were thoroughly studied in [Sch90b] and [LMG12a]. The corresponding Lie superalgebra is denoted by $\mathcal{L}_{0,3}$ and the Jordan superalgebra by $\mathcal{J}_{0,3}$. These two algebras are closely related since $\mathcal{L}_{0,3}$ is the adjoint superalgebra of $\mathcal{J}_{0,3}$. These algebras contain the conformal algebras

$$\mathcal{L}_{0,3} \supset \mathcal{K}(1) \supset \mathrm{osp}(1|2) \qquad \text{and} \qquad \mathcal{J}_{0,3} \supset \mathcal{AK}(1) \supset \mathcal{K}_3.$$

We calculate explicitly the 2-cocycle on the Lie superalgebra $\mathcal{L}_{0,3}$ that is unique up to isomorphism and vanishes on the Lie subalgebra osp(1|2). This 2-cocycle induces a 1-cocycle on $\mathcal{L}_{0,3}$ with values in its dual space. Finally, we give an explicit formula for the unique 1-cocycle up to isomorphism on $\mathcal{J}_{0,3}$ with values in its dual space.

In **chapter** 4, we define the main objects of the two last chapters: the Krichever-Novikov (K-N) Lie and Jordan superalgebras.

Section 4.1 is devoted to the classical K-N Lie algebra, $\mathfrak{g}_{g,N}$ and the K-N associative algebra $\mathfrak{a}_{g,N}$. We set up the geometric background and introduce our main examples on the Riemann sphere; the algebras $\mathfrak{a}_{0,2}$, $\mathfrak{g}_{0,2}$ and $\mathfrak{a}_{0,3}$, $\mathfrak{g}_{0,3}$. The notion of grading is replaced by an almost-grading.

In **Section** 4.2, we recall the notion of Lie superalgebras then focus on the K-N Lie superalgebras, denoted by $\mathcal{L}_{g,N}$. The geometric background needs to be refined and we consider a *theta characteristics.* The main examples $\mathcal{L}_{0,2}$ and $\mathcal{L}_{0,3}$ are continued where explicit bases are given.

In **Section** 4.3, we introduce a particular class of Jordan superalgebras named Lie antialgebras. Deeply connected to Lie superalgebras, Lie antialgebras originate with the existence of an odd osp(1|2)-invariant bivector on $\mathbb{C}^{2|1}$. Lie antialgebras of K-N type, denoted by $\mathcal{J}_{g,N}$, were found by Leidwanger and Morier-Genoud in [LMG12a]. The main examples $\mathcal{J}_{0,2}$ and $\mathcal{J}_{0,3}$ are taken into consideration where explicit bases are given.

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4.1 Algebras of K-N type

In [KN87b], [KN87a] and [KN89], Krichever and Novikov introduced some generalizations of the well known Witt algebra and its central extension called the Virasoro algebra (see [She04] for a global overview of this theory). In this section we first recall some definitions and basic results on algebras. We set up the geometric background on Riemann surfaces and define the Lie (resp. associative) algebras $\mathfrak{g}_{g,N}$ (resp. $\mathfrak{a}_{g,N}$) of K-N type. We introduce and give details of main examples that we will continue throughout Chapters 4 and 5.

All the structures in Chapters 4 and 5 will be considered over the field $\mathbb{C}$.

4.1.1 Generalities on algebras

An *algebra* $\mathfrak{a}$ on $\mathbb{C}$ is a vector space together with a bilinear application, called multiplication or product,

$$(x, y) \in \mathfrak{a} \times \mathfrak{a} \mapsto x \cdot y \in \mathfrak{a}.$$

The algebra $\mathfrak{a}$ is *commutative* (resp. *associative*) if the multiplication is commutative (resp. associative).

An algebra $\mathfrak{g}$ on $\mathbb{C}$ is a *Lie algebra* if the product, called the Lie bracket and written $[,]$, is skewsymmetric and satisfies the Jacobi property, i.e. for every x, y and z in $\mathfrak{g}$, we have

$$[x, y] = -[y, x] \qquad \text{and} \qquad [[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

For any associative algebra $\mathfrak{a}$ (with a multiplication denoted by "$\cdot$"), one can construct a Lie algebra $\mathfrak{g}_{\mathfrak{a}}$. As a vector space, $\mathfrak{g}_{\mathfrak{a}}$ is the same than $\mathfrak{a}$ and the Lie bracket of two elements of $\mathfrak{g}_{\mathfrak{a}}$ is defined to be the *commutator* in $\mathfrak{a}$, i.e. for every x, y in $\mathfrak{g}_{\mathfrak{a}}$, we have

$$[x, y] := x \cdot y - y \cdot x.$$

If we consider V, a non-trivial vector space over $\mathbb{C}$, then $End(V)$ which is the space of all linear transformations from V to itself, is a unital associative algebra on $\mathbb{C}$ where the multiplication is given by the composition. So that, we can regard the space of endomorphisms as a Lie algebra. An endomorphism D of an algebra $\mathfrak{a}$ is a *derivation* if

$$D(x \cdot y) = x \cdot D(y) + D(x) \cdot y$$

for all x, y in $\mathfrak{a}$. The space of all derivations of $\mathfrak{a}$ is a Lie subalgebra of $End(\mathfrak{a})$.

An algebra $\mathfrak{j}$ on $\mathbb{C}$ is a *Jordan algebra* if the product is commutative and satisfies the Jordan identity, i.e. for every x, y and z in $\mathfrak{j}$, we have

$$x.y = y.x \qquad \text{and} \qquad (x.y).x^2 = x.(y.x^2).$$

For any associative algebra $\mathfrak{a}$ (with a multiplication denoted by "$\cdot$"), one can construct a Jordan algebra $\mathfrak{j}_{\mathfrak{a}}$. As a vector space, $\mathfrak{j}_{\mathfrak{a}}$ is the same than $\mathfrak{a}$ and the Jordan product of two elements of $\mathfrak{j}_{\mathfrak{a}}$ is defined to be the *anti-commutator* in $\mathfrak{a}$, i.e. for every x, y in $\mathfrak{j}_{\mathfrak{a}}$, we have

$$x.y := \frac{1}{2}(x \cdot y + y \cdot x).$$

Note that the anti-commutator is also denoted by $[x, y]_+$.

4.1.2 The geometrical set-up I

Let M be a compact Riemann[1] surface of genus[2] g without boundary (i.e. a smooth projective curve over $\mathbb{C}$), see Figure 4.1 for an example. Consider the union of two sets of ordered disjoint points called *punctures*

$$A = \underbrace{(P_1, \ldots, P_K)}_{=:I} \cup \underbrace{(Q_1, \ldots, Q_{N-K})}_{=:O}$$

where $N, K \in \mathbb{N}\backslash\{0\}$ with $N \geq 2$ and $1 \leq K < N$. We call I, the set of *in-points*, and O the set of *out-points*.
Let $\mathcal{K}$ be the canonical line bundle[3] of M (i.e. holomorphic cotangent bundle)

[1] A Riemann surface is a one-dimensional connected complex manifold.
[2] Roughly speaking, the genus gives the number of holes in a surface.
[3] We choose to write $\mathcal{K}$ instead of $\mathcal{K}_M$ to simplify the notation.

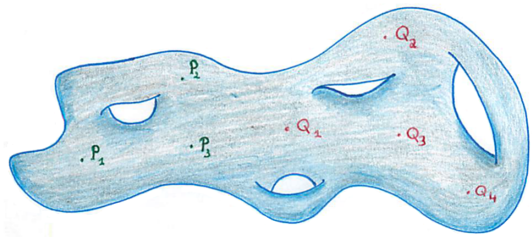

Figure 4.1: Riemann surface of genus 4 with 3 in-points and 4 out-points.

and consider the bundle $\mathcal{K}^\lambda$, $\lambda \in \mathbb{Z}$ where we assert that $\mathcal{K}^{\otimes 0}$ is the trivial bundle, $\mathcal{K}^\lambda := \mathcal{K}^{\otimes\lambda}$ for $\lambda > 0$ and $\mathcal{K}^\lambda := (\mathcal{K}^*)^{\otimes(-\lambda)}$ for $\lambda < 0$, in particular we have $\mathcal{K}^{\otimes -1} := \mathcal{K}^*$.

The local sections of the bundle are the local holomorphic differentials. If $P \in M$ is a point and z a local holomorphic coordinate[4] at P then a local holomorphic differential can be written as $f(z)dz$ with a local holomorphic function defined in a neighbourhood of P. A global holomorphic section can be described locally for a covering by coordinate charts $(U_\alpha, z_\alpha)_{\alpha\in J}$ by a system of local holomorphic functions $(f_\alpha)_{\alpha\in J}$. They are related by the transformation rule induced by the transition functions $z_\beta = g_{\beta\alpha}(z_\alpha)$ and the condition $f_\alpha dz_\alpha = f_\beta dz_\beta$ yielding

$$f_\beta = f_\alpha \cdot (g'_{\beta\alpha})^{-1}, \tag{4.1}$$

where $'$ denote differentiation with respect to the coordinate z_α. A meromorphic section of $\mathcal{K}$ is given as a collection of local meromorphic functions $(f_\alpha)_{\alpha\in J}$ satisfying the transformation law (4.1).

4.1.3 Definitions and main examples

Definition 4.1. Denote by $\mathfrak{a}_{g,N}$ the *associative algebra* of meromorphic functions on M which are holomorphic outside of A.

[4] Given a chart (U, φ) such that $P \in U$, by identification of z with $\varphi^{-1}(z)$ this is also a local coordinate on M around the point P.

Definition 4.2. The *Krichever-Novikov algebra* $\mathfrak{g}_{g,N}$ is the Lie algebra of meromorphic vector fields on M which are holomorphic outside of A.

The algebra $\mathfrak{g}_{g,N}$ is equipped with the usual Lie bracket of vector fields. We will use the same symbol for the vector field and its local representation so that the Lie bracket is

$$\left[e(z)\frac{d}{dz}, f(z)\frac{d}{dz}\right] = \left(e(z)f'(z) - f(z)e'(z)\right)\frac{d}{dz}.$$

If $g = 0$, one considers the Riemann sphere $\mathbb{CP}^1$ with N punctures. The moduli space $\mathcal{M}_{0,N}$ is of dimension $N-3$, if $N \geq 3$. This means that, for $N \leq 3$, the points can be chosen in an arbitrary way providing isomorphic algebraic structures. Note also that $\mathbb{CP}^1$ can be equipped with a "quasi-global" coordinate z.

Example 4.1. In the case $g = 0$ and $N = 2$, one can take $I = \{0\}$ and $O = \{\infty\}$.

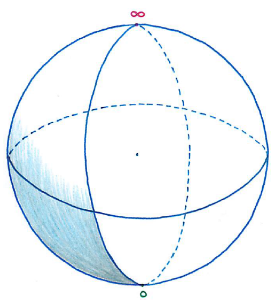

Figure 4.2: Riemann sphere with one in-point 0 and one out-point ∞.

The K-N algebra $\mathfrak{g}_{0,2}$ is nothing but the Witt algebra. It admits a basis $\{e_n = z^{n+1}\frac{d}{dz} : n \in \mathbb{Z}\}$ satisfying the following relations

$$[e_n, e_m] = (m-n)e_{n+m}.$$

The (unique) non-trivial central extension of the Witt algebra is well-known, it is called the Virasoro algebra. This algebra has the following basis

$\{e_n = z^{n+1}\frac{d}{dz} : n \in \mathbb{Z}\}$ together with the central element c, such that

$$[e_n, e_m] = (m-n)e_{n+m} + \frac{1}{12}(m^3 - m)\delta_{n,-m}c \ , \qquad [e_n, c] = 0.$$

The algebra of functions $\mathfrak{a}_{0,2}$ is the algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$.

Example 4.2. Another simple example considered in [Sch93] and further in [FS03] is the case $g = 0$ and $N = 3$. The marked points are then chosen as follows: $I = \{\alpha, -\alpha\}$ and $O = \{\infty\}$, where $\alpha \in \mathbb{C}\backslash\{0\}$.

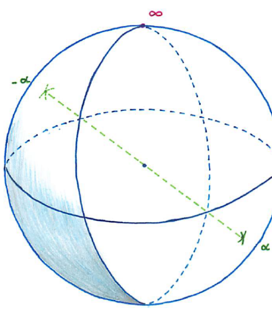

Figure 4.3: Riemann sphere with in-points α and $-\alpha$ and out-point ∞.

The Lie algebra $\mathfrak{g}_{0,3}$ is spanned by the following vector fields

$$\begin{aligned} V_{2k}(z) &= z(z-\alpha)^k(z+\alpha)^k\frac{d}{dz}, \\ V_{2k+1}(z) &= (z-\alpha)^{k+1}(z+\alpha)^{k+1}\frac{d}{dz}, \end{aligned} \tag{4.2}$$

for all $k \in \mathbb{Z}$ and satisfying

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & \text{if } n, m \text{ odd}, \\ (m-n)V_{n+m} \\ +(m-n-1)\alpha^2 V_{n+m-2}, & \text{if } n \text{ odd}, m \text{ even}, \\ (m-n)\left(V_{n+m} + \alpha^2 V_{n+m-2}\right), & \text{if } n, m \text{ even}. \end{cases} \tag{4.3}$$

The corresponding function algebra $\mathfrak{a}_{0,3}$ has the basis $\{G_n : n \in \mathbb{Z}\}$ where the functions are locally defined by

$$\begin{aligned} G_{2k}(z) &= (z-\alpha)^k(z+\alpha)^k, \\ G_{2k+1}(z) &= z(z-\alpha)^{k+1}(z+\alpha)^{k+1}, \end{aligned} \tag{4.4}$$

for all $k \in \mathbb{Z}$ and satisfying

$$G_n.G_m = \begin{cases} G_{n+m} + \alpha^2 G_{n+m-2}, & \text{if } n, m \text{ odd}, \\ G_{n+m}, & \text{otherwise}. \end{cases} \tag{4.5}$$

4.1.4 Almost-graded structure

In the classical situation of $\mathbb{CP}^1$ and the punctures $A = \{0\} \cup \{\infty\}$, the algebras $\mathfrak{g}_{0,2}$ and $\mathfrak{a}_{0,2}$ are graded algebras. The algebras $\mathfrak{g}_{0,3}$ and $\mathfrak{a}_{0,3}$ are not graded algebras, as realized by Krichever and Novikov [KN87b] and later by Schlichenmaier [Sch90a]. In the higher genus case and even in the genus zero with more than two points, an almost-grading which extend the notion of grading was introduced. Let $\mathfrak{a}$ be an algebra such that $\mathfrak{a} = \oplus_{n\in\mathbb{Z}}\mathfrak{a}_n$ is a vector space direct sum, then $\mathfrak{a}$ is an *almost-graded* algebra if

(i) $\dim \mathfrak{a}_n < \infty \qquad \forall n \in \mathbb{Z}$,

(ii) there exist constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathfrak{a}_n \cdot \mathfrak{a}_m \subset \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathfrak{a}_h \qquad \forall n, m \in \mathbb{Z}.$$

This crucial notion of almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O. The algebras $\mathfrak{g}_{g,N}$ and $\mathfrak{a}_{g,N}$ are almost-graded.

4.2 Lie superalgebras of K-N type

We recall the notion of Lie superalgebras in a general setting then we focus on the case of Krichever-Novikov type $\mathcal{L}_{g,N}$. Lie superalgebras of K-N type were studied in [LMG12a] and later in [Sch13]. The geometrical set up is refined since we consider $\mathcal{K}^\lambda$ with λ a half integer. The main Examples 4.1 and 4.2 are continued and we introduce the superalgebras $\mathcal{L}_{0,2}$ and $\mathcal{L}_{0,3}$.

4.2.1 Generalities on Lie superalgebras

A *superalgebra* $\mathcal{A}$ on $\mathbb{C}$ is a $\mathbb{Z}_2$-graded vector space $\mathcal{A}_0 \oplus \mathcal{A}_1$ with a bilinear product

$$(x,y) \in \mathcal{A} \times \mathcal{A} \mapsto x \cdot y \in \mathcal{A} \quad \text{such that} \quad \mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j},$$

where the subscripts are read modulo 2. The subspace $\mathcal{A}_0$ is the space of *even elements* and the subspace $\mathcal{A}_1$ is that of *odd elements*. A *homogeneous element* is an element that belongs either to $\mathcal{A}_0$ or to $\mathcal{A}_1$. The *degree* of a homogeneous element x is denoted by $\bar{x}$, i.e. $\bar{x} = i$ for $x \in \mathcal{L}_i$.

The superalgebra $\mathcal{A}$ is *supercommutative* if for every homogeneous elements x and y in $\mathcal{A}$ one has

$$x \cdot y = (-1)^{\bar{x}\bar{y}} y \cdot x.$$

A *Lie superalgebra* is a superalgebra, $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$, where the bilinear product called the *Lie superbracket* and denoted by $], [$, is such that

(LS1) super skewsymmetry:

$$[x,y] = -(-1)^{\bar{x}\bar{y}}[y,x]$$

(LS2) super Jacobi identity:

$$(-1)^{\bar{x}\bar{z}}\,[x,[y,z]] + (-1)^{\bar{y}\bar{x}}\,[y,[z,x]] + (-1)^{\bar{z}\bar{y}}\,[z,[x,y]] = 0$$

for all homogeneous elements x, y, z in $\mathcal{L}$.

Any Lie algebra $\mathcal{L}_0$ may be regarded as a purely even superalgebra; that is, by taking $\mathcal{L}_1$ to be trivial.

Given an associative superalgebra, we can define a Lie superalgebra $\mathcal{L}_{\mathcal{A}}$. As a super vector space, $\mathcal{L}_{\mathcal{A}}$ is the same than $\mathcal{A}$ and the Lie superbracket of two homogeneous elements of $\mathcal{L}_{\mathcal{A}}$ is defined by the *supercommutator* in $\mathcal{A}$, i.e. for every homogeneous elements x and y, one has

$$[x,y] := x \cdot y - (-1)^{\bar{x}\bar{y}} y \cdot x$$

If $V_0 \oplus V_1$ is a non-trivial supervector space over $\mathbb{C}$, then the Lie superbracket of two endomorphisms S and S' of weight $\bar{S}$ and $\bar{S}'$ respectively[5] is given by

$$[S, S'] := S \circ S' - (-1)^{\bar{S}\bar{S}'} S' \circ S.$$

An endomorphism D of a superalgebra $\mathcal{A}$ is a *derivation* if for all homogeneous elements x and y in $\mathcal{A}$, we have

$$D(x \cdot y) = D(x) \cdot y + (-1)^{\bar{D}\bar{x}} x \cdot D(y).$$

This formula then extends by linearity for arbitrary $x, y \in \mathcal{A}$. The space of all derivations of $\mathcal{A}$ is a Lie superalgebra denoted by $Der(\mathcal{A})$.

Example 4.3. The *conformal Lie superalgebra* $\mathcal{K}(1)$ is an infinite-dimensional Lie superalgebra with basis $\{e_n,\ n \in \mathbb{Z}\}$ of the even part and $\{b_i,\ i \in \mathbb{Z} + \frac{1}{2}\}$ of the odd part satisfying the relations following relations

$$[e_n, e_m] = (m - n)\, e_{n+m}\ , \qquad [e_n, b_i] = \left(i - \frac{n}{2}\right) b_{i+n}\ , \qquad [b_i, b_j] = e_{i+j}.$$

The even part of $\mathcal{K}(1)$ coincides thus with the Witt algebra $\mathfrak{g}_{0,2}$.
The set of elements $\{b_{-\frac{1}{2}}, b_{\frac{1}{2}}, e_{-1}, e_0, e_1\}$ span the classical simple Lie superalgebra osp(1|2).

4.2.2 The geometrical set-up II

To consider superspaces we need to assume one more geometric hypothesis. A *theta characteristics*[6] of $\mathcal{K}$ (the canonical line bundle of M) is a line bundle $\mathcal{S}$ such that $\mathcal{S}^{\otimes 2} = \mathcal{K}$. On a Riemann surface of genus g the number of different square roots equals 2^{2g}. Except for $g = 0$, the theta characteristics is not unique. So, let us fix one on M for the rest of the script. We will drop mentioning S, but we should keep in mind this choice. Now we can consider the bundle $\mathcal{K}^{\otimes\lambda}$ where $\lambda \in \mathbb{Z} \cup \frac{1}{2} + \mathbb{Z}$. Denote $\mathcal{F}_\lambda$ the (infinite-dimensional)

[5] In $End(V_0 \oplus V_1)$ the even elements and odd elements, are those morphisms belonging to $End(V_0) \oplus End(V_1)$ and $Hom(V_0, V_1) \oplus Hom(V_1, V_0)$, respectively.

[6] It is also called a *square root*.

vector space of global meromorphic sections of $\mathcal{K}^{\otimes\lambda}$ which are holomorphic on $M\backslash A$, also called space of tensor densities of weight λ in [LMG12a].

For the spaces $\mathcal{F}_\lambda$, Schlichenmaier introduced for $m \in \mathbb{Z}$ or $m \in \frac{1}{2} + \mathbb{Z}$ depending whether λ is integer or half integer, subspaces $\mathcal{F}_{\lambda,m}$ of dimension K. He exhibited certain elements $f^p_{\lambda,m}$, $p = 1, \ldots, K$ which constitute a basis of $\mathcal{F}_{\lambda,m}$ and it was shown that

$$\mathcal{F}_\lambda = \bigoplus_m \mathcal{F}_{\lambda,m}.$$

The almost-grading[7] is fixed by exhibiting certain basis elements in the space $\mathcal{F}_\lambda$ as homogeneous. Recall that such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O. For the zero-order at the point $P_i \in I$ of the element $f^p_{\lambda,m}$, we have that

$$\mathrm{ord}_{P_i}(f^p_{\lambda,m}) = (m + 1 - \lambda) - \delta_{i,p}.$$

4.2.3 Definitions and main examples

The space $\mathcal{F} = \oplus_\lambda \mathcal{F}_\lambda$ is a Poisson algebra[8] with the following bilinear operations (given in local coordinates)

$$\begin{array}{cccccc} \bullet & : & \mathcal{F}_\lambda \times \mathcal{F}_\mu & \longrightarrow & \mathcal{F}_{\lambda+\mu} \\ & & (e(z)dz^\lambda, f(z)dz^\mu) & \longmapsto & e(z)f(z)dz^{\lambda+\mu}, \end{array}$$

$$\begin{array}{cccccc} \{,\} & : & \mathcal{F}_\lambda \times \mathcal{F}_\mu & \longrightarrow & \mathcal{F}_{\lambda+\mu+1} \\ & & (e(z)dz^\lambda, f(z)dz^\mu) & \longmapsto & (\mu e'(z)f(z) - \lambda e(z)f'(z))\, dz^{\lambda+\mu+1}, \end{array}$$

where $dz^\lambda := (dz)^{\otimes\lambda}$. We have the Lie algebra isomorphism $\mathfrak{g}_{g,N} \simeq \mathcal{F}_{-1}$, and the natural action of the Lie algebra $\mathfrak{g}_{g,N}$ on $\mathcal{F}_{-1/2}$ is given by the above Poisson bracket.

Definition 4.3. The *Krichever-Novikov Lie superalgebra*, denoted by $\mathcal{L}_{g,N}$, is the vector space $(\mathcal{L}_{g,N})_0 \oplus (\mathcal{L}_{g,N})_1 = \mathfrak{g}_{g,N} \oplus \mathcal{F}_{-1/2}$ with the Lie superbracket

[7] The definition of almost-grading makes sense also for more general index set $\frac{1}{2}\mathbb{Z}$.

[8] A *Poisson algebra* over $\mathbb{C}$ is a triplet $(\mathcal{P}, \bullet, \{,\})$ where $(\mathcal{P}, \bullet)$ is a commutative and associative algebra, $(\mathcal{P}, \{,\})$ is a Lie algebra such that the Poisson bracket, denoted by $\{,\}$, acts as a derivations of the associative product, i.e., $\{x \bullet y, z\} = \{x, z\} \bullet y + x \bullet \{y, z\}$ for all $x, y, z \in \mathcal{P}$.

defined by

$$[e(z)(dz)^{-1}, f(z)(dz)^{-1}] := \{e(z)(dz)^{-1}, f(z)(dz)^{-1}\},$$

$$[e(z)(dz)^{-1}, \psi(z)(dz)^{-1/2}] := \{e(z)(dz)^{-1}, \psi(z)(dz)^{-1/2}\},$$

$$[\varphi(z)(dz)^{-1/2}, \psi(z)(dz)^{-1/2}] := \frac{1}{2}\varphi(z)(dz)^{-1/2} \bullet \psi(z)(dz)^{-1/2}.$$

The axioms of Lie superalgebras can be easily checked. More precisely, we can write in local coordinates

$$[e(z)(dz)^{-1}, f(z)(dz)^{-1}] = \left(-e'f + ef'\right)(z)(dz)^{-1},$$

$$[e(z)(dz)^{-1}, \psi(z)(dz)^{-1/2}] = \left(-\frac{1}{2}e'\psi + e\psi'\right)(z)(dz)^{-1/2},$$

$$[\varphi(z)(dz)^{-1/2}, \psi(z)(dz)^{-1/2}] = \frac{1}{2}\,(\varphi\psi)(z)(dz)^{-1}.$$

Note that $\mathcal{L}_{g,N}$ is an almost-graded algebra as pointed in [Kre13] and in [Sch13] by Schlichenmaier.

Example 4.4. In the case of two marked points $A = \{0\} \cup \{\infty\}$ on the Riemann sphere, we can identify $\mathcal{L}_{0,2}$ with $\mathcal{K}(1)$. We have the following identification

$$e_n = z^{n+1}(dz)^{-1} \quad \text{and} \quad b_i = \sqrt{2}\, z^{i+1/2}(dz)^{-1/2}.$$

Example 4.5. Consider the Lie superalgebra $\mathcal{L}_{0,3}$ associated with the Riemann sphere with three punctures $A = \{-\alpha, \alpha\} \cup \{\infty\}$, where $\alpha \in \mathbb{C}\backslash\{0\}$. According to [FS03], the even part of $\mathcal{L}_{0,3}$, namely $\mathfrak{g}_{0,3}$, has the basis (4.2). The odd part, $\mathcal{F}_{-1/2}$, according to [LMG12a], has the basis

$$\begin{aligned} \varphi_{2k+\frac{1}{2}}(z) &= \sqrt{2}z(z-\alpha)^k(z+\alpha)^k dz^{-1/2}, \\ \varphi_{2k-\frac{1}{2}}(z) &= \sqrt{2}(z-\alpha)^k(z+\alpha)^k dz^{-1/2}, \end{aligned} \tag{4.6}$$

for all $k \in \mathbb{Z}$. The Lie superbracket of this algebra is given, from [LMG12a], by the relations (4.3) together with

$$[V_n, \varphi_i] = \begin{cases} (i - \frac{n}{2})\varphi_{n+i}, & \text{if } n \text{ odd}, i - \frac{1}{2} \text{ odd}, \\ (i - \frac{n}{2})\varphi_{n+i} + (i - \frac{n}{2} - 1)\alpha^2 \varphi_{n+i-2}, & \text{if } n \text{ odd}, i - \frac{1}{2} \text{ even}, \\ (i - \frac{n}{2})\varphi_{n+i} + (i - \frac{n}{2} + \frac{1}{2})\alpha^2 \varphi_{n+i-2}, & \text{if } n \text{ even}, i - \frac{1}{2} \text{ odd}, \\ (i - \frac{n}{2})\varphi_{n+i} + (i - \frac{n}{2} - \frac{1}{2})\alpha^2 \varphi_{n+i-2}, & \text{if } n \text{ even}, i - \frac{1}{2} \text{ even}, \end{cases}$$

and

$$[\varphi_i, \varphi_j] = \begin{cases} V_{i+j} + \alpha^2 V_{i+j-2}, & \text{if } i - \frac{1}{2} \text{ even}, j - \frac{1}{2} \text{ even}, \\ V_{i+j}, & \text{otherwise}. \end{cases} \tag{4.7}$$

In [LMG12a], it is shown in particular that the sub-superalgebra $\mathcal{L}^-_{0,3}$, defined as $\left\langle V_n \,:\, n \leq 1\,;\ \varphi_i \,:\, i \leq \frac{1}{2} \right\rangle$, of $\mathcal{L}_{0,3}$ is isomorphic to $\mathcal{K}(1)$.

4.3 Lie Antialgebras: particular class of Jordan Superalgebras

A particular type of Jordan superalgebras, studied by Kaplansky and McCrimmon [McC94], has been rediscovered by Ovsienko in [Ovs11] under the name of "Lie antialgebras". Appearing in the context of symplectic geometry, see Subsection 4.3.3, Lie antialgebras are deeply connected to Lie superalgebras and link together commutative and Lie algebras, see Subsection 4.3.2. Taking into account the general theory of Jordan superalgebras and the specificities of Lie antialgebras, a theory of Lie antialgebra was developed in [Ovs11].

Properties of Lie antialgebras were further studied in [MG09] and [LMG12b]. In [LMG12a], Leidwanger and Morier-Genoud introduced Lie antialgebras of K-N type, denoted by $\mathcal{J}_{g,N}$. The main Examples 4.1 and 4.2 are continued and detailed in Subsection 4.3.4.

4.3.1 Definition and main examples

Definition 4.4. A *Lie antialgebra* on $\mathbb{C}$ is a $\mathbb{Z}_2$-graded supercommutative superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with a product

$$x \centerdot y = (-1)^{\bar{x}\bar{y}} y \centerdot x,$$

for all homogeneous elements $x, y \in \mathcal{A}$, satisfying the following conditions.

(i) The subalgebra $\mathcal{A}_0$ is *associative*.

(ii) For every $a \in \mathcal{A}_1$, the operator of right multiplication by a is an (odd) derivation of $\mathcal{A}$, i.e.

$$(x \centerdot y) \centerdot a = (x \centerdot a) \centerdot y + (-1)^{\bar{x}}\, x \centerdot (y \centerdot a)\,, \tag{4.8}$$

for all homogeneous elements $x, y \in \mathcal{A}$.

(iii) $\mathcal{A}_0$ acts commutatively on $\mathcal{A}_1$, i.e.

$$\alpha \centerdot (\beta \centerdot a) = \beta \centerdot (\alpha \centerdot a),$$

for all elements $\alpha, \beta \in \mathcal{A}_0$ and $a \in \mathcal{A}_1$.

In the following, we will use the notation $x, y, z, \ldots$ to denote any homogeneous element in $\mathcal{A}$, the Greek letters $\alpha, \beta, \varepsilon, \ldots$ to denote any even element in $\mathcal{A}_0$ and the letters $a, b, c, \ldots$ to denote any odd element in $\mathcal{A}_1$.

Remark 4.1. Let us give some remarks on the axioms.

- Note that, in the case where $\mathcal{A}$ is generated by its odd part[9] $\mathcal{A}_1$, the first axiom of associativity is a corollary of (4.8), cf. [Ovs11], [LMG12a].

- Thanks to the axiom *(ii)*, the axiom *(iii)* is equivalent to

$$\alpha \centerdot (\beta \centerdot a) = \frac{1}{2}(\alpha \centerdot \beta) \centerdot a$$

[9] We say that the Lie antialgebra is *ample* if every even element is a linear combination of products of odd elements.

for all elements $\alpha, \beta \in \mathcal{A}_0$ and $a \in \mathcal{A}_1$. This half-action is essential to the theory even if at first sight it seems unnatural.

- The identities of the axioms of Lie antialgebras are cubic.

- The axioms (i) and (ii) imply that $\mathcal{A}$ is a Jordan superalgebra. The identities of Jordan superalgebra are quartic and one has to find the right sequence of transformations. It is not as simple as we could expected and we can find more details in [McC94]. Recal that a *Jordan superalgebra* is a supercommutative superalgebra $\mathcal{J} = \mathcal{J}_0 \oplus \mathcal{J}_1$ satisfying the Jordan identity, i.e. for all $x, y, z \in \mathcal{J}$ homogeneous, we have

$$(x.y).(z.t) + (-1)^{\bar{z}\bar{y}}(x.z).(y.t) + (-1)^{(\bar{z}+\bar{y})\bar{t}}(x.t).(z.y)$$
$$= ((x.y).z).t + (-1)^{(\bar{y}+\bar{z})\bar{t}+\bar{y}\bar{z}}((x.t).z).y + (-1)^{\bar{x}(\bar{y}+\bar{z}+\bar{t})+\bar{z}\bar{t}}((y.t).z).x.$$

Example 4.6. The first example of finite-dimensional Lie antialgebra is the famous tiny Kaplansky superalgebra, denoted by $\mathcal{K}_3$. It was first studied by McCrimmon in [McC94] and after by Morier-Genoud in [MG09] under the name of asl_2. The basis is $\{\varepsilon; a, b\}$ where ε is even and a, b are odds. It is defined by the following relations

$$\varepsilon \centerdot \varepsilon = \varepsilon\,, \qquad \varepsilon \centerdot a = \frac{1}{2}a\,, \qquad \varepsilon \centerdot b = \frac{1}{2}b\,, \qquad a \centerdot b = \frac{1}{2}\varepsilon.$$

The algebra $\mathcal{K}_3$ is an example of exceptional simple Jordan superalgebra.

Example 4.7. The second important example is an infinite-dimensional algebra, denoted by $\mathcal{AK}(1)$. Its geometric origins are related to the contact structure on the supercircle $S^{1|1}$. The basis of $\mathcal{AK}(1)$ is given by $\{\varepsilon_n : n \in \mathbb{Z}\} \oplus \{a_i : i \in \mathbb{Z} + \frac{1}{2}\}$ and the relations are

$$\varepsilon_n \centerdot \varepsilon_m = \varepsilon_{n+m}\,, \qquad \varepsilon_n \centerdot a_i = \frac{1}{2}a_{i+n}\,, \qquad a_i \centerdot a_j = \frac{1}{2}(j-i)\varepsilon_{i+j}.$$

Note that $\langle \varepsilon_0, a_{-1/2}, a_{1/2} \rangle$ as a subalgebra of $\mathcal{AK}(1)$ isomorphic to $\mathcal{K}_3$.

4.3.2 Relation to Lie superalgebras

A natural way to link Lie antialgebras and Lie superalgebras is to consider the Lie superalgebra of derivations $Der(\mathcal{A})$. In particular, one has, see [Ovs11] $Der(\mathcal{K}_3) \simeq \mathrm{osp}(1|2)$ and $Der(\mathcal{AK}(1)) \simeq \mathcal{K}(1)$.

Another way to associate a Lie superalgebra $\mathcal{G}_\mathcal{A}$ to an arbitrary Lie antialgebra $\mathcal{A}$, called the *adjoint Lie superalgebra*, was elaborated in [Ovs11]. Consider the $\mathbb{Z}_2$-graded space $\mathcal{G}_\mathcal{A} = \mathcal{G}_0 \oplus \mathcal{G}_1$ where, $\mathcal{G}_1 := \mathcal{A}_1$ and $\mathcal{G}_0 := (\mathcal{A}_1 \otimes \mathcal{A}_1)/S$ and where S is the ideal generated by

$$\{\, a \otimes b - b \otimes a,\; a \centerdot \alpha \otimes b - a \otimes b \centerdot \alpha \;\mid\; a, b \in \mathcal{A}_1, \alpha \in \mathcal{A}_0 \,\}.$$

If we denote by $a \odot b$ the image of $a \otimes b$ in $\mathcal{G}_0$, we have

$$\begin{cases} a \odot b & = & b \odot a, \\ a \centerdot \alpha \odot b & = & a \odot b \centerdot \alpha. \end{cases}$$

One can write the Lie superbracket, announced in [Ovs11] and then proved in [LMG12b], as follows

$$\begin{aligned} [a, b] &:= a \odot b, \\ [a \odot b, c] &:= a \centerdot (b \centerdot c) + b \centerdot (a \centerdot c), \\ [a \odot b, c \odot d] &:= 2a \centerdot (b \centerdot c) \odot d + 2b \centerdot (a \centerdot d) \odot c. \end{aligned}$$

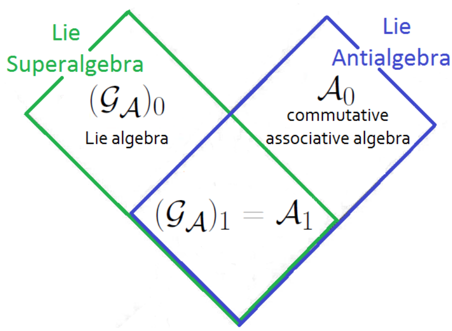

Figure 4.4: Link between Lie antialgebra and its adjoint Lie superalgebra.

There is a natural action of $\mathcal{G}_{\mathcal{A}}$ on the corresponding Lie antialgebra $\mathcal{A}$, so that there is a Lie algebra homomorphism

$$R : \mathcal{G}_{\mathcal{A}} \to Der(\mathcal{A}).$$

Indeed, the action of the odd part $\mathcal{G}_1$ is given by the right multiplication $R_a(x) = x \centerdot a$ for all $a \in \mathcal{A}_1$ and for all $x \in \mathcal{A}$. The action of the odd part generates the action of $\mathcal{G}_0$ that is defined by $R_{a\odot b}(x) = (x \centerdot a) \centerdot b + (x \centerdot b) \centerdot a$ for all $a, b \in \mathcal{A}_1$ and for all $x \in \mathcal{A}$.

Note that, one has $\mathcal{G}_{\mathcal{K}_3} \simeq \mathrm{osp}(1|2)$ and $\mathcal{G}_{\mathcal{AK}(1)} \simeq \mathcal{K}(1)$. In general, the adjoint Lie superalgebra is not isomorphic to the Lie superalgebra of derivations.

4.3.3 Origin of Lie antialgebras

Lie antialgebras took hold in symplectic geometry. The structure of Lie antialgebra, introduced in [Ovs11], is due to the existence of an odd bivector osp(1|2)-invariant. Let us explain briefly the origin of Lie antialgebras, see [Ovs11] or [MG14] for more details.

Consider the supermanifold $\mathbb{C}^{2|1}$ where p, q denote the even coordinates and τ denotes the odd coordinate[10] equipped with the standard symplectic form

$$w = dp \wedge dq + \frac{1}{2} d\tau \wedge d\tau.$$

The Poisson bivector $\mathcal{P}$ on $\mathbb{C}^{2|1}$, being the inverse of the symplectic form w, is given by

$$\mathcal{P} = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \frac{1}{2} \frac{\partial}{\partial \tau} \wedge \frac{\partial}{\partial \tau}. \tag{4.9}$$

The bivector (4.9) is the unique (up to a multiplicative constant) even bivector invariant with respect to the action of osp(1|2). However, there exists another odd osp(1|2)-invariant bivector on $\mathbb{C}^{2|1}$ given by

$$\Lambda = \frac{\partial}{\partial \tau} \wedge \mathcal{E} + \tau \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}, \tag{4.10}$$

[10] The odd coordinate τ is also called the Grassmann variable and satisfies $\tau^2 = 0$.

where $\mathcal{E} = p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} + \tau\frac{\partial}{\partial \tau}$ is the Euler field. It seems that this bivector was first found by V. Ovsienko in [Ovs11].

Theorem 4.1. *([Ovs11]) The bivectors Λ and $\mathcal{P}$ and their linear combinations are the only bivectors invariant with respect to the action of* osp(1|2).

Any bivector defines an algebraic structure on the space of functions $C^\infty(\mathbb{C}^{2|1})$. Consider F an arbitrary function on $\mathbb{C}^{2|1}$, that is $F = F_0(p,q) + \tau F_1(p,q)$ where F_0, F_1 are smooth functions on $\mathbb{C}^2$ and $\overline{F_0} = 0$, $\overline{\tau F_1} = 1$. We construct bilinear operations associated with the bivectors (4.9) and (4.10) given in local coordinates by

$$\{F, G\} := \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} + \frac{1}{2}\frac{\partial F}{\partial \tau}\frac{\partial G}{\partial \tau} \tag{4.11}$$

and

$$]F, G[:= -\frac{(-1)^{\overline{F}}}{2}\left(\frac{\partial F}{\partial \tau}\mathcal{E}(G) - (-1)^{\overline{F}}\mathcal{E}(F)\frac{\partial G}{\partial \tau} + \tau\left(\frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial p}\right)\right) \tag{4.12}$$

respectively. The product (4.11) is the standard Poisson bracket on $\mathbb{C}^{2|1}$ while the other product (4.12) was first called the *ghost Poisson bracket*, see [GO08].

It is known that the space of quadratic functions on $\mathbb{C}^{2|1}$, also denoted by $< p^2, pq, q^2, p\tau, q\tau >$, equipped with the bracket (4.11) is a Lie superalgebra isomorphic to osp(1|2). We have the following result.

Proposition 4.1. *([Ovs11]) The space of linear functions on $\mathbb{C}^{2|1}$ equipped with the bracket (4.12) is a Lie antialgebra isomorphic to $\mathcal{K}_3$.*

Note that we recover Lie antialgebra structures using the parity inversion. In Proposition 4.1, we use the following parity inverting identification

$$\{\varepsilon; a, b\} \leftrightarrow \{\tau; p, q\}.$$

It is known that the space of homogeneous functions F of degree two according to the Euler field, i.e. $\mathcal{E}(F) = 2F$ is stable under $\{,\}$. Restricting this subspace to rational functions with poles at $p = 0$ and $q = 0$, also denoted by $< p^2\left(\frac{q}{p}\right)^{n+1}, \tau p\left(\frac{q}{p}\right)^{i+\frac{1}{2}} >$, we obtain a Lie superalgebra isomorphic to the conformal Lie superalgebra $\mathcal{K}(1)$.

Proposition 4.2. *([Ovs11]) The space of homogeneous functions of degree one on $\mathbb{C}^{2|1}$ according to the Euler field with poles only at $p = 0$ and $q = 0$, equipped with the bracket (4.12), is a Lie antialgebra isomorphic to $\mathcal{AK}(1)$.*

In proposition 4.2, we use the following parity inverting identification

$$\{\varepsilon_n; a_i\} \leftrightarrow \{\tau \left(\frac{q}{p}\right)^n ; p \left(\frac{q}{p}\right)^{i+\frac{1}{2}}\} \qquad \text{where} \qquad n \in \mathbb{Z}, i \in \mathbb{Z} + \frac{1}{2}.$$

The algebras $\mathcal{K}_3$ and $\mathcal{AK}(1)$ are Lie antialgebras, but $C^\infty(\mathbb{C}^{2|1})$ equipped with the product $], [$ is not a Lie antialgebra.

4.3.4 Jordan superalgebras of K-N type

Lie antialgebras are particular types of Jordan superalgebras. A new series of Lie antialgebras extended $\mathcal{AK}(1)$ was found by Leidwanger and Morier-Genoud; see [LMG12a]. These algebras are related to Riemann surfaces with marked points and are called *Krichever-Novikov Jordan superalgebras*, $\mathcal{J}_{g,N}$. Remind that a splitting and a theta characteristics are still fixed, the even part of $\mathcal{J}_{g,N}$ is the space of meromorphic functions, $\mathfrak{a}_{g,N} \simeq \mathcal{F}_0$, while the odd part is the space of $-1/2$-densities.

Definition 4.5. The Lie antialgebra $\mathcal{J}_{g,N}$ is the vector superspace $\mathfrak{a}_{g,N} \oplus \mathcal{F}_{-1/2}$ equipped with the product

$$\begin{aligned} e(z) \centerdot f(z) &:= e(z) \bullet f(z), \\ e(z) \centerdot \psi(z)(dz)^{-1/2} &:= \frac{1}{2} e(z) \bullet \psi(z)(dz)^{-1/2}, \\ \varphi(z)(dz)^{-1/2} \centerdot \psi(z)(dz)^{-1/2} &:= \{\varphi(z)(dz)^{-1/2}, \psi(z)(dz)^{-1/2}\}. \end{aligned}$$

It is shown in [LMG12a], that the adjoint Lie superalgebra of $\mathcal{J}_{g,N}$ coincides with $\mathcal{L}_{g,N}$. Note that Jordan superalgebras $\mathcal{J}_{g,N}$ becomes an almost-graded algebra depending on a splitting $A = I \cup O$.

Example 4.8. In the case of two marked points $A = \{0\} \cup \{\infty\}$ on the Riemann sphere, the algebra $\mathcal{J}_{0,2}$ can be identified with $\mathcal{AK}(1)$. This algebra is also known as the *full derivative superalgebra.*

Example 4.9. A beautiful example in the case of three punctures on the Riemann sphere is considered in [LMG12a]. One can fix $A = \{-\alpha, \alpha\} \cup \{\infty\}$, where $\alpha \in \mathbb{C}\backslash\{0\}$. The generators on the even part of the Jordan superalgebra $\mathcal{J}_{0,3}$ are the same than the generators on $\mathfrak{a}_{0,3}$, see (4.4). The odd part, has the basis

$$\begin{aligned}\varphi_{2k+\frac{1}{2}}(z) &= \sqrt{2}z(z-\alpha)^k(z+\alpha)^k\, dz^{-1/2},\\ \varphi_{2k-\frac{1}{2}}(z) &= \sqrt{2}(z-\alpha)^k(z+\alpha)^k\, dz^{-1/2},\end{aligned}$$

where $k \in \mathbb{Z}$. Remark that the generators of the odd parts of $\mathcal{L}_{g,N}$ and $\mathcal{J}_{g,N}$ are the same. The product is given by the relations (4.5) and (4.7) together with

$$G_n \centerdot \varphi_i = \begin{cases} \dfrac{1}{2}\varphi_{n+i}, & \text{if } n \text{ even or } i - \frac{1}{2} \text{ odd,}\\ \dfrac{1}{2}(\varphi_{n+i} + \alpha^2\varphi_{n+i-2}), & n \text{ odd and } i - \frac{1}{2} \text{ even.}\end{cases}$$

The sub-superalgebra $\mathcal{J}_{0,3}^{-} := \left\langle G_n : n \leq 0\,, \varphi_i : i \leq \frac{1}{2}\right\rangle$ is isomorphic to $\mathcal{AK}(1)$. More precisely, the embedding $\iota : \mathcal{AK}(1) \hookrightarrow \mathcal{J}_{0,3}$ is defined on the generators as follows

$\iota(\varepsilon_{-1}) = G_0 + 2\alpha G_{-1} + 2\alpha^2 G_{-2}$, $\quad$ $\iota(\varepsilon_1) = G_0 - 2\alpha G_{-1} + 2\alpha^2 G_{-2}$,

$\iota(a_{-\frac{1}{2}}) = \frac{1}{2\sqrt{\alpha}}(\varphi_{1/2} + \alpha\varphi_{-1/2})$, $\quad$ $\iota(a_{\frac{1}{2}}) = \frac{1}{2\sqrt{\alpha}}(\varphi_{1/2} - \alpha\varphi_{-1/2})$,

$\iota(\varepsilon_0) = G_0$,

see [LMG12a] for the details.

Chapter 5

One and Two Cocycles on Algebras of K-N type

The existence of (local) 2-cocycles and central extensions for Lie algebras $\mathfrak{g}_{g,N}$ was showed by Sclichenmaier in [Sch03], extending the explicit formula of 2-cocycles due to Krichever and Novikov. In **Section** 5.1, we consider 2-cocycles on these Lie algebras $\mathfrak{g}_{g,N}$ and recall some tools that we will use in the computation of cocycles in the case of the Riemann sphere.

In **Section** 5.2, we first give a local 2-cocycle on Lie superalgebras $\mathcal{L}_{g,N}$ (Theorem 5.2). This result is published in [Kre13]. We construct a 1-cocycle on $\mathcal{L}_{g,N}$ with value in the dual space related to the 2-cocycle (Proposition 5.2). We exhibit explicit formulas in the particular case of $\mathcal{L}_{0,3}$ for the unique (up to isomorphism) 2-cocycle and the corresponding 1-cocycle.

In **Section** 5.3, we give a local 1-cocycle on $\mathcal{J}_{g,N}$ (Theorem 5.3) and construct the unique (up to isomorphism) 1-cocycle on $\mathcal{J}_{0,3}$ that vanishes on $\mathcal{K}_3$.

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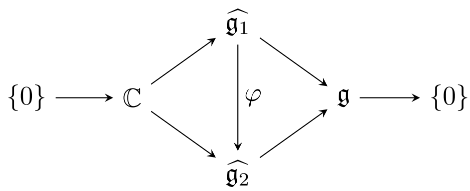
5.1 Construction of a 2-cocycle on Lie algebras of K-N type

Central extensions of dimension one are classified up to equivalence by 2-cocycles (with coefficient $\mathbb{C}$) up to coboundaries. We exhibit a 2-cocycle on $\mathfrak{g}_{g,2}$ due to Krichever and Novikov [KN87b], [KN87a] and further generalized on $\mathfrak{g}_{g,N}$ by Schlichenmaier [Sch03]. This well defined cocycle, that generalize the Gelfind-Fuchs cocycle, has properties of locality and uniqueness.

A (one dimension) *central extension* of a Lie algebra $\mathfrak{g}$ is the middle term of a short exact sequence of Lie algebras

$$\{0\} \longrightarrow \mathbb{C} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow \{0\}$$

such that $\mathbb{C}$ is central in $\widehat{\mathfrak{g}}$ i.e. $\mathbb{C} \subset Z(\widehat{\mathfrak{g}}) = \{x \in \widehat{\mathfrak{g}} : [x, \widehat{\mathfrak{g}}] = 0\}$. Two central extensions $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ are *equivalent* if there is a Lie isomorphism $\varphi : \widehat{\mathfrak{g}}_1 \longrightarrow \widehat{\mathfrak{g}}_2$ such that the diagram



is commutative.

The space of 2-*cochain* on $\mathfrak{g}$ with value in $\mathbb{C}$, denoted by $C^2(\mathfrak{g})$, is the space of functions $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ that are bilinear and skewsymmetric. A 2-cochain is a 2-*cocycle* if it satisfies the following equation

$$\phi(x, [y, z]) + \phi(y, [z, x]) + \phi(z, [x, y]) = 0.$$

The space of 2-cocycles is denoted by $Z^2(\mathfrak{g})$. A 2-cocycle ϕ is a 2-*coboundary* if there exists a linear function f such that $\phi(x, y) = f([x, y])$. The space of 2-coboundaries is denoted by $B^2(\mathfrak{g})$. We define, the second Lie algebra cohomology $H^2(\mathfrak{g}) := Z^2(\mathfrak{g})/B^2(\mathfrak{g})$ of $\mathfrak{g}$ with value in the trivial module $\mathbb{C}$.

It is known that there is a one to one correspondence between the equivalent classes of central extensions of a Lie algebra $\mathfrak{g}$ by $\mathbb{C}$ and the elements of $H^2(\mathfrak{g})$. Given a 2-cocycle ϕ, we can construct $\widehat{\mathfrak{g}}_\phi := \mathfrak{g} \oplus \mathbb{C}$ as vector space such that

$$[(x,z),(y,z')] := ([x,y],\phi(x,y)).$$

Then $\widehat{\mathfrak{g}}_\phi$ is a central extension. Conversely, $\widehat{\mathfrak{g}}$ be such an extension, $\pi : \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g}$ the canonical projection and $\sigma : \mathfrak{g} \longrightarrow \widehat{\mathfrak{g}}$ a section of $\widehat{\mathfrak{g}}$, i.e. a linear map satisfying $\pi \circ \sigma = Id_{\mathfrak{g}}$. It follows that there exists a map ϕ from $\mathfrak{g} \times \mathfrak{g}$ to $\mathbb{C} \subset \widehat{\mathfrak{g}}$ such that

$$\sigma([x,y]) = [\sigma(x),\sigma(y)] + \phi(x,y),$$

where ϕ measure the extent in which the section σ fails to be a homomorphism. One deduces that ϕ is a 2-cocycle.

Given a Riemann surface and $(U_\alpha, z_\alpha)_{\alpha\in J}$ a covering by holomorphic coordinates with transition functions $z_\beta = g_{\beta\alpha}(z_\alpha)$, a *projective connection* is a system of functions $R = (R_\alpha(z_\alpha))_{\alpha\in J}$ transforming as

$$R_\beta(z_\beta).\left(g'_{\beta\alpha}\right)^2 = R_\alpha(z_\alpha) + S(g_{\beta\alpha}), \qquad \text{where} \quad S(g) = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2$$

is the *Schwarzian derivative* (see [OT05]) and where $'$ denotes differentiation with respect to the coordinate z_α. It is a classical result that every Riemann surfaces admits a holomorphic projective connection; see [Gun66] or [HS66] (p. 202).

For every smooth differentiable curve $\mathcal{C}$, with respect to a chosen projective connection R, there is a 2-cocycle on $\mathfrak{g}_{g,N}$ defined by

$$\gamma_{\mathcal{C},R}\left(e(z)\frac{d}{dz}, f(z)\frac{d}{dz}\right) = \frac{1}{2i\pi}\int_{\mathcal{C}}\left(\frac{1}{2}(e'''f - ef''') - R(e'f - ef')\right)dz. \quad (5.1)$$

Thanks to the projective connection the integral is independent of the chosen coordinate. Another choice of a projective connection leads to a cohomologous cocycle. Note that, in the case $g = 0$, one can take $R \equiv 0$. We call a *separating cycle* $\mathcal{C}_S$ a smooth differentiable closed curve that separates the points in I from the points in O. The 2-cocycle (5.1) can be understood as a generalization of the famous Gelfand-Fuchs cocycle; see [Fuk86]. Unlike the classical situation (on the Riemann sphere with two points), for the higher genus and/or multi-point situation there are many different closed smooth curves leading to many non-equivalent central extensions defined by integration.

Given a splitting $A = I \cup O$, recall that we have an almost-grading on $\mathfrak{a}_{g,N}$ and on $\mathfrak{g}_{g,N}$, as well as on the modules of tensor densities $\mathcal{F}_\lambda$, $\lambda \in \mathbb{Z}$, see [Sch03] pp. 58–61 for more details. Krichever and Novikov introduced the notions of *local* and *bounded* cocycles in [KN87b] in the two point case, then widely used by Schlichemnaier in the general case. The definitions are the following.

Definition 5.1. Let $\mathfrak{g} = \oplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ be an almost-graded Lie algebra. A cocycle γ for $\mathfrak{g}$ is called

- *local* (or *almost-graded*) if there exist $M_1, M_2 \in \mathbb{Z}$ such that

$$c\,((\mathcal{L}_{g,N})_n, (\mathcal{L}_{g,N})_m) \neq 0 \quad \Longrightarrow \quad M_1 \leq n+m \leq M_2,$$

- *bounded* (from above) if there exists $M \in \mathbb{Z}$ such that

$$c\,((\mathcal{L}_{g,N})_n, (\mathcal{L}_{g,N})_m) \neq 0 \quad \Longrightarrow \quad n+m \leq M.$$

It is important to note that the locality is defined in term of the grading, and the grading itself depends on the splitting $A = I \cup 0$. If the cocycle is local, then the almost grading of $\mathfrak{g}$ can be extended to $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$. We call such an extension an *almost-graded extension.* Note that local cocycles are globally defined in contrast to their names.

Results about the cocycle (5.1) are the following.

Theorem 5.1. *([Sch03]) Let $\mathfrak{g}_{g,N}$ be the Krichever-Novikov Lie algebra.*

i) The space of bounded cohomology classes is K dimensional ($K = \#I$). A basis is given by setting the integration path in (5.1) to $\mathcal{C}_i$, the little (deformed) circles around the points $P_i \in I$, $i = 1, \dots, K$.

ii) The space of local cohomology classes is one-dimensional. A generator is given by integrating (5.1) over a separating cycle $\mathcal{C}_S$.

iii) Up to equivalence and rescaling there is only one-dimensional central extension of the vector field algebra $\mathfrak{g}_{g,N}$ which allows an extension of the almost-grading.

In the following, we restrict our study to local cocycles. If the cocycle is local (i.e., preserves the almost-grading) with respect to the splitting, then $\mathcal{C}_S$ can be taken as a sum of (small) circles around the points in I: $\mathcal{C}_S = \sum_i^K \mathcal{C}_i$. The integral in (5.1) can be written in the complex analytic setting in terms of the residues. The Riemann sphere (g=0) can be viewed as the structure of the extended complex plane $\widehat{\mathbb{C}}$; see [Mar09]. In the next section, we calculate the residue at ∞ and considering the function $f_{1/z} : z \mapsto f(\frac{1}{z})$, one has

$$Res_{\infty}(f) = -Res_0 \left(\frac{f_{1/z}}{z^2} \right),$$

and moreover, if $z_0 \in \mathbb{C}$ is a pole of f, of order $p \in \mathbb{N}\backslash\{0\}$, then

$$Res_{z_0} f = \frac{1}{(p-1)!} \lim_{z \to z_0} D^{p-1} \left((z - z_0)^p f(z) \right).$$

5.2 Lie superalgebras of K-N type and their central extensions

We show the existence of a local non-trivial 2-cocycle on $\mathcal{L}_{g,N}$ satisfying similar properties to those of the cocycle (5.1). We consider, in particular, the case $g = 0$ and $N = 3$, namely, the Lie superalgebra $\mathcal{L}_{0,3}$ and compute the corresponding 2-cocycle explicitly. A 2-cocycle induces a 1-cocycle with values in the dual space. Such a 1-cocycle is constructed on $\mathcal{L}_{g,N}$ and explicitly computed on $\mathcal{L}_{0,3}$.

5.2.1 A non trivial 2-cocycle on $\mathcal{L}_{g,N}$

We show that every Lie superalgebra $\mathcal{L}_{g,N}$ has a non-trivial central extension according to the splitting (recall also that a theta characteristics is fixed).

To this end, we construct a non-trivial 2-cocycle quite similar to (5.1). Note that this problem has already been studied by Bryant in [Bry90] for the case $N = 2$ with an arbitrary genus g. The result in the general case $\mathcal{L}_{g,N}$ for a separate cycle $\mathcal{C}_S$ is given in the paper [Kre13]. Independently of this work, Schlichenmaier showed identical results in the case of a general curve $\mathcal{C}$ together with uniqueness in the case of a separate cycle $\mathcal{C}_S$, see [Sch13] or [Sch14].

Recall that a 2-*cocycle* on a Lie superalgebra $\mathcal{L}$ is an even bilinear function $c : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{C}$ satisfying the following conditions

(C1) super skewsymmetry: $c(u, v) = -(-1)^{\bar{u}\bar{v}} c(v, u)$

(C2) super Jacobi identity: $c(u, [v, w]) = c([u, v], w) + (-1)^{\bar{u}\bar{v}} c(v, [u, w])$

for every homogeneous elements $u, v, w \in \mathcal{L}$. As in the usual Lie case, a 2-cocycle defines a central extension of $\mathcal{L}$. A 2-cocycle is called *trivial*, or a *coboundary* if it is of the form $c(u, v) = f([u, v])$, where f is a linear function on $\mathcal{L}$. Otherwise, c is called *non-trivial*. The space of all 2-cocycles is denoted by $Z^2(\mathcal{L})$ and the space of 2-coboundaries by $B^2(\mathcal{L})$, the quotient-space $H^2(\mathcal{L}) = Z^2(\mathcal{L})/B^2(\mathcal{L})$ is called the second cohomology space of $\mathcal{L}$. This space classifies non-trivial central extensions of $\mathcal{L}$.

The first result of this chapter, from [Kre13] is the following.

Theorem 5.2. *(i) The even bilinear map $c : \mathcal{L}_{g,N} \times \mathcal{L}_{g,N} \to \mathbb{C}$ given by*

$$\begin{aligned} c\left(e\tfrac{d}{dz}, f\tfrac{d}{dz}\right) &= \frac{-1}{2i\pi} \int_{\mathcal{C}_S} \frac{1}{2}(e''' f - e f''') - R(e'f - ef')dz, \\ c\left(\varphi dz^{-\frac{1}{2}}, \psi dz^{-\frac{1}{2}}\right) &= \frac{1}{2i\pi} \int_{\mathcal{C}_S} \frac{1}{2}\left(\varphi'' \psi + \varphi \psi''\right) - \frac{1}{2} R \varphi \psi dz, \\ c\left(e\tfrac{d}{dz}, \psi dz^{-\frac{1}{2}}\right) &= 0, \end{aligned} \tag{5.2}$$

where $\mathcal{C}_S$ is a separating cycle, is a local non-trivial 2-cocycle.

(ii) The expressions in (5.2) does not depend on the choice of the projective connection.

Proof. Part (i). To show that the above integral is well defined, one notices that, after a coordinate change $z_\beta = g_{\beta\alpha}(z_\alpha)$, the expressions in both parts of (5.2) are transformed as 1-forms. Furthermore, the expression (5.2) is globally defined. The cocycle condition is then straightforward. Since c is cohomologically non-trivial on the even part (see [FS03], p933) it is also the case on $\mathcal{L}_{g,N}$.

The locality of the cocycle on the even part was proved in [Sch03]. To show it on the odd part, the main idea is to evaluate the cocycle on the elements of the basis of the form $\varphi^p_{-1/2,m}$ and $\psi^p_{-1/2,m}$, where $p = 1, \dots K$ and $m \in \frac{1}{2} + \mathbb{Z}$ then take the residues either at the points $P_i \in I$ or at the points $Q_j \in O$.

Part (ii). On the even part, the result is due to Schlichenmaier; see [Sch03], p64. Let R' be a different projective connection, then $R - R'$ is a well-defined quadratic differential. The 2-cocycle $c - c'$ depends only on the Lie bracket of the elements, on the odd part, we have

$$
\begin{aligned}
c_R(\varphi,\psi) - c'_{R'}(\varphi,\psi) &= \frac{1}{2i\pi}\int_{\mathcal{C}_S} -\frac{1}{2}(R-R')\varphi\psi dz \\
&= \frac{1}{2i\pi}\int_{\mathcal{C}_S} \{(R'-R)(dz)^2 \bullet [\varphi,\psi](dz)^{-1}\}
\end{aligned}
$$

and therefore the above expression is a coboundary. □

Remark that a different splitting yields a different almost-grading and hence also a different notion of locality and a different cohomology class.

Aside form the work in [Kre13], given a splitting and a corresponding separating curve $\mathcal{C}_S$, it has been proved in [Sch13] uniqueness of the local cohomology class up to multiplication with a scalar. A representing element for the cohomology class is given by formula (5.2). In [Sch13], Schlichenmaier considered a more general case of the integral on any closed differentiable curve $\mathcal{C}$ not meeting the points in A.

Note that the central element is an even element (c is of even parity). In [Sch13], Schlichenmaier considered central odd extension. He showed that all bounded from above cocycles for odd central extensions of the Lie superalgebra will split. In other words, There are no non-trivial central extensions of the Lie superalgebras $\mathcal{L}_{g,N}$ with odd central element coming from a bounded cocycle. We only consider even cocycles as we want to extend central extensions of vector field algebras to superalgebras.

5.2.2 The case of genus zero

Let us now assume that $g = 0$ and consider the Lie superalgebra $\mathcal{L}_{0,N}$. Choose the projective connection $R \equiv 0$ (in the standard flat coordinate z) adapted to the standard projective structure on $\mathbb{CP}^1$.

An important property of $\mathcal{L}_{0,N}$ is that it contains a subalgebra isomorphic to osp(1|2) that consists in holomorphic vector fields and $-1/2$-densities. The Lie superalgebra $\mathcal{L}_{0,N}$ also contains many copies of the conformal Lie superalgebra $\mathcal{K}(1)$ consisting in densities holomorphic outside two points of the set A.

In the case $N = 2$, there is exactly one splitting on $\mathcal{L}_{0,2}$, hence one almost-grading. So that, we only have one local cocyle, up to a coboundary (and one equivalence class of almost-graded central extension). The case where $N = 3$ is also special since the role of the points in A can be switched by a PSL(2)-action. If we fix one splitting then by a biholomorphic mapping of $\mathbb{CP}^1$ any other splitting can be mapped to the one $A = \{-\alpha, \alpha\} \cup \{\infty\}$. Hence, up to isomorphism (but not equivalence) there is only one class of almost-graded central extension. Moreover, we are interested in the cocycle that vanishes on the Lie subalgebra osp(1|2), let us now compute the explicit formula for the 2-cocycle (5.2) with $R \equiv 0$ on $\mathcal{L}_{0,3}$.

Proposition 5.1. *Up to isomorphism, the* 2*-cocycle on the Lie superalgebra* $\mathcal{L}_{0,3}$ *vanishing on* osp(1|2) *is given by*

$$\begin{aligned}
c\left(\varphi_{2k+\frac{1}{2}}, \varphi_{2l-\frac{1}{2}}\right) &= 4k(2k+1)\delta_{k+l,0} + 8\alpha^2 k(k-1)\delta_{k+l,1} \\
c\left(V_{2k}, V_{2l}\right) &= -2k(4k^2-1)\delta_{k+l,0} \\
&\quad -8\alpha^2 k(k-1)(2k-1)\delta_{k+l,1} \\
&\quad -8\alpha^4 k(k-1)(k-2)\delta_{k+l,2} \\
c\left(V_{2k+1}, V_{2l+1}\right) &= -8\alpha^2 (k+1)k(k-1)\delta_{k+l,0} \\
&\quad -4k(k+1)(2k+1)\delta_{k+l,-1}
\end{aligned}$$

$$c\left(\varphi_{2k+\frac{1}{2}}, \varphi_{2l+\frac{1}{2}}\right) = c\left(\varphi_{2k-\frac{1}{2}}, \varphi_{2l-\frac{1}{2}}\right) = c\left(V_{2k}, V_{2l+1}\right) = 0,$$

for all $k, l \in \mathbb{Z}$.

Proof. Let us give the details of the calculation for the even generators, the others cases are similar.

$$\begin{aligned}
& c\left(V_{2k+1}, V_{2l+1}\right) \\
= & -\frac{1}{2i\pi}\int_{\mathcal{C}_\alpha\cup\mathcal{C}_{-\alpha}} ((z^2-\alpha^2)^{k+1})'''(z^2-\alpha^2)^{l+1} \\
& \qquad -((z^2-\alpha^2)^{l+1})'''(z^2-\alpha^2)^{k+1}dz \\
= & \frac{1}{2i\pi}\int_{\mathcal{C}_\infty} 6z((k+1)k-(l+1)l)(z^2-\alpha^2)^{k+l} \\
& \qquad +4z^3((k+1)k(k-1)-(l+1)l(l-1))(z^2-\alpha^2)^{k+l-1}dz \\
= & -6((k+1)k-(l+1)l)Res_0\left(\frac{(1-z^2\alpha^2)^{k+l}}{z^{2k+2l+3}}\right) \\
& -4((k+1)k(k-1)-(l+1)l(l-1))Res_0\left(\frac{(1-z^2\alpha^2)^{k+l-1}}{z^{2k+2l+3}}\right).
\end{aligned}$$

Consider the residues. If $k+l\leq -2$, then the functions are holomorphics near 0 and the residues vanish, and if $k+l\geq 1$ they also vanish taking into account the Taylor development. Consider the remaining following cases.

$$\begin{aligned}
&\text{if } k+l=0, \quad \text{then} \quad Res_0\left(\frac{1}{z^3}\right)=0 \quad \text{and} \quad Res_0\left(\frac{(1-z^2\alpha^2)^{-1}}{z^3}\right)=\alpha^2 \\
&\text{if } k+l=-1, \quad \text{then} \quad Res_0\left(\frac{(1-z^2\alpha^2)^{-1}}{z}\right)=1 \quad \text{and} \\
&\qquad\qquad\qquad\qquad\qquad Res_0\left(\frac{(1-z^2\alpha^2)^{-2}}{z}\right)=1.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
c\left(V_{2k+1}, V_{2l+1}\right) &= -6((k+1)k-(l+1)l)\delta_{k+l,-1} \\
&\quad -4((k+1)k(k-1)-(l+1)l(l-1))(\alpha^2\delta_{k+l,0}+\delta_{k+l,-1}) \\
&= -8\alpha^2(k+1)k(k-1)\delta_{k+l,0}-4k(k+1)(2k+1)\delta_{k+l,-1}.
\end{aligned}$$

Hence the given 2-cocycle. The 2-cocycle vanishes on the sub-superalgebra $\left\langle V_{-1}, V_0, V_1, \varphi_{-1/2}, \varphi_{1/2}\right\rangle$ that is isomorphic to osp(1|2). The Uniqueness (up to isomorphism) follows from the uniqueness of this cocycle on $\mathcal{K}(1)$ and that $\mathcal{K}(1)$ is isomorphic to the sub-superalgebra $\mathcal{L}^-_{0,3}$ of $\mathcal{L}_{0,3}$. □

5.2.3 1-cocycle on $\mathcal{L}_{g,N}$

In this subsection, we construct a 1-cocycle on $\mathcal{L}_{g,N}$ with values in the dual space. In the Lie case, existence of such a 1-cocycle is almost equivalent to the existence of a 2-cocycle with trivial coefficients (5.2).

Given a 2-cocycle on a Lie (super) algebra $c : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$, one can define a 1-cocycle C, on $\mathcal{L}$ with values in the dual space $\mathcal{L}^*$. The definition is as follow

$$\langle C(x), y\rangle := c(x, y), \tag{5.3}$$

for all $x, y \in \mathcal{L}$. The 1-cocycle condition

$$C([x, y]) = ad^*_x(C(y)) - (-1)^{\bar{x}\bar{y}} ad^*_y(C(x))$$

follows from the 2-cocycle condition for c. Note that the converse construction does not work since c is not necessarily skewsymmetric.

The 2-cocycle (5.2) defines, therefore, a 1-cocycle on every Lie superalgebra $\mathcal{L}_{g,N}$. Since the splitting is fixed, the separating cycle $\mathcal{C}_S$ defined a natural pairing (called the *K-N pairing* in [Sch03], p58) between $\mathcal{F}_\lambda$ and $\mathcal{F}_{1-\lambda}$ which is given by

$$\mathcal{F}_\lambda \times \mathcal{F}_{1-\lambda} \longrightarrow \mathbb{C} \quad : \quad (f, g) \longmapsto \langle f, g\rangle := \frac{1}{2i\pi} \int_{\mathcal{C}_S} f \bullet g.$$

So that, the space $\mathcal{F}_2 \oplus \mathcal{F}_{3/2}$ will be seen has a nice geometric sub-space of the dual space $\mathcal{L}^*_{g,N}$ thanks to the pairing.

Since for every $a, b \in \mathcal{L}$ and for every $u \in \mathcal{L}^*$ we must have[1]

$$\langle ad^*_a u, b\rangle := -(-1)^{\bar{a}\bar{u}} \langle u, ad_a b\rangle\,,$$

we can see that the coadjoint action of $\mathcal{L}_{g,N}$ is given by

$$\begin{aligned} & ad^*_{\varphi(z)(dz)^{-1/2}} \left(u(z)(dz)^2 \oplus w(z)(dz)^{3/2}\right) \\ = \;& -\{\varphi, w\} \oplus -\frac{1}{2}\varphi \bullet u \\ = \;& -\left(\frac{3}{2}\varphi' w + \frac{1}{2}\varphi w'\right) (dz)^2 \oplus -\frac{1}{2}\varphi u \, (dz)^{3/2} \end{aligned}$$

[1] $ad_a : \mathcal{L} \longrightarrow \mathcal{L} : b \mapsto [a, b]$.

and

$$\begin{aligned} & ad^*_{e(z)(dz)^{-1}}\left(u(z)(dz)^2 \oplus w(z)(dz)^{3/2}\right) \\ = \; & \{e,u\} \oplus \{e,w\} \\ = \; & \left(2e'u + eu'\right)\;(dz)^2 \oplus \left(\frac{3}{2}e'w + ew'\right)\;(dz)^{3/2} \end{aligned}$$

where u, w, e and φ are some meromorphic functions on the surface.

Note that we consider a 2-cocycle c as even function, then the 1-cocycle C defined from c is also an even function.

Proposition 5.2. *A local 1-cocycle on $\mathcal{L}_{g,N}$ is given by*

$$\begin{aligned} C\left(e(z)\tfrac{d}{dz}\right) &= -\left(e''' - 2Re' - R'e\right)dz^2, \\ C\left(\varphi(z)dz^{-1/2}\right) &= \left(\varphi'' - \tfrac{1}{2}R\varphi\right)dz^{3/2}. \end{aligned} \tag{5.4}$$

Proof. Straightforward from (5.2). The locality of C follows from the locality of c. □

Example 5.1. In the case of Riemann sphere ($g = 0$) with respect to the splitting, the 1-cocycle (5.3) related to (5.2) with $R \equiv 0$, reads simply

$$C\left(e(z)\frac{d}{dz}\right) = -e'''(z)\,dz^2 \quad \text{and} \quad C\left(\psi(z)\frac{d}{dz^{1/2}}\right) = \psi''(z)\,dz^{3/2}, \tag{5.5}$$

where z is the standard coordinate.

5.2.4 An explicit formula of the 1-cocycle on $\mathcal{L}_{0,3}$

In the case of the Lie superalgebra $\mathcal{L}_{0,3}$ (and further with the Jordan superalgebra $\mathcal{J}_{0,3}$) the constructed 1-cocycle can be calculated explicitly. The space $\mathcal{L}^*_{0,3}$ has the following basis

$$\begin{aligned} \varphi^*_{2k-1/2} &= \frac{1}{\sqrt{2}}z(z^2-\alpha^2)^{-k-1}(dz)^{3/2}, & V^*_{2k} &= (z^2-\alpha^2)^{-k-1}(dz)^2, \\ \varphi^*_{2k+1/2} &= \frac{1}{\sqrt{2}}(z^2-\alpha^2)^{-k-1}(dz)^{3/2}, & V^*_{2k+1} &= z(z^2-\alpha^2)^{-k-2}(dz)^2, \end{aligned}$$

dual to (4.2) and (4.6).

Proposition 5.3. *Up to isomorphism, the 1-cocycle on the Lie superalgebra $\mathcal{L}_{0,3}$ related to (5.5) that vanishes on* osp(1|2) *is given by*

$$\begin{aligned}
C(V_n) = &\ - n(n-1)(n+1)V^*_{-n} - 2\alpha^2 n(n-2)(n-1)V^*_{-n+2} \\
&- \alpha^4 n(n-2)(n-4)V^*_{-n+4}, \\
C(V_m) = &\ - (m+1)m(m-1)V^*_{-m} - \alpha^2(m+1)(m-1)(m-3)V^*_{-m+2}, \\
C(\varphi_i) = &\ 2(i+\tfrac{1}{2})(i-\tfrac{1}{2})\varphi^*_{-i} + 2\alpha^2(i-\tfrac{1}{2})(i-\tfrac{5}{2})\varphi^*_{-i+2}, \\
C(\varphi_j) = &\ 2(j+\tfrac{1}{2})(j-\tfrac{1}{2})\varphi^*_{-j} + 2\alpha^2(j+\tfrac{1}{2})(j-\tfrac{3}{2})\varphi^*_{-j+2},
\end{aligned}$$

where $i - \frac{1}{2}$, n are even and $j - \frac{1}{2}$, m are odd.

Proof. This is a simple application of the general formulas (5.4) together with $R \equiv 0$. □

5.3 Jordan superalgebras of K-N type and 1-cocycles with values in the dual space

In the Jordan case, the situation is different than in the Lie case. It was proved in [Ovs11] that a Lie antialgebra has no non-trivial central extensions, provided the even part contains a unit element. Therefore, there is no 2-cocycle on $\mathcal{J}_{g,N}$ analogous to (5.2). However, there exists a nice construction of 1-cocycle that has very similar properties than (5.4).

5.3.1 Modules and 1-cocycles on Lie Antialgebras

Let $\mathcal{B}$ be a $\mathbb{Z}_2$-graded vector space and $\rho : \mathcal{A} \to End(\mathcal{B})$ an even linear function. If $\mathcal{A} \oplus \mathcal{B}$ equipped with the product

$$(x, b) \centerdot (x', b') = \Big(x \centerdot x', \rho_x(b') + (-1)^{\bar{x}'\bar{b}}\rho_{x'}(b)\Big) \tag{5.6}$$

for all homogeneous elements $x, x' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, is a Lie antialgebra then $(\mathcal{B}, \rho)$ is called an $\mathcal{A}$-*module.* The structure (5.6) is called a *semi-direct sum* and denoted by $\mathcal{A} \ltimes \mathcal{B}$. Given an $\mathcal{A}$-module $\mathcal{B}$, the dual space $\mathcal{B}^*$ is also an $\mathcal{A}$-module, the $\mathcal{A}$-action being given by

$$\langle \rho_x^* u, b \rangle := (-1)^{\bar{x}\bar{u}} \langle u, \rho_x b \rangle ,$$

for all homogeneous elements $x \in \mathcal{A}$, $b \in \mathcal{B}$ and $u \in \mathcal{B}^*$.

A 1-*cocycle* on a Lie antialgebra $\mathcal{A}$ with coefficients in an $\mathcal{A}$-module $\mathcal{B}$, is an even linear map $\mathrm{C} : \mathcal{A} \longrightarrow \mathcal{B}$ such that

$$\mathrm{C}\,(x \centerdot y) = \rho_x\,(\mathrm{C}(y)) + (-1)^{\bar{x}\bar{y}} \rho_y\,(\mathrm{C}(x))\,, \tag{5.7}$$

For all homogeneous elements $x, y \in \mathcal{A}$. A 1-cocycle is a $1-$*coboundary* if there exists $b_0 \in \mathcal{B}$ such that $\mathrm{C}(x) = \rho_x(b_0)$, for all $x \in \mathcal{A}$.

A Lie antialgebra is tautologically a module over itself, the adjoint action $ad : \mathcal{A} \longrightarrow End(\mathcal{A})$ defined such that $ad_a(a') = a \centerdot a'$ for all $a, a' \in \mathcal{A}$. So that, the dual space, $\mathcal{A}^*$, is an $\mathcal{A}$-module as well.

5.3.2 One-cocycles on $\mathcal{J}_{g,N}$

Since the algebras $\mathcal{J}_{g,N}$ contain a unit element, according to a result in [Ovs11] on Lie antialgebras, these algebras have no non-trivial 2-cocycles. However, coming from conversations with Ovsienko and Lecomte, they found a 1-cocycle on $\mathcal{AK}(1)$. We denote by $\{\varepsilon_n^*, a_i^*\}$ the basis of $\mathcal{AK}(1)^*$ dual to $\{\varepsilon_n, a_i\}$. One has the following result.

Proposition 5.4. *The linear map* $\mathrm{C}_{LO} : \mathcal{AK}(1) \longrightarrow \mathcal{AK}(1)^*$ *given by*

$$\mathrm{C}_{LO}(\varepsilon_n) = -n\varepsilon_{-n}^*, \qquad \mathrm{C}_{LO}(a_i) = \left(i - \tfrac{1}{2}\right)\left(i + \tfrac{1}{2}\right) a_{-i}^*, \tag{5.8}$$

is a non-trivial 1*-cocycle on* $\mathcal{AK}(1)$.

More general than Proposition 5.4, the second result of this chapter from [Kre13] is the following.

Theorem 5.3. *(i) With respect to the splitting, the expression*

$$\mathrm{C}\,(\varepsilon(z)) = -\varepsilon'(z)dz, \quad \mathrm{C}\left(\psi(z)dz^{-1/2}\right) = \left(\psi''(z) - \frac{1}{2}R\,\psi(z)\right) dz^{3/2} \tag{5.9}$$

defines a local 1*-cocycle on* $\mathcal{J}_{g,N}$ *with coefficients in* $\mathcal{J}_{g,N}^*$.

(ii) Consider the cocycle (5.9) with $R \equiv 0$ and vanishing on the subalgebra $\mathcal{K}_3$, then if $N = 2$ it is unique up to a multiplicative constant and if $N = 3$ it is unique up to isomorphism.

Proof. Part (i). Similarly to formula (5.2), the first expression in the right-hand-side of (5.9) is independent of the choice of the coordinate z. One now easily checks that this expression indeed satisfies the condition (5.7) of 1-cocycle. This follows from the relations

$$\begin{aligned} ad^*_{\varphi(z)(dz)^{-1/2}}\left(u(z)dz \oplus w(z)(dz)^{3/2}\right) &= -\tfrac{1}{2}\varphi w\, dz \oplus -\tfrac{1}{2}\varphi u' - \varphi' u\, (dz)^{3/2}, \\ ad^*_{\varepsilon(z)}\left(u(z)dz \oplus w(z)(dz)^{3/2}\right) &= \varepsilon u\, dz \;\oplus\; \tfrac{1}{2}\varepsilon w\, (dz)^{3/2}, \end{aligned}$$

where u, w, ε and φ are some meromorphic functions on the surface. The locality is proved by evaluate the cocycle (5.9) on the basis elements introduced in Section 4.2.2 according to the fact that the basis elements of $\mathcal{J}_{g,N}$ and $\mathcal{J}^*_{g,N}$ are dual to each other. Note that this cocycle is non-trivial.

Part (ii). Let us first consider the case $N = 2$ and show that the cocycle (5.9) with $R \equiv 0$ from $\mathcal{AK}(1)$ to $\mathcal{AK}(1)^*$ is the unique (up to a multiplicative constant) 1-cocycle that vanishes on $\mathcal{K}_3$. Assume that $\mathrm{C} : \mathcal{AK}(1) \longrightarrow \mathcal{AK}(1)^*$ is a 1-cocycle. Since C is even, it is of the form

$$\mathrm{C}\left(\varepsilon_n \oplus a_i\right) = \mathrm{C}\left(\varepsilon_n\right) \oplus \mathrm{C}\left(a_i\right) = \sum_{r\in\mathbb{Z}} \lambda^r_n \varepsilon^*_r \oplus \sum_{k\in\mathbb{Z}+\frac{1}{2}} \mu^k_i a^*_k.$$

The condition of 1-cocycle (5.7) gives

$$\begin{aligned} \mathrm{C}\left(\varepsilon_n \cdot \varepsilon_m\right) &= ad^*_{\varepsilon_n}\mathrm{C}\left(\varepsilon_m\right) + ad^*_{\varepsilon_m}\mathrm{C}\left(\varepsilon_n\right) &\Leftrightarrow\quad \lambda^r_{n+m} &= \lambda^{r+n}_m + \lambda^{r+m}_n, \\ \mathrm{C}\left(\varepsilon_n \cdot a_i\right) &= ad^*_{\varepsilon_n}\mathrm{C}\left(a_i\right) + ad^*_{a_i}\mathrm{C}\left(\varepsilon_n\right) &\Leftrightarrow\quad \mu^k_{i+n} &= \mu^{k+n}_i + (k-i)\lambda^{i+k}_n, \\ \mathrm{C}\left(a_i \cdot a_j\right) &= ad^*_{a_i}\mathrm{C}\left(a_j\right) - ad^*_{a_j}\mathrm{C}\left(a_i\right) &\Leftrightarrow\quad (j-i)\lambda^r_{i+j} &= -\mu^{r+i}_j + \mu^{r+j}_i, \end{aligned}$$

for all $n, m, r \in \mathbb{Z}$ and all $i, j, k \in \mathbb{Z} + \frac{1}{2}$. Since this cocycle vanishes on the Lie antialgebra $\mathcal{K}_3$ generated by $\langle \varepsilon_0, a_{-1/2}, a_{1/2} \rangle$, by induction one has the following (unique up to a constant) solution

$$\lambda^r_n = -n\delta_{r,-n} \quad \text{and} \quad \mu^k_i = (k^2 - \tfrac{1}{4})\delta_{k,-i} \qquad \forall n, r \in \mathbb{Z};\ \forall i, k \in \mathbb{Z} + \tfrac{1}{2},$$

and thus obtains the cocycle (5.8).

Now, let us show the uniqueness for $N = 3$. As proved in [LMG12a], the subalgebra $\mathcal{J}_{0,3}^{-} = \langle G_n : n \leq 0;\ \varphi_i : i \leq 1/2\rangle$ is isomorphic to $\mathcal{AK}(1)$. Suppose that we have a 1-cocycle C : $\mathcal{J}_{0,3} \longrightarrow \mathcal{J}_{0,3}^{*}$ and writing it with the elements of the basis as the same way than in the first part of the proof (*ii*). Using the 1-cocycle condition (5.7), we can can show that if we know the 1-cocycle C on $\mathcal{J}_{0,3}^{-}$ (i.e. on $\mathcal{AK}(1)$), then the cocycle is uniquely (up to isomorphism) determined on $\mathcal{J}_{0,3}$ entirely. Hence the result on $\mathcal{J}_{0,3}$ since the 1-cocycle on $\mathcal{AK}(1)$ is unique when it vanishes on $\mathcal{K}_3$. □

Remark 5.1. The 1-cocycle (5.9) has a very simple and, geometrically, a very natural form : this is the De Rham differential of a function combined with the Sturm-Liouville equation associated to a projective connection, applied to a $-1/2$-density.

5.3.3 An explicit formula of the 1-cocycle on $\mathcal{J}_{0,3}$

We finish the chapter with an explicit formula of the 1-cocycle (5.9) in the case of 3 marked points.

Proposition 5.5. *Up to isomorphism, the 1-cocycle (5.9) on the algebra $\mathcal{J}_{0,3}$ is given by*

$$C(G_m) = -mG^*_{-m} - \alpha^2(m-1)G^*_{-m+2}, \qquad C(G_n) = -nG^*_{-n},$$

$$C(\varphi_i) = 2(i+\tfrac{1}{2})(i-\tfrac{1}{2})\varphi^*_{-i} + 2\alpha^2(i-\tfrac{1}{2})(i-\tfrac{5}{2})\varphi^*_{-i+2},$$

$$C(\varphi_j) = 2(j+\tfrac{1}{2})(j-\tfrac{1}{2})\varphi^*_{-j} + 2\alpha^2(j+\tfrac{1}{2})(j-\tfrac{3}{2})\varphi^*_{-j+2},$$

where $i - \frac{1}{2}$ and n are even and $j - \frac{1}{2}$, m are odd.

Proof. The elements of the basis of the dual space $\mathcal{J}_{0,3}^{*}$ are given by

$$\varphi^*_{2k-1/2} = \frac{1}{\sqrt{2}}z(z^2-\alpha^2)^{-k-1}(dz)^{3/2}, \qquad G^*_{2k} = z(z^2-\alpha^2)^{-k-1}dz,$$

$$\varphi^*_{2k+1/2} = \frac{1}{\sqrt{2}}(z^2-\alpha^2)^{-k-1}(dz)^{3/2}, \qquad G^*_{2k+1} = (z^2-\alpha^2)^{-k-1}dz.$$

The computations are straightforward. □

Complement to Chapter 3 - A

Proofs of Lemmas 3.1 to 3.3

We are going to prove in details the four important lemmas of Section 3.3. They are the key of the periodicity on the cubic forms $\alpha_{p,q}$ and therefore of the algebras $\mathbb{O}_{p,q}$. There are four sections in this **Complement to Chapter 3** A depending on the residue of n modulo 4.

One of the biggest difficulties was to find suitable changes of coordinates. After simulations with `Mathematica` a periodicity appears, as beautiful as expected. Programming was essential in the discovery of these results. Appearing later, another difficulty was the diversity of changes of coordinates required to prove these lemmas according to the signature (p, q) and the parity of k where $n = 4k + r$ with $r = 0, 1, 2, 3$ and $k \in \mathbb{N}$. The work of calculation and mathematical proofs is presented in **Complement** A, while the simulations with `Mathematica` is presented in **Complement** B.

In the first two **Sections** A.1 and A.2, where $n = 4k$ and $n = 4k + 2$ respectively, the aim is to find an equivalent form to $\alpha_{p,q}$ where the last variable x_n is (partially) removed. In the proofs, we exhibit suitable changes of coordinates.

In **Sections** A.3 and A.4, where $n = 4k + 1$ and $n = 4k + 3$ respectively, the aim is to find an equivalent form to $\alpha_{p,q}$ where the first variable x_1 is nearly factorized. Different changes of coordinates are required according to the parity of k.

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In each section, we first focus on the case $\alpha_{0,n}$ that is invariant under permutations. Secondly, the other cubic forms (where $p+q=n$) are easily obtained by the following formula

$$\alpha_{p,q}(x_1,\ldots,x_n)=\alpha_{0,n}(x_1,\ldots,x_n)+\sum_{i=1}^{p}x_i, \tag{A.1}$$

by taking into account the linear part depending on p. We are dealing with polynomials (of degree at most three) in n variables $x_1,\ldots,x_n$, such that $x_i\in\mathbb{Z}_2$ for all $i\in\{1,\ldots,n\}$, with values in $\mathbb{Z}_2$. If $x_i\in\mathbb{Z}_2$, then we obviously have $x_i^2=x_i$. We introduce here some useful notations for the calculations in Complement A. Let us denote by k an element in $\mathbb{N}$ and define

$$\chi_{0,k}=\begin{cases}1 & \text{if } k\equiv 0 \mod 2,\\ 0 & \text{if } k\equiv 1 \mod 2.\end{cases}$$

We can interpret $\chi_{0,k}$ as the *kronecker's delta* congruent modulo 2 or as the *indicator function*[1] of the subset of even positive integers of the set of positive integers. Let us denote the sum $x_1+\cdots+x_n$ by X, where $x=(x_1,\ldots,x_n)$ is an element of $\mathbb{Z}_2^n$. We have the following straightforward result.

Lemma A.1.

$$\sum_{i=j}^{n} i=\begin{cases}0 \mod 2 & \textit{if } (j\equiv 0,1 \mod 4 \;\&\; n\equiv 0,3 \mod 4)\\ & \textit{or } (j\equiv 2,3 \mod 4 \;\&\; n\equiv 1,2 \mod 4),\\ 1 \mod 2 & \textit{if } (j\equiv 0,1 \mod 4 \;\&\; n\equiv 1,2 \mod 4)\\ & \textit{or } (j\equiv 2,3 \mod 4 \;\&\; n\equiv 0,3 \mod 4).\end{cases} \tag{A.2}$$

[1]The indicator function of a subset A of a set Y is a function

$$\mathbf{1}_A(x)=\begin{cases}1 & \text{if } x\in A,\\ 0 & \text{if } x\notin A.\end{cases}$$

A.1 Proof of Lemma 3.1, the case $n = 4k$

Lemma 3.1 states that $\alpha_{p,q}$ (with $p + q \equiv 0 \bmod 4$) is equivalent to another cubic form where the variable x_n is (partially) removed. The point of this section is to exhibit suitable changes of variables in the different cases. Let us start with some general and simple calculations on special polynomials in $\mathbb{Z}_2[\mathbb{Z}_2^n]$ that we will be useful in the following.

Lemma A.2. *We have the following equalities of polynomials in* $\mathbb{Z}_2[\mathbb{Z}_2^n]$,

$$\sum_{2\leq i<j\leq 2k+1} (1 + x_i + x_j) = 1 + \chi_{0,k} + \sum_{i=2}^{2k+1} x_i, \tag{A.3}$$

$$\sum_{2k+2\leq i<j\leq 4k-1} (1 + x_i + x_j) = \chi_{0,k} + \sum_{i=2k+2}^{4k-1} x_i, \tag{A.4}$$

$$\sum_{2\leq i<j<l\leq 2k+1} (1 + x_i + x_j + x_l + x_ix_j + x_ix_l + x_jx_l) = \chi_{0,k} \sum_{i=2}^{2k+1} x_i, \tag{A.5}$$

$$\begin{aligned} &\sum_{2k+2\leq i<j<l\leq 4k-1} (1 + x_i + x_j + x_l + x_ix_j + x_ix_l + x_jx_l) \\ = \;& (1 + \chi_{0,k}) \sum_{i=2k+2}^{4k-1} x_i. \end{aligned} \tag{A.6}$$

Proof. The first sum (A.3) is given by

$$\begin{aligned} \sum_{i=2}^{2k} \sum_{j=i+1}^{2k+1} (1 + x_i + x_j) &= \sum_{i=2}^{2k} (1 + i)(1 + x_i) + \sum_{j=3}^{2k+1} \sum_{i=2}^{j-1} x_j \\ &= 1 + \chi_{0,k} + \sum_{i=2}^{2k} (1 + i)x_i + \sum_{j=3}^{2k+1} jx_j \\ &= 1 + \chi_{0,k} + \sum_{i=2}^{2k+1} x_i, \end{aligned}$$

by using the result (A.2). The second sum (A.4) can be deduced directly from (A.3) or we can explicit the calculation as the same way than (A.3). The calculations for the third sum (A.5) and the fourth one (A.6) are a bit more tricky. Let us expose the details for the fourth sum. It is given by

$$\begin{aligned}
LHS &:= \sum_{2k+2\leq i<j<l\leq 4k-1} (1 + x_i + x_j + x_l + x_i x_j + x_i x_l + x_j x_l) \\
&= \sum_{i=2k+2}^{4k-3} \sum_{j=i+1}^{4k-2} (1+j) + \sum_{i=2k+2}^{4k-3} \sum_{j=i+1}^{4k-2} (1+j)(x_i + x_j + x_i x_j) \\
&\quad + \sum_{j=2k+3}^{4k-2} \sum_{l=j+1}^{4k-1} j(x_l + x_j x_l) + \sum_{j=2k+2}^{4k-3} \sum_{l=i+2}^{4k-1} (l+1+i)(x_i x_l).
\end{aligned}$$

The term coloured in gray means this term is vanishing by itself. After distribution and simplification we have

$$\begin{aligned}
LHS &= \sum_{i=2k+2}^{4k-3} x_i \sum_{j=i+1}^{4k-2} (1+j) + \sum_{i=2k+2}^{4k-3} \sum_{j=i+1}^{4k-2} (1+j)x_j \\
&\quad + \Big[\sum_{i=2k+2}^{4k-3} x_i \sum_{j=i+1}^{4k-2} (1+j)x_j \Big]_1 + \sum_{j=2k+3}^{4k-2} j \sum_{l=j+1}^{4k-1} x_l \\
&\quad + \Big[\sum_{j=2k+3}^{4k-2} j x_j \sum_{l=j+1}^{4k-1} x_l \Big]_2 + \Big[\sum_{i=2k+3}^{4k-3} i x_i \sum_{l=j+2}^{4k-1} x_l \Big]_2 \\
&\quad + \Big[\sum_{i=2k+2}^{4k-3} x_i \sum_{l=i+2}^{4k-1} (1+l)x_l \Big]_1 \\
&= \sum_{i=2k+2}^{4k-3} x_i \sum_{j=i+1}^{4k-2} (1+j) + \sum_{j=2k+3}^{4k-2} j \sum_{l=j+1}^{4k-1} x_l \\
&\quad + \Big[\sum_{i=2k+2}^{4k-3} (i x_i x_{i+1}) \Big]_1 + \Big[\sum_{i=2k+3}^{4k-2} (i x_i x_{i+1}) \Big]_2 \\
&= \sum_{i=2k+2}^{4k-3} x_i \sum_{j=i+1}^{4k-2} (1+j) + \sum_{j=2k+3}^{4k-2} j \sum_{l=j+1}^{4k-1} x_l \\
&= x_{2k+2} \sum_{j=2k+3}^{4k-2} (1+j) + x_{2k+3} \sum_{j=2k+4}^{4k-2} (1+j) + x_{4k-2} \sum_{j=2k+3}^{4k-3} j \\
&\quad + x_{4k-1} \sum_{j=2k+3}^{4k-2} j + \sum_{i=2k+4}^{4k-3} x_i \left(\sum_{j=i+1}^{4k-2} (1+j) + \sum_{j=2k+3}^{i-1} j \right).
\end{aligned}$$

The terms in the brackets $[\ \ ]_1$ (resp. in the brackets $[\ \ ]_2$) are treated together and simplified. We use the result (A.2) and we obtain the result

$$LHS = (1+\chi_{0,k})(x_{2k+2}+x_{2k+3}+x_{4k-2}+x_{4k-1})+(1+\chi_{0,k})\sum_{2k+4}^{4k-3} x_i.$$

□

A.1.1 The signature $(0, n)$

Let us consider the proof of Lemma 3.1. The goal is to find an equivalent form to $\alpha_{0,n}$ where the last variable x_n is removed. We choose the following coordinate transformation

$$\begin{aligned} x'_1 &= x_1 + x_n, \\ x'_i &= x_1 + x_i + x_n, \qquad \text{for } i = 2, \dots, 2k+1, \\ x'_i &= x_1 + X + x_i, \qquad \text{for } i = 2k+2, \dots, n-1, \\ x'_n &= x_n, \end{aligned} \tag{A.7}$$

where $X = x_1 + \cdots + x_n$ and $n = 4k$, $k \in \mathbb{N}\backslash\{0\}$. Partial results on the linear, quadratic and cubic parts are the following.

Lemma A.3. *Applying the coordinate transformation (A.7), if $n = 4k$, then we have the following equality of the linear part*

$$\sum_{i=1}^{n} x'_i = \sum_{i=1}^{n-1} x_i. \tag{A.8}$$

Proof. Straightforward. □

Lemma A.4. *Applying the coordinate transformation (A.7), if $n = 4k$, then we have the following equality of the quadratic part*

$$\begin{aligned} \sum_{1\le i<j\le n} x'_i x'_j = &\sum_{2\le i<j\le 2k+1} x_i x_j + \sum_{2k+2\le i<j\le n-1} x_i x_j + x_1 \sum_{i=2k+2}^{n-1} x_i \\ &+ x_n \sum_{i=1}^{n-1} x_i + \chi_{0,k} \sum_{i=2}^{2k+1} x_i + (1+\chi_{0,k}) \left(x_1 + \sum_{i=2k+2}^{n-1} x_i \right). \end{aligned} \tag{A.9}$$

Proof. Let us decompose the sum according to the coordinate transformation, the quadratic part is given by

$$\begin{aligned}
\sum_{1\le i<j\le n} x_i'x_j' &= (x_1+x_n)\sum_{i=2}^{n} x_i + \sum_{2\le i<j\le 2k+1} (x_1+x_i+x_n)(x_1+x_j+x_n) \\
&\quad + \sum_{i=2}^{2k+1}\sum_{j=2k+2}^{4k-1} (x_1+x_i+x_n)(x_1+x_j+X) \\
&\quad + \sum_{2k+2\le i<j\le 4k-1} (x_1+x_i+X)(x_1+x_j+X) + x_n\sum_{i=2}^{n-1} x_j \\
&= x_1\sum_{i=2}^{n} x_i + x_n + (x_1+x_n)\sum_{2\le i<j\le 2k+1}(1+x_i+x_j) \\
&\quad + \sum_{2\le i<j\le 2k+1} x_ix_j + \sum_{i=2}^{2k+1}\sum_{j=2k+2}^{4k-1} x_ix_j + \sum_{2k+2\le i<j\le 4k-1} x_ix_j \\
&\quad +(x_1+X)\sum_{2k+2\le i<j\le 4k-1}(1+x_i+x_j).
\end{aligned}$$

Using the results (A.3) and (A.4), we obtain

$$\begin{aligned}
\sum_{1\le i<j\le n} x_i'x_j' &= x_1\sum_{i=2}^{n} x_i + x_n + (x_1+x_n)\left(1+\chi_{0,k}+\sum_{j=2}^{2k+1} x_j\right) \\
&\quad + \sum_{2\le i<j\le 2k+1} x_ix_j + \sum_{i=2}^{2k+1}\sum_{j=2k+2}^{4k-1} x_ix_j + \sum_{2k+2\le i<j\le 4k-1} x_ix_j \\
&\quad + \left(x_n + \sum_{i=2}^{2k+1} x_i + \sum_{j=2k+2}^{4k-1} x_j\right)\left(\chi_{0,k} + \sum_{j=2k+2}^{4k-1} x_j\right).
\end{aligned}$$

Hence the result, by simplifying this last expression. □

Lemma A.5. *Applying the coordinate transformation (A.7), if $n = 4k$, then we have the following equality of the cubic part*

$$\begin{aligned}
\sum_{1\le i<j<l\le n} x_i'x_j'x_l' &= \sum_{1\le i<j<l\le n-1} x_ix_jx_l + x_n\sum_{i=1}^{n-1} x_i + x_1\sum_{i=2}^{2k+1} x_i \\
&\quad + \sum_{i=2}^{2k+1} x_i \sum_{j=2k+2}^{n-1} x_j + \chi_{0,k}\sum_{i=2}^{2k+1} x_i \qquad \text{(A.10)} \\
&\quad +(1+\chi_{0,k})\left(x_1 + \sum_{i=2k+2}^{n-1} x_i\right).
\end{aligned}$$

Proof. Let us decompose the sum according to the coordinate transformation, the cubic part (CP) is given by

$$\begin{aligned}
CP &:= \sum_{1\le i<j<l\le n} x'_i x'_j x'_l \\
&= (x_1+x_n)\sum_{2\le i<j\le n} x'_i x'_j + \sum_{2\le i<j<l\le 2k+1} x_i x_j x_l \\
&\quad +(x_1+x_n)\sum_{2\le i<j<l\le 2k+1}(1+x_i+x_j+x_l+x_ix_j+x_ix_l+x_jx_l) \\
&\quad +\sum_{2\le i<j\le 2k+1}((x_1+x_n)(1+x_i+x_j)+x_ix_j)\sum_{l=2k+2}^{4k-1}(x_1+x_l+X) \\
&\quad +\sum_{i=2}^{2k+1}(x_1+x_i+x_n)\sum_{2k+2\le j<l\le 4k-1}((x_1+X)(1+x_j+x_l)+x_jx_l) \\
&\quad +\sum_{2k+2\le i<j<l\le 4k-1} x_ix_jx_l \\
&\quad +(x_1+X)\sum_{2k+2\le i<j<l\le 4k-1}(1+x_i+x_j+x_l+x_ix_j+x_ix_l+x_jx_l) \\
&\quad +x_n\sum_{2\le i<j\le n-1} x'_i x'_j .
\end{aligned}$$

Using the results (A.3), (A.4), (A.5), (A.6) and the result on the quadratic part (A.9), we obtain

$$\begin{aligned}
CP = {}& x_1\sum_{2\le i<j\le 2k+1} x_ix_j + x_1\sum_{2k+2\le i<j\le n-1} x_ix_j + \left[x_1\sum_{i=2k+2}^{n-1} x_i\right]_1 \\
&+\left[x_1x_n\sum_{i=1}^{n-1} x_i\right]_1 + x_1\chi_{0,k}\sum_{i=2}^{2k+1} x_i + x_1(1+\chi_{0,k})\left(x_1+\sum_{i=2k+1}^{n-1}\right) \\
&+\left[x_1(x_1+x_n)\sum_{i=2}^{n} x_i\right]_1 + x_n\sum_{i=2}^{n-1} x_i + \sum_{2\le i<j<l\le n-1} x_ix_jx_l \\
&+(x_1+x_n)\chi_{0,k}\sum_{i=2}^{2k+1} x_i + (x_1+x_n)\left(1+\chi_{0,k}+\sum_{i=2}^{2k+1} x_i\right)\sum_{l=2k+2}^{n-1} x_l \\
&+(x_1+X)\sum_{i=2}^{2k+1} x_i\left(\chi_{0,k}+\sum_{j=2k+2}^{n-1} x_j\right) + (x_1+X)(1+\chi_{0,k})\sum_{i=2k+2}^{n-1} x_i .
\end{aligned}$$

If we first simplify the terms in the brackets $[\ ]_1$, then join together the cubic terms, distribute and simplify again, the cubic part (CP) is given by

$$\sum_{1\le i<j<l\le n-1} x_i x_j x_l + x_1 \sum_{i=2}^{2k+1} x_i + x_1 x_n + \left[x_1 \chi_{0,k} \sum_{i=2}^{2k+1} x_i\right]_2$$
$$+ x_1(1+\chi_{0,k}) + \left[x_1(1+\chi_{0,k}) \sum_{i=2k+2}^{n-1} x_i\right]_3 + x_n \sum_{i=2}^{n-1} x_i + \left[x_1 \chi_{0,k} \sum_{i=2}^{2k+1} x_i\right]_2$$
$$+ \left[x_n \chi_{0,k} \sum_{i=2}^{2k+1} x_i\right]_4 + \left[x_1(1+\chi_{0,k}) \sum_{l=2k+2}^{n-1} x_l\right]_3 + \left[x_n(1+\chi_{0,k}) \sum_{l=2k+2}^{n-1} x_l\right]_5$$
$$+ \left[x_n \sum_{i=2}^{2k+1} x_i \sum_{l=2k+2}^{n-1} x_l\right]_6 + x_n \sum_{i=2}^{2k+2} x_i \left(\left[\chi_{0,k}\right]_4 + \left[\sum_{j=2k+2}^{n-1} x_j\right]_6 \right)$$
$$+ \sum_{l=2}^{n-1} x_l \sum_{i=2}^{2k+1} x_i \left(\chi_{0,k} + \sum_{j=2k+2}^{n-1} x_j \right) + \left[x_n(1+\chi_{0,k}) \sum_{i=2k+2}^{n-1} x_i\right]_5$$
$$+ (1+\chi_{0,k}) \sum_{l=2}^{n-1} x_l \sum_{i=2k+2}^{n-1} x_i.$$

Simplifying the different terms according to the terms in the brackets $[\ ]_2$, $[\ ]_3$, $[\ ]_4$, $[\ ]_5$ and $[\ ]_6$ then distribute the latest terms, we have

$$\begin{aligned}
CP &= \sum_{1\le i<j<l\le n-1} x_i x_j x_l + x_1 \sum_{i=2}^{2k+1} x_i + x_1(1+\chi_{0,k}) + x_n \sum_{i=1}^{n-1} x_i \\
&\quad + \sum_{i=2}^{2k+1} x_i \left(\chi_{0,k} + \sum_{j=2k+2}^{n-1} x_j \right) + \left[(1+\chi_{0,k}) \sum_{l=2}^{2k+1} x_l \sum_{i=2k+2}^{n-1} x_i\right]_7 \\
&\quad + \left[\sum_{l=2k+2}^{n-1} x_l \sum_{i=2}^{2k+2} x_i\right]_7 + \left[\chi_{0,k} \sum_{l=2k+2}^{n-1} x_l \sum_{i=2}^{2k+2} x_i\right]_7 + (1+\chi_{0,k}) \sum_{i=2k+2}^{n-1} x_i \\
&= \sum_{1\le i<j<l\le n-1} x_i x_j x_l + x_1 \sum_{i=2}^{2k+1} x_i + x_n \sum_{i=1}^{n-1} x_i \\
&\quad + \sum_{i=2}^{2k+1} x_i \left(\chi_{0,k} + \sum_{j=2k+2}^{n-1} x_j \right) + (1+\chi_{0,k}) \left(x_1 + \sum_{i=2k+2}^{n-1} x_i \right).
\end{aligned}$$

By simplifying again according to the terms in the brackets $[\ ]_7$, we obtain the correct result. □

According to the three previous lemmas and the formulas (A.8), (A.9) and (A.10), Lemma 3.1 is proved for the signature $(0, n)$. In other words, we have

$$\alpha_{0,n}(x'_1, \ldots, x'_n) = \widehat{\alpha}_{0,n-1}(x_1, \ldots, x_n).$$

Recall that $(x'_1, \ldots, x'_n)$ is given in (A.7) and that $\widehat{\alpha}$ is the cubic form on $\mathbb{Z}_2^n$ obtained by the pull-back of a cubic form α on $\mathbb{Z}_2^{n-1}$ of the projection (defined by "forgetting" the last coordinate x_n) of $\mathbb{Z}_2^n$ on $\mathbb{Z}_2^{n-1}$.

A.1.2 The signatures $(n, 0)$ and (p, q), where $pq > 0$

Consider the signature $(n, 0)$ and (p, q), with p and q odd, according to the change of variables (A.7) and the formula (A.1), we have

$$\begin{aligned}
\alpha_{n,0}(x'_1, \ldots, x'_n) &= \alpha_{0,n}(x'_1, \ldots, x'_n) + \sum_{i=1}^{n} x'_i \\
&= \widehat{\alpha}_{0,n-1}(x_1, \ldots, x_n) + \sum_{i=1}^{n-1} x_i \\
&= \widehat{\alpha}_{n-1,0}(x_1, \ldots, x_n),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{p,q}(x'_1, \ldots, x'_n) &= \widehat{\alpha}_{0,n-1}(x'_1, \ldots, x'_n) + \sum_{i=1}^{p} x'_i \\
&= \widehat{\alpha}_{0,n-1}(x_1, \ldots, x_n) + x_1 + x_n + \sum_{i=2}^{p} x_i \\
&= \widehat{\alpha}_{p,q-1}(x_1, \ldots, x_n) + x_n.
\end{aligned}$$

In the case where $pq > 0$ are even numbers, the coordinate transformation is given by

$$\begin{aligned}
x''_1 &= x_n, & & \\
x''_i &= x_1 + x_i + x_n, & & \text{for } i = 2, \ldots, 2k+1, \\
x''_i &= x_1 + X + x_i, & & \text{for } i = 2k+2, \ldots, n-1, \\
x''_n &= x_1 + x_n. & &
\end{aligned} \tag{A.11}$$

Note that we just exchange the coordinate transformation of x_1 and x_n compared to the transformation given by (A.7). Since the cubic form $\alpha_{0,n}$ is invariant under permutation, the transformation (A.11) still gives the result $\alpha_{0,n} \simeq \widehat{\alpha}_{0,n-1}$. Let us first restrict the values of signature (p, q) to the one where $p \leq q$ are even, we have

$$\begin{aligned} \alpha_{p,q}(x''_1, \ldots, x''_n) &= \widehat{\alpha}_{0,n-1}(x_1, \ldots, x_n) + x_n + \sum_{i=2}^{p}(x_1 + x_i + x_n) \\ &= \widehat{\alpha}_{p,q-1}(x_1, \ldots, x_n). \end{aligned}$$

If $p > q$ are even numbers, then from the properties of Chapter 2, we have

$$\begin{aligned} \alpha_{p,q}(x''_1, \ldots, x''_n) &\simeq \alpha_{q,p}(x''_1, \ldots, x''_n) && \text{due to Theorem 2.1}(i) \\ &\simeq \widehat{\alpha}_{q,p-1}(x_1, \ldots, x_n) && \text{since} \quad q < p, \\ &\simeq \widehat{\alpha}_{q-1,p}(x_1, \ldots, x_n) && \text{due to Lemma 2.7}, \\ &\simeq \widehat{\alpha}_{p,q-1}(x_1, \ldots, x_n) && \text{due to Theorem 2.1}(i). \end{aligned}$$

Hence the result of Lemma 3.1 for all signatures.

A.2 Proof of Lemma 3.2, the case $n = 4k + 2$

Lemma 3.2 states that $\alpha_{p,q}$ (with $p + q \equiv 2 \bmod 4$) is equivalent to another cubic form where the variable x_n is (partially) removed. As in Section A.1, we exhibit suitable coordinate transformations. The proofs in this section are very similar to the ones in Section A.1, therefore we give here the different lemmas without proof and explain the different steps. Let us start with some general and simple calculations that will be useful in the following.

Lemma A.6. *We have the following equalities of polynomials in $\mathbb{Z}_2[\mathbb{Z}_2^n]$,*

$$\sum_{1 \leq i < j \leq 2k+1} (1 + x_i + x_j) = 1 + \chi_{0,k}, \tag{A.12}$$

$$\begin{aligned}
&\sum_{1\le i<j<l\le 2k+1} (1 + x_i + x_j + x_l + x_ix_j + x_ix_l + x_jx_l) \\
= &\ (1+\chi_{0,k})\left(1 + \sum_{i=1}^{2k+1} x_i\right) + \sum_{1\le i<j\le 2k+1} x_ix_j,
\end{aligned} \tag{A.13}$$

A.2.1 The signature $(0, n)$

Let us consider the proof of Lemma 3.2 where the goal is to find an equivalent form to $\alpha_{0,n}$ where the last variable x_n is removed from the cubic part. We choose the following coordinate transformation

$$\begin{aligned}
x_i' &= x_i + x_n, && \text{for } i = 1, \dots, 2k+1, \\
x_i' &= X + x_i, && \text{for } i = 2k+2, \dots, n-1, \\
x_n' &= x_n,
\end{aligned} \tag{A.14}$$

where $X = x_1 + \cdots + x_n$ and $n = 4k+2$, $k \in \mathbb{N}\backslash\{0\}$. Partial results on the linear, quadratic and cubic parts are the following.

Lemma A.7. *Applying the coordinate transformation (A.14), if $n = 4k+2$, then we have the following equality of the linear part*

$$\sum_{i=1}^{n} x_i' = \sum_{i=1}^{n-1} x_i. \tag{A.15}$$

Proof. Straightforward. □

Lemma A.8. *Applying the coordinate transformation (A.14), if $n = 4k+2$, then we have the following equality of the quadratic part*

$$\begin{aligned}
\sum_{1\le i<j\le n} x_i'x_j' &= \sum_{1\le i<j\le 2k+1} x_ix_j + \sum_{2k+2\le i<j\le n-1} x_ix_j \\
&\quad + x_n + x_n\sum_{i=1}^{n-1} x_i + \sum_{i=1}^{2k+1} x_i + \chi_{0,k}\sum_{i=1}^{n-1} x_i.
\end{aligned} \tag{A.16}$$

Proof. The proof is similar to the one of Lemma A.4. Note that we use the result (A.12). □

Lemma A.9. *Applying the coordinate transformation (A.14), if $n = 4k+2$, then we have the following equality of the cubic part*

$$\begin{aligned}\sum_{1\le i<j<l\le n} x'_i x'_j x'_l &= \sum_{1\le i<j<l\le n-1} x_i x_j x_l + \chi_{0,k}\sum_{i=1}^{n-1} x_i + \sum_{i=1}^{2k+1} x_i \\ &\quad + \sum_{i=1}^{2k+1} x_i \sum_{j=2k+2}^{n-1} x_j.\end{aligned} \tag{A.17}$$

Proof. The proof is similar to the one of Lemma A.5. Note that we use the result (A.13). □

According to the three previous lemmas and the formulas (A.15), (A.16) and (A.17), Lemma 3.1 is proved for the signature $(0, n)$. In other words, for $n = 4k + 2$, we have

$$\alpha_{0,n}(x'_1,\dots,x'_n) = \widehat{\alpha}_{0,n-1}(x_1,\dots,x_n) + x_n + x_n \sum_{i=1}^{n-1} x_i,$$

where $(x'_1,\dots,x'_n)$ is given in (A.14).

A.2.2 The signatures $(n, 0)$ and (p, q), where $pq > 0$

For the other signatures $(n, 0)$ and (p, q) with $pq > 0$, we are going to consider the following coordinate transformation

$$\begin{aligned} x''_1 &= x_n, & \\ x''_i &= x_{i-1} + x_n, & \text{for } i = 2,\dots,2k+1, \\ x''_i &= X + x_{n+2k+1-j}, & \text{for } i = 2k+2,\dots,n. \end{aligned} \tag{A.18}$$

In comparison with the coordinate transformation (A.14), it is the same up to a permutation. Since the cubic form $\alpha_{0,n}$ is invariant under permutation, the transformation (A.14) still gives the result $\alpha_{0,n} \simeq \widehat{\alpha}_{0,n-1} + x_n + x_n \sum_{i=1}^{n-1} x_i$. Applying (A.18), we have for the signature $(n, 0)$

$$\alpha_{n,0}(x''_1,\dots,x''_n) = \alpha_{0,n}(x''_1,\dots,x''_n) + \sum_{i=1}^{n} x''_i = \widehat{\alpha}_{n-1,0}(x_1,\dots,x_n).$$

Suppose that $p \leq q$ are odd numbers, then

$$\begin{aligned} \alpha_{p,q}(x''_1,\ldots,x''_n) &= \widehat{\alpha}_{0,n-1}(x_1,\ldots,x_n) + x_n + x_n\sum_{i=1}^{n-1} x_i + x_n + \sum_{i=1}^{p-1} x_i \\ &= \widehat{\alpha}_{p-1,q}(x_1,\ldots,x_n) + x_n \sum_{1\leq i\leq n-1} x_i. \end{aligned}$$

Suppose that $p \leq q$ are even numbers, then

$$\begin{aligned} \alpha_{p,q}(x''_1,\ldots,x''_n) &= \widehat{\alpha}_{0,n-1}(x_1,\ldots,x_n) + x_n + x_n\sum_{i=1}^{n-1} x_i + \sum_{i=1}^{p-1} x_i \\ &= \widehat{\alpha}_{p-1,q}(x_1,\ldots,x_n) + x_n + x_n \sum_{1\leq i\leq n-1} x_i. \end{aligned}$$

The case where $p > q$ is deduced from the properties of Chapter 2 on the classification. Suppose that $p > q$ are odd numbers, then

$$\begin{aligned} \alpha_{p,q}(x''_1,\ldots,x''_n) &\simeq \alpha_{q,p}(x''_1,\ldots,x''_n) && \text{due to Theorem 2.1}(i), \\ &\simeq \widehat{\alpha}_{q-1,p}(x_1,\ldots,x_n) && \text{since} \quad q<p, \\ &\simeq \widehat{\alpha}_{q,p-1}(x_1,\ldots,x_n) && \text{due to Lemma 2.5}, \\ &\simeq \widehat{\alpha}_{p-1,q}(x_1,\ldots,x_n) && \text{due to Theorem 2.1}(i). \end{aligned}$$

Suppose that $p > q$ are even numbers, then

$$\begin{aligned} \alpha_{p,q}(x''_1,\ldots,x''_n) &\simeq \alpha_{q,p}(x''_1,\ldots,x''_n) && \text{due to Theorem 2.1}(i), \\ &\simeq \alpha_{q+2,p-2}(x''_1,\ldots,x''_n) && \text{due to Lemma 2.5}, \\ &\simeq \widehat{\alpha}_{q+1,p-2}(x_1,\ldots,x_n) && \text{since} \quad q<p, \\ &\simeq \widehat{\alpha}_{p-2,q+1}(x_1,\ldots,x_n) && \text{due to Theorem 2.1}(i), \\ &\simeq \widehat{\alpha}_{p-1,q}(x_1,\ldots,x_n) && \text{due to Lemma 2.5}. \end{aligned}$$

Hence the result of Lemma 3.2 for all signatures.

A.3 Proof of Lemma 3.3, the case $n = 4k + 1$

Lemma 3.3 states that $\alpha_{p,q}$ (with $p + q \equiv 1 \bmod 4$) is equivalent to another cubic form where the variable x_1 is nearly factorized. In the decomposition of this later cubic form appears quadratic forms associated to the Clifford algebras. The point of this section is to exhibit suitable coordinate transformations in the different cases. The proof of Lemma 3.3 is divided in two cases, the first where $k > 1$ is odd and the second where $k > 0$ is even.

A.3.1 The subcase where k is odd

Consider the case where k is odd ($n = 4k + 1$) and give some details on the proof of Lemma 3.3. We choose the following coordinate transformation

$$\begin{aligned} x'_1 &= X, & & \\ x'_i &= x_1 + x_i, & &\text{for } i = 2, \dots, 2k+1, \\ x'_i &= x_1 + X + x_i, & &\text{for } i = 2k+2, \dots, n. \end{aligned} \tag{A.19}$$

Partial results on the linear, quadratic and cubic parts are the following.

Lemma A.10. *Applying the coordinate transformation (A.19), if $n = 4k+1$ and k is odd, then we have the following equality of the linear part*

$$\sum_{i=1}^{n} x'_i \quad = \quad x_1. \tag{A.20}$$

Proof. Straightforward. □

Lemma A.11. *Applying the coordinate transformation (A.19), if $n = 4k+1$ and k is odd, then we have the following equality of the quadratic part*

$$\begin{aligned} \sum_{1 \le i < j \le n} x'_i x'_j \quad = \quad & \sum_{2 \le i < j \le 2k+1} x_i x_j + \sum_{2k+2 \le i < j \le n} x_i x_j \\ & + x_1 + x_1 \sum_{i=2k+2}^{n} x_i + \sum_{i=2k+2}^{n} x_i. \end{aligned} \tag{A.21}$$

Proof. Let us decompose the sum, according to the coordinate transformation, the quadratic part (QP) is given by

$$\begin{aligned} QP \quad := \quad & \sum_{1 \le i < j \le n} x'_i x'_j \\ = \quad & \sum_{2 \le i < j \le 2k+1} (x_1 + x_i)(x_1 + x_j) + \sum_{i=2}^{2k+1} \sum_{j=2k+2}^{4k+1} (x_1 + x_i)(x_1 + x_j + X) \\ & + X(x_1 + X) + \sum_{2k+2 \le i < j \le 4k+1} (x_1 + x_i + X)(x_1 + x_j + X). \end{aligned}$$

We use the result (A.3) and simplify the terms in the brackets $[\ ]_1$ and $[\ ]_2$, then we obtain the expected result after distribution and simplification.

$$
\begin{aligned}
QP = {} & \left[x_1X\right]_1 + \left[X\right]_2 + \sum_{2\leq i<j\leq 2k+1} x_ix_j + \left[x_1\right]_1 + \left[x_1 \sum_{i=2}^{2k+1} x_i\right]_1 + x_1 + \left[X\right]_2 \\
& + (\left[x_1\right]_1 + X) \sum_{j=2k+2}^{n} x_j + \sum_{i=2}^{2k+1} x_i \sum_{j=2k+2}^{n} x_j + \sum_{2k+2\leq i<j\leq n} x_ix_j.
\end{aligned}
$$

□

Lemma A.12. *Applying the coordinate transformation (A.19), if $n = 4k+1$ and k is odd, then we have the following equality of the cubic part*

$$
\sum_{1\leq i<j<l\leq n} x'_ix'_jx'_l = x_1 \left(\sum_{2\leq i<j\leq 2k+1} x_ix_j + \sum_{2k+2\leq i<j\leq n} x_ix_j \right) + x_1. \qquad \text{(A.22)}
$$

Proof. Let us decompose the sum according to the coordinate transformation then use the result on the quadratic part (A.21) and the preliminaries results for k odd (A.3) and (A.5), the cubic part (CP)

$$
CP := \sum_{1\leq i<j<l\leq n} x'_ix'_jx'_l
$$

is given by

$$
\begin{aligned}
& X \left(\left[x_1\right]_1 + \left[\sum_{2\leq i<j\leq 2k+1} x_ix_j\right]_2 + \left[\sum_{2k+2\leq i<j\leq n} x_ix_j\right]_3 + (x_1+1) \sum_{j=2k+2}^{n} x_j \right) \\
& + XX(\left[x_1\right]_1 + X) + \left[\sum_{2\leq i<j<l\leq 2k+1} x_ix_jx_l\right]_2 + \left[\sum_{2\leq i<j\leq 2k+1} x_ix_j \sum_{l=2k+2}^{n} x_l\right]_2 \\
& + x_1 \left(1 + \sum_{i=2}^{2k+1} x_i\right) \sum_{l=2k+2}^{n} x_l + \left[\sum_{i=2}^{2k+1} x_i \sum_{2k+2\leq j<l\leq n} x_jx_l\right]_3 \\
& + \sum_{i=2}^{2k+2} x_i(x_1+X) \left(1 + \sum_{j=2k+2}^{n} x_j\right) + \left[\sum_{2k+2\leq i<j<l\leq n} x_ix_jx_l\right]_3.
\end{aligned}
$$

If we first simplify the terms in the brackets $[\ ]_1$, $[\ ]_2$ and $[\ ]_3$, then we obtain according to the colors

$$\begin{aligned} CP &= X(x_1+1)\sum_{j=2k+2}^{n} x_j + X + x_1\left(1+\sum_{i=2}^{2k+2} x_i\right)\sum_{j=2k+2}^{n} x_j \\ &\quad +(x_1+X)\sum_{i=2}^{2k+2} x_i\left(1+\sum_{i=2}^{2k+2} x_i\right) \\ &\quad +x_1\left(\left[\sum_{2\le i<j\le 2k+1} x_i x_j\right]_2 + \left[\sum_{2k+2\le i<j\le n} x_i x_j x_l\right]_3\right). \end{aligned}$$

Then distribute and simplify again according the terms in the brackets $[\ ]_4$, $[\ ]_5$ and $[\ ]_6$, the cubic part (CP) is given by

$$\begin{aligned} &\left[x_1\sum_{j=2k+2}^{n} x_j\right]_4 + \left[\sum_{i=2}^{2k+1} x_i \sum_{j=2k+2}^{n} x_j\right]_5 + \sum_{j=2k+2}^{n} x_j + \left[x_1\sum_{i=2}^{2k+1} x_i \sum_{j=2k+2}^{n} x_j\right]_6 \\ &+X + \left[x_1\sum_{l=2k+2}^{n} x_l\right]_4 + \left[x_1\sum_{i=2}^{2k+1} x_i \sum_{l=2k+2}^{n} x_l\right]_6 + \sum_{i=2}^{2k+1} x_i \\ &+\left[\sum_{i=2}^{2k+1} x_i \sum_{j=2k+2}^{n} x_j\right]_5 + x_1\left(\sum_{2\le i<j\le 2k+1} x_i x_j + \sum_{2k+2\le i<j\le n} x_i x_j x_l\right). \end{aligned}$$

Hence the result. □

Thanks to the three previous lemmas and the formulas (A.20), (A.21) and (A.22), Lemma 3.3 is proved for the signature $(0,n)$ if k is odd. In other words, due to the coordinate transformation (A.19) for $n = 4k+1$ and k odd, the form $\alpha_{0,n}$ is equivalent to

$$x_1 + (x_1+1)\left(\alpha^{\mathrm{Cl}}_{2k,0}(x_2,\ldots,x_{2k+1}) + \alpha^{\mathrm{Cl}}_{0,2k}(x_{2k+1},\ldots,x_{4k+1})\right).$$

According to the linear part, the form $\alpha_{n,0}$ is equivalent to

$$(x_1+1)\left(\alpha^{\mathrm{Cl}}_{2k,0}(x_2,\ldots,x_{2k+1}) + \alpha^{\mathrm{Cl}}_{0,2k}(x_{2k+2},\ldots,x_{4k+1})\right).$$

To obtain the two last expressions for the signatures $(2k+1, 2k+2)$ and $(2k, 2k+3)$, we have to permute the indices of the coordinate transformation

(A.19) has follow

$$\begin{aligned} \sigma(i) &= 2k+1+i, && \text{for } i = 1,\dots,2k,\\ \sigma(2k+1) &= 1, \\ \sigma(j) &= j-2k, && \text{for } j = 2k+2,\dots,n, \end{aligned} \tag{A.23}$$

and we obtain a new coordinate transformation

$$\begin{aligned} x''_i &= x_1 + X + x_{2k+1+i}, && \text{for } i = 1,\dots,2k,\\ x''_{2k+1} &= X,\\ x''_i &= x_1 + x_{i-2k}, && \text{for } i = 2k+2,\dots,n. \end{aligned}$$

Since the form $\alpha_{0,n}$ is invariant under permutation and that

$$\sum_{i=1}^{2k} x''_i = \sum_{i=2k+2}^{4k+1} x_i \qquad \text{and} \qquad \sum_{i=1}^{2k-2} x''_i = \sum_{i=2k+2}^{4k-1} x_i,$$

we obtain the attended results on $\alpha_{2k,2k+1}$ and $\alpha_{2k-2,2k+3}$. In other words, the form $\alpha_{2k,2k+1}$ is equivalent to

$$x_1 + (x_1+1)\left(\alpha^{\mathrm{Cl}}_{2k,0}(x_2,\dots,x_{2k+1}) + \alpha^{\mathrm{Cl}}_{0,2k}(x_{2k+2},\dots,x_{4k+1})\right) + \sum_{i=2k+2}^{4k+1} x_i$$

and the form $\alpha_{2k-2,2k+3}$ is equivalent to

$$x_1 + (x_1+1)\left(\alpha^{\mathrm{Cl}}_{2k,0}(x_2,\dots,x_{2k+1}) + \alpha^{\mathrm{Cl}}_{0,2k}(x_{2k+2},\dots,x_{4k+1})\right) + \sum_{i=2k+2}^{4k-1} x_i.$$

A.3.2 The subcase where k is even

We continue the proof of Lemma 3.3 and consider k even, $k \in \mathbb{N}\backslash\{0\}$. We choose the following coordinate transformation

$$\begin{aligned} x'_1 &= X,\\ x'_i &= X + x_1 + x_i, && \text{for } i = 2,\dots,2k+1,\\ x'_i &= x_1 + x_i, && \text{for } i = 2k+2,\dots,n. \end{aligned} \tag{A.24}$$

Partial results on the linear, quadratic and cubic parts are the following.

Lemma A.13. *Applying the coordinate transformation (A.24), if $n = 4k+1$ and k is even, then we have the following equality of the linear part*

$$\sum_{i=1}^{n} x'_i \quad = \quad x_1. \tag{A.25}$$

Proof. Straightforward. □

Lemma A.14. *Applying the coordinate transformation (A.24), if $n = 4k+1$ and k is even, then we have the following equality of the quadratic part*

$$\begin{aligned}\sum_{1\le i<j\le n} x'_i x'_j \quad = \quad & \sum_{2\le i<j\le 2k+1} x_i x_j + \sum_{2k+2\le i<j\le n} x_i x_j \\ & + x_1 \sum_{i=2}^{2k+1} x_i + \sum_{i=2k+2}^{n} x_i.\end{aligned} \tag{A.26}$$

Proof. The proof is similar to the one of Lemma A.11. □

Lemma A.15. *Applying the coordinate transformation (A.24), if $n = 4k+1$ and k is even, then we have the following equality of the cubic part*

$$\begin{aligned}\sum_{1\le i<j<l\le n} x'_i x'_j x'_l \quad = \quad & x_1 \left(\sum_{2\le i<j\le 2k+1} x_i x_j + \sum_{2k+2\le i<j\le n} x_i x_j \right) \\ & + x_1 \sum_{i=2}^{n} x_i.\end{aligned} \tag{A.27}$$

Proof. The proof is similar to the one of Lemma A.12. □

According to the three previous lemmas and the formulas (A.25), (A.26) and (A.27), Lemma 3.3 is proved for the signature $(0, n)$ if k is even. In other words, due to the coordinate transformation (A.24) for $n = 4k + 1$ and k even, the form $\alpha_{0,n}$ is equivalent to

$$x_1 + (x_1 + 1)\left(\alpha^{\mathrm{Cl}}_{2k,0}(x_2, \ldots, x_{2k+1}) + \alpha^{\mathrm{Cl}}_{0,2k}(x_{2k+1}, \ldots, x_{4k+1})\right).$$

Due to the linear part, it is direct that the form $\alpha_{n,0}$ is equivalent to

$$(x_1+1)\left(\alpha^{\mathrm{Cl}}_{2k,0}(x_2,\ldots,x_{2k+1})+\alpha^{\mathrm{Cl}}_{0,2k}(x_{2k+2},\ldots,x_{4k+1})\right).$$

To obtain the two last expressions for the signatures $(2k+1, 2k+2)$ and $(2k, 2k+3)$, we have to permute the indices of the coordinate transformation (A.24) as in the case where k is odd, see (A.23), and obtain a new coordinate transformation

$$\begin{aligned} x''_i &= x_1 + x_{2k+1+i}, && \text{for } i=1,\ldots,2k,\\ x''_{2k+1} &= X,\\ x''_i &= X + x_1 + x_{i-2k}, && \text{for } i=2k+2,\ldots,n. \end{aligned}$$

We obtain the expected result where k is even, as in the case where k is odd.

A.4 Proof of Lemma 3.4, the case $n = 4k+3$

Lemma 3.4 states that $\alpha_{p,q}$ (with $p+q \equiv 3 \bmod 4$) is equivalent to another cubic form where the first variable x_1 is nearly factorized. In the decomposition of this later cubic form appears quadratic forms associated to the Clifford algebras. As in Section A.3, we exhibit suitable coordinate transformations. The proofs in this section are very similar to the ones in Section A.3, therefore we give here the different lemmas without proofs and explain the different steps.

A.4.1 The subcase where k is odd

Let us consider the proof of Lemma 3.4, with $n = 4k+4$ and k is odd ($k \in \mathbb{N}\backslash\{0\}$). Choose the following coordinate transformation

$$\begin{aligned} x'_1 &= x_{2k+2} + x_{2k+3},\\ x'_i &= x_{2k+2} + x_i, && \text{for } i=2,\ldots,2k+1,\\ x'_{2k+2} &= x_{2k+2},\\ x'_{2k+3} &= X + x_{2k+2},\\ x'_i &= X + x_{2k+2} + x_i, && \text{for } i=2k+4,\ldots,n. \end{aligned} \tag{A.28}$$

Partial results on the linear, quadratic and cubic parts are the following.

Lemma A.16. *Applying the coordinate transformation (A.28), if $n = 4k+3$ and k is odd, then we have the following equality of the linear part*

$$\sum_{i=1}^{n} x'_i \quad = \quad x_1. \tag{A.29}$$

Proof. Straightforward. □

Lemma A.17. *Applying the coordinate transformation (A.28), if $n = 4k+3$ and k is odd, then we have the following equality of the quadratic part*

$$\begin{aligned} \sum_{1\leq i<j\leq n} x'_i x'_j \quad = \quad & \sum_{1\leq i<j\leq 2k+3} x_i x_j + \sum_{2k+4\leq i<j\leq n} x_i x_j \\ & + x_1 + x_1 x_{2k+2} + \sum_{i=2k+4}^{n} x_i. \end{aligned} \tag{A.30}$$

Proof. The proof is similar to the one of Lemma A.11. □

Lemma A.18. *Applying the coordinate transformation (A.28), if $n = 4k+3$ and k is odd, then we have the following equality of the cubic part*

$$\begin{aligned} \sum_{1\leq i<j<l\leq n} x'_i x'_j x'_l \quad = \quad & x_1 \left(\sum_{2\leq i<j\leq 2k+3} x_i x_j + \sum_{2k+4\leq i<j\leq n} x_i x_j \right) \\ & + x_1 \sum_{i=2}^{n} x_i + x_1 x_{2k+2} + x_1. \end{aligned} \tag{A.31}$$

Proof. The proof is similar to the one of Lemma A.12. □

According to the three previous lemmas and the formulas (A.29), (A.30) and (A.31), Lemma 3.4 is proved for the signature $(0, n)$ if k is odd. So, due to the coordinate transformation (A.28) for $n = 4k+3$ and k odd, the form $\alpha_{0,n}$ is equivalent to

$$x_1 + (x_1 + 1) \left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2, \dots, x_{2k+3}) + \alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4}, \dots, x_{4k+3}) \right).$$

Due to the linear part, it is direct that the form $\alpha_{n,0}$ is equivalent to

$$(x_1 + 1) \left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2, \dots, x_{2k+3}) + \alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4}, \dots, x_{4k+3}) \right),$$

that the form $\alpha_{2k+1,2k+2}$ is equivalent to

$$x_1 + (x_1+1)\left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2,\ldots,x_{2k+3}) + \alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4},\ldots,x_{4k+3})\right) + \sum_{i=2}^{2k+3} x_i$$

and that the form $\alpha_{2k,2k+3}$ is equivalent to

$$x_1+(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2,\ldots,x_{2k+3}) + \alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4},\ldots,x_{4k+3})\right)+\sum_{i=2}^{2k} x_i+x_{2k+3}.$$

A.4.2 The subcase where k is even

Let us continue the proof of Lemma 3.4 by giving the different steps of the proof. If k is even ($k \in \mathbb{N}$), then choose the following coordinate transformation

$$\begin{aligned}
x'_1 &= x_{2k+2} + x_n, \\
x'_i &= x_{2k+2} + x_i + X, && \text{for } i = 2,\ldots,2k+1, \\
x'_{2k+2} &= x_{2k+2}, \\
x'_i &= x_{2k+2} + x_i, && \text{for } i = 2k+3,\ldots,n-1, \\
x'_n &= X + x_{2k+2}.
\end{aligned} \tag{A.32}$$

Let us first give partial results on the linear, quadratic and cubic parts.

Lemma A.19. *Applying the coordinate transformation (A.32), if $n = 4k+3$ and k is even, then we have the following equality of the linear part*

$$\sum_{i=1}^{n} x'_i = x_1. \tag{A.33}$$

Proof. Straightforward. □

Lemma A.20. *Applying the coordinate transformation (A.32), if $n = 4k+3$ and k is even, then we have the following equality of the quadratic part*

$$\begin{aligned}\sum_{1\le i<j\le n} x'_i x'_j &= \sum_{2\le i<j\le 2k+1} x_i x_j + \sum_{2k+2\le i<j\le n} x_i x_j \\ &\quad + x_1 x_{2k+2} + (x_1+1)\sum_{i=2k+2}^{n} x_i.\end{aligned} \tag{A.34}$$

Proof. The proof is similar to the one of Lemma A.11. □

Lemma A.21. *Applying the coordinate transformation (A.32), if $n = 4k+3$ and k is even, then we have the following equality of the cubic part*

$$\begin{aligned}\sum_{1\le i<j<l\le n} x'_i x'_j x'_l &= x_1 \left(\sum_{2\le i<j\le 2k+1} x_i x_j + \sum_{2k+2\le i<j\le n} x_i x_j \right) \\ &\quad + x_1 x_{2k+2}\end{aligned} \tag{A.35}$$

Proof. The proof is similar to the one of Lemma A.12. □

According to the three previous lemmas and the formulas (A.33), (A.34) and (A.35), Lemma 3.4 is proved for the signature $(0, n)$ if k is even. So, due to the coordinate transformation (A.32) for $n = 4k + 3$ and k even, the form $\alpha_{0,n}$ is equivalent to

$$x_1 + (x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\dots,x_{2k+1}) + \alpha^{\mathcal{C}l}_{0,2k+2}(x_{2k+2},\dots,x_{4k+3})\right).$$

Due to the linear part, it is direct that the form $\alpha_{n,0}$ is equivalent to

$$(x_1+1)\left(\alpha^{\mathcal{C}l}_{2k,0}(x_2,\dots,x_{2k+1}) + \alpha^{\mathcal{C}l}_{0,2k+2}(x_{2k+2},\dots,x_{4k+3})\right).$$

To obtain the two last expressions for the signatures $(2k+1, 2k+2)$ and $(2k, 2k+3)$, we have to permute the indices of the coordinate transformation (A.32) has follow

$$\begin{aligned}\sigma(1) &= 1, \\ \sigma(i) &= 2k+1+i, && \text{for } i = 2,\dots,2k+1, \\ \sigma(2k+2) &= 2k+2, \\ \sigma(i) &= i-(2k+1), && \text{for } i = 2k+3,\dots,4k+2, \\ \sigma(n) &= n,\end{aligned}$$

and obtain a new coordinate transformation

$$\begin{aligned}
x''_1 &= x_{2k+2} + x_n, \\
x''_i &= x_{2k+2} + x_{2k+1+i}, && \text{for } i = 2, \dots, 2k+1, \\
x''_{2k+2} &= x_{2k+2}, \\
x''_i &= X + x_{2k+2} + x_{i-(2k+1)}, && \text{for } i = 2k+3, \dots, n-1, \\
x''_n &= X + x_{2k+2}.
\end{aligned}$$

Since the form $\alpha_{0,n}$ is invariant under permutation and

$$\sum_{i=1}^{2k+1} x''_i = \sum_{i=2k+2}^{4k+3} x_i \qquad \text{and} \qquad \sum_{i=1}^{2k} x''_i = x_{4k+3} + \sum_{i=2k+3}^{4k+1} x_i,$$

we obtain the attended results on $\alpha_{2k+1,2k+2}$ and $\alpha_{2k,2k+3}$. In other words, the form $\alpha_{2k+1,2k+2}$ is equivalent to

$$x_1 + (x_1+1)\left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2,\dots,x_{2k+3}) + \alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4},\dots,x_{4k+3})\right) + \sum_{j=2k+2}^{4k+3} x_j$$

and the form $\alpha_{2k,2k+3}$ is equivalent to

$$\begin{aligned}
& x_1 + (x_1+1)\left(\alpha^{\mathcal{C}l}_{2k+2,0}(x_2,\dots,x_{2k+3}) + \alpha^{\mathcal{C}l}_{0,2k}(x_{2k+4},\dots,x_{4k+3})\right) \\
& + x_{4k+3} + \sum_{j=2k+3}^{4k+1} x_j.
\end{aligned}$$

We obtain the expected result where k is even.

Appendix - B

Mathematica code

In this **Appendix** B, we first give in **Section** B.1 the `Mathematica` code to compute the function $\alpha_{p,q}$ with formal variables. Then in **Section** B.2, we give the `Mathematica` code to compute the statistic of $\alpha_{p,q}$, meaning that we count the number of elements $x \in \mathbb{Z}_2^n$ such that $\alpha_{p,q}(x) = 1$.

B.1 The cubic form $\alpha_{p,q}$

Programming helps a lot to guess the correct coordinate transformations of Lemmas 3.1, 3.2, 3.4 and 3.3 according to the four cases and the parity of k where $n = 4k + r$ with $r = 0, 1, 2$ or 3 and $k \in \mathbb{N}$. One of the computational goal is to see the emergence of properties on the different parts (cubic, quadratic and linear) of the functions $\alpha_{p,q}$ applied to `Mv` where `v` is in $\mathbb{Z}_2^n$ and `M` is a non singular matrix with coefficients in $\mathbb{Z}_2$. To do this, we encode the result of $\alpha_{p,q}$ (`Mv`) in a vector divided in three vectors corresponding respectively to the cubic, quadratic and linear parts. Computational examples for the different Lemmas 3.1, 3.2, 3.4 and 3.3 are explicitly described.

We denote by `vn` the element `(x1,...,xn)` of $\mathbb{Z}_2^n$ assimilated to a vector or also called the "elementary" element. We first generate the elementary element as formal variables of, for examples, $\mathbb{Z}_2^5$, $\mathbb{Z}_2^6$, $\mathbb{Z}_2^7$, $\mathbb{Z}_2^8$, and $\mathbb{Z}_2^9$.

```
:> v5 = Table[Symbol["x" <> ToString[i]], {i, 5}]
   v6 = Table[Symbol["x" <> ToString[i]], {i, 6}]
```

```
   v7 = Table[Symbol["x" <> ToString[i]], {i, 7}]
   v8 = Table[Symbol["x" <> ToString[i]], {i, 8}]
   v9 = Table[Symbol["x" <> ToString[i]], {i, 9}]

Out[]= {x1, x2, x3, x4, x5}
       {x1, x2, x3, x4, x5, x6}
       {x1, x2, x3, x4, x5, x6, x7}
       {x1, x2, x3, x4, x5, x6, x7, x8}
       {x1, x2, x3, x4, x5, x6, x7, x8, x9}
```

The function `alphapq` computes the form $\alpha_{p,q}$ for any given element $\mathtt{x} \in \mathbb{Z}_2^n$. The arguments in the function `alphapq` are the following: `v` is the elementary element defined above which is used to simplify the expression modulo 2; `z` is any element of $\mathbb{Z}_2^n$ usually obtained by coordinate transformation of `v`; and finally `p` and `q` correspond to the signature of the form.

```
:> alphapq[v_, z_, p_, q_] := Module[{t, u, u1, u2, u3, l },
   t = 0; u = 0;
```

We compute the cubic, quadratic and linear parts.

```
   For[i = 1, i <= p + q, i++,
      For[j = i + 1, j <= p + q, j++,
         For[k = j + 1, k <= p + q, k++,
           t = t + z[[i]]*z[[j]]*z[[k]]
    ]; ]; ];
   For[i = 1, i <= p + q, i++,
      For[j = i + 1, j <= p + q, j++,
        t = t + z[[i]]*z[[j]]
    ]; ];
   For[i = p + 1, i <= p + q, i++,
     t = t + z[[i]]
    ];
```

The variables `xi` are in $\mathbb{Z}_2$, hence we have $\mathtt{xi}^2 = \mathtt{xi}$ for $\mathtt{i} = 1, \ldots, \mathtt{n}$. We simplify the squares and cubes in the terms of the sum (encoded in the variable `t`) according to the elementary element `v`.

```
t = Expand[t, Modulus -> 2];
For[i = 1, i <= p + q, i++,
  t = t /. {(v[[i]])^2 -> v[[i]], (v[[i]])^3 -> v[[i]]}
 ];
```

We put in order the terms of the sum in a vector according the cubic, quadratic and linear parts which depends on the length of the terms.

```
t = Expand[t, Modulus -> 2];
u = Sort[MonomialList[t], Length[#1] > Length[#2] &];
```

We arrange the terms of the sum into a vector which contains three vectors themselves containing respectively the cubic, quadratic and linear part. We take care of the particular cases when some parts are empty and put in order again the different vectors.

```
t = 0; l = 1; u1 = 0; u2 = 0; u3=0;
If[Length[u] == t, u = {{}, {}, {}},
    While[l<(Length[u]+1) && Length[u[[l]]]==3,t=t+1;l++];
    u3 = u[[1 ;; t]];
   If[Length[u] == t, u = {Sort[u3], {}, {}},
       While[l<(Length[u]+1) && Length[u[[l]]]==2,t=t+1;l++];
       u2 = u[[Length[u3] + 1 ;; t]];
      If[Length[u] == t, u = {Sort[u3], Sort[u2], {}},
          u1 = u[[t+1 ;; Length[u]]];
          u = {Sort[u3], Sort[u2], Sort[u1]}];
    ]; ];
  u
 ]
```

Let us go through examples and explicit calculations. Note that the arrangement of the result of `alphapq` into a vector divided into three vectors helps us to compare two functions and more precisely to compare the cubic, quadratic and linear parts independently from each other.

Example B.1. Consider a computational example of Lemma 3.1.

```
:> M8 = {{1, 0, 0, 0, 0, 0, 0, 1},
        {1, 1, 0, 0, 0, 0, 0, 1},
        {1, 0, 1, 0, 0, 0, 0, 1},
        {1, 0, 0, 1, 0, 0, 0, 1},
        {1, 0, 0, 0, 1, 0, 0, 1},
        {0, 1, 1, 1, 1, 0, 1, 1},
        {0, 1, 1, 1, 1, 1, 0, 1},
        {0, 0, 0, 0, 0, 0, 0, 1}};
   z8 = M8.v8;
   alphapq[v8, z8, 0, 8] == alphapq[v7, v7, 0, 7]

Out[]= True
```

The output means that the two functions $\alpha_{0,8}$(`z8`) and $\alpha_{0,7}$(`v7`) are equal.

Example B.2. Consider a computational example of Lemma 3.2.

```
:> M6 = {{1, 0, 0, 0, 0, 1},
        {0, 1, 0, 0, 0, 1},
        {0, 0, 1, 0, 0, 1},
        {1, 1, 1, 0, 1, 1},
        {1, 1, 1, 1, 0, 1},
        {0, 0, 0, 0, 0, 1}};
   z6 = M6.v6;
   alphapq[v6, z6, 0, 6][[1]] == alphapq[v5, v5, 0, 5][[1]]
   alphapq[v6, z6, 0, 6]

Out[]= True
Out[]= {{x1 x2 x3, x1 x2 x4, x1 x3 x4, x2 x3 x4, x1 x2 x5,
       x1 x3 x5, x2 x3 x5, x1 x4 x5, x2 x4 x5, x3 x4 x5},
       {x1 x2, x1 x3, x2 x3, x1 x4, x2 x4, x3 x4, x1 x5,
       x2 x5, x3 x5, x4 x5, x1 x6, x2 x6, x3 x6, x4 x6, x5 x6},
       {x1, x2, x3, x4, x5, x6}}
```

The first output means that the cubic part of $\alpha_{0,6}$(`z6`) is equal to the cubic part of $\alpha_{0,5}$(`v5`). The second output is just the result of $\alpha_{0,6}$(`z6`) where the variable `x6` appears only in the linear and in the quadratic part.

Example B.3. Consider a computational example of Lemma 3.4.

```
:> M7 = {{0, 0, 0, 1, 1, 0, 0},
         {0, 1, 0, 1, 0, 0, 0},
         {0, 0, 1, 1, 0, 0, 0},
         {0, 0, 0, 1, 0, 0, 0},
         {1, 1, 1, 0, 1, 1, 1},
         {1, 1, 1, 0, 1, 0, 1},
         {1, 1, 1, 0, 1, 1, 0}};
   z7 = M7.v7;
   alphapq[v7, z7, 0, 7]

Out[]= {{x1 x2 x3, x1 x2 x4, x1 x3 x4, x1 x2 x5, x1 x3 x5,
       x1 x4 x5, x1 x6 x7}, {x2 x3, x2 x4, x3 x4, x2 x5,
       x3 x5, x4 x5, x1 x6, x1 x7, x6 x7}, {x1, x6, x7}}
```

The variable x1 takes an important role in $\alpha_{0,7}$(z7). The output shows that the variable x1 in $\alpha_{0,7}$(z7) is nearly factorized.

Example B.4. Consider a computational example of Lemma 3.3.

```
:> M9 = {{1, 1, 1, 1, 1, 1, 1, 1, 1},
         {0, 0, 1, 1, 1, 1, 1, 1, 1},
         {0, 1, 0, 1, 1, 1, 1, 1, 1},
         {0, 1, 1, 0, 1, 1, 1, 1, 1},
         {0, 1, 1, 1, 0, 1, 1, 1, 1},
         {1, 0, 0, 0, 0, 1, 0, 0, 0},
         {1, 0, 0, 0, 0, 0, 1, 0, 0},
         {1, 0, 0, 0, 0, 0, 0, 1, 0},
         {1, 0, 0, 0, 0, 0, 0, 0, 1}};
   z9 = (M9).v9;
   alphapq[v9, z9, 0, 9]

Out[]= {{x1 x2 x3, x1 x2 x4, x1 x3 x4, x1 x2 x5, x1 x3 x5,
       x1 x4 x5, x1 x6 x7, x1 x6 x8, x1 x7 x8, x1 x6 x9,
       x1 x7 x9, x1 x8 x9}, {x2 x3, x2 x4, x3 x4, x2 x5, x3 x5,
```

```
x4 x5, x1 x6, x1 x7, x6 x7, x1 x8, x6 x8, x7 x8, x1 x9,
x6 x9, x7 x9, x8 x9}, {x1, x6, x7, x8, x9}}
```

The variable `x1` takes an important role in $\alpha_{0,9}$(`z9`). As in Example B.3, the output shows that the variable `x1` in $\alpha_{0,9}$(`z9`) is nearly factorized.

B.2 The statistic $s(p,q)$ of the algebras $\mathbb{O}_{p,q}$

We compute the statistics $s(p,q)$ of the algebras $\mathbb{O}_{p,q}$ and get results for large numbers $p+q$. It helps a lot at first time to convince ourself that the statistics of the algebras $\mathbb{O}_{p,q}$ can be used as obstruction of isomorphism preserving the structure of $\mathbb{Z}_2^n$-graded algebra.

First, we encode in a vector of vectors, `En`, all the elements of $\mathbb{Z}_2^n$. As an illustration, let us do it for $n=3$ and $n=4$.

```
:> E3 = Tuples[{0, 1}, 3]
   E4 = Tuples[{0, 1}, 4]

Out[]= {{0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {0, 0, 1, 1},
        {0, 1, 0, 0}, {0, 1, 0, 1}, {0, 1, 1, 0}, {0, 1, 1, 1},
        {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 1, 1},
        {1, 1, 0, 0}, {1, 1, 0, 1}, {1, 1, 1, 0}, {1, 1, 1, 1}}
Out[]= {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 1}, {0, 0, 0, 1, 0},
        {0, 0, 0, 1, 1}, {0, 0, 1, 0, 0}, {0, 0, 1, 0, 1},
        {0, 0, 1, 1, 0}, {0, 0, 1, 1, 1}, {0, 1, 0, 0, 0},
        {0, 1, 0, 0, 1}, {0, 1, 0, 1, 0}, {0, 1, 0, 1, 1},
        {0, 1, 1, 0, 0}, {0, 1, 1, 0, 1}, {0, 1, 1, 1, 0},
        {0, 1, 1, 1, 1}, {1, 0, 0, 0, 0}, {1, 0, 0, 0, 1},
        {1, 0, 0, 1, 0}, {1, 0, 0, 1, 1}, {1, 0, 1, 0, 0},
        {1, 0, 1, 0, 1}, {1, 0, 1, 1, 0}, {1, 0, 1, 1, 1},
        {1, 1, 0, 0, 0}, {1, 1, 0, 0, 1}, {1, 1, 0, 1, 0},
        {1, 1, 0, 1, 1}, {1, 1, 1, 0, 0}, {1, 1, 1, 0, 1},
        {1, 1, 1, 1, 0}, {1, 1, 1, 1, 1}}
```

The function `s` evaluates how much time the function $\alpha_{p,q}$ is equal to one on each element of $\mathbb{Z}_2^n$ for a given signature.

```
:> s[x_, p_, q_] := Module[{t = 0, u = 0},
    For[m = 1, m <= 2^(p + q), m++,
     t = 0;
     For[i = 1, i <= p + q, i++,
        For[j = i + 1, j <= p + q, j++,
          For[k = j + 1, k <= p + q, k++,
              t = t + x[[m, i]]*x[[m, j]]*x[[m, k]]
      ]; ]; ];
     Expand[t, Modulus -> 2]
      For[i = 1, i <= p + q, i++,
       For[j = i + 1, j <= p + q, j++,
           t = t + x[[m, i]]*x[[m, j]]
       ]; ];
     Expand[t, Modulus -> 2]
      For[i = p + 1, i <= p + q, i++,
       t = t + x[[m, i]]
       ];
     t = Expand[t, Modulus -> 2];
     u = u + t
     ];
    u
    ]
```

As examples, we have the following results for $n = 3$ and $n = 4$.

```
:> s[E3,3,0]      Out[]= 3
   s[E3,2,1]      Out[]= 3
   s[E3,1,2]      Out[]= 3
   s[E3,0,3]      Out[]= 7
   s[E4,4,0]      Out[]= 6
   s[E4,3,1]      Out[]= 8
   s[E4,2,2]      Out[]= 6
   s[E4,1,3]      Out[]= 8
   s[E4,0,4]      Out[]= 14
```

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