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Different possible behaviors of wavelet leaders of the Brownian motion

Antoine Ayache [a], Céline Esser [a,b,*,1], Thomas Kleyntssens [b]

[a] University of Lille, Laboratoire Paul Painlevé, Cité Scientifique, F-59655 Villeneuve d'Ascq, France
[b] University of Liège, Department of Mathematics, Quartier Polytech 1, Allée de la Découverte 12, Bâtiment B37, B-4000 Liège (Sart-Tilman), Belgium

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ABSTRACT

The aim of this paper is to prove that wavelet leaders allow to get very fine properties of the trajectories of the Brownian motion: we show that the three well-known behaviors of its oscillations, namely to be ordinary, rapid and slow, are also present in the behavior of the size of its wavelet leaders.

1. Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. In order to quantify its regularity around some fixed point $t \in \mathbb{R}$, it is natural to try to determine as sharply as possible the asymptotic behavior of the quantity

$$\operatorname{Osc}(f, I(t,\rho)) = \sup_{t'',t' \in I(t,\rho)} \left| f(t'') - f(t') \right| \quad \text{where} \quad I(t,\rho) = [t-\rho, t+\rho], \tag{1}$$

when $\rho \to 0^+$. This quantity is called the *oscillation* of f on $I(t,\rho)$.

In the last decades, wavelet methods have become a very powerful tool to finely study regularity properties of functions (Meyer and Salinger, 1995; Jaffard, 2004) and to obtain numerical methods to study real-life signals (see among others Jaffard, 2004; Arneodo et al., 1995; Jaffard et al., 2007; Lashermes et al., 2008; Esser et al., 2017; Deliège et al., 2017). A compactly supported function $\psi \in L^1(\mathbb{R})$ whose first moment vanishes, i.e.

$$\int_{\mathbb{R}} \psi(x)\, dx = 0\,, \tag{2}$$

is called a *compactly supported wavelet*. Let N be a positive integer such that the support of ψ is included in $[-N, N]$. The *wavelet coefficients* $c_{J,K}$, $(J,K) \in \mathbb{N} \times \mathbb{Z}$, of the function f are defined by

$$c_{J,K} = 2^J \int_{\mathbb{R}} f(t)\psi(2^J t - K)\, dt = \int_{\mathbb{R}} \left(f\left(\frac{x+K}{2^J}\right) - f\left(\frac{K}{2^J}\right)\right) \psi(x)\, dx \tag{3}$$

$$= \int_{-N}^{N} \left(f\left(\frac{x+K}{2^J}\right) - f\left(\frac{K}{2^J}\right)\right) \psi(x)\, dx. \tag{4}$$

* Corresponding author at: University of Liège, Department of Mathematics, Quartier Polytech 1, Allée de la Découverte 12, Bâtiment B37, B-4000 Liège (Sart-Tilman), Belgium.

E-mail address: celine.esser@uliege.be (C. Esser).

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In what follows, we will use the notation c_λ to denote the wavelet coefficient $c_{J,K}$, where λ is the dyadic interval $\lambda = \lambda_{J,K} = [2^{-J}K, 2^{-J}(K+1))$, which provides the location of the wavelet $\psi(2^J\cdot -K)$. We denote by Λ the set of all dyadic intervals of $\mathbb{R}$ and for any $J \in \mathbb{N}$, we denote by Λ_J the set of all dyadic intervals of size 2^{-J}. Amplitudes of wavelet coefficients $c_{\lambda'}$ located in some fixed interval λ (that is $\lambda' \subseteq \lambda$) can be very fluctuating from one scale to another. In order to avoid such a drawback, the so-called *wavelet leaders*, which among other things offer the advantage of stability, have been introduced in Jaffard et al. (2007). For every $\lambda_{J,K} \in \Lambda$, the *wavelet leader* $d_{\lambda_{J,K}}$ of f is defined by

$$d_{\lambda_{J,K}} = \max_{\lambda\in N(\lambda_{J,K})} \sup_{\lambda'\subseteq\lambda} \left|c_{\lambda'}\right| \quad \text{where} \quad N(\lambda_{J,K}) = \left\{\lambda_{J,K-1}, \lambda_{J,K}, \lambda_{J,K+1}\right\}. \tag{5}$$

Remark 1.1. In a probabilistic framework, the supremum on $N(\lambda_{J,K})$ appearing in (5) can create correlation between wavelet leaders, even if it does not exist between wavelet coefficients. It might seem more natural to consider only the dyadic intervals $\lambda' \subseteq \lambda_{J,K}$ in the definition of the wavelet leaders so that the supremum at a given scale is taken on non-overlapping intervals. Let us point out that the methodology of our article can easily be adapted to this setting. However we will work with the classical wavelet leaders defined as in (5) since they provide a characterization of pointwise regularity of functions (Jaffard, 2004).

Note that there exists a link between the notions of oscillation and wavelet leaders. First, for any $t \in \mathbb{R}$ and $J \in \mathbb{N}$, let $\lambda_J(t)$ denote the unique dyadic interval of size 2^{-J} that contains t, and let $d_J(t)$ be the corresponding wavelet leader. From (4) and (1), if $\lambda_{j,k} \subseteq \lambda$ for some $\lambda \in N\big(\lambda_J(t)\big)$, one has

$$|c_{j,k}| \le \int_{-N}^{N} \left| f\left(\frac{x+k}{2^j}\right) - f\left(\frac{k}{2^j}\right)\right| |\psi(x)|\,dx \le c_0 \operatorname{Osc}\Big(f, I\big(t, (2+N)2^{-J}\big)\Big),$$

where $c_0 = \|\psi\|_{L^1(\mathbb{R})}$. As a consequence, one has that

$$d_J(t) \le c_0 \operatorname{Osc}\Big(f, I\big(t, (2+N)2^{-J}\big)\Big). \tag{6}$$

Conversely, Jaffard (1998) has shown that, under some general assumptions on f, the reverse inequality holds up to a logarithmic factor, i.e.

$$\operatorname{Osc}\Big(f, \lambda_J(t)\Big) \le c_1 d_J(t) \left|\log(d_J(t))\right| \tag{7}$$

for some constant $c_1 > 0$. It can be derived from (6) and (7) that there is a priori a loss of information if one considers wavelet leaders instead of oscillations.

In the present paper, we treat the particular case of Brownian motion. Despite Inequality (7), a first numerical work (Kleyntssens and Nicolay, 2017) has led to the idea that, in this case, working with wavelet leaders instead of oscillations does not imply a logarithmic loss. We confirm in this paper that wavelet leaders are precise enough to reflect very fine properties of the trajectories of Brownian motion, namely the coexistence in them of slow, rapid and ordinary points. Section 2 contains useful recalls on the Brownian motion and the statement of the main result (Theorem 2.3). Section 3 is devoted to the proof of this result.

2. Brownian motion and statement of the main result

The Brownian motion is the unique real-valued centered Gaussian process $B = \{B(t)\}_{t\in\mathbb{R}}$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with independent and stationary increments satisfying $B(t)-B(s) \sim \mathcal{N}(0, |t-s|)$ for any $t, s \in \mathbb{R}$ and such that almost surely, B has continuous paths and $B(0) = 0$. From now on, if the value of $B(t)$ has to be explicitly associated to an elementary event $\omega \in \Omega$, the notation $B(t,\omega)$ will be used. The following theorem (Kahane, 1985) summarizes some properties of the regularity of B: it shows the existence of three different possible behaviors of the oscillations of the Brownian motion.

Theorem 2.1. *There exists an event $\Omega^* \subseteq \Omega$ of probability 1 such that for every $\omega \in \Omega^*$ and every non-empty open interval A of $\mathbb{R}$, there are $t_o(\omega), t_r(\omega), t_s(\omega) \in A$ such that*

1. *$t_o(\omega)$ is an ordinary point of $B(\cdot,\omega)$, i.e.*

$$0 < \limsup_{\rho\to 0^+} \left\{ \frac{\operatorname{Osc}\Big(B(\cdot,\omega), I\big(t_o(\omega),\rho\big)\Big)}{\rho^{1/2}\sqrt{\log\log(\rho^{-1})}} \right\} < +\infty\,;$$

2. *$t_r(\omega)$ is a rapid point of $B(\cdot,\omega)$, i.e.*

$$0 < \limsup_{\rho\to 0^+} \left\{ \frac{\operatorname{Osc}\Big(B(\cdot,\omega), I\big(t_r(\omega),\rho\big)\Big)}{\rho^{1/2}\sqrt{\log(\rho^{-1})}} \right\} < +\infty\,;$$

3. *$t_s(\omega)$ is a slow point of $B(\cdot,\omega)$, i.e.*

$$0 < \limsup_{\rho\to 0^+} \left\{ \frac{\mathrm{Osc}\Big(B(\cdot,\omega), I\big(t_s(\omega),\rho\big)\Big)}{\rho^{1/2}} \right\} < +\infty\,.$$

Moreover, for every $\omega\in\Omega^*$, almost every $t\in\mathbb{R}$ is an ordinary point of $B(\cdot,\omega)$, and for every $t\in\mathbb{R}$,

$$\limsup_{\rho\to 0^+} \left\{ \frac{\mathrm{Osc}\Big(B(\cdot,\omega), I\big(t,\rho\big)\Big)}{\rho^{1/2}\sqrt{\log(\rho^{-1})}} \right\} < +\infty\,. \tag{8}$$

From now on, we assume that $c_{J,K}$ (resp. $d_{J,K}$), $(J,K)\in\mathbb{N}\times\mathbb{Z}$, represent the wavelet coefficients (resp. the wavelet leaders) of the Brownian motion B. Theorem 2.1 motivates the following definition.

Definition 2.2. Let $\omega\in\Omega$ and $t\in\mathbb{R}$. We say that

1. t is a *leader-ordinary point* of $B(\cdot,\omega)$ if

$$0 < \limsup_{J\to+\infty} \left\{ \frac{d_J\big(t,\omega\big)}{2^{-J/2}\sqrt{\log(J)}} \right\} < +\infty\,;$$

2. t is a *leader-rapid point* of $B(\cdot,\omega)$ if

$$0 < \limsup_{J\to+\infty} \left\{ \frac{d_J\big(t,\omega\big)}{2^{-J/2}\sqrt{J}} \right\} < +\infty\,;$$

3. t is a *leader-slow point* of $B(\cdot,\omega)$ if

$$0 < \limsup_{J\to+\infty} \left\{ \frac{d_J\big(t,\omega\big)}{2^{-J/2}} \right\} < +\infty\,.$$

Our main result stated below is a reminiscent of Theorem 2.1 in our context.

Theorem 2.3. *There exists an event $\Omega_0^*\subseteq\Omega$ of probability 1 such that for every $\omega\in\Omega_0^*$ and every non-empty open interval A of $\mathbb{R}$, there are $t_o(\omega), t_r(\omega), t_s(\omega)\in A$ such that $t_o(\omega)$ is a leader-ordinary point, $t_r(\omega)$ is a leader-rapid point and $t_s(\omega)$ is a leader-slow point of $B(\cdot,\omega)$. Moreover, for every $\omega\in\Omega_0^*$, almost every $t\in\mathbb{R}$ is a leader-ordinary point of $B(\cdot,\omega)$.*

Remark 2.4. Theorem 2.1 together with the characterization (6) implies that for any $\omega\in\Omega^*$, if t is an ordinary point of $B(\cdot,\omega)$ (resp. a rapid or a slow point), then the upper limit appearing in the definition of the leader-ordinary points of $B(\cdot,\omega)$ (resp. leader-rapid or leader-slow points) is bounded from above. The main difficulty is then to obtain the bound from below.

3. Proof of Theorem 2.3

The definition of B and Equalities (3) and (4) imply that, for every $(J,K)\in\mathbb{N}\times\mathbb{Z}$, one has $c_{J,K}\sim\mathcal{N}(0,2^{-J}\mathbb{E}(c_{0,0}^2))$ (see Abry et al. (2003) for more details). Moreover, since the increments of B are independent and since the support of ψ is included in $[-N,N]$, Equality (4) shows that the wavelet coefficients $c_{J_1,K_1},\ldots,c_{J_n,K_n}$ are independent as soon as

$$\left(\frac{K_i-N}{2^{J_i}},\frac{K_i+N}{2^{J_i}}\right)\cap\left(\frac{K_l-N}{2^{J_l}},\frac{K_l+N}{2^{J_l}}\right)=\emptyset,\quad \forall 1\le i<l\le n. \tag{9}$$

In particular, the coefficients $c_{J,K}$ and $c_{J,K'}$ are independent if $|K'-K|\geq 2N$. This leads us to define the following condition.

Definition 3.1. Let $n\geq 2$. We say that the dyadic intervals $\lambda_{J_1,K_1},\ldots,\lambda_{J_n,K_n}$ satisfy Condition $(\mathcal{C}_N)$ if (9) is satisfied.

Remark 3.2. Clearly, the sequence $\{\varepsilon_\lambda:\lambda\in\Lambda\}$ defined by

$$\varepsilon_\lambda=\frac{1}{\sqrt{2^{-J}\mathbb{E}\big(c_{0,0}^2\big)}}c_\lambda, \tag{10}$$

is a sequence of real-valued $\mathcal{N}(0,1)$ random variables such that for every $n\geq 2$ and every dyadic intervals $\lambda_1,\ldots,\lambda_n$ satisfying Condition $(\mathcal{C}_N)$, the random variables $\varepsilon_{\lambda_1},\ldots,\varepsilon_{\lambda_n}$ are independent.

In the sequel, we fix an arbitrary sequence $\{\varepsilon_\lambda : \lambda \in \Lambda\}$ of real-valued $\mathcal{N}(0,1)$ random variables verifying the property of Remark 3.2. Moreover, if $(J,K) \in \mathbb{N}\times\mathbb{Z}$ and $\lambda = \lambda_{J,K}$, then for any $m \in \mathbb{N}$, we denote by $\mathcal{S}_{J,K,m}$ or $\mathcal{S}_{\lambda,m}$ the finite set of cardinality 2^m whose elements are the dyadic intervals of scale $J+m$ included in $\lambda_{J,K}$. The following lemma allows to obtain a general lower bound for the size of the wavelet leaders of the Brownian motion.

Lemma 3.3. *There exists an event $\Omega_1^* \subseteq \Omega$ of probability 1 such that, for every $\omega \in \Omega_1^*$ and every $t \in \mathbb{R}$, one has*

$$\limsup_{j\to+\infty} \left\{ \max_{\substack{\lambda' \in \mathcal{S}_{\lambda,\lfloor \log_2(N)\rfloor+2} \\ \lambda \in N(\lambda_j(t))}} \left|\varepsilon_{\lambda'}(\omega)\right| \right\} > 0\,. \tag{11}$$

Proof. Let us fix $(J,K) \in \mathbb{N}\times\mathbb{Z}$. For any $m \in \mathbb{N}$ and any $S \in \mathcal{S}_{J,K,m}$, there is a unique finite sequence $(I_n)_{0\le n\le m}$ of dyadic intervals which is decreasing in the sense of the inclusion and satisfies $I_0 = \lambda_{J,K}$, $I_m = S$ and $I_n \in \mathcal{S}_{J,K,n}$ for all $n \in \{1,\dots,m\}$. Next, we consider the sequence $(T_n)_{1\le n\le m}$ of dyadic intervals constructed as follows: for every $n \in \{1,\dots,m\}$, T_n is the unique dyadic interval of $\mathcal{S}_{J,K,n}$ such that $I_{n-1} = T_n \cup I_n$. Note that, since the sequence $(I_n)_{0\le n\le m}$ is decreasing, this construction ensures that the intervals $(T_n)_{1\le n\le m}$ are pairwise disjoint. Let us also note that, for every $n \in \{1,\dots,m\}$, one has $T_n \in N\big(I_n\big)$. Moreover, for every $n \in \{1,\dots,m\}$, there is a dyadic interval $T_n' \in \mathcal{S}_{T_n,\lfloor \log_2(N)\rfloor+2}$ such that

$$\left(\frac{k_n - N}{2^{j_n}}, \frac{k_n+N}{2^{j_n}}\right) \subseteq T_n$$

where $T_n' = \lambda_{j_n,k_n}$. Consequently, by assumption, the corresponding Gaussian random variables $(\varepsilon_{T_n'})_{1\le n\le m}$ are independent. In the sequel, the set $\{T_n' : 1\le n\le m\}$ is denoted by $\mathcal{T}'_{J,K,m}(S)$.

Let $c_0 = 2^{-3/2}\sqrt{\pi}$. Then, the probability p_0 that a real-valued $\mathcal{N}(0,1)$ random variable belongs to the interval $(-c_0, c_0)$ belongs to $(0, 1/2)$. For all $S \in \mathcal{S}_{J,K,m}$, we denote by $B_{J,K,m}(S)$ the Bernoulli random variable defined as

$$B_{J,K,m}(S) = \prod_{T' \in \mathcal{T}'_{J,K,m}(S)} \mathbf{1}_{\{|\varepsilon_{T'}|<c_0\}}\,. \tag{12}$$

Notice that, using the definition of p_0 and the independence property of the random variables $\varepsilon_{T'}$ for $T' \in \mathcal{T}'_{J,K,m}(S)$, one has $\mathbb{E}\big(B_{J,K,m}(S)\big) = p_0^m$. Next, let $G_{J,K,m}$ be the random variable with values in $\{0,\dots,2^m\}$ defined as

$$G_{J,K,m} = \sum_{S\in\mathcal{S}_{J,K,m}} B_{J,K,m}(S)\,.$$

Since the cardinality of $\mathcal{S}_{J,K,m}$ equals 2^m, one gets that $\mathbb{E}\big(G_{J,K,m}\big) = (2p_0)^m$. It follows from Fatou Lemma that

$$0 \le \mathbb{E}\left(\liminf_{m\to+\infty} G_{J,K,m}\right) \le \lim_{m\to+\infty} \mathbb{E}\big(G_{J,K,m}\big) = 0$$

Hence, the events

$$\Omega_{1,J,K}^* = \left\{\omega\in\Omega : \liminf_{m\to+\infty} G_{J,K,m}(\omega) = 0\right\} \quad \text{and} \quad \Omega_1^* = \bigcap_{(J,K)\in\mathbb{N}\times\mathbb{Z}} \Omega_{1,J,K}^* \tag{13}$$

have a probability equal to 1.

Let us now consider $\omega \in \Omega_1^*$ and $t\in\mathbb{R}$. We fix $J\in\mathbb{N}$ and $K = \lfloor 2^J t\rfloor$, so that $\lambda_{J,K} = \lambda_J(t)$. Since for every $m\in\mathbb{N}$, $G_{J,K,m}$ takes values in $\{0,\dots,2^m\}$, (13) implies that there are infinitely many m such that $B_{J,K,m}(S) = 0$ for every $S \in \mathcal{S}_{J,K,m}$, i.e. using (12), there exists $T' \in \mathcal{T}'_{J,K,m}(S)$ such that $|\varepsilon_{T'}| \ge c_0$. In particular, we have this result for $S = \lambda_{J+m}(t)$. In this case, $T' \in \mathcal{S}_{\lambda,\lfloor\log_2(N)\rfloor+2}$ with $\lambda \in N(\lambda_{J+m}(t))$. Hence (11) is satisfied. □

Proposition 3.4. *Let Ω_1^* denote the event of probability 1 of Lemma 3.3. For every $\omega\in\Omega_1^*$ and every $t\in\mathbb{R}$, one has*

$$\limsup_{J\to+\infty}\left\{\frac{d_J(t,\omega)}{2^{-J/2}}\right\} > 0\,. \tag{14}$$

Proof. Note that using (5) and (10), one has

$$d_J(t,\omega) \ge \max_{\substack{\lambda' \in \mathcal{S}_{\lambda,\lfloor \log_2(N)\rfloor+2} \\ \lambda \in N(\lambda_J(t))}} \left|c_{\lambda'}(\omega)\right| = \sqrt{2^{-(J+\lfloor\log_2(N)\rfloor+2)}\mathbb{E}\big(c_{0,0}^2\big)} \max_{\substack{\lambda' \in \mathcal{S}_{\lambda,\lfloor \log_2(N)\rfloor+2} \\ \lambda \in N(\lambda_J(t))}} \left|\varepsilon_{\lambda'}(\omega)\right|\,.$$

Remark 3.2 and Inequality (11) imply then that (14) is satisfied. □

As we will see in the proof of Theorem 2.3, this result allows to get the existence of leader-slow points. Let us now focus on leader-ordinary points. First, let us recall the following classical lemma which provides asymptotic estimates on the tail behavior of a standard Gaussian distribution.

Lemma 3.5. *If ε is a real-valued $\mathcal{N}(0,1)$ random variable, then one has*

$$\lim_{x\to+\infty} \frac{\mathbb{P}\big(|\varepsilon|>x\big)}{(2\pi^{-1})^{1/2}x^{-1}e^{-x^2/2}} = 1\,.$$

Lemma 3.6. *There exists an event $\Omega_2^*\subseteq\Omega$ of probability 1 such that, for every $\omega\in\Omega_2^*$ and almost every $t\in\mathbb{R}$, one has*

$$\limsup_{j\to+\infty}\left\{\frac{1}{\sqrt{\log(j)}}\max_{\substack{\lambda'\in\mathcal{S}_{\lambda,\lfloor\log_2(N)\rfloor+2}\\ \lambda\in N(\lambda_j(t))}}|\varepsilon_{\lambda'}|\right\}>0\,. \tag{15}$$

Proof. Within this proof, we will use the same notations as in the proof of Lemma 3.3. Let us fix $t\in\mathbb{R}$. If $J\in\mathbb{N}$, we set $K=\lfloor 2^J t\rfloor$. For every $m\in\mathbb{N}$, we consider the dyadic interval $S=\lambda_{J+m}(t)\in\mathcal{S}_{J,K,m}$ and the associated sequence $(T'_n)_{1\le n\le m}$ of dyadic intervals. Next, we set

$$\mathcal{E}_{J,m}(t):=\left\{\omega\in\Omega:\max_{1\le n\le m}|\varepsilon_{T'_n}|\ge\sqrt{\log(2m)}\right\}$$

By construction, the Gaussian random variables $\varepsilon_{T'_n}$, $1\le n\le m$, are independent. Therefore, one has

$$\mathbb{P}\big(\mathcal{E}_{J,m}(t)\big)=1-\prod_{1\le n\le m}\mathbb{P}\left(|\varepsilon_{T'_n}|<\sqrt{\log(2m)}\right)=1-\left(1-\mathbb{P}\big(|\varepsilon|>\sqrt{\log(2m)}\big)\right)^m$$

where $\varepsilon\sim\mathcal{N}(0,1)$. Let us set $C=1/2\,(2\pi^{-1})^{1/2}>0$. Using Lemma 3.5 and the fact that $\log(1-x)\le -x$ if $x\in(0,1)$, there exists $M\in\mathbb{N}$ such that for any $m>M$, we have

$$\begin{aligned}\mathbb{P}\big(\mathcal{E}_{J,m}(t)\big)&\ge 1-\left(1-C\frac{e^{-\frac12\log(2m)}}{\sqrt{\log(2m)}}\right)^m\ge 1-\exp\left(-Cm\frac{e^{-\frac12\log(2m)}}{\sqrt{\log(2m)}}\right)\\&\ge 1-\exp\left(-C\sqrt{\frac{m}{2\log(2m)}}\right)\\&\ge 1-\exp(-m^\gamma)\end{aligned}$$

for $\gamma\in(0,1/2)$. Consequently, one has in particular

$$\sum_{M\in\mathbb{N}}\mathbb{P}\left(\mathcal{E}_{2^M,2^M}(t)\right)=+\infty\,.$$

In view of the fact that the events $\mathcal{E}_{2^M,2^M}(t)$, $M\in\mathbb{N}$, are independents, it follows from the Borel–Cantelli lemma that

$$\mathbb{P}\left(\bigcap_{M\in\mathbb{N}}\bigcup_{m\ge M}\mathcal{E}_{2^m,2^m}(t)\right)=1\,.$$

Hence for a fixed $t\in\mathbb{R}$, almost surely, there are infinitely many $j\in\mathbb{N}$ such that

$$\max_{\substack{\lambda'\in\mathcal{S}_{\lambda,\lfloor\log_2(N)\rfloor+2}\\ \lambda\in N(\lambda_j(t))}}|\varepsilon_{\lambda'}|\ge\sqrt{\log j}\,.$$

Fubini's theorem implies then that there is an event $\Omega_2^*\subseteq\Omega$ of probability 1 on which for almost every $t\in\mathbb{R}$, (15) holds true. □

Proposition 3.7. *Let Ω_2^* denote the event of probability 1 of Lemma 3.6. For every $\omega\in\Omega_2^*$ and almost every $t\in\mathbb{R}$, one has*

$$\limsup_{J\to+\infty}\left\{\frac{d_J(t,\omega)}{2^{-J/2}\sqrt{\log(J)}}\right\}>0\,.$$

Proof. It suffices to proceed as in the proof of Proposition 3.4, using Lemma 3.6. □

Let us end with a result which will be useful for rapid points.

Lemma 3.8. *There exists an event $\Omega_3^* \subseteq \Omega$ of probability 1 such that, for every $\omega \in \Omega_3^*$ and every non-empty open interval A of $\mathbb{R}$, there is $t \in A$ such that*

$$\limsup_{j\to+\infty} \left\{ \frac{|\varepsilon_{\lambda_j(t)}|}{\sqrt{j}} \right\} > 0 .$$

Proof. To avoid making the notations heavier, we suppose that $A = (0, 1)$. The proof can be easily adapted in the general case. The conclusion follows then by covering $\mathbb{R}$ with all open intervals with rational endpoints.

Let us fix $a \in (0, 1)$ and $C > 0$ such that $C^2 < 2a \log 2$. Let us also consider for every $(j, l) \in \mathbb{N} \times \{0, \ldots, \lfloor 2^{j(1-a)} \rfloor - 1\}$, the event

$$\mathcal{F}_{j,l} := \left\{ \omega \in \Omega : \max_{k\in\{l\lfloor 2^{aj}/(2N)\rfloor,\ldots,(l+1)\lfloor 2^{aj}/(2N)\rfloor-1\}} |\varepsilon_{j,2kN}(\omega)| \geq C\sqrt{j} \right\} .$$

Let j_0 be the smallest j such that $\lfloor 2^{aj}/(2N)\rfloor \geq 1$. Assume for a while that

$$\mathbb{P}\left(\complement \left(\bigcap_{l\in\{0,\ldots,\lfloor 2^{j(1-a)}\rfloor-1\}} \mathcal{F}_{j,l} \right)\right) \tag{16}$$

is the general term of a convergent series; Borel–Cantelli lemma implies then that

$$\Omega_3^* := \bigcup_{J\geq j_0} \bigcap_{j\geq J} \bigcap_{l\in\{0,\ldots,\lfloor 2^{j(1-a)}\rfloor-1\}} \mathcal{F}_{j,l} \tag{17}$$

is an event of probability 1. Let us consider $\omega \in \Omega_3^*$. For every $j \geq j_0$, let us set

$$G_j(\omega) := \left\{ k \in \{0, \ldots, 2^j - 1\} : |\varepsilon_{j,k}(\omega)| \geq C\sqrt{j} \right\} . \tag{18}$$

Moreover, for every $n \geq j_0$, one considers

$$O_n(\omega) := \bigcup_{j\geq n} U_j(\omega), \quad \text{where} \quad U_j(\omega) := \bigcup_{k\in G_j(\omega)} \left(\frac{k}{2^j}, \frac{k+1}{2^j} \right) . \tag{19}$$

The open subset $O_n(\omega)$ is dense in $(0, 1)$. Indeed, let us consider $t \in (0, 1)$, $j \geq j_0$ and k such that $\lambda_j(t) = \lambda_{j,k}$. Then, either there is $l \in \{0, \ldots, \lfloor 2^{j(1-a)}\rfloor - 1\}$ such that $k \in \left\{ l\lfloor 2^{ja}\rfloor, \ldots, (l+1)\lfloor 2^{ja}\rfloor - 1 \right\}$, or $k \in \{\lfloor 2^{j(1-a)}\rfloor \lfloor 2^{ja}\rfloor, \ldots, 2^j - 1\}$. In the first case, using (17) and (18), there is $k' \in \{l\lfloor 2^{aj}/(2N)\rfloor, \ldots, (l+1)\lfloor 2^{aj}/(2N)\rfloor - 1\}$ such that $2k'N \in G_j(\omega)$. From (19), we get that t is at a distance at most $2 \cdot 2^{j(a-1)}$ of $U_j(\omega)$. In the second case, there is $k' \in \{(\lfloor 2^{j(1-a)}\rfloor - 1)\lfloor 2^{aj}/(2N)\rfloor, \ldots, \lfloor 2^{j(1-a)}\rfloor \lfloor 2^{aj}/(2N)\rfloor - 1\}$ such that $2k'N \in G_j(\omega)$, and similarly, we get that t is at a distance at most $c \cdot 2^{j(a-1)}$ of $U_j(\omega)$, for some constant $c > 0$ depending only on N and a. The density follows. Hence, Baire's theorem gives that the set $\bigcap_{n\geq j_0} O_n(\omega)$ is not empty. Let t be an element of this set. For every $n \geq j_0$, there is $j \geq n$ such that $|\varepsilon_{\lambda_j(t)}| \geq C\sqrt{j}$, and it leads to the conclusion.

It remains then to show that (16) is the general term of a convergent series. Note that the variables $\varepsilon_{j,2kN}$, $k \in \{l\lfloor 2^{aj}/(2N)\rfloor, \ldots, (l+1)\lfloor 2^{aj}/2N\rfloor - 1\}$ and $l \in \{0, \ldots, \lfloor 2^{j(1-a)}\rfloor - 1\}$, are independent. Consequently, one has

$$\begin{aligned}
&\mathbb{P}\left(\complement \left(\bigcap_{l\in\{0,\ldots,\lfloor 2^{j(1-a)}\rfloor-1\}} \mathcal{F}_{j,l} \right)\right) \\
&= 1 - \prod_{l\in\{0,\ldots,[2^{j(1-a)}]-1\}} \left(1 - \prod_{k\in\{l\lfloor 2^{aj}/(2N)\rfloor,\ldots,(l+1)\lfloor 2^{aj}/2N\rfloor-1\}} \mathbb{P}\left(|\varepsilon_{j,2kN}| < C\sqrt{j} \right) \right) \\
&= 1 - \left(1 - \left(1 - \mathbb{P}\left(|\varepsilon| \geq C\sqrt{j} \right)\right)^{\lfloor 2^{aj}/(2N)\rfloor} \right)^{\lfloor 2^{j(1-a)}\rfloor} \\
&\leq 1 - \exp\left(2^{j(1-a)} \log(1 - x_j) \right)
\end{aligned} \tag{20}$$

where $\varepsilon \sim \mathcal{N}(0, 1)$ and $x_j = \left(1 - \mathbb{P}\left(|\varepsilon| \geq C\sqrt{j}\right)\right)^{\lfloor 2^{aj}/(2N)\rfloor}$. Let us remark that x_j is always positive and tends to 0 as $j \to +\infty$. Indeed, let us set $C' = (1/2)(2\pi^{-1})^{1/2}$. Using Lemma 3.5 and the fact that $\log(1 - x) \leq -x$ if $x \in (0, 1)$, there exists $J \in \mathbb{N}$ such that for any $j \geq J$,

$$\begin{aligned}
0 \leq x_j &\leq (1 - C'\sqrt{j}\exp(-C^2 j/2))^{\lfloor 2^{aj}/(2N)\rfloor} \leq \exp\left(-\left\lfloor \frac{2^{aj}}{2N} \right\rfloor C'\sqrt{j}\exp(-C^2 j/2) \right) \\
&\leq \exp\left(-C''\sqrt{j}\exp(j(a\log 2 - C^2/2)) \right) ,
\end{aligned} \tag{21}$$

where $C'' > 0$ depends only on a, N and C. Expression (21) tends to 0 since $C^2 < 2a \log 2$. Moreover, the same argument shows that $2^{j(1-a)} x_j$ tends to 0. Using the fact that $\log(1-x) = -x + o(x)$ and $\exp(x) = 1 + x + o(x)$ as $x \to 0$, we obtain that, for any $\epsilon > 0$, (20) is upper bounded by $2^{j(1-a)}(\epsilon(x_j + \epsilon x_j) + x_j + \epsilon x_j)$ for j large enough. Together with (21), this implies that (20) is indeed the general term of a convergent series. □

Proposition 3.9. *Let Ω_3^* denote the event of probability 1 of Lemma 3.8. For every $\omega \in \Omega_3^*$ and every non-empty open interval A of $\mathbb{R}$, there is $t(\omega) \in A$ such that*

$$\limsup_{J \to +\infty} \left\{ \frac{d_J\big(t(\omega), \omega\big)}{2^{-J/2}\sqrt{J}} \right\} > 0 \,.$$

Proof. We proceed as in Proposition 3.4, using Lemma 3.8 and Remark 3.2. □

We are now able to prove Theorem 2.3.

Proof of Theorem 2.3. Inequalities (8) and (6) imply that for every $\omega \in \Omega^*$,

$$\limsup_{J \to +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2}\sqrt{J}} \right\} < +\infty \tag{22}$$

holds for every $t \in \mathbb{R}$, where Ω^* is the event given in Theorem 2.1. Let us consider the event $\Omega_0^* := \Omega^* \cap \Omega_1^* \cap \Omega_2^* \cap \Omega_3^*$ of probability 1, where the events Ω_1^*, Ω_2^* and Ω_3^* are the events of Lemmas 3.3, 3.6 and 3.8 respectively. Let us fix $\omega \in \Omega_0^*$ and consider a non-empty open interval A of $\mathbb{R}$.

Let us first show that almost every $t \in \mathbb{R}$ is a leader-ordinary point of $B(\cdot, \omega)$. Using Theorem 2.1, we know that almost every $t \in \mathbb{R}$ is an ordinary point of $B(\cdot, \omega)$. Together with Remark 2.4 and Proposition 3.7, this implies that for almost every $t \in \mathbb{R}$,

$$0 < \limsup_{J \to +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2}\sqrt{\log(J)}} \right\} < +\infty \,.$$

In particular, there exist leader-ordinary points of $B(\cdot, \omega)$ in A.

Secondly, Proposition 3.9 shows that there exists $t_r(\omega) \in A$ such that

$$\limsup_{J \to +\infty} \left\{ \frac{d_J\big(t_r(\omega), \omega\big)}{2^{-J/2}\sqrt{J}} \right\} > 0.$$

This result combined with Eq. (22) implies that the point $t_r(\omega)$ is a leader-rapid point of $B(\cdot, \omega)$.

Finally, Theorem 2.1 and Remark 2.4 give $t_s(\omega) \in I$ such that

$$\limsup_{J \to +\infty} \left\{ \frac{d_J\big(t_s(\omega), \omega\big)}{2^{-J/2}} \right\} < +\infty \,.$$

Using Proposition 3.4, we obtain that $t_s(\omega)$ is a leader-slow point of $B(\cdot, \omega)$. □

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