

Contribution to Spectral Line Formation in Moving Stellar Envelopes

Radiation Field and Statistical Equilibrium Equations

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Summary. Using the escape probability method introduced by Sobolev we derive the mean intensity of the radiation field at any point of an envelope which expands spherically with a positive or negative radial velocity gradient. When considering outward-decelerating envelopes, a new contribution to the mean intensity of the radiation field must be taken into account when compared with the results obtained separately by Castor and the author in the case of outward-accelerating envelopes.

A general discussion of the net radiative rates which populate the levels of an atom relates the problem of spectral line formation in a moving medium to the one in a transparent atmosphere at rest. Namely, it is shown that dilution effects are expected to play an important role in the envelopes where very large velocity gradients occur.

Key words: moving stellar envelopes — emission-line stars — radiative transfer

1. Introduction

This work is based on the hypothesis of Sobolev (1947, 1957, 1958) who introduced the escape probability concept for treating the transfer of line radiation in a stellar envelope which is in rapid radial expansion. The approximation Sobolev adopted is the following: a line photon emitted at some point in the envelope is either absorbed in the immediate vicinity of that point or escapes entirely due to Doppler shifts across the envelope, assuming that the rate of motion of the envelope is much greater than the mean chaotic (thermal and turbulent) velocities of the atoms.

In the second section we re-derive directly the escape probability β_{ij}^1 of a photon emitted in the line transition $j \rightarrow i$ (j and i designating respectively the upper and lower

level of an atom). In order to calculate the level populations of an atom, the statistical equilibrium equations must be solved. Therefore, in the third section we determine, in a different formalism from previous works, the expression of the radiation field J_{ij} at any point of an outward-accelerating envelope. Our results are shown to be in good agreement with those of Sobolev (1947, 1958) and Castor (1970) concerning the so-called "local contribution" J_{ij}^1 to the mean intensity expression J_{ij} , and with those of Castor (1970) according to the "stellar core" contribution J_{ij}^2 . The expression of the radiation field J_{ij} in the case of an outward-decelerating envelope is derived in Section 4. Section 5 deals with discussions about the net radiative rate by which an atomic level is populated in an expanding envelope. Adopting different approximations, we compare the behaviour of the atomic level populations in the case of a moving envelope and a transparent medium at rest. Discussion and conclusions form the sixth section.

The general hypotheses assumed through all this work are expressed in the second section. The method of determining the different physical quantities is described there once in detail; indeed, the reasoning followed in establishing the escape probability β_{ij}^1 is very similar to that followed in deriving the different mean intensity expressions J_{ij}^1 , J_{ij}^2 , J_{ij}^3 and J_{ij}^4 in Sections 3 and 4.

2. Determination of the Escape Probability β_{ij}^1

We consider spherically expanding envelopes in which the level populations of different atoms have reached a steady state. We assume a complete redistribution in direction and frequency for an emitted photon in the frame of a moving atom.

Let $\Phi(v - v_{ij})$ be an arbitrary function describing the line profile (see Fig. 1), normalized to unity when integrated over frequency and equal to zero outside the interval $[v_{ij} - (\Delta\nu/2), v_{ij} + (\Delta\nu/2)]$. v_{ij} denotes the central frequency of the line and $\Delta\nu$ the maximum width of the line profile due to chaotic motions of the atoms. We

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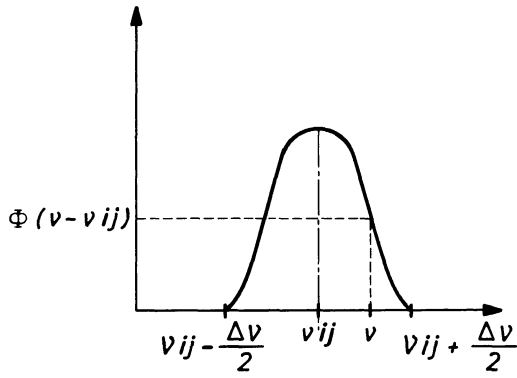


Fig. 1. Line profile function in the frame of an atom, normalized to unity (see text)

have the following relation:

$$\frac{\Delta v}{v_{ij}} = \frac{2u_{\max}}{c}, \quad (1.2)$$

where u_{\max} is the maximum chaotic velocity of the atoms and c the velocity of light.

Because of the large velocity gradients in the envelope, the distance ΔS , across which the Doppler displacement at the line frequency v_{ij} changes by an amount equal to Δv , is very small compared with the dimensions of the envelope. Therefore, over that range of distance we can assume that the volume absorption coefficient α_{ij} , the volume emission coefficient ϵ_{ij} and the velocity gradient are constant and that the line profile function remains the same. Without any restrictions these quantities may vary appreciably over larger distances. If $\partial v_s / \partial s$ denotes the velocity gradient along the direction and at the point we evaluate ΔS , we have:

$$\Delta S = \frac{2u_{\max}}{|\partial v_s / \partial s|}. \quad (2.2)$$

The volume absorption and emission coefficients for an observer moving with the atoms are:

$$\alpha_{ij} = \frac{n_i B_{ij} h v_{ij}}{4\pi} \left(1 - \frac{g_i n_j}{g_j n_i} \right), \quad (3.2)$$

$$\epsilon_{ij} = \frac{n_j A_{ji} h v_{ij}}{4\pi}, \quad (4.2)$$

n_i and n_j being respectively the volume populations of the levels i and j , and g_i and g_j their statistical weights. B_{ij} and A_{ji} are Einstein-Milne transition probabilities and h Planck's constant.

In the absence of scattering and with the assumption of a complete redistribution of the radiation with frequency and direction, the source function S_{ij} of a line transition $j \rightarrow i$ in the frame of the atom is given by

$$S_{ij} = \frac{\epsilon_{ij}}{\alpha_{ij}}. \quad (5.2)$$

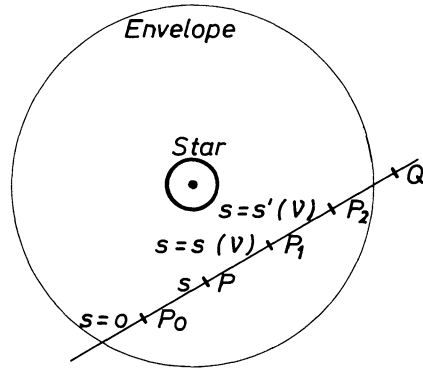


Fig. 2. Arbitrary direction in the envelope along which radiative transfer is studied (see text). No scale is respected

Replacing ϵ_{ij} and α_{ij} by their values from (4.2) and (3.2), we obtain

$$S_{ij} = \frac{\sigma_{ij}}{\frac{g_j n_i}{g_i n_j} - 1}, \quad (6.2)$$

with

$$\sigma_{ij} = \frac{2h v_{ij}^3}{c^2}. \quad (7.2)$$

In the following, we directly determine the expression for the escape probability β_{ij}^1 , defined as the probability that a photon, created in the line transition $j \rightarrow i$, will escape the region around its point of emission along any direction. Let P_0 be a fixed point in the expanding envelope with its abscissa s equal to zero (see Fig. 2). The abscissae axis s is positively directed from P_0 to Q . Let us suppose that a line photon at local frequency¹ v (see Fig. 1) is emitted at P_0 along that direction. Because of the large velocity gradient $\partial v_s / \partial s$, the photon has a certain probability $PR_v(P_0 Q)$ to be absorbed between the points P_0 and P_1 with respective abscissae $s=0$ and $s=s(v)$. Beyond the point P_1 , the medium becomes totally transparent to the photons of frequency v , locally defined at P_0 , because of the Doppler shifts across the envelope.

The expression of the probability $PR_v(P_0 Q)$ is easily found to be

$$PR_v(P_0 Q) = \int_0^{s(v)} \exp \left(- \int_0^s \alpha_{ij} \Phi \left(v - v_{ij} - \frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s' \right) ds' \right) \cdot \alpha_{ij} \Phi \left(v - v_{ij} - \frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s \right) ds. \quad (8.2)$$

The exponential factor in the main integral represents the probability that a photon, at local frequency v

¹ The local frequency of a photon emitted or passing at a given point is the frequency seen by an observer moving with the medium at that point

emitted at P_0 , will reach (unabsorbed) the point P with abscissa s (see Fig. 2). The exponent, with opposite sign, is the opacity of the medium between P_0 and P for the photon considered. The additional term $(-v_{ij}/c)(\partial v_s/\partial s)s'$ in the argument of the profile function expresses the Doppler shift of the local frequency ν at P_0 due to the velocity difference along P_0Q between the points P and P_0 . The remaining factor $\alpha_{ij}\Phi(\nu - v_{ij} - (v_{ij}/c)(\partial v_s/\partial s)s)$ represents the absorption probability of the photon between the points of abscissae s and $s + ds$. We can simplify Equation (8.2) with the following variable transformations

$$x = \nu - v_{ij} - \frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s', \quad dx = -\frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} ds' \quad (9.2)$$

and

$$y = \nu - v_{ij} - \frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s, \quad dy = -\frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} ds. \quad (10.2)$$

Equation (8.2) becomes

$$PR_{\nu}(P_0Q) = \int_{y(s(\nu))}^{\nu - v_{ij}} \exp \left(\int_{\nu - v_{ij}}^y \frac{\alpha_{ij}}{v_{ij}} \frac{c}{\partial v_s} \Phi(x) dx \right) \cdot \frac{\alpha_{ij}}{v_{ij}} \frac{c}{\partial v_s} \Phi(y) dy. \quad (11.2)$$

Because of the basic hypotheses, α_{ij} and $\partial v_s/\partial s$ can be regarded as constants in the integrals and if we define

$$\tau_{ij} = \frac{\alpha_{ij}}{v_{ij}} \frac{c}{\partial v_s}, \quad (12.2)$$

we obtain

$$PR_{\nu}(P_0Q) = \int_{y(s(\nu))}^{\nu - v_{ij}} \frac{d}{dy} \left(\exp \left(\tau_{ij} \int_{\nu - v_{ij}}^y \Phi(x) dx \right) \right) dy, \quad (13.2)$$

which is equivalent to

$$PR_{\nu}(P_0Q) = 1 - \exp \left(\tau_{ij} \int_{\nu - v_{ij}}^{\nu - v_{ij}} \Phi(x) dx \right). \quad (14.2)$$

The absorption probability $PR(P_0Q)$ of a photon emitted at any frequency in the line profile of the atom at P_0 , in the direction P_0Q is given by

$$PR(P_0Q) = \int_{\nu_{ij} - \frac{\Delta\nu}{2}}^{\nu_{ij} + \frac{\Delta\nu}{2}} \Phi(\nu - v_{ij}) PR_{\nu}(P_0Q) d\nu. \quad (15.2)$$

Using the variable transformation

$$l = \nu - v_{ij}, \quad dl = d\nu \quad (16.2)$$

and recalling Equation (14.2), we obtain

$$PR(P_0Q) = \int_{-\frac{\Delta\nu}{2}}^{\frac{\Delta\nu}{2}} \Phi(l) \left(1 - \exp \left(\tau_{ij} \int_i^{y(s(\nu))} \Phi(x) dx \right) \right) dl. \quad (17.2)$$

Because of the normalization condition of the function $\Phi(\nu - v_{ij})$, we have

$$PR(P_0Q) = 1 + \int_{-\frac{\Delta\nu}{2}}^{\frac{\Delta\nu}{2}} \frac{d}{dl} \left(\exp \left(\tau_{ij} \int_i^{y(s(\nu))} \Phi(x) dx \right) \right) \frac{1}{\tau_{ij}} dl. \quad (18.2)$$

If along P_0Q , the velocity gradient is positive, the local line frequency ν at P_0 will be red-shifted to the local line frequency $\nu_{ij} - (\Delta\nu/2)$ at P_1 . In the same way, if the velocity gradient is negative the former frequency ν at P_0 will be blue-shifted to the local line frequency $\nu_{ij} + (\Delta\nu/2)$ at P_1 . With the transformation (10.2), we find

$$\text{if } \frac{\partial v_s}{\partial s} > 0 \quad y(s(\nu)) = \nu - v_{ij} - \frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s(\nu) \quad (19.2)$$

and since the Doppler shift between P_1 and P_0 is expressed by (see Fig. 1)

$$\frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s(\nu) = \nu - \left(\nu_{ij} - \frac{\Delta\nu}{2} \right), \quad (20.2)$$

Equation (19.2) becomes

$$y(s(\nu)) = -\frac{\Delta\nu}{2}. \quad (21.2)$$

Similarly if $\frac{\partial v_s}{\partial s} < 0$

$$y(s(\nu)) = \frac{\Delta\nu}{2}. \quad (22.2)$$

We can simplify Equation (18.2) using the results obtained in (21.2) and (22.2):

$$\text{if } \frac{\partial v_s}{\partial s} > 0 \quad PR(P_0Q) = 1 + \left(\exp \left(-\tau_{ij} \int_{-\frac{\Delta\nu}{2}}^{\frac{\Delta\nu}{2}} \Phi(x) dx \right) - 1 \right) \frac{1}{\tau_{ij}} \quad (23.2)$$

and

$$\text{if } \frac{\partial v_s}{\partial s} < 0 \quad PR(P_0Q) = 1 + \left(1 - \exp \left(\tau_{ij} \int_{-\frac{\Delta\nu}{2}}^{\frac{\Delta\nu}{2}} \Phi(x) dx \right) \right) \frac{1}{\tau_{ij}} \quad (24.2)$$

or, independently of the sign of $\partial v_s/\partial s$

$$PR(P_0Q) = 1 - (1 - \exp(-|\tau_{ij}|)) \frac{1}{|\tau_{ij}|}. \quad (25.2)$$

The probability β_{ij}^1 for a photon created in the line transition $j \rightarrow i$ to escape the medium locally along any direction is obviously given by

$$\beta_{ij}^1 = \int_{\Omega=4\pi} (1 - PR(P_0Q)) \frac{d\omega}{4\pi}, \quad (26.2)$$

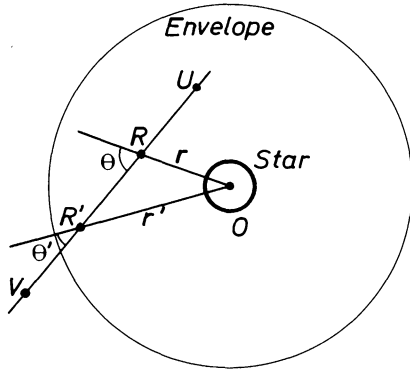


Fig. 3. Arbitrary direction (U, V) in the envelope along which the projected radial velocity ($V(r) \cos(\theta)$) is studied as a function of the distance r (see text). No scale is respected

where the integration is extended over all directions (solid angle $\Omega = 4\pi$).

Replacing Equation (25.2) in (26.2) we obtain

$$\beta_{ij}^1 = \int_{\Omega=4\pi} (1 - \exp(-|\tau_{ij}|)) \frac{1}{|\tau_{ij}|} \frac{d\omega}{4\pi}. \quad (27.2)$$

This last expression is similar to the ones obtained indirectly by Sobolev (1947, 1958) in the case of a rectangular line profile, Sobolev (1957) for an arbitrary coefficient contour in a three dimensional medium composed of plane parallel layers and Castor (1970) under the same hypotheses as ours. However, Castor's restrictive assumption that the function $\Phi(v - v_{ij})$ must be the same throughout the envelope is in fact not necessary.

The probability that a photon emitted in the line transition $j \rightarrow i$ at P_0 will escape the envelope without striking the stellar core is given by

$$\beta_{ij}^2 = \int_{\Omega=4\pi(1-W)} (1 - \exp(-|\tau_{ij}|)) \frac{1}{|\tau_{ij}|} \frac{d\omega}{4\pi}, \quad (28.2)$$

where the integration is now performed over every direction except those joining the point P_0 to the stellar core. The solid angle we integrate over is now $\Omega = 4\pi(1 - W)$, where W is the dilution factor

$$W = \frac{1}{2} \left(1 - \left(1 - \left(\frac{R^*}{r} \right)^2 \right)^{1/2} \right), \quad (29.2)$$

with r being the distance from P_0 to the center of the star and R^* the radius of the stellar core.

3. Radiation Field in an Expanding Envelope with a Positive Radial Velocity Gradient

Let $V(r)$ be the radial velocity field in the envelope as a function of r and similarly $dV(r)/dr$ the radial velocity gradient. If $dV(r)/dr$ is positive, along any direction in the moving medium (see Fig. 3, direction UV) atoms at two points such as R and R' , separated by more than a

distance in which the Doppler displacement at the line frequency changes by an amount equal to the line width, will not be able to interact with each other through the radiation they emit in the line². Indeed, designating the angles between the direction UV and the directions OR and OR' by θ and θ' respectively, we have:

$$\text{if } r' > r \quad V(r') > V(r), \\ \cos(\theta') > \cos(\theta)$$

and so $V(r') \cos(\theta') > V(r) \cos(\theta)$.

The last inequality means that the projected radial velocity along the direction UV is a monotonic function of the distance r so that the preceding statement is demonstrated.

If we consider now the case of a negative radial velocity gradient we find:

$$\text{if } r' > r \quad V(r') < V(r), \\ \cos(\theta') > \cos(\theta)$$

and we do not reach the same conclusion as before. Since the functions $V(r)$ and $\cos(\theta)$ are separately monotonic functions, we can find pairs of distant points such as R and R' for which

$$V(r) \cos(\theta) = V(r') \cos(\theta'). \quad (1.3)$$

In such a case it is possible for atoms at distant points in the envelope to interact with each other through the radiation they emit in the line (see next section).

Let us come back to the case of an expanding envelope with a positive radial velocity gradient. The mean intensity of the radiation field J_{ij} at a point in the medium (for instance P_0 in Fig. 2) is defined as being the intensity of the radiation which can interact with an atom at P_0 along every direction and over frequency in the line transition $j \rightarrow i$. In addition to a "local contribution" J_{ij}^1 due to the radiation emitted by the neighbouring atoms at P_0 , the mean intensity of the radiation field J_{ij} will include another contribution J_{ij}^2 , due to radiation from the stellar core.

We first derive the expression of the local contribution J_{ij}^1 . Let us consider in Figure 2 the point P_0 and the direction QP_0 . The positive sense of the abscissae is now directed from Q to P_0 . The intensity of the radiation $I_{ij}^y(QP_0)$, at the local frequency ν defined at P_0 , which can interact with an atom at P_0 is given by

$$I_{ij}^y(QP_0) = \int_{s'(v)}^0 \varepsilon_{ij} \Phi \left(\nu - \nu_{ij} - \frac{\nu_{ij}}{c} \frac{\partial v_s}{\partial s} s \right) \cdot \exp \left(\int_s^0 -\alpha_{ij} \Phi \left(\nu - \nu_{ij} - \frac{\nu_{ij}}{c} \frac{\partial v_s}{\partial s} s' \right) ds' \right) ds. \quad (2.3)$$

In the integral, the factor $\varepsilon_{ij} \Phi(\nu - \nu_{ij} - (\nu_{ij}/c) (\partial v_s / \partial s) s) ds$ represents the amount of spectral energy emitted to-

² Two points such as R and R' will be called "distant points", otherwise "neighbouring points"

wards P_0 at the point P with abscissa s by the volume ds . The exponential factor expresses the attenuation undergone by that energy on its way between the points P and P_0 . $s'(v)$ is the abscissa of the point P_2 beyond which the frequency of the emitted line radiation is too shifted to contribute energy at the local frequency ν at P_0 .

The intensity of the radiation $I_{ij}^y(QP_0)$ integrated over the line frequency in the frame of an atom at P_0 is merely

$$I_{ij}(QP_0) = \int_{\nu_{ij} - \frac{\Delta\nu}{2}}^{\nu_{ij} + \frac{\Delta\nu}{2}} \Phi(\nu - \nu_{ij}) I_{ij}^y(QP_0) d\nu. \quad (3.3)$$

Following the same reasoning as in the preceding section we can simplify the last expression. We substitute the Equation (2.3) in (3.3) and using the transformation relations defined in the Section 2, with the exception of Equations (21.2) and (22.2) which become respectively

$$\text{if } \frac{\partial v_s}{\partial s} > 0 \quad y(s(\nu)) = \frac{\Delta\nu}{2} \quad (4.3)$$

and

$$\text{if } \frac{\partial v_s}{\partial s} < 0 \quad y(s(\nu)) = -\frac{\Delta\nu}{2} \quad (5.3)$$

we obtain

$$I_{ij}(QP_0) = S_{ij}(1 - (1 - \exp(-|\tau_{ij}|))/|\tau_{ij}|), \quad (6.3)$$

S_{ij} being the source function defined by (5.2).

Finally, the local contribution J_{ij}^1 to the mean intensity of the radiation field at P_0 is obtained by integrating Equation (6.3) over all directions. Recalling formula (27.2), we have successively

$$J_{ij}^1 = \int_{\Omega=4\pi} I_{ij}(QP_0) \frac{d\omega}{4\pi}, \quad (7.3)$$

$$J_{ij}^1 = S_{ij}(1 - \beta_{ij}^1). \quad (8.3)$$

The last expression for J_{ij}^1 is the same as the zero-order term appearing in the power series expansion of the mean intensity expression derived by Castor (1970).

If we assume the stellar core to radiate continuously, without limb darkening and with an intensity I_c constant over the line frequency, the intensity of the radiation $I_v^c(SP_0)$ emitted from a point S of the stellar core (see Fig. 4) and appearing with a frequency ν for an atom at P_0 is given by

$$I_v^c(SP_0) = I_c \exp\left(-\int_{s(v)}^0 \alpha_{ij} \Phi\left(\nu - \nu_{ij} - \frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s'\right) ds'\right). \quad (9.3)$$

The exponential factor expresses the extinction of the continuum radiation between the points S_1 and P_0 , the abscissae s being positively oriented from S to P_0 . Atoms at points with abscissae $s < s(v)$ are too distant from P_0 (important Doppler shifts) to absorb at the line frequency ν locally defined at P_0 . If we integrate Equation (9.3)

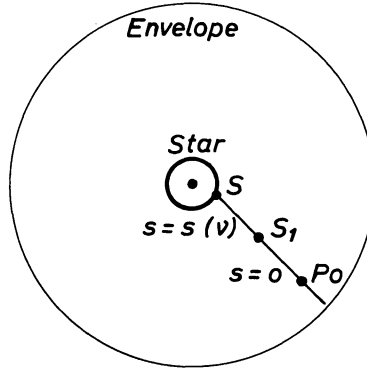


Fig. 4. Arbitrary direction joining the point P_0 and the stellar core (see text). No scale is respected

over the line frequency in the frame of an atom at P_0 , using the preceding transformation relations (Section 3) we obtain successively:

$$I^c(SP_0) = \int_{\nu_{ij} - \frac{\Delta\nu}{2}}^{\nu_{ij} + \frac{\Delta\nu}{2}} \Phi(\nu - \nu_{ij}) I_v^c(SP_0) d\nu, \quad (10.3)$$

$$I^c(SP_0) = I_c(1 - \exp(-|\tau_{ij}|))/|\tau_{ij}|. \quad (11.3)$$

The stellar rays emerging from the core and reaching P_0 are confined within a certain solid angle $\Omega = 4\pi W$, W being the dilution factor given by (29.2). The contribution of the stellar core continuum to the mean intensity of the radiation field at P_0 is given by

$$J_{ij}^2 = I_c \int_{\Omega=4\pi W} (1 - \exp(-|\tau_{ij}|))/|\tau_{ij}| \frac{d\omega}{4\pi} \quad (12.3)$$

or

$$J_{ij}^2 = I_c \beta_{ij}^3 \quad (13.3)$$

with

$$\beta_{ij}^3 = \int_{\Omega=4\pi W} (1 - \exp(-|\tau_{ij}|))/|\tau_{ij}| \frac{d\omega}{4\pi}. \quad (14.3)$$

Castor (1970) obtained the same expression for J_{ij}^2 . β_{ij}^3 denotes the probability that a line photon will escape the region around its point of emission and will strike the stellar core.

4. Radiation Field in an Expanding Envelope with a Negative Radial Velocity Gradient

We have seen at the beginning of the last section that in the case of an outward-decelerating envelope it was possible to find pairs of distant points for which the relative velocity was zero. Considering decelerating velocity distributions of the type that Kuan and Kuhl (1975) adopted in their work on P Cygni stars, namely

$$V(r) = V_0(R^*/r)^l, \quad (14)$$

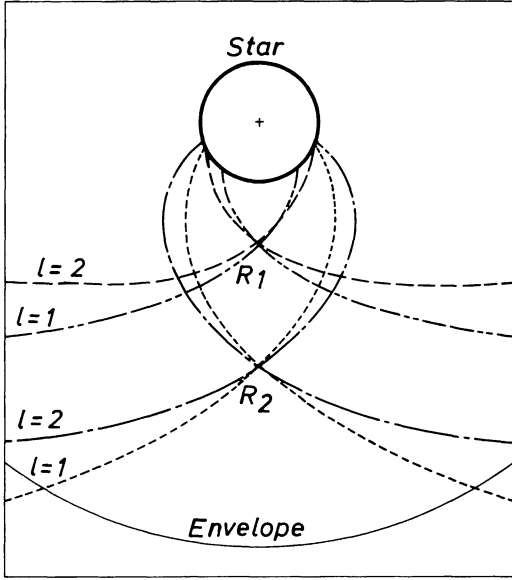


Fig. 5. Geometrical locus of the points R' relative to the points R_1 ($r=2R^*$) and R_2 ($r=4R^*$) for different values of the decelerating parameter l ($l=1, l=2$), see text

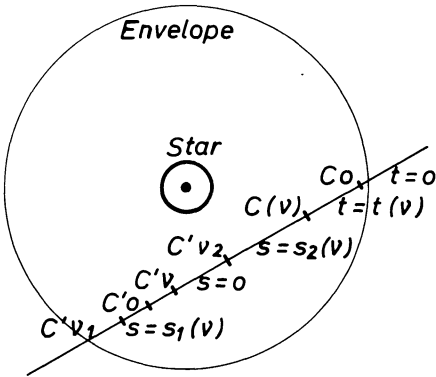


Fig. 6. Pair of distant points C_0 and C'_0 between which is studied the transfer of line radiation (see text). No scale is respected

V_0 being a constant and l a positive parameter, we illustrate in Figure 5 the geometrical locus of the points R' which satisfy Equation (1.3) for different values of the decelerating parameter l and at different points R in the envelope. It easily appears that the mean intensity of the radiation field at a point such as C_0 (see Fig. 6) in the envelope will include a contribution due to distant points C'_0 situated on the geometrical locus defined above.

The intensity of the radiation $I_{ij}^{\nu}(C'_0 C_0)$ emitted by the atoms in the neighbourhood of the point C'_0 towards C_0 , at the local line frequency ν defined at C_0 , is given by

$$I_{ij}^{\nu}(C'_0 C_0) = \int_{s_1(\nu)}^{s_2(\nu)} \varepsilon_{ij} \Phi \left(-\frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s \right) \cdot \exp \left(-\int_s^{s_2(\nu)} \alpha_{ij} \Phi \left(-\frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s' \right) ds' \right) ds, \quad (2.4)$$

where the point with abscissa $s=0$ corresponds to the point C'_0 (see Fig. 6) distant from C_0 by a value

$$\Delta S = \frac{v - v_{ij}}{v_{ij}} c \left/ \frac{\partial v_s}{\partial s} \right. \quad (3.4)$$

The positive sense of the s and t abscissae axes is directed from C'_0 to C_0 . The limits of integration $s_1(\nu)$ and $s_2(\nu)$ are the abscissae of the points $C'_{\nu 1}$ and $C'_{\nu 2}$ limiting the region in which atoms can emit line radiation which contributes to Equation (2.4). Using the following relations

$$y = -\frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s, \quad dy = -\frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} ds \quad (4.4)$$

and

$$x = -\frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} s', \quad dx = -\frac{v_{ij}}{c} \frac{\partial v_s}{\partial s} ds', \quad (5.4)$$

we obtain

$$I_{ij}^{\nu}(C'_0 C_0) = \int_{y(s_2(\nu))}^{y(s_1(\nu))} \varepsilon_{ij} \Phi(y) \exp \left(\int_y^{y(s_2(\nu))} \alpha_{ij} \frac{\Phi(x)c}{v_{ij} \frac{\partial v_s}{\partial s}} dx \right) \cdot \frac{c}{v_{ij} \frac{\partial v_s}{\partial s}} dy. \quad (6.4)$$

Recalling the relation (12.2) and stating

$$\text{if } \frac{\partial v_s}{\partial s} > 0 \quad y(s_2(\nu)) = -\frac{\Delta v}{2}, \quad y(s_1(\nu)) = \frac{\Delta v}{2} \quad (7.4)$$

and

$$\text{if } \frac{\partial v_s}{\partial s} < 0 \quad y(s_2(\nu)) = \frac{\Delta v}{2}, \quad y(s_1(\nu)) = -\frac{\Delta v}{2}, \quad (8.4)$$

we can simplify Equation (6.4) following similar reasoning as in section 3. Thus,

$$I_{ij}^{\nu}(C'_0 C_0) = S_{ij}(C'_0) (1 - \exp(-|\tau_{ij}(C'_0)|)), \quad (9.4)$$

where S_{ij} and τ_{ij} are taken at the point C'_0 .

Because of the proximity of the points C'_0 and C_0 , see (3.4), S_{ij} and τ_{ij} in Equation (9.4) can be evaluated at the point C'_0 . We then have

$$I_{ij}^{\nu}(C'_0 C_0) = S_{ij}(C'_0) (1 - \exp(-|\tau_{ij}(C'_0)|)). \quad (10.4)$$

The intensity of the line radiation $I_{ij}^{\nu}(C'_0 C_0)$ which reaches an atom at C_0 will be attenuated by its neighbouring atoms as follows:

$$I_{ij}^{\nu}(C_0) = I_{ij}^{\nu}(C'_0 C_0) \exp \left(-\int_{t(\nu)}^0 \alpha_{ij} \Phi \left(\nu - v_{ij} - \frac{v_{ij}}{c} \frac{\partial v_t}{\partial t} t' \right) dt' \right), \quad (11.4)$$

$t=t(\nu)$ and $t=0$ being respectively the abscissae of the points $C(\nu)$ and C_0 . These enclose the region of the absorbing atoms in the neighbourhood of C_0 .

The mean value of the radiation intensity $I_{ij}^v(C_0)$ when integrated over the line profile in the frame of an atom at C_0 is

$$I_{ij}(C_0' C_0) = \int_{v_{ij} - \frac{\Delta v}{2}}^{v_{ij} + \frac{\Delta v}{2}} \Phi(v - v_{ij}) I_{ij}^v(C_0) dv. \quad (12.4)$$

Simplifying this type of expression as before, we obtain

$$I_{ij}(C_0' C_0) = S_{ij}(C_0') (1 - \exp(-|\tau_{ij}(C_0')|)) \cdot (1 - \exp(-|\tau_{ij}(C_0)|)) / |\tau_{ij}(C_0)|. \quad (13.4)$$

When deriving this results we have to assume that the physical and kinematical properties of the medium (ϵ_{ij} , α_{ij} , $\partial v_s / \partial s$) do not change over a distance where the velocity range equals four times the maximum chaotic velocity of the atoms. This condition is not restrictive in the context of the Sobolev's approximation which we assume throughout this work. The contribution to the mean intensity of the field radiation at C_0 due to distant points such as C_0' , is given by

$$J_{ij}^3 = \int_{\Omega(C_0, C_0')} S_{ij}(C_0') (1 - \exp(-|\tau_{ij}(C_0')|)) \cdot (1 - \exp(-|\tau_{ij}(C_0)|)) / |\tau_{ij}(C_0)| \frac{d\omega}{4\pi}, \quad (14.4)$$

where the integration is performed over directions for which Equation (1.3) has a solution. The expression J_{ij}^3 evaluated at C_0 depends simultaneously on the physical properties of the medium at the point C_0 and at distant points C_0' .

It is clear that the local contribution J_{ij}^1 to the mean intensity of the radiation field at C_0 remains the same as in the case of an envelope expanding with a positive radial velocity gradient. Concerning the rays emerging from the stellar core towards distant points in the envelope ($r \gg R^*$), we can retain with a good approximation the previous expression for J_{ij}^2 . Indeed, except for the photons emerging from a negligible exterior ring of the stellar disc (see Fig. 5), all of the photons, will reach the neighbourhood of a point such as R in the envelope, unhindered. However, at points situated in the neighbourhood of the star (up to a few stellar radii) we must substitute to J_{ij}^2 the following expression

$$J_{ij}^4 = I_c \int_{\Omega=4\pi W} (1 - \exp(-|\tau_{ij}|)) \exp(-|\tau'_{ij}|) / |\tau_{ij}| \frac{d\omega}{4\pi}, \quad (15.4)$$

in which the factor $\exp(-|\tau'_{ij}|)$ expresses the probability for a stellar photon not to be absorbed in the neighbourhood of a distant point C_0' , solution of Equation (1.3).

5. Statistical Equilibrium Equations

In order to calculate the level populations n_i of an atom, the statistical equilibrium equations must be solved. In a steady state

$$\frac{dn_i}{dt} = R_i + C_i = 0, \quad (1.5)$$

where R_i and C_i are the net volume radiative and collisional rates respectively by which level i is populated.

When considering only the radiative exchanges occurring between two levels i and j of an atom, the net volume radiative rate R_{ij} by which the level i is populated can be written

$$R_{ij} = n_j A_{ji} + (n_j B_{ji} - n_i B_{ij}) J_{ij}, \quad (2.5)$$

recalling that n_i , n_j are the volume populations of the levels i , j ; A_{ji} , B_{ji} and B_{ij} the Einstein-Milne transition probabilities and J_{ij} the mean intensity of the radiation field at the point where R_{ij} is calculated.

The expressions for R_{ij} are evaluated in the next two paragraphs for both cases of an outward-accelerating and decelerating envelopes.

A) Case of an Outward-accelerating Envelope

The mean intensity of the radiation field J_{ij} was derived in Section 3:

$$J_{ij} = S_{ij}(1 - \beta_{ij}^1) + I_c \beta_{ij}^3. \quad (3.5)$$

As a first approximation, considering that the factors appearing in the integral of the escape probability β_{ij}^3 given by Equation (14.3) are independent of direction, we find that

$$\beta_{ij}^3 = W \beta_{ij}^1. \quad (4.5)$$

Taking into account the Formulae (6.2) for S_{ij} and (4.5) for β_{ij}^3 , we can substitute the expression for J_{ij} in (2.5) by the one given in (3.5). We obtain for the rate R_{ij}

$$R_{ij} = n_j A_{ji} + (n_j B_{ji} - n_i B_{ij}) \left(\frac{\sigma_{ij}}{n_i g_j - 1} (1 - \beta_{ij}^1) + W I_c \beta_{ij}^1 \right). \quad (5.5)$$

Recalling the well known relations existing between the Einstein-Milne transition probabilities

$$A_{ji}/B_{ji} = \sigma_{ij} \quad \text{and} \quad g_i B_{ij} = g_j B_{ji}, \quad (6.5)$$

where

$$\sigma_{ij} = 2h\nu_{ij}^3/c^2,$$

Formula (5.5) is simplified to

$$R_{ij} = \beta_{ij}^1 (n_j A_{ji} + (n_j B_{ji} - n_i B_{ij}) W I_c). \quad (7.5)$$

The factor $n_j A_{ji} + (n_j B_{ji} - n_i B_{ij}) W I_c$ in (7.5) represents the rate R_{ij} given in (2.5) for which $J_{ij} = W I_c$. Calling it R_{ij}^p , it represents the volume radiative rate by which the level i is populated when the atom is plunged into a transparent medium at rest, the levels i and j interacting with each other through the dilute stellar radiation field $W I_c$. Our first conclusion is that in the presence of very

large velocity gradients, such as $\beta_{ij}^1 \sim 1$, the radiative rate R_{ij} is simplified to R_{ij}^D .

Supposing now that β_{ij}^1 is not close to unity, the rate R_{ij} given by (7.5) can be rewritten

$$R_{ij} = n_j A'_{ji} + (n_j B'_{ji} - n_i B_{ij}) W I_c \quad (8.5)$$

with

$$A'_{ji} = \beta_{ij}^1 A_{ji}, \quad B'_{ji} = \beta_{ij}^1 B_{ji} \quad \text{and} \quad B'_{ij} = \beta_{ij}^1 B_{ij}.$$

Our second conclusion is that for pairs of levels i, j where the escape probability β_{ij}^1 is not close to unity, the radiative rate R_{ij} is the same as in the case of a transparent medium at rest in the presence of a dilute stellar radiation field $W I_c$. However, the radiative affinity (A_{ij}, B_{ij}, B_{ji}) of the atoms for the transition (i, j) is decreased by the ratio $\beta_{ij}^1/1$.

If we neglect in the expression for J_{ij} the contribution from the stellar core continuum, i.e. assuming

$$J_{ij} = J_{ij}^1, \quad (9.5)$$

we obtain for the radiative rate R_{ij}

$$R_{ij} = n_j A_{ji} \beta_{ij}^1. \quad (10.5)$$

This last expression of R_{ij} is the same as the one entering the statistical equilibrium equations that Sobolev (1947, 1958) derived using an on-the-spot approximation.

The remaining parts of the net radiative and collisional rates, R_i and C_i entering Equations (1.5), are directly independent of the hypothesis of a moving medium. If those Equations (1.5) are solved for various parts of the expanding envelope, we can determine the total amount of energy E_{ij} emitted by the envelope in any spectral line

$$E_{ij} = h\nu_{ij} \int n_j A_{ji} \beta_{ij}^2 dV. \quad (11.5)$$

The integration in (11.5) is extended over the whole volume of the envelope. The escape probability β_{ij}^2 was defined in Section 2 [see Formula (28.2)].

B) Case of an Outward-decelerating Envelope

In Section 4, we derived the expression for the mean intensity of the radiation field J_{ij} :

$$J_{ij} = S_{ij}(1 - \beta_{ij}^1) + I_c \beta_{ij}^3 + \int_{\Omega(C_0, C_0)} S'_{ij}(1 - \exp(-|\tau'_{ij}|)) \cdot (1 - \exp(-|\tau_{ij}|)) \frac{1}{|\tau_{ij}|} \frac{d\omega}{4\pi}. \quad (12.5)$$

The solid angle over which the integration is performed in the last term is approximately $2\pi - 4\pi W$ (see Fig. 5).

Assuming, for simplicity, that the physical properties of the medium are approximately isotropic and homogeneous we obtain for J_{ij}

$$J_{ij} = S_{ij}(1 - \beta_{ij}^1) + I_c W \beta_{ij}^1 + S'_{ij}(1 - \exp(-|\tau_{ij}|)) (1/2 - W) \beta_{ij}^1. \quad (13.5)$$

Following the same reasoning as in the previous paragraph, the expression for R_{ij} in (2.5) can be simplified to

$$R_{ij} = \beta_{ij}^1 (n_j A_{ji} (1 - (1 - \exp(-|\tau_{ij}|)) (1/2 - W)) + (n_j B_{ji} - n_i B_{ij}) I_c W). \quad (14.5)$$

In the presence of very large velocity gradients such as $\beta_{ij}^1 \sim 1$ ($|\tau_{ij}| \ll 1$), the radiative rate R_{ij} evaluated can be simplified to R_{ij}^D defined in the preceding paragraph.

Supposing now that β_{ij}^1 is not close to unity the rate R_{ij} given by (14.5) can be rewritten as follows

$$R_{ij} = n_j A''_{ji} + (n_j B''_{ji} - n_i B_{ij}) W I_c \quad (15.5)$$

with

$$A''_{ji} = A_{ji} \beta_{ij}^1 (1/2 + W),$$

$$B''_{ji} = B_{ji} \beta_{ij}^1 \quad \text{and} \quad B''_{ij} = B_{ij} \beta_{ij}^1.$$

We conclude that for pairs of levels i, j where the escape probability β_{ij}^1 is not close to unity, the volume radiative rate R_{ij} is the same as in the case of a transparent medium at rest in the presence of a dilute stellar radiation field $W I_c$. However the effective transition probabilities B_{ij} and B_{ji} are decreased by the ratio $\beta_{ij}^1/1$ and the effective spontaneous emission probability A_{ji} by $\beta_{ij}^1(1/2 + W)/1$.

If we solve Equation (1.5) for various parts of the envelope, we are able to calculate the total amount of energy E_{ij} emitted by the envelope in any spectral line

$$E_{ij} = h\nu_{ij} \int n_j A_{ji} \beta_{ij}^4 dV. \quad (16.5)$$

The integration in (16.5) is performed over the whole volume of the envelope. The escape probability β_{ij}^4 is easily found to be

$$\beta_{ij}^4 = \int_{\Omega=4\pi(1-W)} (1 - \exp(-|\tau_{ij}|)) \frac{1}{|\tau_{ij}|} \exp(-|\tau'_{ij}|) \frac{d\omega}{4\pi}, \quad (17.5)$$

where the integration is extended over all directions except those joining the stellar core to the point C_0 , where β_{ij}^4 is evaluated.

The escape probability β_{ij}^4 is the probability that a photon emitted in the line transition $j \rightarrow i$ will escape the outward-decelerating envelope without striking the stellar core. Indeed, if we fix one direction, the factor $(1 - \exp(-|\tau_{ij}|))/|\tau_{ij}|$ in the integral of Equation (17.5) represents the escape probability along that direction of a photon around its point of emission (C_0) and the second factor $\exp(-|\tau'_{ij}|)$ expresses the probability for the photon not to be absorbed in the neighbourhood of the distant point C'_0 , which satisfies Equation (1.3).

6. Discussion

Castor (1970) has given an approximate method for treating the transfer of spectral line radiation in a rapidly expanding spherical envelope with increasing outward

velocity. In a different formalism, we obtained in Section 3 similar expressions for the "local" J_{ij}^1 and "stellar core" J_{ij}^2 mean intensities of the radiation field in one point of such an envelope. Furthermore, it is shown in § A [case $(dV(r)/dr) > 0$] of the fifth section that the on-the-spot approximation Sobolev (1947, 1958) used for deriving the statistical equilibrium equations in a moving medium consists in taking into account just the "local" contribution J_{ij}^1 in the expression of the mean intensity J_{ij} .

In Section 4 we treated the case of an expanding envelope with outward-decreasing velocity and showed that it was necessary to add a new contribution J_{ij}^3 to the expression for the mean intensity J_{ij} previously found. The quite different conclusions reached by de Groot (1969) and Kuan and Kuhi (1975) in their studies of P Cygni, as pointed out by Oegerle and Van Blerkom (1976b), could be partly due to the inadequate model $(dV(r)/dr) < 0$ which Kuan and Kuhi applied. Indeed, they neglected the contribution of line radiation emitted from distant points in the mean intensity expression J_{ij} .

In Section 5 we simplified the exact form of the statistical equilibrium equations to give an intuitive idea about the general behaviour of atoms in a moving medium.

We showed that in the presence of very large velocity gradients, such as $\beta_{ij}^1 \sim 1$, the rate R_{ij} , by which radiative transitions $i \rightleftharpoons j$ populate level i , is very similar within a good approximation to the rate R_{ij}^p obtained in the case of a transparent medium at rest, the radiation field being due only to the dilute stellar continuum. If the assumption $\beta_{ij}^1 \sim 1$ is valid for the principal series transitions of an atom and a fortiori for every other transition, the system of statistical equilibrium equations is simplified to the one describing the behaviour of every atomic level in a transparent medium at rest including the dilute stellar radiation field. Struve and Wurm (1938), Wellmann (1952), Ghobros (1962), etc. solved approximately similar sets of equations in the case of a helium atom and accounted for the overpopulation of the metastable levels $2 \cdot 1S$, $2 \cdot 3S$ and others ($2 \cdot 2P$, etc.) as due to dilution effects in extended atmospheres at rest. Similarly the abnormal strengths of some displaced absorption lines arising from metastable levels (Swings and Struve, 1941; Hutchings, 1968; Sobolev, 1958; etc.) and others (Hutchings, 1968; Walborn, 1975; etc.) in the spectra of W-R, Of, etc. type stars can be simply interpreted as due to dilution effects in those regions of the expanding envelope where the condition $\beta_{ij}^1 \sim 1$ is satisfied for every transition between atomic levels. Of course this is just one among the possible explanations to the observed features, but it appears very simple.

For both cases of outward-decelerating and accelerating envelopes, we have concluded in Section 5 that the rate of radiative transitions occurring between two levels i, j , where the escape probability β_{ij}^1 is small

($\beta_{ij}^1 < 1$), is decreased approximately by the ratio $\beta_{ij}^1/1$ in comparison with the rate at which levels i and j interact with each other through the dilute stellar radiation field when the atom is plunged into a transparent medium at rest. For instance, if at one point of such a medium the main factor depopulating a level i is due to spontaneous emissions toward the ground level ($i=1$), then in an expanding medium where $\beta_{1i}^1 < 1$, those spontaneous transitions will be decreased by the ratio $\beta_{1i}^1/1$, and consequently level i will appear more populated than in the former case.

When stating in Section 5, § B, the expression of the mean intensity J_{ij}^3 at one point of the envelope, we had to suppose that the physical conditions existing at some distant points were known in order to calculate the amount of energy emitted from them. Similarly for solving rigorously the statistical equilibrium equations at one point we need to know the physical properties (degree of excitation, etc.) of the medium at some others [see Equations (14.4), (2.5) and (12.5)]. An iterative method which neglects the contribution of J_{ij} in the radiation field expression J_{ij} as the first step and later takes it into account in the n^{th} step on the basis of the results from the $n^{\text{th}}-1$ step should easily allow the determination of the level populations of an atom at every point of an outward-decelerating envelope.

We will not discuss here the case of outward-accelerating envelopes for which several authors have already applied the model in a simplified form (Castor and Van Blerkom, 1970; Oegerle and Van Blerkom, 1975a, b; etc.).

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