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Determination of the pole orientation of an asteroid. The amplitude-aspect relation revisited

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Summary. Near the opposition of a minor planet the geometrical approximation essentially constitutes a good representation of the Hapke-Irvine relation for describing the scattering properties of a surface layer, and we show that the normalized light curve – i.e. I_r^2 versus $\cos^2(\psi)$, where I_r is the relative intensity of a measurement observed at phase ψ – of a three-axes ellipsoid model reduces to a straight line whose slope depends only on the aspect angle A and on the semi-axes ratios a/b, b/c of the ellipsoid. A set of non-linear equations – including at least one equation per opposition – is then solved by a least squares method in order to derive the four unknown parameters λ_0, β_0 (ecliptic coordinates of the pole) and a/b, b/c. The use of realistic standard deviations as well as of a covariance matrix allows one to fully estimate the degree of accuracy, correlation, etc. affecting the derivation of these parameters. Among several other advantages offered by the present technique, let us mention that: (i) it allows one to test meaningfully the applicability of the amplitude-aspect relation for a given asteroid; (ii) it does not make use of additional and *uncertain* magnitude-aspect relations; (iii) every single photometric measurement within a light curve is taken into account, rather than just the amplitude. Furthermore, it is found that the occultation effect can always be neglected as long as the phase angle $\alpha \lesssim 15°$. As a practical example, we have applied this technique to published observations of two asteroids. For 624 Hektor, two possible solutions are found:

$\lambda_0 = 314\overset{\circ}{.}6 \pm 2\overset{\circ}{.}0$, $\beta_0 = 15\overset{\circ}{.}9 \pm 4\overset{\circ}{.}1$, $a/b = 2.27 \pm 0.03$ and $b/c = 1.41 \pm 0.16$ for the first pole (P_1), and

$\lambda_0 = 151\overset{\circ}{.}5 \pm 2\overset{\circ}{.}1$, $\beta_0 = 27\overset{\circ}{.}0 \pm 4\overset{\circ}{.}2$, $a/b = 2.26 \pm 0.03$ and $b/c = 1.29 \pm 0.12$ for the second one (P_2).

For the case of 44 Nysa, we show that additional observations are needed in order to derive a self-consistent pole orientation.

Key words: asteroids: pole orientation – asteroids: model – photometry: light curve – solar system: minor planets

1. Introduction

Our understanding of the collisional evolution of asteroids in the main belt is intimately connected with our knowledge of the distribution of their rotation axes in space. Whereas there are actually more than 3000 minor planets which are catalogued, only about 15 of them do have a reliable pole orientation. This justifies the vast campaigns of photometric observations of asteroids that are organized at various observatories.

Among the numerous techniques used when deriving the pole orientation of a minor planet, two major approaches can be distinguished. One of these, called "photometric astrometry", enables one to find the pole orientation, sidereal period and sense of rotation of an asteroid by dividing the number of rotational cycles, corrected to a sidereal frame of reference, into the observed synodic intervals between light curve maxima (Gehrels, 1967; Taylor, 1979). This technique is not explicitly model dependent but requires an appreciable amount of good photometric observations. Indeed, in order to derive a reliable pole orientation, it is necessary to obtain six to ten high quality light curves from one opposition that span as wide a longitude range as possible, plus at least one high quality light curve from each of four additional and different oppositions (Taylor, 1984).

In the second of these approaches, one assumes that the observed light curve amplitude and/or maximum and/or minimum brightness of an asteroid are a function of the pole position. This class of techniques has either led to empirical or analytic relations between quantities, such as the amplitude of a light curve and the aspect – angle between the line-of-sight and the rotation axis – (see Vesely, 1971 and references therein; Gehrels and Owings, 1962; Zappalà, 1981; Zappalà et al., 1983), the magnitude at maximum brightness and the aspect (Sather, 1976; Zappalà, 1981), etc. All these techniques are model dependent and very often the implicit or explicit assumption of a three-axes ellipsoid model is made.

Although the amplitude-aspect relationship has been cast in doubt by several workers (Dunlap, 1971; Vesely, 1971; Surdej and Surdej, 1978), we show in the present work that, whenever applied carefully to selected cases, this technique provides one with an easy and good means of deriving the pole orientation (λ_0, β_0) as well as the semi-axes ratios a/b and b/c of the ellipsoid.

2. The three-axes ellipsoid model

In order to simulate the light curve of a spinning asteroid, we adopt a three-axes ($a > b \geqq c$) ellipsoid model whose rotation around the shortest axis is fully described by the angle ψ (cf. Fig. 1 in Surdej and Surdej, 1978, referred to hereafter as Paper I).

Veverka (1971) has shown that the generalized Lommel-Seliger law

$$dE \propto \frac{\cos(i)\cos(\varepsilon)}{\cos(i)+\cos(\varepsilon)}\, ds, \qquad (1)$$

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also known as the Hapke-Irvine relation, is very appropriate in order to calculate the amount of sunlight dE reflected by an asteroid surface element ds, which normally makes an angle i (resp. ε) with the direction of incidence (resp. scattering) of a solar ray. Comparison between laboratory and theoretical light curves has confirmed the good approximation provided by this relation for describing the scattering properties of a dark asteroid surface, intricate in texture (cf. Paper I).

Surdej and Louis (1982) have pointed out that at opposition ($\alpha = 0°$) the angles i and ε are equal and that consequently Eq. (1) reduces to

$$dE \propto \cos\varepsilon \, ds, \tag{2}$$

justifying, a posteriori, the use of the geometrical approximation – also known as the cross section approximation – for calculating asteroid light curves. In the remainder of this paper, we assume that the observed brightness of an asteroid (ellipsoid) is proportional to the instantaneous cross section seen by a distant observer.

Following Barsuhn (1983), Barucci and Fulchignoni (1982) or Ostro and Connelly (1984), we easily find that the orthogonal projection of the ellipsoid onto a plane perpendicular to the line-of-sight is an ellipse, the area of which is (see the Appendix)

$$S = \pi(abc)\left[\sin^2 A\left(\frac{\sin^2(\psi)}{a^2}+\frac{\cos^2(\psi)}{b^2}\right)+\frac{\cos^2(A)}{c^2}\right]^{1/2}, \tag{3}$$

where A denotes the aspect angle.

Intensities at maximum (S_M) and minimum (S_m) brightness occur at $\psi = 0$ and $\psi = \pi/2$, and are given by

$$S_M = \pi(abc)\left[\frac{\sin^2(A)}{b^2}+\frac{\cos^2(A)}{c^2}\right]^{1/2}, \tag{4}$$

and

$$S_m = \pi(abc)\left[\frac{\sin^2(A)}{a^2}+\frac{\cos^2(A)}{c^2}\right]^{1/2}. \tag{5}$$

Defining the relative intensity of a point in the light curve as

$$I_r = r \cdot S, \tag{6}$$

where r is an arbitrary constant, and using the variables

$$y = I_r^2 \quad \text{and} \quad x = \cos^2(\psi), \tag{7}$$

we easily obtain from Eq. (3) the relation

$$y = Bx + C, \tag{8}$$

such that the quantity $D = B/C$, i.e.

$$D = \frac{(1-\cos^2(A))\,(a^2/b^2-1)}{1+\cos^2(A)\,(a^2/c^2-1)} \tag{9}$$

only depends on three unknown parameters: the module of the cosine of the aspect angle and the semi-axes ratios a/b and a/c. Let us note that if I_r is defined with respect to the minimum brightness (S_m), we should have $C = 1$ and $D = B$, i.e., D would represent the slope of the linear x-y relation.

We have illustrated in Fig. 1 the x-y relation (see the dots) calculated from the photoelectric photometric observations of 624 Hektor observed on 7 March, 1967 at Catalina observatory (Dunlap and Gehrels, 1969). Adopting $t_M = 8^h.385$ for the epoch of the light maximum, $P = 6^h.9225$ for the sidereal period, a simple linear regression between the y and x variables gives $D = 0.814$

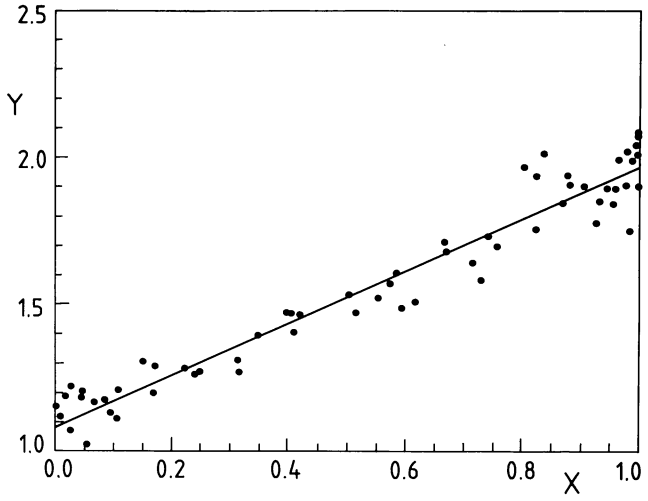

Fig. 1. This x–y relation (dots) was constructed from the photoelectric measurements of 624 Hektor observed on 7 March, 1967 (Dunlap and Gehrels, 1969). The straight line represents the best least squares linear fit of the form $y = Bx + C$, for which $D = B/C$ is found to be 0.814 ± 0.037 (see text)

± 0.037 (the straight line in Fig. 1 represents the corresponding least squares linear fit) with a Pearson product-moment correlation $R = 0.972$. Similar calculations have been performed from nine additional photoelectric light curves of 624 Hektor obtained during five different oppositions. The results are summarized in Table 1 which also contains aspect data, references, etc., pertaining to these observations.

Whereas the shape of 624 Hektor is not perfectly ellipsoidal (see $R < 1$ in Table 1), the rather good x-y linear fits derived from the light curves of this minor planet indicate that the three-axes ellipsoid model essentially constitutes a first good approximation for modelling its light curves. Conversely, the absence of a linear relationship between the x and y quantities calculated from the light curves of an asteroid would denote the non-applicability of the ellipsoid model for that particular object.

Noticing that the full amplitude Δm of an ellipsoidal light curve is related to the quantity D via the relation

$$\Delta m = 1.25 \log_{10}(1+D), \tag{10}$$

two obvious conclusions can be drawn:

(i) All amplitude-aspect type relations should only be applied to asteroid light curves that are observed near opposition.

(ii) These light curves should be nearly ellipsoidal, i.e. there should be an approximate linear relationship between the x and y quantities.

Compared to the classical amplitude-aspect relations (cf. Gehrels and Owings, 1962; Zappalà, 1981; Zappalà and Knežević, 1984), the present method enables one to take into account every single measurement – rather than just the two most extreme points (Δm) – of an observed light curve when deriving the parameter D, as well as to judge the applicability of such relations.

3. Determination of the pole orientation(s)

Defining the parameters

$$\gamma = \left(\frac{a}{b}\right)^2 \quad \text{and} \quad \delta = \left(\frac{a}{c}\right)^2, \tag{11}$$

Table 1. Date, reference, ecliptic coordinates, phase angle and characteristics of the $x-y$ relation for different light curves of the minor planet 624 Hektor (see text)

Date of Observations	Reference	λ (1950.0)	β	α	t_M	D	R
28 April 1957	Fig. 1 (DG)	249°.91	-21°.82	7°.0	22^h.591	2.855±0.565	0.968
4 May	Fig. 2 (DG)	249.28	-22.08	6.2	23.829	2.490 0.367	0.953
30 May	Fig. 3 (DG)	245.85	-22.59	4.4	22.769	2.238 0.176	0.982
31 May	Fig. 4 (DG)	245.71	-22.59	4.5	23.068	3.009 0.498	0.939
4 February 1965	Fig. 5 (DG)	119.38	+14.64	4.1	4.382	0.189 0.009	0.958
7 March 1967	Fig. 6 (DG)	192.90	-9.90	5.5	8.385	0.814 0.037	0.972
29 April 1968	Fig. 7 (DG)	226.02	-20.24	4.2	7.487	4.026 0.122	0.994
1 May	Fig. 8 (DG)	225.74	-20.29	4.1	4.487	3.676 0.102	0.994
13 February 1977	Fig. 2 (HC)	132.93	+10.51	3.0	9.103	0.084 0.011	0.789
14 February	Fig. 2 (HC)	132.80	+10.47	3.2	12.660	0.088 0.015	0.696

DG : Dunlap and Gehrels, 1969

HC : Hartmann and Cruikshank, 1978

and with the help of Eq. (9), the well-known relation between the aspect angle A_i, the ecliptic coordinates λ_i, β_i of an asteroid observed at a time t_i, and the ecliptic coordinates λ_0, β_0 of its rotation axis (cf. Taylor, 1979) may be written in the form

$$\sin(\beta_i)\sin(\beta_0)+\cos(\beta_i)\cos(\beta_0)\cos(\lambda_i-\lambda_0) + \operatorname{sign}(\cos(A_i))\left[\frac{(\gamma-1)-D_i}{D_i(\delta-1)+(\gamma-1)}\right]^{1/2}=0, \quad (12)$$

where D_i represents the parameter D derived for light curve $n°$ i and $\operatorname{sign}(\cos(A_i))$ is the undetermined + or − sign.

In principle, a minimum number of four ($i=1, \ldots, 4$) such non-linear equations is needed in order to determine the four unknown parameters λ_0, β_0, γ, and δ. We have adopted a least-squares adjustment technique for solving such a system of transcendental equations.

3.1. 624 Hektor

For the case of 624 Hektor, we have considered a system of ten ($i=1, \ldots, 10$) non-linear equations for which the λ_i, β_i, and D_i are listed in Table 1. This system has been solved by means of the FIT numerical program (see MIDAS, 1984), designed and kindly put at our disposal by Otto-Georg Richter at the European Southern Observatory. Considering first the 16 possible combinations of $\operatorname{sign}(\cos(A_i))$ for a system of 4 equations – corresponding to four different oppositions – and adding successively each of the six remaining equations, we find that there are only two sets of $\operatorname{sign}(\cos(A_i))$ ($i=1, \ldots, 10$) for each of which two solutions exist. Adopting the order of the observations in Table 1, the first set of $\operatorname{sign}(\cos(A_i))$ appears to be $-,-,-,-,+,+,+,+,+,+$. Because the poles (λ_0, β_0) and ($\lambda_0+\pi$, $-\beta$) define an identical orientation of the rotation axis, the second set of $\operatorname{sign}(\cos(A_i))$ naturally corresponds to $+,+,+,+,-,-,-,-,-,-$. For a low inclination orbit ($\beta_i \sim 0°$), the two pole solutions (λ_0, β_0) and ($\lambda_0+\pi, \beta_0$) equally satisfy equation (12). This ambiguity in the determination of a pole has been fully discussed by Taylor (1979) and Zappalà et al. (1983). The two resulting solutions found for the pole of 624 Hektor are listed in Table 2. Unfortunately, the reduced Chi-squares ($\chi^2/(n-p)$ *where* $n=10$ and $p=4$) characterizing the two solutions are found to be very similar, and there remains a real ambiguity. A look at the covariance matrix in Table 3 also clearly indicates that there is no correlation between the parameters λ_0, β_0, γ, and δ.

Assigning an equal weight (1/5) to each of the five oppositions and an additional weight to each equation that is proportional to the number of extrema seen in the relevant light curve, we have computed the two new relevant pole positions of 624 Hektor (see Table 2). Within the mean standard deviations, no significant differences are noticed between these and the previous solutions.

The fact of deleting one or more (up to 6) equations in the system usually results in

(i) altering somewhat the values of the four parameters;

(ii) increasing their mean standard deviations;

(iii) creating possible correlations between some of the parameters.

However, let us remark that the new calculated pole solutions always lie, within the larger uncertainties, near the previous ones. Since most asteroids are orbiting in low inclination orbits ($\beta_i \sim 0°$), two light curves observed at longitudes λ_i and $\lambda_i+180°$ will essentially provide a same value for D_i (see Eq. (12)). Therefore, it

Table 2. The different pole solutions for 624 Hektor

	λ_o	β_o	a/b	b/c	χ^2_{red}
1) Unweighted calculations					
P_1	315°.3±2°.5	16°.2±5°.9	2.30±0.07	1.43±0.28	$1.84\ 10^{-3}$
P_2	152.5±3.0	27.0±5.3	2.28±0.05	1.32±0.20	$1.91\ 10^{-3}$
2) Weighted calculations (see Text)					
P_1	314°.6±2°.0	15°.9±4°.1	2.27±0.03	1.41±0.16	$1.06\ 10^{-4}$
P_2	151.5±2.1	27.0±4.2	2.26±0.03	1.29±0.12	$1.28\ 10^{-4}$

Table 3. Correlation factors between the parameters λ_0, β_0, γ, and δ

1) Unweighted calculations

a) Covariance matrix for P_1

	λ_o	β_o	γ
β_o	0.876		
γ	-0.227	-0.418	
δ	-0.617	-0.820	0.732

b) Covariance matrix for P_2

	λ_o	β_o	γ
β_o	-0.885		
γ	0.381	-0.223	
δ	0.779	-0.670	0.672

2) Weighted calculations (see Text)

a) Covariance matrix for P_1

	λ_o	β_o	γ
β_o	0.870		
γ	0.030	-0.198	
δ	-0.714	-0.854	0.453

b) Covariance matrix for P_2

	λ_o	β_o	γ
β_o	-0.825		
γ	0.250	-0.001	
δ	0.727	-0.746	0.362

would be ideal to observe an asteroid at four distinct oppositions such that the corresponding ecliptic longitudes (λ_i or $\lambda_i + 180°$) are equally distributed in the range $\lambda_i \varepsilon [0°, 180°]$. Additional light curves from more oppositions will in general be required in order to improve the solution(s), to minimize the mean standard deviations as well as to destroy any possible correlation existing between some of the parameters.

3.2. 44 Nysa

In order to derive the pole orientation (λ_0, β_0) and the semi-axes ratios a/b and b/c of 44 Nysa, we have analyzed nine light curves of this asteroid, pertaining to six different oppositions (see Table 4). However, we can immediately note that during the 1949, 1964, and 1979 oppositions, the ecliptic longitude of 44 Nysa has remained in the very narrow range $\lambda \varepsilon [3°, 19°]$. Consequently, it is as if 44 Nysa had been observed at only four distinct oppositions. Assigning an equal weight of 1/12 to each of the 1949, 1964 and 1979 oppositions and an equal weight of 1/4 to the three remaining ones, we have solved a system of nine ($i = 1, \ldots, 9$) non-linear equations [cf. Eq. (12)] for which the λ_i, β_i, and D_i are listed in Table 4. Our main result is the following: for the only two possible sets of sign$(\cos(A_i))$ being $+,+,-,-,+,+,+,+,+$ and $-,-,+,+,-,-,-,-,-$, we find that the parameter δ gets to unrealistic high values and that it is strongly correlated with λ_0 and β_0, which are also significantly correlated between each other. Such a non-convergence of the solution(s) can be due either to the

Table 4. Date, reference, ecliptic coordinates, phase angle and characteristics of the $x-y$ relation for different light curves of the minor planet 44 Nysa (see text)

Date of Observations	Reference	λ (1950.0)	β	α	t_M	D	R
6 November 1949	Fig. 1 (S)	18°.54	-6°.03	10°.5	4^h.474	1.083±0.041	0.984
7 November	Fig. 2 (S)	18.33	-6.01	11.0	6.148	1.090 0.041	0.982
13 January 1958	Fig. 17 (GO)	98.82	-3.09	6.6	5.099	0.327 0.012	0.956
2 March 1962	Fig. 7 (CC)	143.75	2.41	8.3	16.496	0.632 0.024	0.976
8 October 1964	Fig. 6 (Y)	12.18	-5.56	2.6	15.117	0.921 0.040	0.988
29 October	Fig. 6 (Y)	7.41	-5.56	11.6	15.100	0.922 0.054	0.957
16 May 1974	Fig. 4 (ZV)	247.04	5.59	5.0	22.989	0.596 0.023	0.960
25 September 1979	Fig. 2B (B)	3.63	-4.85	2.2	5.970	0.781 0.023	0.975
26 September	Fig. 2C (B)	3.28	-4.88	2.0	17.207	0.821 0.021	0.974

S : Shatzel, 1954
GO : Gehrels and Owings, 1962
CC : Chang and Chang, 1962
Y : Yang et al., 1965
ZV : Zappalà and Van Houten - Groeneveld, 1979
B : Birch et al., 1983

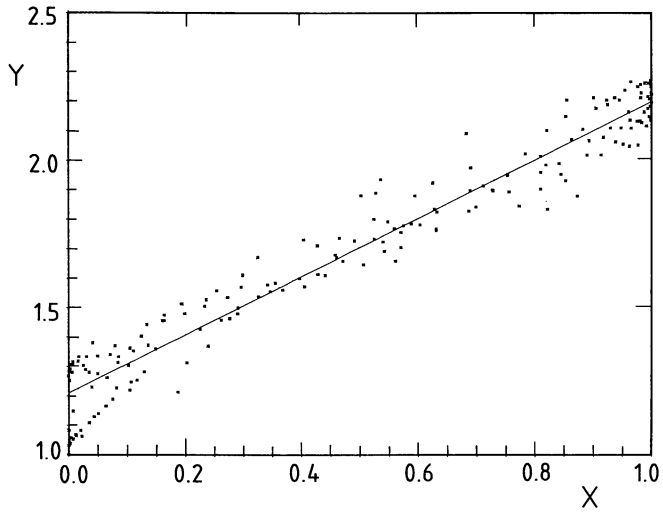

Fig. 2. This x–y relation (dots) was constructed from the photoelectric measurements of 44 Nysa observed on 26 September, 1979 (Birch et al., 1983). The straight line represents the best least squares linear fit of the form $y = Bx + C$, for which $D = B/C$ is found to be 0.821 ± 0.021. Let us note at the lower left side of the figure a small group of dots presenting a systematic departure from the linear fit (see text)

non-applicability of the ellipsoid model, or to a lack of additional equation(s).

We have illustrated in Fig. 2 the x-y relation (see Sect. 2) calculated from the photoelectric photometric observations of 44 Nysa obtained on 26 September, 1979 (Birch et al., 1983). We clearly see that a small group of dots, located at the lower left side of the figure, presents a systematic departure from the linear relation. These dots correspond in fact to the photoelectric measurements of the deepest of the two recorded minima in the light curve of 44 Nysa (Fig. 2c in Birch et al., 1983). Similar, but less pronounced, features are also seen in some of the other x–y relations. It thus seems possible that the non-convergence of the pole solutions of 44 Nysa is *partly* due to its shape being somewhat non-ellipsoidal.

Since a lack of observations (D_i) suitably spaced along the ecliptic could also account for the non-convergence of the pole solution(s) of 44 Nysa, we have tried to better condition our system of equations in another way. Constraining the parameter b/c to a fixed value, we find that there is always a good convergence of the solutions for $b/c \geqq 1$. Considering plausible values for the ratio b/c $\varepsilon [1, 3]$, we have illustrated in Fig. 3 the resulting values of λ_0 and β_0 for the two possible pole solutions P_1 and P_2. For both these solutions, the ratio $a/b = 1.446 \pm 0.006$ is found to be irrespective of the value of b/c.

We naturally conclude that additional observations (D_i) of 44 Nysa at other ecliptic longitudes are needed in order to derive a more self-consistent pole orientation.

4. The pole(s) of Hektor

By means of photometric astrometry, Dunlap and Gehrels (1969) have first derived the pole orientation of 624 Hektor. Using the light curves from the first four oppositions (see Table 1), they found the most likely solution to be

P_1: $\lambda_0 = 324° \pm 3°$, $\beta_0 = 10° \pm 2°$,

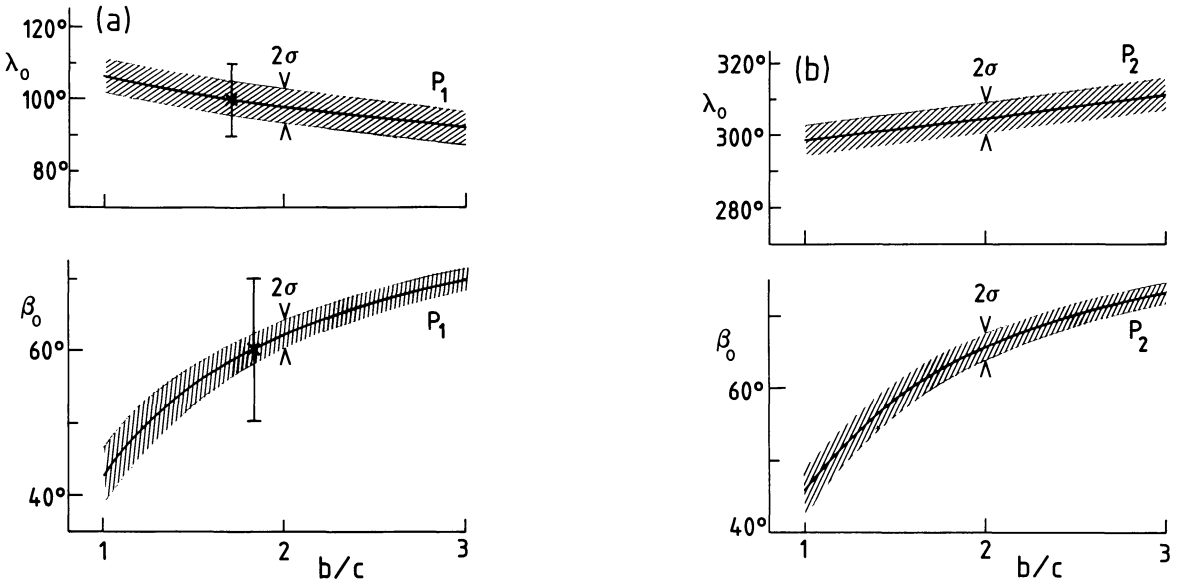

Fig. 3. a Resulting values of the coordinates λ_0 and β_0 for the pole solution P_1 of 44 Nysa as a function of the semi-axes ratio b/c. The crosses and error bars correspond to the pole solution P_1 determined by Tedesco and Taylor (1983) on the basis of photometric astrometry. **b** Similar relations for the second pole solution P_2. The mean standard deviations affecting the determination of λ_0 and β_0 are indicated. The reduced Chi-square characterizing each pair of solutions (P_1, P_2) are found to be very similar. Furthermore, the covariance matrix derived for each solution indicates that there is no correlation between the parameters λ_0, β_0 and $a/b = 1.446 \pm 0.006$

rejecting the spurious pole at

P_2: $\lambda_0 \sim 165°$, $\beta_0 \sim 15°$.

Furthermore, Dunlap and Gehrels report a direct sense of rotation and a sidereal period $P = 6^h55^m21^s115 \pm 0^s004$. Within about 15°, there is a very good agreement between their results and those determined by the revisited amplitude-aspect relation in Table 2.

Combining an amplitude-aspect type description and an approach similar to the photometric astrometry, Magnusson (1983) has derived the pole solutions of 624 Hektor to be

P_1: $\lambda_0 = 322° \pm 10°$, $\beta_0 = -4° \pm 10°$

and

P_2: $\lambda_0 = 144° \pm 10°$, $\beta_0 = 10° \pm 10°$.

However, his adopted linear amplitude-aspect-phase relationship (cf. Eq. (1) in Magnusson, 1983) is not found to be appropriate for accounting for changes of amplitude with aspect [see our Eqs. (9) and (10)]. This inadequate treatment partly explains the larger deviations found between his solutions and those listed in Table 2.

Finally, Zappalá and Knežević (1984) have recently determined the pole solutions of Hektor on the basis of amplitude-magnitude-aspect relations. They report the values

P_1: $\lambda_0 = 314° \pm 7°$, $\beta_0 = 15° \pm 5°$

and

P_2: $\lambda_0 = 152° \pm 4°$, $\beta_0 = 29° \pm 7°$,

in very good agreement with our results in Table 2. Their semi-axes ratios $a/b = 2.66$ and $b/c = 1.13$ are based on a more uncertain magnitude-aspect relation.

5. The pole(s) of Nysa

On the basis of amplitude-magnitude-aspect relations, Gehrels and Owings (1962) have reported the first pole determination of 44 Nysa:

$\lambda_0 = 105°$ and $\beta_0 = 30°$.

They also predicted the maximum amplitude of the light curve to be 0.48 mag. Four additional and independent calculations of the pole(s) of 44 Nysa have been reported meanwhile. These are summarized in Table 5. Applying the photometric astrometry technique, Taylor and Tedesco (1983) have firmly established that the rotation of 44 Nysa is prograde – with a sidereal period $P = 0^d.26755902$ – and that consequently the pole solution P_2 at $\lambda_0 \sim 290°$ can be definitely rejected. For a ratio value $b/c \sim 1.75$, the results of our calculations in Fig. 3a are found to be in good agreement with the pole solution P_1 of Taylor and Tedesco (1983).

6. The occultation effects

For phase angles $\alpha \neq 0$, Eq. (3) should contain a correction term for occultation effects (cf. Barucci and Fulchignoni, 1982; Ostro and Connelly, 1984). As seen by an observer located on the earth, the orthogonal projection of the ellipsoid onto a plane perpendicular to the sun-asteroid direction – defining the terminator – is also an ellipse whose area S_2 (see the Appendix) is a complicated function depending on the aspect (A), the obliquity (0), the phase angle (α), the rotation angle (ψ) and the semi-axes a, b, and c (see Fig. 1 in Paper I for a visualization of these angles). A development of S_2 in a Taylor series around $\alpha = 0$ gives (see the Appendix)

$$S_2 = S_1 - \frac{\alpha^2}{2} \cdot f(A, 0, \psi, a, b, c), \qquad (13)$$

Table 5. Previous pole determinations of 44 Nysa

λ_o	β_o	a/b	b/c	Technique(s)	Reference
105°± ?	30°± ?	____	____	A-M-A	Gehrels and Owings (1962)
100 ± 10	50 ± 10	1.58	1.30	A-M-A	Zappalà and Van Houten - Groeneveld (1979)
94 ± 3	59 ± 3	____	____	A-M-A	Magnusson (1983)
288 ± 3	63 ± 3			+ PA	
100 ± 10	60 ± 10	____	____	PA	Taylor and Tedesco (1983)
99 ± 7	49 ± 6	1.51	1.18	A-M-A	Zappalà and Knezević (1984)
295 ± 8	54 ± 6				

Note: A-M-A stands for Amplitude-Magnitude-Aspect relations, PA for Photometric Astrometry

with

$$f(A,0,\psi,a,b,c)=\pi(abc)\left\{\frac{\sin^2(A)\cos^2(0)}{a^2b^2}+\frac{(\sin(\psi)\sin(0)+\cos(\psi)\cos(0)\cos(A))^2}{a^2c^2}+\frac{(\cos(\psi)\sin(0)-\sin(\psi)\cos(0)\cos(A))^2}{b^2c^2}\right\}\Bigg/\left(\sin^2(A)\left(\frac{\sin^2(\psi)}{a^2}+\frac{\cos^2(\psi)}{b^2}\right)+\frac{\cos^2(A)}{c^2}\right)^{3/2}, \quad (14)$$

and where S_1 is the area given by Eq. (3). Under the assumption that the geometrical approximation still holds, the unocculted area of the asteroid seen by a terrestrial observer is then given by (cf. Barucci and Fulchignoni, 1982; Ostro and Connelly, 1984)

$$S=\frac{S_1+S_2}{2}, \quad (15)$$

which, in our case, reduces to

$$S=S_1-\frac{\alpha^2}{4}\cdot f(A,0,\psi,a,b,c). \quad (16)$$

This result clearly demonstrates that for sufficiently small phase angles ($\alpha \lesssim 15°$) the occultation effects can be neglected to the second order in α – the first order term being identically equal to zero – when calculating the light curve reflected by an ellipsoidal asteroid. This result is also in good agreement with the small scatter experimentally observed between different amplitude-aspect plots as a function of the obliquity for phase angles $\alpha \lesssim 20°$ (see Barucci et al., 1982).

Our main conclusion is that for $\alpha \neq 0$, Eqs. (3)–(12) remain a very good approximation for deriving the pole orientation of an asteroid as long as the relevant photoelectric light curves were obtained at phase angles $\alpha \lesssim 15°$.

7. Discussion

The main conclusions of the present work are concisely summarized in the abstract. We shall here briefly point out some pitfalls that should be avoided when deriving the pole orientation of an asteroid by means of amplitude-aspect type relations.

Whereas it is attractive to use additional magnitude-aspect relations [cf. Eqs. (4) and (5)] in order to derive parameters such as the b/c ratio (Zappalà, 1981; Zappalà et al., 1983; Zappalà and Knežević, 1984), the a/c ratio, etc., it is absolutely necessary to correct the observed magnitudes at maximum, minimum, etc. brightness to a common phase angle. Unfortunately, the phase coefficient β of an elongated asteroid cannot be interpreted unambiguously due to a dependence of β upon the geometric configuration of the observations (Surdej and Surdej, 1978). Making use of Eqs. (4) and (5), it is straightforward to show that the magnitude difference Δm between the maxima (resp. minima) of two different light curves observed at the same phase angle but for extreme values of the aspect $A_1 = 0°$ and $A_2 = 90°$ amounts to

$$(\Delta m)_M = 2.5\log_{10}(b/c), \quad (17)$$

and

$$(\Delta m)_m = 2.5\log_{10}(a/c), \quad (18)$$

respectively. As an example, we obtain $(\Delta m)_M \sim 0.5$ mag and $(\Delta m)_m \sim 1.6$ mag for the particular case of 624 Hektor (see Table 2). Unless the phase coefficient has been properly corrected for the aspect changes (cf. Sather, 1976) – and obliquity changes if $\alpha \gtrsim 15°$ – no attempt should be made to derive semi-axes ratios by means of magnitude-aspect relations. Furthermore, due to many unknown factors, such as possibly poor weather conditions, variability of the comparison star, slight differences between photometric systems, etc., it is really dangerous to base a physical model on absolute photometric measurements that were collected by different observers at various observatories. The technique

described in Sects. 2 and 3 is entirely based on relative intensities which are not only more reliable but also easier to use.

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Appendix

Considering the aspect (A), obliquity (0) and phase (α) angles as defined in Paper I (see Fig. 1), we easily find that the coordinates transformation

$$X' = TX, \tag{A1}$$

where X' (resp. X) is a vector defined in the reference system of the ellipsoid (resp. terrestrial observer), is given by

$$T = \begin{pmatrix} -\sin(\psi)\sin(A) & \cos(\psi)\cos(0)+\sin(\psi)\cos(A)\sin(0) & -\cos(\psi)\sin(0)+\sin(\psi)\cos(A)\cos(0) \\ -\cos(\psi)\sin(A) & -\sin(\psi)\cos(0)+\cos(\psi)\cos(A)\sin(0) & \sin(\psi)\sin(0)+\cos(\psi)\cos(A)\cos(0) \\ \cos(A) & \sin(A)\sin(0) & \sin(A)\cos(0) \end{pmatrix}. \tag{A2}$$

One can then show that the intersection between the ellipsoid surface

$$\left(\frac{x'}{a}\right)^2+\left(\frac{y'}{b}\right)^2+\left(\frac{z'}{c}\right)^2=1 \tag{A3}$$

and the tangent cylinder having a generatrix parallel to the line-of-sight is an ellipse (cf. Barucci and Fulchignoni, 1982) having an area

$$S_1 = \pi(abc)\sqrt{V_{11}}, \tag{A4}$$

where

$$V = \tilde{R}R, \tag{A5}$$

$$R_{ij} = \frac{T_{ij}}{a_i}, \tag{A6}$$

$$\tilde{R}_{ij} = R_{ji}, \tag{A7}$$

and

$$a_i = a,\ b,\ c \tag{A8}$$

for $i = 1, 2, 3$, respectively.

Elementary algebra leads to the result

$$V_{11} = \sin^2(A)\left(\frac{\sin^2(\psi)}{a^2}+\frac{\cos^2(\psi)}{b^2}\right)+\frac{\cos^2(A)}{c^2}. \tag{A9}$$

Combining (A4) and (A9) gives Eq. (3).

For a given phase angle $\alpha \neq 0$, the intersection between the ellipsoid surface and the tangent cylinder having a generatrix parallel to the sun-asteroid direction, is an ellipse in both the sun and observer rest frames (cf. Barucci and Fulchignoni, 1982). In the frame of the observer, the area of this ellipse is given by

$$S_2 = \pi(abc)\,(W_{11}\cos(\alpha) - W_{12}\sin(\alpha))/\sqrt{W_{11}}, \tag{A10}$$

where

$$W = \tilde{S}S, \tag{A11}$$

$$S_{ij} = \frac{U_{ij}}{a_i}, \tag{A12}$$

and

$$U = TA, \tag{A13}$$

with

$$A = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A14}$$

being the transformation matrix between coordinates defined in the frame of the observer and that of the sun. In terms of the matrix elements V_{ij}, Eq. (A10) may be rewritten in the convenient form

$$S_2 = \pi(abc)\,(V_{11}\cos(\alpha)+V_{12}\sin(\alpha))/ (V_{11}\cos^2(\alpha)+V_{22}\sin^2(\alpha)+2V_{12}\sin(\alpha)\cos(\alpha))^{1/2}. \tag{A15}$$

Developing this last expression of S_2 in a Taylor series around $\alpha = 0$, we obtain

$$S_2 = \pi(abc)\left\{\sqrt{V_{11}} - \frac{\alpha^2}{2}((R_{11}R_{22}-R_{12}R_{21})^2 + (R_{11}R_{32}-R_{12}R_{31})^2+(R_{21}R_{32}-R_{22}R_{31})^2)/V_{11}^{3/2}\right\}. \tag{A16}$$

Combining the different results (A2), (A4)–(A8), and (A16), we finally obtain Eq. (13).

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