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Numerical Applications to Radiative Transfer in Expanding Envelopes: The Two-level Atom Model

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Summary. We apply the escape probability method introduced by V.V. Sobolev in order to solve the statistical equilibrium equations relative to a two-level atom model in envelopes which expand spherically with positive or negative radial velocity gradients.

In this latter case, a generalization of the Sobolev approximation (local approximation) appears necessary in order to consider the possible transfer of photons in space.

It is shown that the level populations behave very differently in outward-accelerating and outward-decelerating envelopes, the degree of excitation appearing much higher in the second case.

Key words: moving stellar envelopes — emission-line stars — radiative transfer

1. Introduction

The escape probability method, initially developed by Sobolev, for treating the transfer of line radiation in a stellar envelope which is in rapid radial expansion is applied here to the case of a two-level atom. We recall that the approximation Sobolev (1947, 1957, 1958) adopted is the following: a line photon emitted at one point in the envelope is either absorbed in the immediate vicinity of that point or escapes entirely due to Doppler shifts across the envelope, assuming that the rate of motion of the envelope is much greater than the mean chaotic (thermal and turbulent) velocities of the atoms.

The numerical applications presented hereafter are based on a previous work (Surdej, 1977, referred to below as Paper I) which contains a general development of the theory about spectral line formation in outward-accelerating and outward-decelerating stellar envelopes. In the second section, we consider only the radiative exchanges occuring between the atomic levels in order to bring clearly the specific role of spectral line formation in those different media. The statistical equilibrium equations are then solved rigorously in the cases of outward-accelerating and outward-decelerating envelopes. Within these cases and under different physical conditions we compare the populations of the atomic levels.

Section 3 deals with discussions about those applications. Last section contains general conclusions.

2. Numerical Applications to the Two-level Atom Model

In the following, we often refer the reader to the fifth section of Paper I: "Contribution to spectral line formation in moving stellar envelopes. Radiation field and statistical equilibrium equations". We consider spherically expanding envelopes in which the level populations of different atoms have reached a steady state. Taking into account only radiative exchanges occuring between the two levels (1 refers to the ground level and 2 to the upper one) the net volume radiative rate by which the ground level is populated can be written

$$dn_1/dt = n_2A_{21} + (n_2B_{21} - n_1B_{12})J_{12}, \qquad (1)$$

recalling that n_1, n_2 are the volume populations of the levels 1,2; A_{21}, B_{21} and B_{12} the Einstein-Milne transition probabilities and J_{12} the mean intensity of the radiation field at the point where dn_1/dt is calculated. In a steady state

$$dn_1/dt = 0, \qquad (2)$$

and it follows

$$S_{12} = J_{12}, \qquad (3)$$

with

$$S_{12} = \sigma_{12}/(g_2n_1/g_1n_2 - 1), \qquad (4)$$

and

$$\sigma_{12} = 2h\nu_{12}^3/c^2, \qquad (5)$$

where

ν_{12} denotes the central frequency of the line transition; g_1, g_2 the statistical weights of the levels 1 and 2; h is Planck's constant and c the velocity of light. When considering only line processes in the transition 1→2 and under the assumption of a complete redistribution of the radiation with frequency and direction, the second member of relation (4) represents the source function S_{12} for the line transition in the frame of the atom.

In this section we will consider velocity distributions of the type Kuan and Kuhi (1975) adopted in their work on P Cygni stars, namely

$$V(r)=V_0(R^*/r)^l, \tag{6}$$

V_0 being the velocity at the stellar surface, R^* the stellar radius, r the distance from the center of the star to the point we evaluate $V(r)$ and l, a decelerating (resp. accelerating) parameter when positive (resp. negative). We define the following relation

$$L=r/R^*. \tag{7}$$

Supposing that the amount of matter ejected by the star is constant in time, the density $n(r)$ at the distance r from the center of the core is given by

$$n(r)=n_0 L^{l-2}, \tag{8}$$

n_0 representing the density in numbers per unit volume at the stellar surface. In the numerical applications, we consider that the envelope extends up to 20 stellar radii ($L=20$). The determination of the physical properties in the envelopes are evaluated at subsequent points distant from each other by one stellar radius.

A) Transparent Medium at Rest

We shall deal first with the case of a transparent medium at rest, referred later by T.M.R. Assuming the stellar core to radiate continuously, like a black body at temperature T, without limb darkening and with an intensity I_c constant over the line frequency, the expression for J_{12} entering Equation (3) is given by

$$J_{12}=WI_c, \tag{9}$$

where W is the dilution factor

$$W=0.5(1-(1-(1/L)^2)^{1/2}), \tag{10}$$

and I_c the Planck function at the core temperature. If we define x as being the population ratio of the two atomic levels, i.e. $x=n_1/n_2$, we obtain

$$x=(g_1/g_2)((\exp(h\nu_{12}/kT)-1)/W+1), \tag{11}$$

with k being Boltzmann's constant.

B) Outward-accelerating Envelope

In an outward-accelerating envelope, a two-level atom interacts with the radiation from the stellar core and from its neighbouring atoms. In this case (Castor 1970, see Paper I) the mean intensity expression J_{12} reduces to

$$J_{12}=S_{12}(1-\beta^1_{12})+I_c\beta^3_{12}. \tag{12}$$

We recall that β^1_{12} is the probability for a photon created in the line transition $2\rightarrow1$ to escape locally the medium along any direction

$$\beta^1_{12}=\int_{\Omega=4\pi}(1-\exp(-|\tau_{12}|))/|\tau_{12}|d\omega/4\pi, \tag{13}$$

where the integration is extended over all directions (solid angle $\Omega=4\pi$). τ_{12} is defined by

$$\tau_{12}=(\alpha_{12}\cdot c)/(\nu_{12}\cdot\partial v_s/\partial s), \tag{14}$$

where α_{12} is the volume absorption coefficient for an observer moving with the atoms

$$\alpha_{12}=(n_1B_{12}h\nu_{12}/4\pi)(1-(g_1n_2/g_2n_1)), \tag{15}$$

and $\partial v_s/\partial s$ represents the velocity gradient along the direction we evaluate τ_{12}.

β^3_{12} is the probability that a line photon will escape the region around its point of emission and will strike the stellar core,

$$\beta^3_{12}=\int_{\Omega=4\pi W}(1-\exp(-|\tau_{12}|))/|\tau_{12}|d\omega/4\pi, \tag{16}$$

the integration being extended over the only directions which join the point we evaluate β^3_{12} and the stellar core. The probability β^2_{12} that a photon emitted in the line transition $2\rightarrow1$ will escape the envelope without striking the stellar core is similarly given by

$$\beta^2_{12}=\int_{\Omega=4\pi(1-W)}(1-\exp(-|\tau_{12}|))/|\tau_{12}|d\omega/4\pi. \tag{17}$$

Combining Equations (3) and (12), we get successively

$$S_{12}=(\beta^3_{12}/\beta^1_{12})I_c \tag{18}$$

and

$$x\ =(g_1/g_2)((\beta^1_{12}/\beta^3_{12})(\exp(h\nu_{12}/kT)-1)+1). \tag{19}$$

The following expressions are defined here below

$$\mathrm{LGX}\ =\log_{10}(x), \tag{20}$$

$$\mathrm{LGS1}=\log_{10}(S_{12}(1-\beta^1_{12})), \tag{21}$$

$$\mathrm{LGS3}=\log_{10}(I_c\,\beta^3_{12}), \tag{22}$$

$$\mathrm{LGB1}=\log_{10}(\beta^1_{12}), \tag{23}$$

$$\mathrm{LGB2}=\log_{10}(\beta^2_{12}), \tag{24}$$

$$\mathrm{LGB3}=\log_{10}(\beta^3_{12}), \tag{25}$$

and later used for illustrating in some of the figures the behaviour of the physical conditions in the envelopes.

C) Outward-decelerating Envelope

At any point of an outward-decelerating envelope a two-level atom interacts with the radiation from the stellar core, from its neighbouring atoms and from atoms situated at distant points[1] (see Paper I). We recall here the expression derived for the mean intensity J_{12}

$$J_{12}=S_{12}(1-\beta^1_{12})+J^5_{12}+\int_{\Omega(C_0,C_0')} S'_{12}(1-\exp(-|\tau'_{12}|)) \cdot(1-\exp(-|\tau_{12}|))/|\tau_{12}|d\omega/4\pi. \quad (26)$$

J^5_{12} is the mean intensity expression specific to the stellar photons which can interact with an atom at the point we calculate it:

$$J^5_{12}=I_c\,\beta^5_{12} \quad (27)$$

with

$$\beta^5_{12}=\int_{\Omega=4\pi W}(1-\exp(-|\tau_{12}|))\exp(-|\tau'_{12}|)/|\tau_{12}|d\omega/4\pi. \quad (28)$$

β^5_{12} is the probability for a photon emitted from the stellar core to reach unhindered the point P_0 where we calculate its expression. In formula (28) the factor $\exp(-|\tau'_{12}|)$ expresses the probability for a stellar photon not to be absorbed in the neighbourhood of a distant point P'_0, whose relative velocity to P_0 is null.

The third term in Equation (26) is the contribution to the mean intensity expression at P_0 from distant points such as P'_0. The integration is extended over all directions from P_0 such as a couple of points (P_0, P'_0) exists.

Combining Equations (3) and (26) we find successively

$$S_{12}=(\beta^5_{12}/\beta^1_{12})I_c+(1/\beta^1_{12})\int_{\Omega(C_0,C_0')} S'_{12}(1-\exp(-|\tau'_{12}|)) \cdot(1-\exp(-|\tau_{12}|))/|\tau_{12}|d\omega/4\pi \quad (29)$$

and $x=(g_1/g_2)(\sigma_{12}/S_{12}+1)$. (30)

The probability β^4_{12} that a photon emitted in the line transition will escape the outward-decelerating envelope without striking the stellar core is expressed by

$$\beta^4_{12}=\int_{\Omega=4\pi(1-W)}(1-\exp(-|\tau_{12}|))/|\tau_{12}|\exp(-|\tau'_{12}|)d\omega/4\pi. \quad (31)$$

In the case of outward-decelerating envelopes the following expressions are here defined

$$\mathrm{LGX}=\log_{10}(x), \quad (32)$$

$$\mathrm{LGS1}=\log_{10}(S_{12}(1-\beta^1_{12})), \quad (33)$$

$$\mathrm{LGS5}=\log_{10}(I_c\,\beta^5_{12}), \quad (34)$$

$$\mathrm{LGS2}=\log_{10}\Big(\int_{\Omega(C_0,C_0')} S'_{12}(1-\exp(-|\tau'_{12}|)) \cdot(1-\exp(-|\tau_{12}|))/|\tau_{12}|d\omega/4\pi, \quad (35)$$

$$\mathrm{LGB1}=\log_{10}(\beta^1_{12}), \quad (36)$$

$$\mathrm{LGB5}=\log_{10}(\beta^5_{12}), \quad (37)$$

$$\mathrm{LGB4}=\log_{10}(\beta^4_{12}) \quad (38)$$

and will be later used in some of the figures to illustrate the behaviour of the physical properties they represent. To solve Equation (30) in order to determine the relative level populations x at a point such as P_0 requires à priori the knowledge of the physical conditions existing at P_0 and at distant points such as P'_0 [see second member of Eq. (29)]. Therefore in the calculations of the quantity x at P_0, we used an iterative method which as a first step considers initial values of the x quantities throughout the envelope and later replaces them by successive approximations. This method converges rapidly and when computing all integrals $(\beta^1_{12}, \beta^2_{12}, \ldots)$ with 100 steps of integration five iterations give a precision better than 0.005 for the x values.

D) Parameters of the Model

Before presenting the results of our calculations, let us describe the main parameters which determine the physical conditions throughout a moving envelope. A look at the Equations (18) and (29) first suggests to develop the optical depth τ_{12}, quantity which enters the different escape probability expressions and the further radiative coupling term in Equation (29).

As in Paper I let $dV(r)/dr$ be the radial velocity gradient as a function of r, and θ, the angle between the radial direction and the direction along which is evaluated the optical depth τ_{12} [see Eq. (14)]. In this system of coordinates (r,θ) the velocity gradient $\partial v_s/\partial s$ reduces to

$$\partial v_s/\partial s=(dV(r)/dr)\cos^2\theta+(V(r)/r)\sin^2\theta, \quad (39)$$

and for the velocity distributions [Eq. (6)] assumed here, we obtain

$$\partial v_s/\partial s=(V(r)/r)(1-(l+1)\cos^2\theta). \quad (40)$$

In the following we prefer to express the volume absorption coefficient α_{12} by using the oscillator strength f_{12} for the transition $1\rightarrow 2$ rather than the Einstein-Milne transition probabilities. We have the relation

$$f_{12}=\frac{mh}{\pi e^2}\frac{B_{12}}{\lambda_{12}}\frac{c^2}{4\pi}, \quad (41)$$

m and e representing the mass and charge of the electron and λ_{12} the wavelength of the line transition.

Combining Equations (8), (14), (15), (40) and (41) we find

$$|\tau_{12}|=\tau^l_{12}L^{2l-1}F(\theta)\frac{(1-(g_1/g_2x))}{(1+(1/x))}, \quad (42)$$

where τ^l_{12} is the radial $(\theta=0)$ optical depth at the stellar surface $(L=1)$ supposing that $x \gg 1$

$$\tau^l_{12}=\frac{\pi e^2}{mc}f_{12}\lambda_{12}\frac{Rn_0}{V_0|l|}. \quad (43)$$

The second term in Equation (42) expresses the radial dependence of $|\tau_{12}|$ as a function of L. Finally the function $F(\theta)$ gives the angular correction to apply to the

[1] Atoms situated at two points (P_0, P'_0) which are separated by more than a distance in which the Doppler displacement at the line frequency changes by an amount equal to the line width and which are able to interact radiatively with each other are called "distant points"

Table 1. References to the presentation of the figures (see § E)

Figure number	Physical quantities (ordinate)	Nature of the envelope	l	τ_{12}^l
1	LGX	A.E	−0.5, −1, −2	10^3
2	LGX	T.M.R, A.E	−0.5, −1, −2	1
3	LGS1, LGS3	A.E	−0.5	10^3
4	LGS1, LGS3	A.E	−1	10^3
5	LGS1, LGS3	A.E	−2	10^3
6	LGB1, LGB2, LGB3	A.E	−0.5	10^3
7	LGB1, LGB2, LGB3	A.E	−1	10^3
8	LGB1, LGB2, LGB3	A.E	−2	10^3
9	LGS1, LGS3	A.E	−0.5	1
10	LGB1, LGB2, LGB3	A.E	−0.5	1
11	LGX	D.E.−, D.E.+	0.5	10^3
12	LGX	D.E.−, D.E.+	1	10^3
13	LGX	D.E.−, D.E.+	2	10^3
14	LGX	T.M.R, D.E.−, D.E.+	0.5, 1	10^{-3}
15	LGX	T.M.R, D.E.−, D.E.+	2	10^{-3}
16	LGS1, LGS2, LGS5	D.E.+	0.5	10^3
17	LGS1, LGS2, LGS5	D.E.+	2	10^3
18	LGB1, LGB4, LGB5	D.E.+	0.5	10^3
19	LGB1, LGB4, LGB5	D.E.+	2	10^3
20	LGS1, LGS2, LGS5	D.E.+	0.5	10^{-3}
21	LGS1, LGS2, LGS5	D.E.+	2	10^{-3}
22	LGB1, LGB4, LGB5	D.E.+	0.5	10^{-3}
23	LGB1, LGB4, LGB5	D.E.+	2	10^{-3}

radial optical depth in order to determine $|\tau_{12}|$ along any direction θ.

$$F(\theta)=\left|\frac{l}{1-(l+1)\cos^2\theta}\right|. \qquad (44)$$

From all preceding relations we conclude that the geometrical and physical parameters l and τ_{12}^l fix entirely the optical depth $|\tau_{12}|$ as a function of the variables L and θ.

Finally, the stellar core intensity I_c is another important quantity on which depends the source function S_{12} [see Eqs. (18) and (29)]. In all applications we considered I_c as the Planck function evaluated for $\lambda_{12}T = 0.28979$ cm · deg, i. e. the constant appearing in Wien's law.

E) Results

We present here below in Figures 1−23 the results of our calculations. In all following applications the ratio of the statistical weights was taken to be $g_1/g_2=1$ and the stellar core intensity I_c equal to the Planck function evaluated for $\lambda_{12}T=0.28979$ cm · deg. Each of the figures displays the plot of different physical quantities (LGX; LGS1, ..; LGB1, ...) as a function of the variable $L=r/R^*$ for different values of the parameter l and the radial optical depth τ_{12}^l at the stellar surface.

Table 1 contains a summary of those different parameters for all the figures. In that table, Column 1 gives the figure number, Column 2 the physical quantities plotted in that figure, Column 3 informs about the nature of the envelope as follows: T. M. R refers to a transparent medium at rest, A. E to an outward-accelerating envelope, D. E. + to an outward-decelerating envelope and D. E. − to an outward-decelerating envelope for which we neglected the radiative interactions between distant atoms. Finally, in Column 4 and 5 appear respectively the values of the parameters l and τ_{12}^l.

3. Discussion

Though we considered a rough application (two-level atom, radiative processes) of the radiative transfer in moving envelopes, it appears very significant from Figures 1 to 23 that a rigorous treatment of the problem, specific to every case (A. E, D. E. + . . .), is fundamental in order to determine accurate physical properties of the different media.

Let us recall briefly the physical processes which determine the population ratio of the two atomic levels throughout an expanding envelope. A stellar core radiates continuously and without limb darkening a certain quantity of photons, the same in number over the line frequency. At a fixed point P_0 in the expanding atmosphere a certain part of these photons can interact with the atoms. The escape probability expression β_{12}^3 [Eq. (14.3) in Paper I] accounts for these stellar photons arriving unhindered at P_0. The remaining part of the stellar radiation leaves directly the medium. Those first absorbed photons will then diffuse locally in the envelope with a probability $P=(1-\beta_{12}^1)$ or will escape locally the expanding envelope when striking $(P=\beta_{12}^3)$ or not

Fig. 1—23. See Table 1 for captions

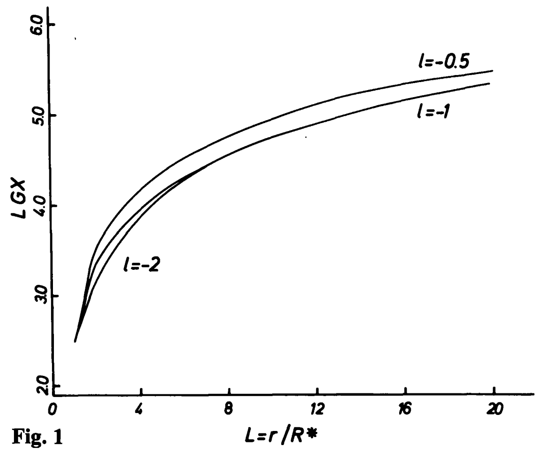


Fig. 1

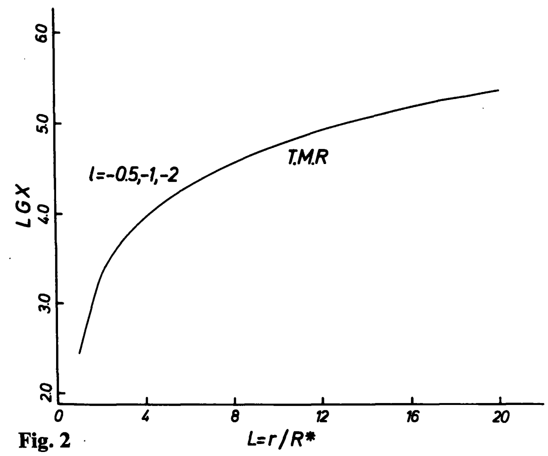


Fig. 2

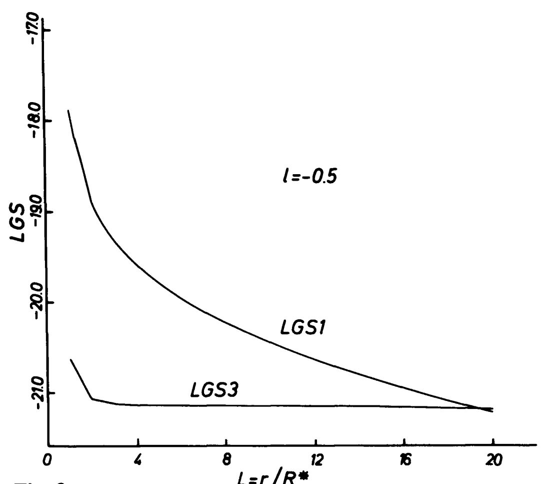
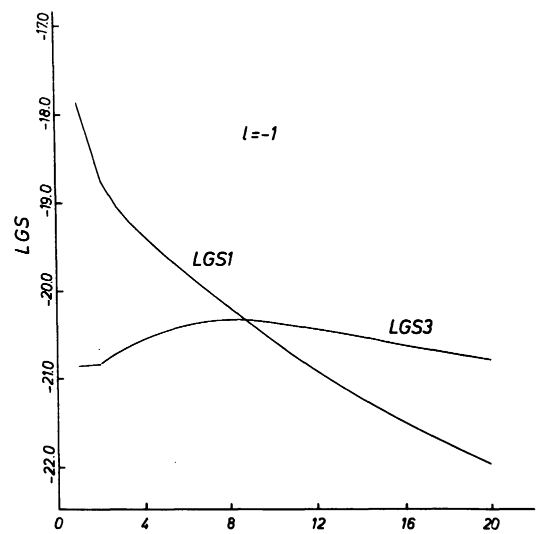


Fig. 3



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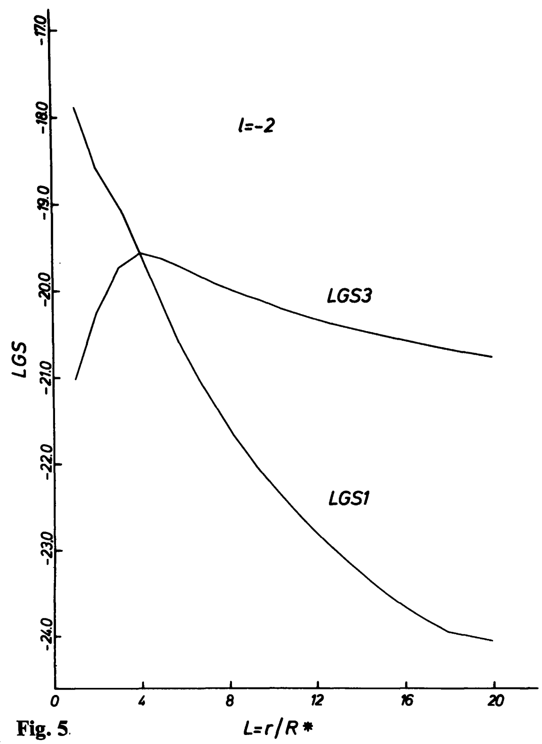

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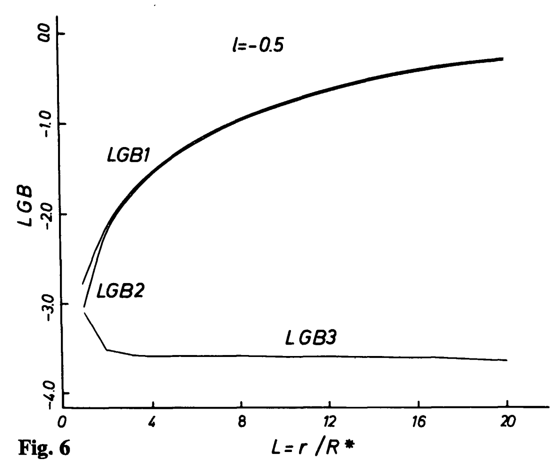

Fig. 6

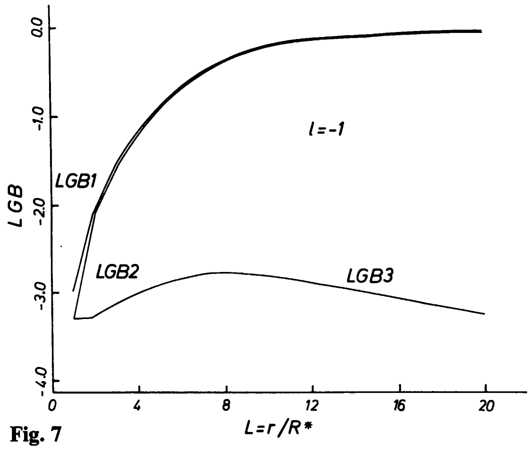

Fig. 7

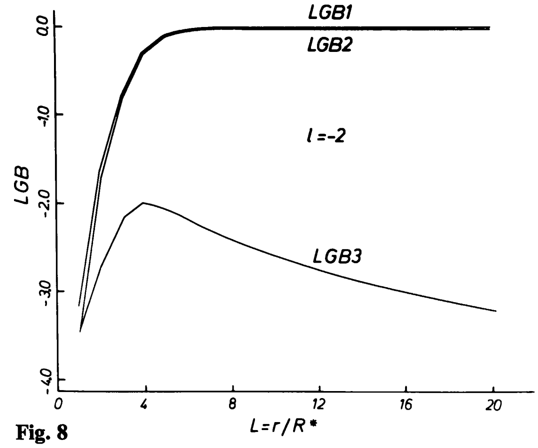

Fig. 8

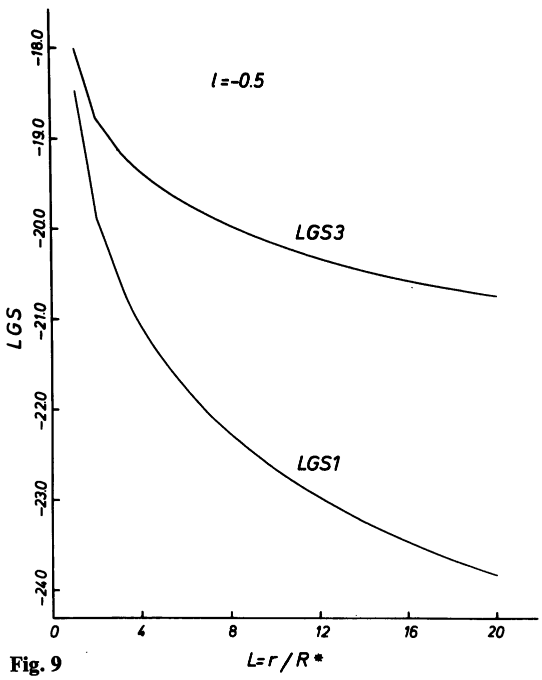

Fig. 9

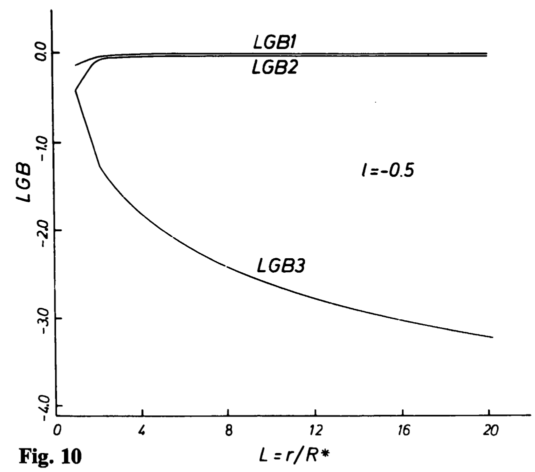

Fig. 10

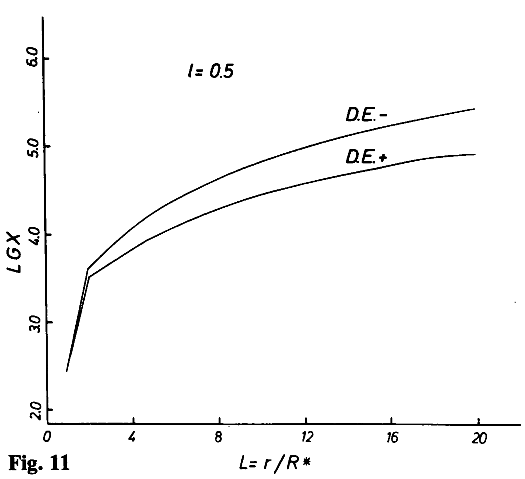

Fig. 11

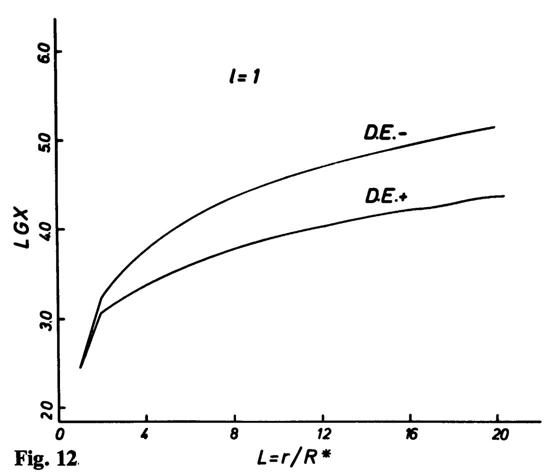

Fig. 12

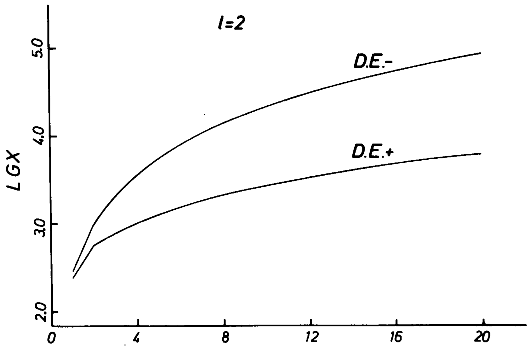

Fig. 13

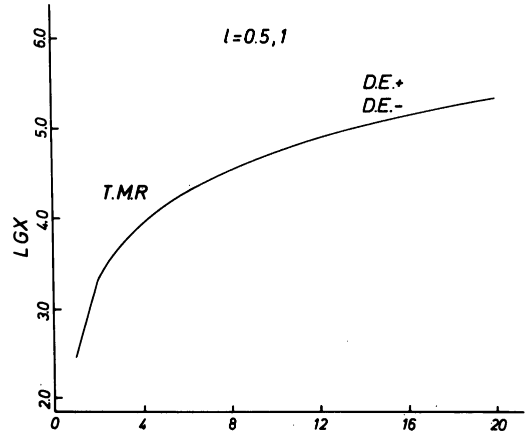

Fig. 14

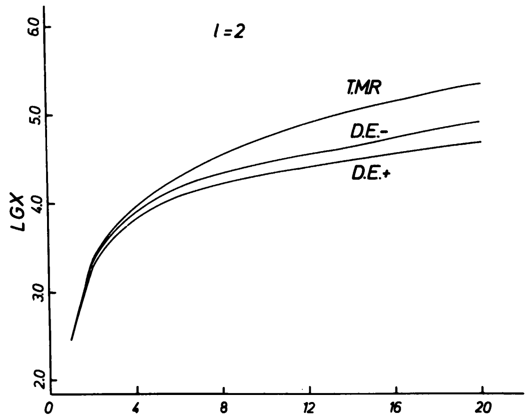

Fig. 15

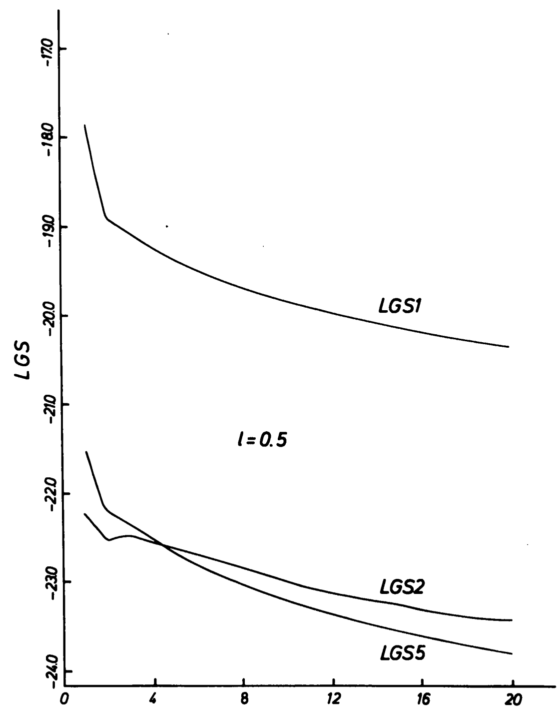

Fig. 16

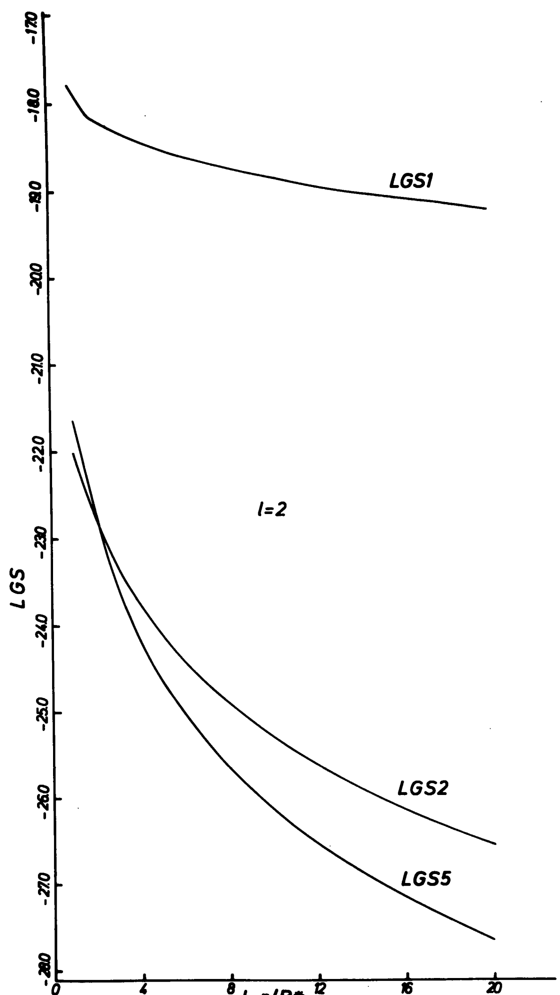

Fig. 17

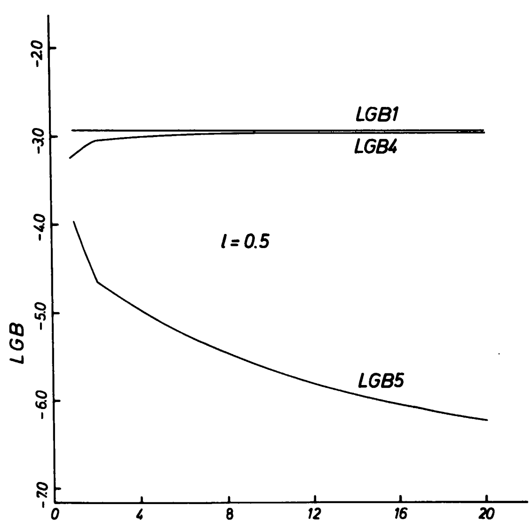

Fig. 18

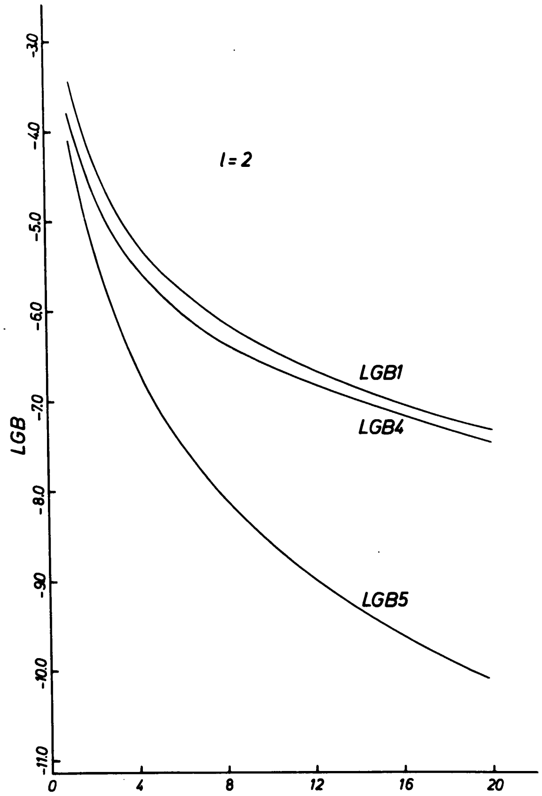

Fig. 19

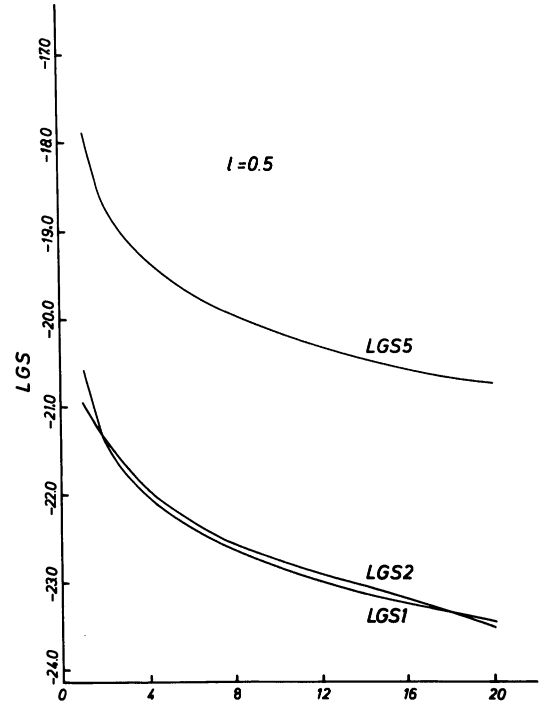

Fig. 20

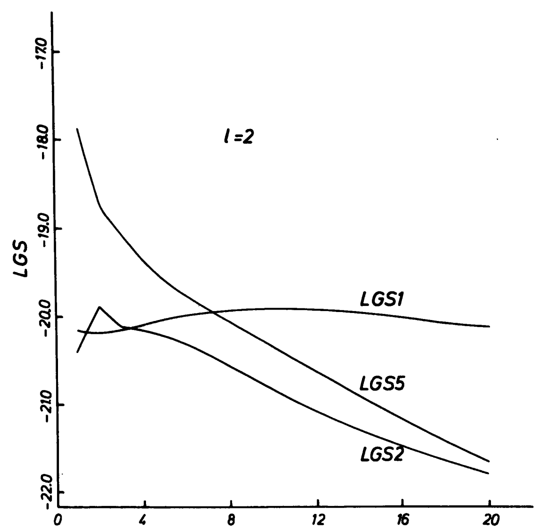

Fig. 21

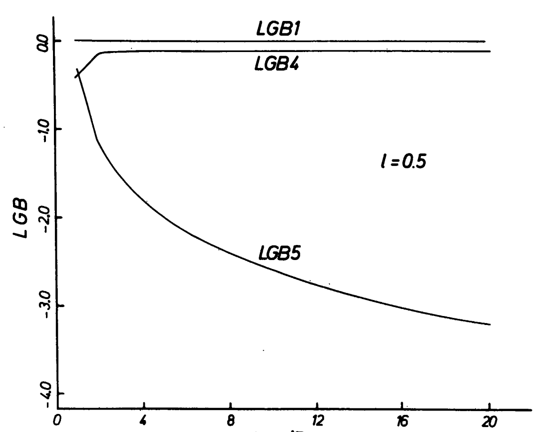

Fig. 22

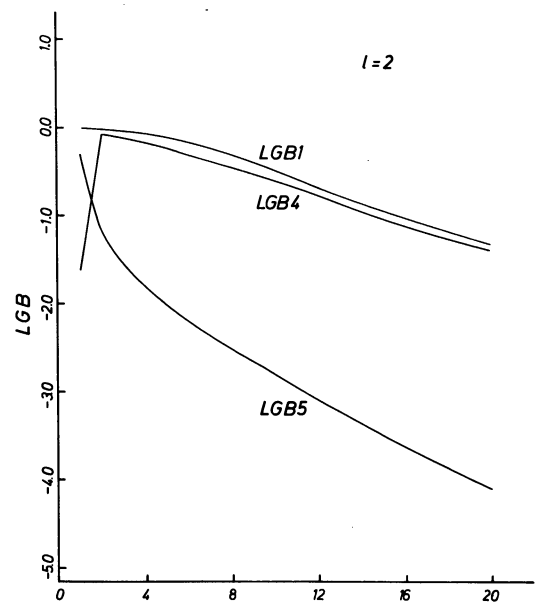

Fig. 23

($P=\beta^2_{12}$) the stellar core. Of course, further diffusions of the photons can take place locally in the medium. We find easily that the probability P^2 a photon created in the envelope has to leave locally the medium after any number n of local diffusions ($n=0, 1, 2, \ldots$) is given by

$$P^2=\beta^2_{12}+(1-\beta^1_{12})\,\beta^2_{12}+(1-\beta^1_{12})^2\,\beta^2_{12}+\ldots \text{ or} \quad (45)$$

$$P^2=\beta^2_{12}/\beta^1_{12}\,.$$

Similarly the probability P^3 that a photon created in the envelope will leave the medium and will strike the stellar core after any number n of local diffusions ($n=0,1,2,\ldots$) is

$$P^3=\beta_{12}^3/\beta_{12}^1. \quad (46)$$

We have the relation

$$P^2+P^3=1, \quad (47)$$

which means that any photon created in the envelope will finally leave locally the medium.

In the case of D.E.+ envelopes the probabilities β_{12}^2 and β_{12}^3 should be replaced respectively by β_{12}^4 and β_{12}^5. Furthermore, let us remark that in this latter case the photons created at a point P_0 will diffuse not only locally but also towards distant points such as P_0'.

Recalling that $1/\beta_{12}^1$ represents the mean number $\bar{n}$ of local diffusions at P_0

$$\bar{n}=\frac{(1-\beta_{12}^1)+2(1-\beta_{12}^1)^2+3(1-\beta_{12}^1)^3+\ldots}{(1-\beta_{12}^1)+(1-\beta_{12}^1)^2+(1-\beta_{12}^1)^3+\ldots}, \quad (48)$$

the quantity $I_c\,\beta_{12}^3/\beta_{12}^1$ [see Eqs. (18) and (29)] accounts directly for an effective number of local diffusions from these stellar photons arriving at the point P_0. The source function S_{12} and thus the population ratio of the two levels are fully determined by that last quantity in the cases of A.E and D.E.− envelopes. When considering D.E.+ envelopes the part played by the photons emitted from distant atoms appears in the second term of Equation (29). That term due to the "radiative coupling" in the envelope gives the additional effective number of local diffusions at P_0 undergone by those photons emitted from distant atoms in the envelope.

A) Outward-accelerating Envelopes

In Figures 1 to 10 we determined numerically the physical conditions in A.E envelopes for different values of the parameters $l=-0.5$, -1, -2 and $\tau_{12}^l=1,10^3$. The behaviour of the different physical quantities (LGX, LGS1, ...) illustrated by these figures can be easily interpreted when using asymptotic expressions for the escape probabilities β_{12}^1, β_{12}^3. In the optical thick approximation ($|\tau_{12}|\gg 1$) the escape probabilities β_{12}^1 and β_{12}^3 reduce to

$$\beta_{12}^{1,3}\simeq \int_{\Omega=4\pi,4\pi W} 1/|\tau_{12}|\,d\omega/4\pi. \quad (49)$$

Using the expression (42) for $|\tau_{12}|$ we find

$$\beta_{12}^{1,3}\simeq 1/(2|l|\tau_{12}^l L^{2l-1})\int_0^{\pi,\ \arcsin(1/L)} |(1-(l+1)\cos^2\theta)|\sin\theta d\theta \quad (50)$$

for $x\gg 1$.

The last integration can be easily performed over the variable θ and if $L\gg 1$, we obtain

$$\beta_{12}^1\simeq(2-l)/(3|l|\tau_{12}^l L^{2l-1}), \quad (51)$$

and

$$\beta_{12}^3\simeq W/(\tau_{12}^l L^{2l-1}), \quad (52)$$

where the dilution factor W is now approximated by

$$W\simeq(1/4)\,(1/L)^2. \quad (53)$$

As stated before the population ratio x is fixed by the following quantity

$$\beta_{12}^1/\beta_{12}^3\simeq(2-l)/(3|l|W). \quad (54)$$

In the second approximation ($|\tau_{12}|\ll 1$) we find straightly

$$\beta_{12}^1\simeq 1, \quad (55)$$

$$\beta_{12}^3\simeq W \quad (56)$$

and

$$\beta_{12}^1/\beta_{12}^3\simeq 1/W. \quad (57)$$

For the values of the parameters $l=-0.5$ and $\tau_{12}^l=10^3$ (Figs. 1, 3 and 6) we have $|\tau_{12}|\gg 1$ throughout the A.E envelope ($L=1{,}20$). We can derive from Equations (51), (52) and (54) the following approximations

LGB1 $\simeq -2.778+2\log(L)$,

LGB3 $\simeq -3.602$,

and

LGX $\simeq$ $2.977+2\log(L)$.

These relations are in good agreement with our results.

If the accelerating parameter is $l=-1$, the rate of expansion of the envelope is the same in all directions (isotropy). As a consequence and only in that case we have $\beta_{12}^3=W\beta_{12}^1$ and the population ratio x is found to vary exactly as $1/W$ (Figs 1 and 2). For $\tau_{12}^l=10^3$ and $l=-1$ we have $|\tau_{12}|>1$ up to $L=10$ and $|\tau_{12}|<1$ between $L=10$ and $L=20$. This explains why the escape probability β_{12}^3 in Figure 7 first increases with L [cf. Eq. (52)] and later on decreases [cf. Eq. (56)]. The case $\tau_{12}^l=10^3$ and $l=-2$ is illustrated by Figures 1, 5 and 8. Because of the very high rate of expansion of the envelope the optical depth $|\tau_{12}|$ decreases very fast with L. Around $L=4$ we have already $|\tau_{12}|\simeq 1$. Therefore the escape probability β_{12}^3 in Figure 8 behaves qualitatively the same as in Figure 7. Similarly, the escape probability β_{12}^1 is well approximated by Equations (51) for $L<4$ and (55) for $L>4$. Let us notice that because of the high transparency in the envelope for $L>4$ the population ratio x in Figure 1 varies also as $1/W$.

It is interesting to compare between Figures 3–5 the mean intensity expressions of the local radiation field LGS1 and of the stellar core LGS3. Let us recall here that in an hypothetical T. M. R medium no transfer of line radiation is involved. The radiation field LGS1 is absent and the stellar core field is simply diluted by the geometry [cf. Eq. (9)]. Those parts of the A. E envelopes for which $|\tau_{12}| \ll 1$ are well approximating a T. M. R medium (LGS1 $\ll$ LGS3, $I_c\, \beta_{12}^3 \simeq I_c W$).

Figure 2 illustrates the logarithmic values of the population ratio x for the parameters $l=-0.5$, -1, -2 and $\tau_{12}^l=1$. Throughout these A. E envelopes we have $|\tau_{12}| \ll 1$ for $L>1$. The different physical quantities plotted in Figures 2, 9 and 10 are therefore well approximated by relations (55–57).

B) Outward-decelerating Envelopes

Inspection of Figures 11–23 leads to the following conclusion: It is primordial for the transfer of line radiation in D. E envelopes to take into account the radiative exchanges occuring between distant atoms (cf. Paper I). Comparison of the results between D. E. + and D. E. − envelopes in Figures 11–13 and 15 shows that radiative interactions between distant atoms have the effect of enhancing the degree of excitation in the envelope by important factors (up to 10 and more).

As precedingly we shall derive asymptotic expressions for the different physical quantities in both cases of D. E. − and D. E. + envelopes.

For D. E. − envelopes and in the optical thick approximation the escape probabilities β_{12}^1 and β_{12}^3 are obtained from Equation (50). Being aware that the function $(1-(l+1)\cos^2\theta)$ changes sign between $\theta=0$ and $\theta=\pi/2$ we obtain now

$$\beta_{12}^1 \simeq (4/(1+l)^{1/2}-(2-l))/(\tau_{12}^l L^{2l-1} 3l), \tag{58}$$

although β_{12}^3 remains the same as before [cf. Eq. (52)].

For D. E. + envelopes we must substitute the expression β_{12}^5 to β_{12}^3. These escape probabilities differ because in the first case distant atoms can absorb a part of the stellar core radiation before it reaches atoms at P_0. However, under the assumption $L \gg 1$ (see Fig. 5 in Paper I) we can neglect the part played by these distant atoms and take the approximation $\beta_{12}^5 \simeq \beta_{12}^3$. So we obtain

$$\beta_{12}^1/\beta_{12}^{3,5} \simeq (4/(1+l)^{1/2}-(2-l))/(3lW). \tag{59}$$

The logarithmic values of the population ratio in Figures 11–13 are well approximated by relation (59) for the case of D. E. — envelopes ($|\tau_{12}|>10^3$).

For $\tau_{12}^l=10^3$ we can derive from Equations (58) and (52) the following approximations

if $l=0.5$ (see Fig. 18) LGB1 $\simeq -2.929$,

LGB5 $\simeq -3.602-2\log(L)$

and

if $l=2$ (see Fig. 19) LGB1 $\simeq -3.415-3\log(L)$,

LGB5 $\simeq -3.602-5\log(L)$.

These relations are in good agreement with our results for $L \gg 1$. For the values of the parameters $\tau_{12}^l=10^{-3}$ and $l=0.5,1$ the optical depth is such as $|\tau_{12}| \ll 1$ throughout the envelopes. The results obtained in Figures 14, 20 and 22 are well understood from the previous relations (55), (56) and (57). For $\tau_{12}^l=10^{-3}$ and $l=2$ we have parts of the envelope where $|\tau_{12}| \ll 1$ ($L=1$—4) and parts where $|\tau_{12}| \gg 1$ ($L=15$—20). Relations (57) and (59) fit respectively those parts of the curve in Figures 15, 21 and 23.

We shall now account for the role played by the radiative coupling term in D. E. + envelopes. Using the asymptotic expressions derived before, under the assumptions $|\tau_{12}|$, $|\tau_{12}'| \gg 1$ and $L \gg 1$, Equation (29) can be reduced to

$$S_{12} \simeq \frac{I_c 3l/4L^2 + 3l \int_{\Omega(C_0,C_0')} S_{12}'/F(\theta)\, d\omega/4\pi}{4/(1+l)^{1/2}-(2-l)}. \tag{60}$$

Numerically, we have seen that successive iterations of the quantity S_{12}' in the last equation leads to a rapid convergence of the source function S_{12} throughout the envelope. In this iterative method, the zero order approximation consists in neglecting any spatial transfer of the photons. We have consequently

$$S_{12}^0 = S_{12}(\text{D. E.} -). \tag{61}$$

The first order approximation can be easily performed when replacing from relation (61) S_{12}' in Equation (60). Physically, this is equivalent to consider a first possible diffusion of the photons from a point C_0 towards distant points C_0'. It then results

$$S_{12}^1 = \frac{I_c 3l/4L^2}{4/(1+l)^{1/2}-(2-l)} + \frac{1}{2}\, \frac{I_c(3l)^2}{(4/(1+l)^{1/2}-(2-l))^2} \cdot \int_{\theta_{inf}}^{\pi/2} \frac{1}{4L'^2}\, \frac{|1-(l+1)\cos^2\theta|}{l} \sin\theta\, d\theta. \tag{62}$$

Referring to Paper I (see Fig. 5) we easily find for the case $l=1$,

$\theta_{\text{inf}} = \text{arctg}\,(1/L)$

and

$L' = L \cdot \text{tg}\,(\theta)$.

Under these conditions, Equation (62) simplifies to

$$S_{12}^1 = S_{12}^0 \left(1 + 0.820 \cdot \int_{\text{arctg}\,(1/L)}^{\pi/2} \frac{|1-2\cos^2\theta| \sin\theta\, d\theta}{\text{tg}^2\,\theta}\right). \tag{63}$$

In Table 2 we enclosed the values S_{12}^1/S_{12}^0 as a function of the variable $L (L \gg 1)$ for the case $l=1$. In the last column of that table are listed similarly the exact ratios S_{12}/S_{12}^0 as determined from our numerical applications for $\tau_{12}^l=10^3$ (cf. Fig. 12). From Table 2, we realize that the first diffusion of photons towards distant atoms (first iteration) accounts already for a part of the enhancement of the excitation throughout the D.E.+ envelope but can not yet explain the observed ratios.

When a photon escapes locally the medium at a point C_0, it can be absorbed at a distant point C_0' with a probability $P\sim 1/2$ or escape entirely the D.E.+ envelope with an equal probability (see Fig. 5 in Paper I, recalling that $|\tau_{12}'|\gg 1$ and $L\gg 1$). Similarly the probability a photon will undergo two distant diffusions is about $(1/2)^2$, etc. Therefore, we can expect a factor $(1/2+(1/2)^2+\ldots+(1/2)^n)/(1/2)$ between the first and the n^{th} iteration. When $n\to\infty$, we have approximately

$$S_{12}^{\infty}\simeq 2\cdot S_{12}^1\,. \tag{64}$$

This result, reported in Table 2, shows a quite good agreement with the observed ratios S_{12}/S_{12}^0.

Similar conclusions can be reached for the different values of the decelerating parameter $l=0.5$ and $l=2$.

From Figures 16, 20 and 17, 21 it appears obvious that the two terms which determine the source function S_{12} [cf. Eq. (29)] are very competitive in the optical thick approximation rather than in the optical thin case the contribution due to the "radiative coupling" term becomes negligible before the other one, i.e. $I_c\,\beta_{12}^S/\beta_{12}^1$.

The fact of limiting the D.E.+ envelopes to $L=20$ in all our applications appears as a restriction. Indeed, the atoms situated in those regions of the medium for $L>20$ are able to interact radiatively with those situated at distances $L<20$. As a result the degree of excitation should appear enhanced. Comparison between the curves obtained for LGX when $L_{\text{Max}}=40$ and $L_{\text{Max}}=20$ ($\tau_{12}^l=10^3$ and $l=0.5$) confirms this result. The enhancement of the degree of excitation $1/x$ does not exceed 2% in this case. However, we can wonder about the validity of the escape probability method used for treating the transfer of line radiation at very great distances where all velocity gradients $\partial v_s/\partial s$ become negligible in the case of D.E.+ envelopes.

4. Conclusions

In order to apply the transfer of line radiation in moving media to more realistic cases, we outline in this paper the importance to solve rigorously the statistical equilibrium equations specific to every individual case (A.E, D.E.+ envelopes). Namely, it was shown numerically in the case of D.E.+ envelopes that the contribution to the mean intensity of line radiation due to distant atoms has the effect of enhancing appreciably the degree of excitation throughout the envelope.

Very recently, Bertout (1977) published a work on spectral line formation in YY Orionis envelopes. We shall point out here that the determinations of the physical conditions existing in these collapsing envelopes (inward-accelerating envelopes) is formally the same as in the case of D.E.+ envelopes. Indeed, the velocity distributions are of the same type ($dV/dr<0$) with the only exception that the velocities are directly opposite. In the same context, A.E envelopes are analogous to inward-decelerating envelopes.

Table 2. Comparison between estimated and observed ratios of source-functions in a D.E.+ envelope, for the case $l=1$ and $\tau_{12}^l=10^3$, as a function of $L\gg 1$ (see text)

L	S_{12}^1/S_{12}^0	S_{12}^{∞}/S_{12}^0	S_{12}/S_{12}^0
10	2.65	5.31	4.31
12	2.71	5.42	4.64
14	2.76	5.53	4.96
16	2.81	5.62	5.27
18	2.85	5.70	5.58
20	2.88	5.77	5.88

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