Invariant games and non-homogeneous Beatty sequences

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Wythoff’s game defined by a set of moves (or rules)

- 2 players play alternatively,
- first player unable to move loses (normal condition),
- 2 piles of tokens:
  - remove a positive number of tokens from one pile,
  - remove the same positive number of tokens from both piles.

\[ \mathcal{M}_W := \{(i, 0) \mid i > 0\} \cup \{(0, j) \mid j > 0\} \cup \{(k, k) \mid k > 0\}. \]

**Definition**

A game is *invariant* if the same moves can be played from every position (the only restriction is that enough tokens are available).

Invariant take-away games are Golomb’s vector subtraction games (1966).
Combinatorial games

Variant rule sets (example)

- Remove an even number of tokens from one pile whenever the total number of tokens is even;
- remove an odd number of tokens from one pile, otherwise.

Other examples of games usually defined with a variant ruleset

- Rat and mouse game, Fraenkel
- Raleigh game, Fraenkel’07
- Tribonacci game, Duchêne-R.’08
- Pisot cubic games, Duchêne-R.'08
- Flora game, Fraenkel’10
- ...
Wythoff’s game or “catching the queen”
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The first few $P$-positions (up to symmetry)

$$(0, 0), (1, 2), (3, 5), (4, 7), (6, 10), (8, 13), \ldots$$

**Definition of $P$-positions**

A $P$-position is a position $q$ from which the previous player (moving to $q$) can eventually force a win.

Well-known characterizations of the $P$-positions of Wythoff’s game (recursive, morphic, syntactical properties of Fibonacci expansions, ...)

**Theorem**

The $P$-positions of the Wythoff’s game are exactly the pairs

$$([n\tau], [n\tau^2]), \quad n \geq 0$$

where $\tau$ is the golden ratio $(1 + \sqrt{5})/2$. 

Combinatorial games

Remarks

- To a game, i.e., a set of rules, corresponds a set of $P$-positions.
- Several games may have the same set of $P$-positions.
- Hence, a given set of $P$-positions can be associated with invariant as well as variant games.
- One can define the notion of invariant subset of $\mathbb{N}^p$.

N.B. Ho, two variants of Wythoff’s game...

Adjoining a move removing $k$ tokens from the smaller pile (or any pile if the two piles have the same size) and $\ell$ tokens from the other pile where $\ell < k$. 
Rayleigh or Beatty’s theorem

Let $\alpha, \beta > 1$ be irrational numbers such that $\alpha^{-1} + \beta^{-1} = 1$. The sequences $(\lfloor n\alpha \rfloor)_{n>0}$ and $(\lfloor n\beta \rfloor)_{n>0}$ partition $\mathbb{N}_{>0}$.

Two such sequences are complementary (homogeneous) Beatty sequences.

Example

For the golden ratio, $\tau^2 = \tau + 1$ thus $1 = \tau^{-1} + (\tau^2)^{-1}$. Hence, the set of $P$-positions of Wythoff’s game is derived from a pair of complementary homogeneous Beatty sequences.

A000201 and A001950 in OEIS.
**Beatty sequences**

**Question**

Given a pair \((\lfloor n\alpha \rfloor)_{n>0}\) and \((\lfloor n\beta \rfloor)_{n>0}\) of homogeneous Beatty sequences, does there exist an invariant game having

\[
\{(0,0)\} \cup \{ (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor), (\lfloor n\beta \rfloor, \lfloor n\alpha \rfloor) \mid n > 0 \}
\]

as set of \(P\)-positions?

- \(\tau\) has c.f.-expansion \((1; \bar{1})\), Wythoff
- if \(\alpha\) has c.f.-expansion \((1; \bar{k})\), see Fraenkel’82.
- if \(\alpha\) has c.f.-expansion \((1; 1, \bar{k})\), see Duchêne-R’10.

We conjectured that the above question always has a positive answer.
**Beatty sequences**

A sequence \((B_n)_{n \geq 0}\) is \(B_1\)-superadditive if, for all \(m, n > 0\),

\[
B_m + B_n \leq B_{m+n} < B_m + B_n + B_1.
\]

**Theorem (Larsson, Hegarty, Fraenkel’11)**

Let \((A_n, B_n)_{n \geq 0}\) be a pair of complementary sequences with \(A_0 = B_0 = 0\). (Not necessarily Beatty sequences.) If the sequence \((B_n)_{n \geq 0}\) is \(B_1\)-superadditive, then

\[
\{(A_n, B_n), (B_n, A_n) \mid n \geq 0\}
\]

is the set of \(P\)-positions of an invariant game.

⋆ Every pair of homogeneous Beatty sequences satisfy the above condition, thus our conjecture holds (Larsson et al.).
**Beatty sequences**

**Question 1**

What about non-homogeneous Beatty sequences that realize the set of $P$-positions of an invariant game?

$$A_n = \lfloor n\alpha + \gamma \rfloor, \quad B_n = \lfloor n\beta + \delta \rfloor$$

where $\gamma, \delta \in \mathbb{R}$. We set $A_0 = B_0 = 0$.

We want that $\{A_n \mid n \geq 1\}$ and $\{B_n \mid n \geq 1\}$ partition $\mathbb{N}_{>0}$, this means that we look for an extension of Nim, i.e.,

$$\mathcal{M} = \{(i, 0) \mid i \geq 1\} \cup \{(0, i) \mid i \geq 1\} \cup \cdots.$$ 

**Question 2 (Larsson et al.)**

The superadditivity is a sufficient condition to get a set of $P$-positions of an invariant game. Is it also a necessary condition?
**Beatty sequences**

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**Question 2 (Larsson et al.)**

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Beatty sequences

Theorem (Fraenkel’69)

Let \( \alpha, \beta > 1 \) be irrational numbers such that
\[
\alpha^{-1} + \beta^{-1} = 1
\]
(1)

Then \( \{\lfloor n\alpha + \gamma \rfloor \mid n \geq 1 \} \) and \( \{\lfloor n\beta + \delta \rfloor \mid n \geq 1 \} \) partition \( \mathbb{N}_{>0} \) if and only if
\[
\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = 0 \quad \text{and} \quad n\beta + \delta \notin \mathbb{Z}, \text{ for all } n \geq 1.
\]
(2) \quad (3)

We assume moreover that
\[
A_1 = 1 \text{ and } B_1 \geq 3.
\]
(4)

also see, K. O’Bryant (Integers 2003)
Our main result

Roughly stated

We can characterize the 4-tuples \((\alpha, \beta, \gamma, \delta)\) such that the corresponding set

\[
\{(0, 0)\} \cup \{(\lfloor n\alpha + \gamma \rfloor, \lfloor n\beta + \delta \rfloor), (\lfloor n\alpha + \gamma \rfloor, \lfloor n\beta + \delta \rfloor) \mid n > 0\}
\]

is the set of \(P\)-positions of an invariant game.

The correct expression involves combinatorial properties of some infinite word derived from the two Beatty sequences (direct product of two mechanical words, except maybe for the first symbol).
Our main result

\[ \beta = 3.99 + \sqrt{5}/2 \simeq 5.108, \quad \gamma = -0.2, \quad \alpha = \beta/(\beta - 1) \simeq 1.243, \quad \delta = -\beta \gamma/\alpha \simeq 0.821 \]

iterating \( R_{\alpha, \beta} \): translation of \((\{\alpha\}, \{\beta\})\) over \( \mathbb{T}^2 \) starting from \((\{\gamma\}, \{\delta\})\), product of two Sturmian words

| \( A_{n+1} - A_n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| \( B_{n+1} - B_n \)  | 5 | 6 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 6 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| \( w_+ \)            | a | b | a | a | c | a | a | a | a | a | a | c | b | a | a | c | a | a | a | a |
| \( w_- \)            | a | b | a | a | c | a | a | a | a | a | a | c | b | a | a | c | a | a | a | a |
Our combinatorial condition reduces to

Take two intervals $I, J \neq \emptyset$ over $[0, 1)$ interpreted as intervals over the unit circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.

For a given 4-tuple $(\alpha, \beta, \gamma, \delta)$ of real numbers, we ask, whether or not there exists some $i$ such that $R_{\alpha,\beta}^i(\gamma, \delta) \in I \times J$?

Extension of the density theorem of Kronecker

The set $\{R_{\alpha,\beta}^i(\gamma, \delta) = (\{i\alpha + \gamma\}, \{i\beta + \delta\}) \in \mathbb{T}^2 \mid i \in \mathbb{N}\}$

is dense in $\mathbb{T}^2$ if and only if $\alpha, \beta, 1$ are rationally independent.

$\alpha, \beta, 1$ are rationally independent (i.e., linearly independent over $\mathbb{Q}$), if whenever there exist integers $p$ and $q$ such that $p\alpha + q\beta$ is an integer, then $p = q = 0$. 
So, if $\alpha, \beta, 1$ are rationally independent, then there exist infinitely many $i$ such that $R_{\alpha, \beta}^i(\gamma, \delta) \in I \times J$.

**Remark**

If $\alpha$ and $\beta$ are irrational numbers satisfying $\alpha^{-1} + \beta^{-1} = 1$ which are not both algebraic numbers of degree 2, then $\alpha, \beta, 1$ are rationally independent.
If $\alpha, \beta, 1$ are rationally dependent, since $\alpha$ and $\beta$ are irrational numbers, there exist integers $p, q, r$ with $p, q \neq 0$ such that $p\alpha + q\beta = r$.

We deduce that $q\beta^2 + (p - q - r)\beta + r = 0$, i.e., $\beta$ is thus an algebraic number of degree 2. The same holds for $\alpha$.

The set of points $\{R_{\alpha, \beta}^n(\gamma, \delta) \mid n \in \mathbb{N}\}$ is dense on a straight line in $\mathbb{T}^2$ with rational slope.

The initial question is reduced to determine whether or not a line intersect a rectangle.
Test

$\alpha = \frac{3 + \sqrt{17}}{2}, \quad \beta = \frac{7 + \sqrt{17}}{8}, \quad \alpha = 4\beta - 2, \quad \text{rational slope} \ 1/4$
Thanks to our main result, we can produce examples as the following one.

**Counter-example to Question 2**

The $4$-tuple given by

\[
\beta = 1.99 + \frac{\sqrt{5}}{2}, \quad \alpha = \frac{\beta}{\beta - 1}, \quad \gamma = -0.2 \quad \text{and} \quad \delta = -\frac{\beta \gamma}{\alpha}
\]

satisfies the characterization given by our main result, i.e., leads to a set of $P$-positions of an invariant game, but the sequence is not superadditive.