Contributions to combinatorics on words in an abelian context and covering problems in graphs

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UNIVERSITÉ DE GRENOBLE

maths à modeler

Université de Liège

Problem 1

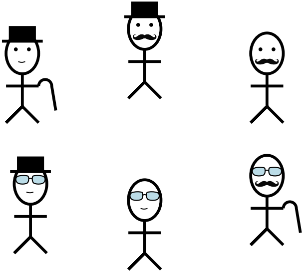
Identifying codes

Guess Who?

Milton Bradley
Does your person have a big nose?
No
Is your person wearing a hat?
Yes
I know!
It's Maria!
Ages 6 and Up
2 Players
The mystery face game
GUESS WHO ?

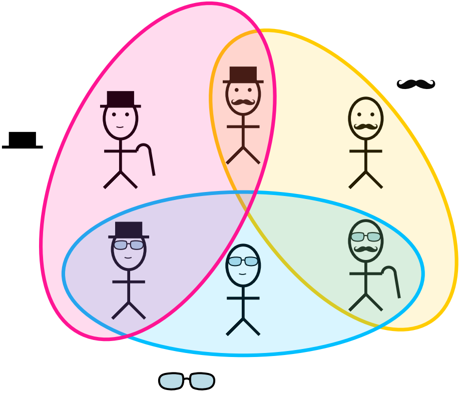
Herman

Guess Who?



- Individuals: 
- Attributes:

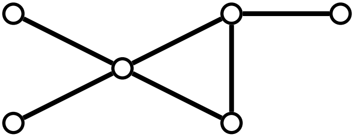
Guess Who?



- Individuals: 
- Attributes:

Translation in terms of graphs

- Individuals:  ⟶ vertices
- Attributes:  ⟶ closed neighbourhoods



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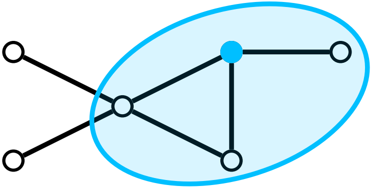

How many neighbourhoods/vertices to identify these points?

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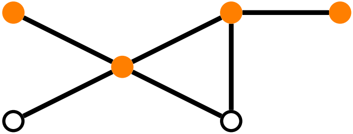


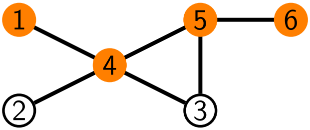
How many neighbourhoods/vertices to identify these points?

Formal definition

An identifying code C is a subset of vertices such that

- $\forall u \in V, \quad N[u] \cap C \neq \emptyset$ (domination)
- $\forall u \neq v \in V, \quad N[u] \cap C \neq N[v] \cap C$ (separation)

[Karpovsky–Chakrabarty–Levitin 1998]



V \ C	1	4	5	6
1	•	•	-	-
2	-	•	-	-
3	-	•	•	-
4	•	•	•	-
5	-	•	•	•
6	-	-	•	•

Given a graph G, what is the minimum size $\gamma^{ID}(G)$ of an identifying code of G?

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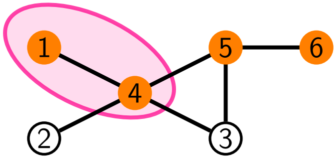

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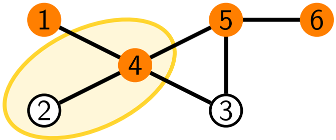

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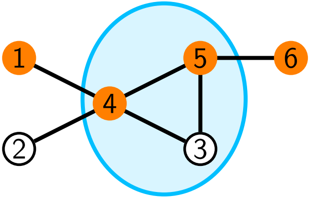

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Linear programming formulation

- A variable x_u for each vertex u
- Goal: minimize $\sum_{u \in V} x_u$
- Constraints: domination and separation

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Fractional relaxation

Let $\gamma_f^{ID}(G)$ be the optimal solution of the fractional problem.

Theorem (Gravier–Parreau–Rottey–Storme–V.)

For any graph G, $\gamma_f^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2\ln|V|)\,\gamma_f^{ID}(G)$.

Can we compute a closed formula for $\gamma_f^{ID}(G)$?

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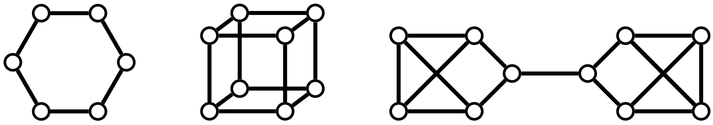

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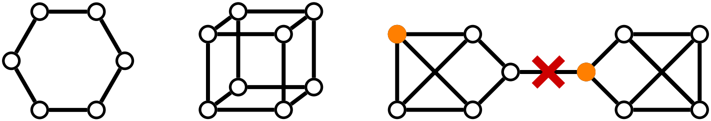

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Properties:

- All vertices have the same degree.
- There is an optimal solution to the fractional program with all the variables equal.

Fractional value for vertex-transitive graphs

There is an optimal solution with $x_u = \lambda$ for all $u \in V$.

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where k is the degree of vertices

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d is the smallest size of sets $N[u] \Delta N[v]$

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Proposition (Gravier–Parreau–Rottey–Storme–V.)

If G is vertex-transitive, $\gamma_f^{ID}(G) = \lambda \cdot |V| = \frac{|V|}{\min(k+1,d)}$.

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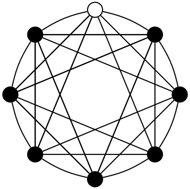
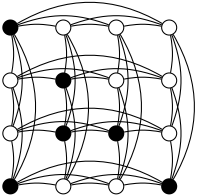
If $d < k + 1$, the separation condition prevails.



power of cycles



Cartesian product of cliques

Proposition (Gravier–Parreau–Rottey–Storme–V.)

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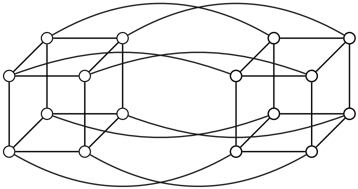
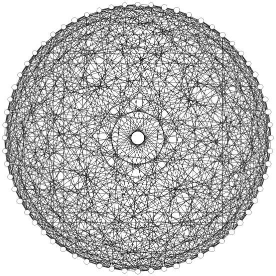
If $k+1 < d$, the domination condition prevails.



hypercubes



generalized quadrangles

Generalized quadrangles

A generalized quadrangle $\mathrm{GQ}(s, t)$ is an incidence structure of points and lines such that:

- each line contains $s + 1$ points,
- each point is on $t + 1$ lines,
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Adjacency graph: points are vertices, lines are cliques.

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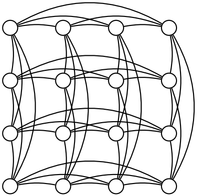
Example: the square grid $n \times n$ i.e. $K_n \square K_n$ is a $\mathrm{GQ}(n-1, 1)$.



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If G is a vertex-transitive $\mathrm{GQ}(s,t)$,

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If G is a vertex-transitive $\mathrm{GQ}(s,t)$, with $s>1,\ t>1$,

$$2^{-5/4}\cdot|V|^{1/4}\leq\gamma_f^{ID}(G)=\frac{s^2t}{st+s+1}+1\leq 2\cdot|V|^{2/5}$$

$\mathrm{GQ}(q-1, q+1)$ with $q = 2^\ell$

Proposition (Gravier–Parreau–Rottey–Storme–V.)

There exists a graph G that is a $\mathrm{GQ}(q-1, q+1)$ with an identifying code of size $3q$.

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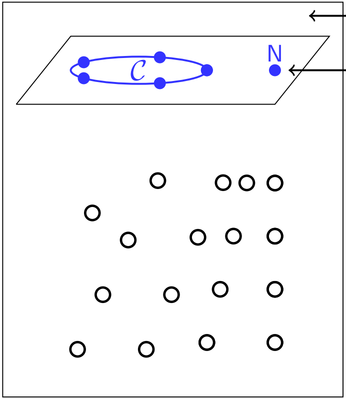

Projective space on $\mathbb{F}_q$, (X_0, X_1, X_2, X_3)

H_∞: projective plane $X_0 = 0$

Points: all except H_∞

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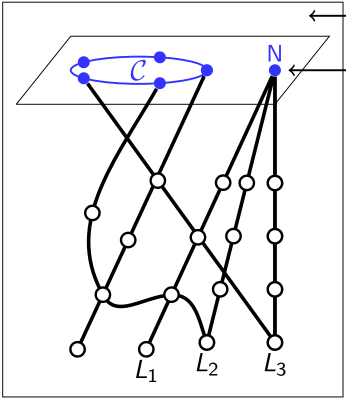

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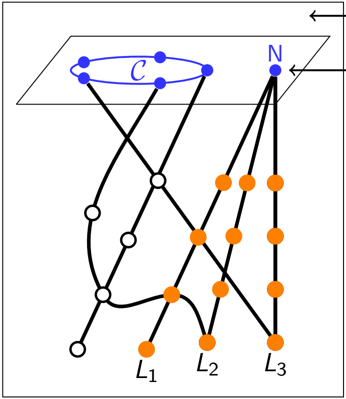

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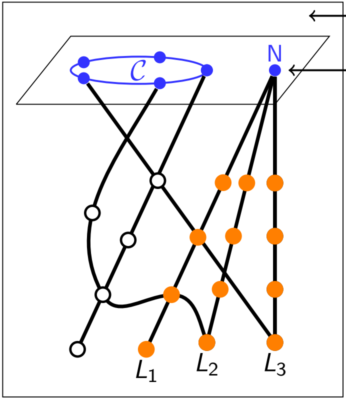
Three non coplanar lines through N form an identifying code.

$\mathrm{GQ}(q-1, q+1)$ with $q = 2^\ell$

Proposition (Gravier–Parreau–Rottey–Storme–V.)

There exists a graph G that is a $\mathrm{GQ}(q-1, q+1)$ with an identifying code of size $3q - 3$.



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Lower bound?

- Using the fractional value: $\frac{q^3}{q^2+q-1} \leq \gamma^{ID}(G)$

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Lower bound?

- Using the fractional value: $\frac{q^3}{q^2+q-1} \leq \gamma^{ID}(G)$
- With discharging methods: $3q - 7 \leq \gamma^{ID}(G)$

Finally,

$$\gamma^{ID}(G) \in \Theta(|V|^{1/3}).$$

Other results

Let q be a prime power.

- There exists a $\mathrm{GQ}(q, q)$ with identifying code of size

$$5q \in \Theta(|V|^{1/3}).$$

- There exists a $\mathrm{GQ}(q, q^2)$ with identifying code of size

$$5q + 5 \in \Theta(|V|^{1/4}).$$

- There exists a $\mathrm{GQ}(q^2, q)$ with identifying code of size

$$5q^2 + 3 \in \Theta(|V|^{2/5}).$$

Perspectives

Proposition (Gravier–Parreau–Rottey–Storme–V.)

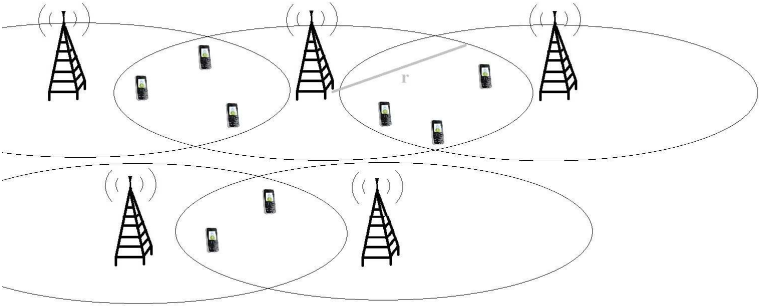
For any graph G, $\gamma_f^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2\ln|V|) \cdot \gamma_f^{ID}(G)$.

- New families with γ^{ID} and γ_f^{ID} of the same order $|V|^\alpha$ with $\alpha \in \{1/3, 1/4, 2/5\}$.
- There exists graphs with γ^{ID} and γ_f^{ID} not of the same order, but γ_f^{ID} is constant for them!
- Existence of graphs with γ_f^{ID} not constant and γ^{ID} not of the same order?

Problem 2

Covering codes

Mobile Network



(r, a, b)-covering codes with

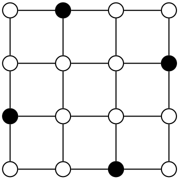
- r: reach of the emitting stations
- a: number of emitting stations within reach of an emitting station
- b: number of emitting stations that reach of a phone

Translation in terms of graphs

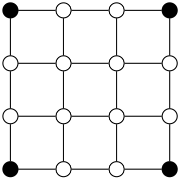
A set $S \subseteq V$ is an (r, a, b)-covering code of $G = (V, E)$ if for any $u \in V$

$$\left|\{B_r(v) \mid u \in B_r(v),\ v \in S\}\right| = \begin{cases} a & \text{if } u \in S \\ b & \text{if } u \notin S. \end{cases}$$

If $a = 1 = b$, they are called r-perfect code. [Biggs 1973]



$r = 1$

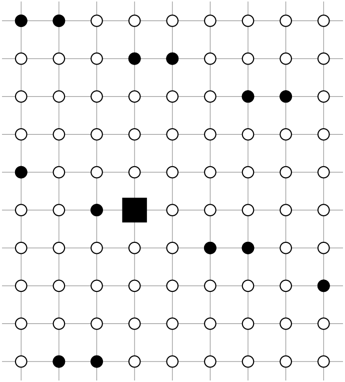


$r = 2$

Finding an r-prefect code is NP-complete. [Kratochvíl 1988]

In the case of the infinite square grid $\mathbb{Z}^2$

$r = 3$



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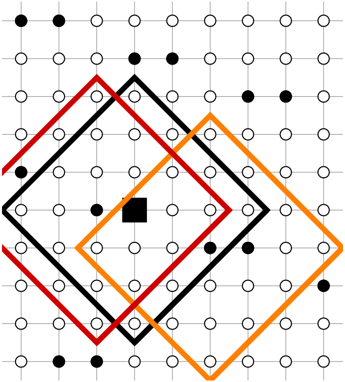

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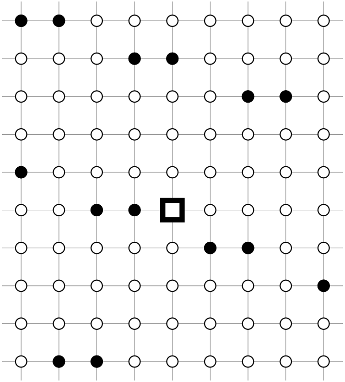

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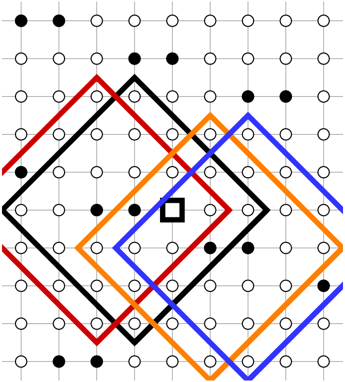

$a = 3$ and $b = 4$

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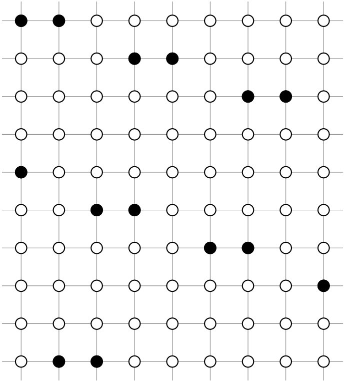

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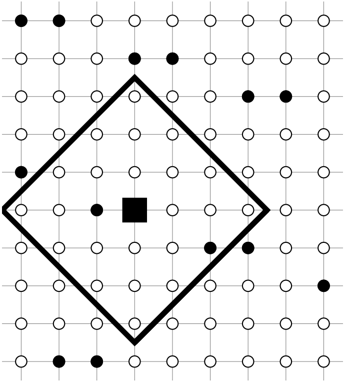

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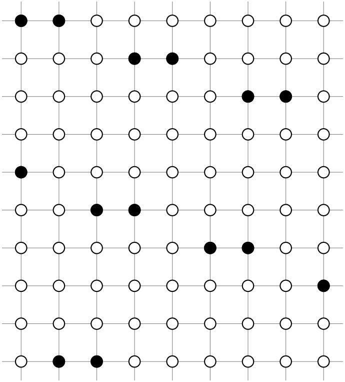

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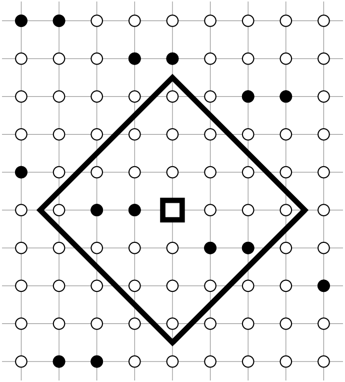

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Theorem (Puzynina 2008)

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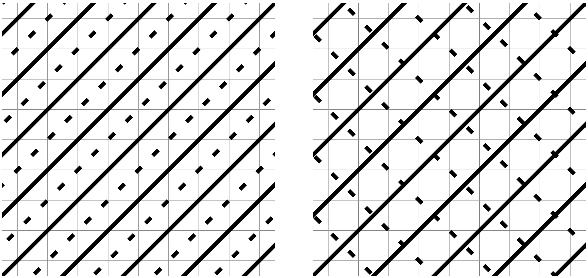
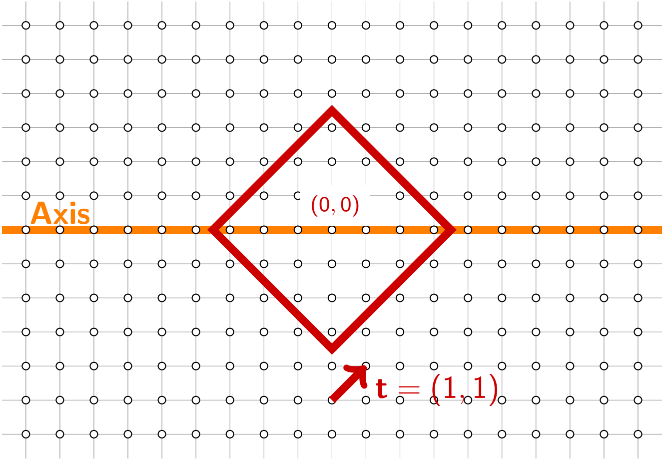


Projection and folding



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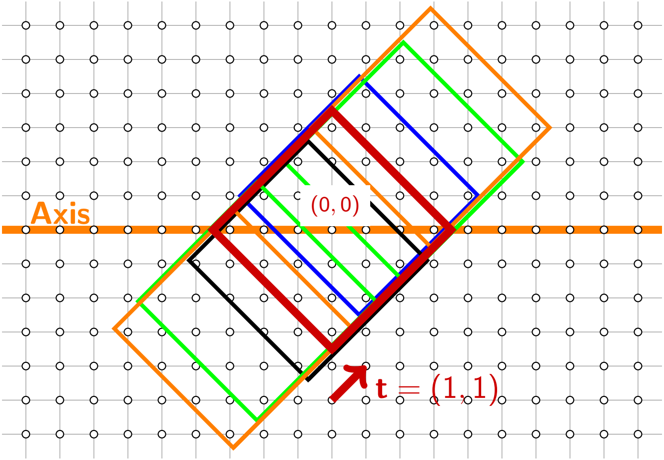

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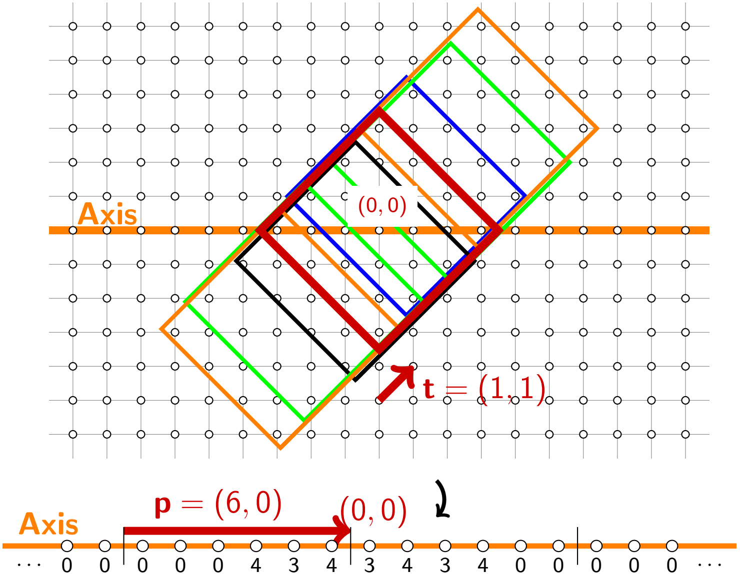

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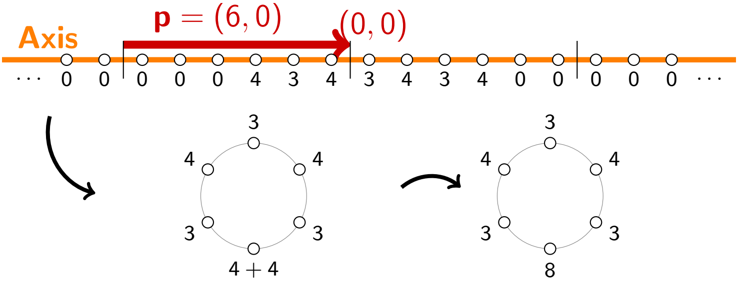

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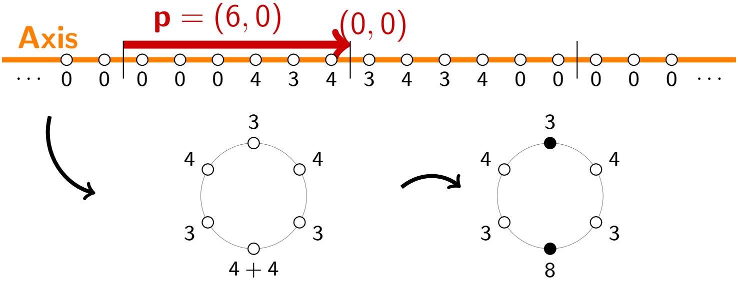

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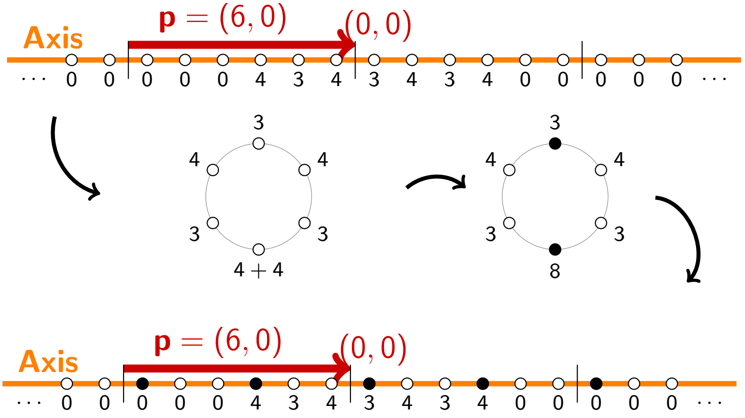

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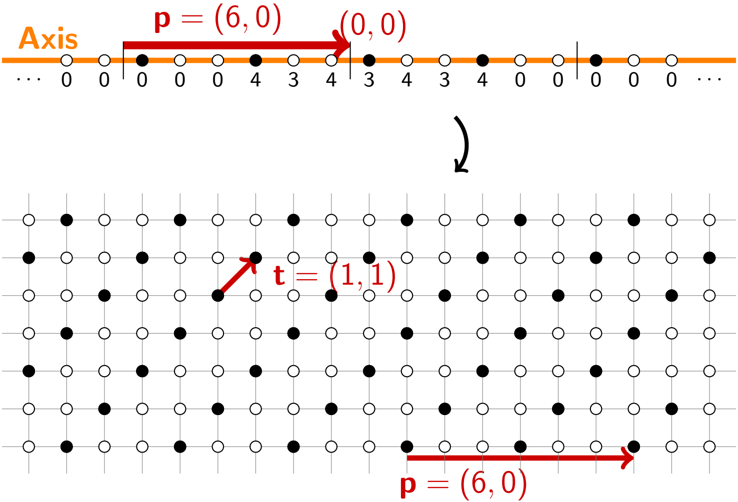
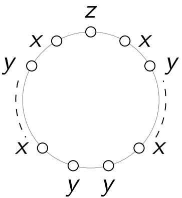


$\exists$ a $(3, 11, 7)$-covering code of $\mathbb{Z}^2$

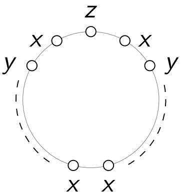
Constant 2-labellings

We only have to study particular colourings in 4 types of cycles!

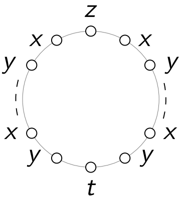
Type1mod



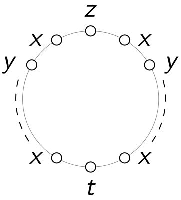
Type3mod



Type2mod

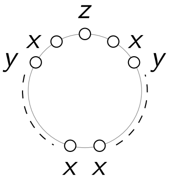


Type4mod



Example of results

Lemma (Gravier–V.)



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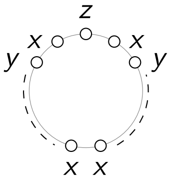

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Perspectives:

Many $(1, a, b)$-covering codes of $\mathbb{Z}^d$ are periodic.

[Dorbec–Gravier–Honkala–Mollard 2009]

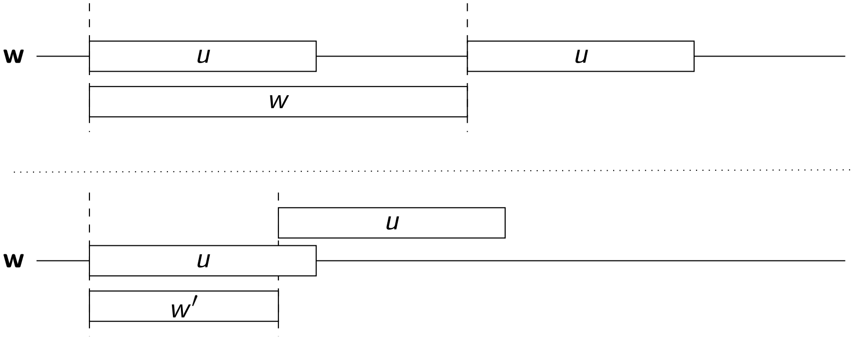
- Similar periodicity result?
- Which kind of weighted cycles?

Problem 3

Abelian return words

Abelian return words

Classical return word to a factor u



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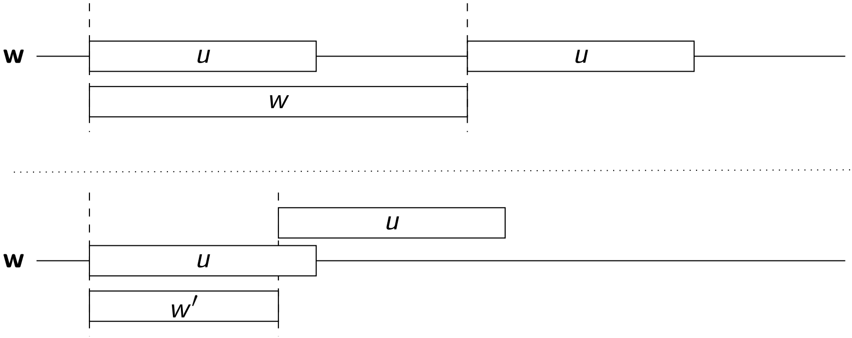

Abelian equivalence: *silent* $\sim_{ab}$ *listen* 00011 $\sim_{ab}$ 10010

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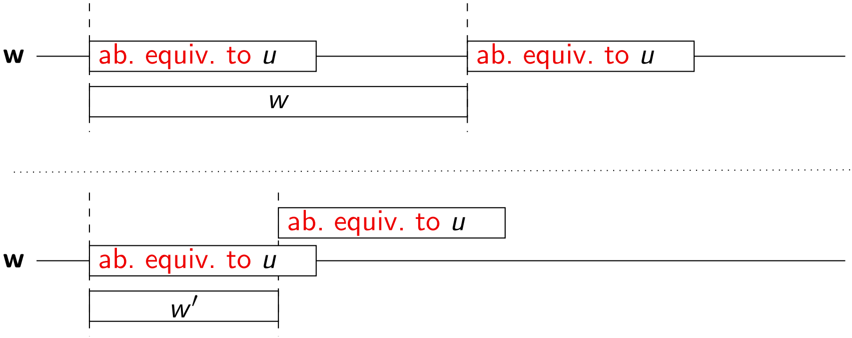

Abelian equivalence: *silent* $\sim_{ab}$ *listen* 00011 $\sim_{ab}$ 10010

Example for the Thue–Morse word

$\mathbf{t} = \texttt{0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1} \cdots$

Return words to $u = \texttt{011}$:

Example for the Thue–Morse word

$$\mathbf{t} = |0\,1\,1\,0\,1\,0|0\,1\,1\,0\,0\,1|0\,1\,1\,0\,1\,0\,0\,1|0\,1\,1\,0|0\,1\,1\,0\,1\,0|0\,1\,1\cdots$$

Return words to $u = 011$:

$$\mathcal{R}_{\mathbf{t},u} = \{011010,\ 011001,\ 01101001,\ 0110\}$$

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$\mathbf{t} = |0\ 1\ 1\ 0\ 1\ 0|0\ 1\ 1\ 0\ 0\ 1|0\ 1\ 1\ 0\ 1\ 0\ 0\ 1|0\ 1\ 1\ 0|0\ 1\ 1\ 0\ 1\ 0|0\ 1\ 1 \cdots$

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1 2 3 1 4 2 1 2 5 2 1 6 1 2 3 1

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1 2 3 1 4 2 1 2 5 2 1 6 1 2 3 1

Abelian derived sequence: $\mathcal{E}_u(\mathbf{t}) = 1231421252161231421\cdots$

Theorem (Durand 1998)

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$\Rightarrow$ The Thue–Morse word $\mathbf{t}$ has finitely many derived sequences.

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Proposition (Rigo–Salimov–V. 2013)

For the Thue–Morse word $\mathbf{t}$, the set $\{\mathcal{E}_u(\mathbf{t}) \mid u \in \mathrm{Pref}(\mathbf{t})\}$ is infinite.

Characterization of Sturmian words

A word $\mathbf{w}$ is Sturmian if for all n, the number of factors of length n is $n+1$.

Example: Fibonacci word

$$\mathbf{f} = 0100101001001010010100100101001001\cdots$$

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Theorem (Puzynina–Zamboni 2013)

An aperiodic recurrent infinite word is Sturmian if and only if each of its factors has two or three abelian return words.

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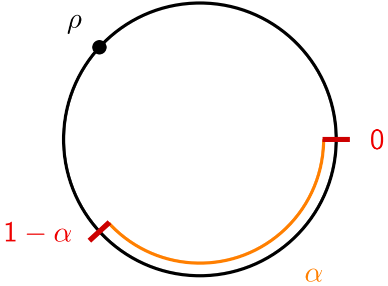

$$\mathbf{f} = St(\alpha, \rho) = 0$$

where

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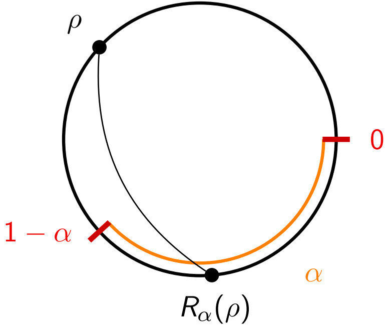

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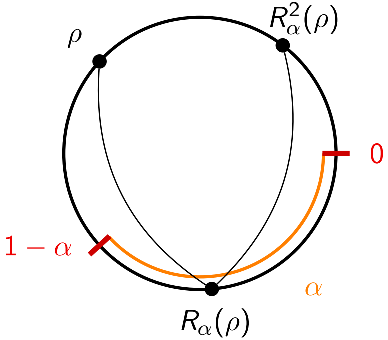

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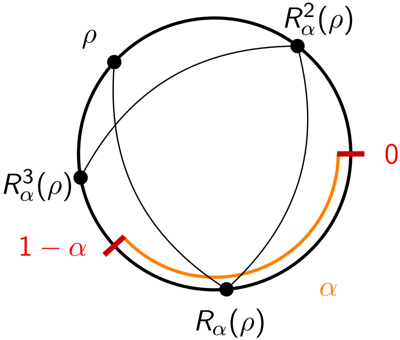

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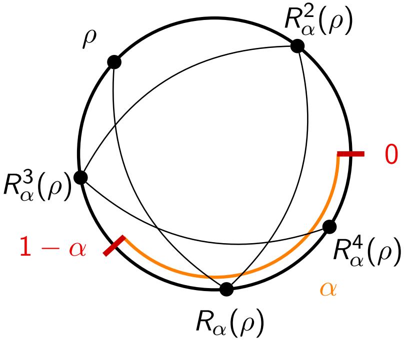

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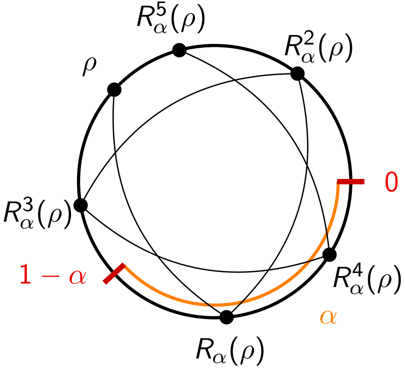

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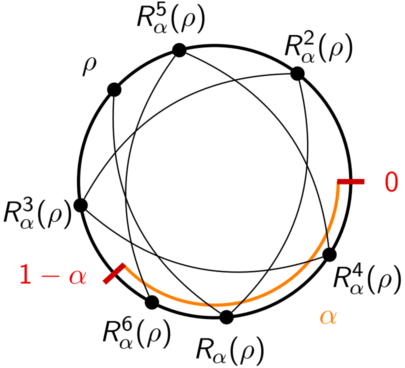
Theorem (Rigo–Salimov–V. 2013)

Let $\mathbf{w}$ be a Sturmian word. The set $\mathcal{APR}_{\mathbf{w}}$ is finite if and only if $\mathbf{w}$ does not have a null intercept.



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$$= 0100101$$

where

- $\alpha = 1/\phi^2$,
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- $\phi = (1+\sqrt{5})/2$.

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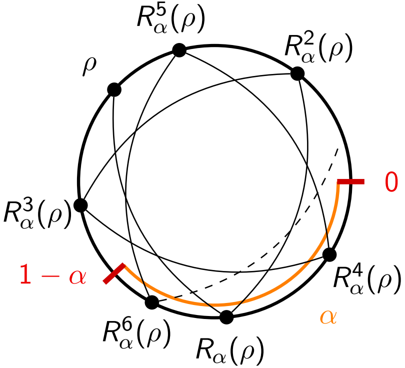

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Other result

Proposition (Rigo–Salimov–V. 2013)

If $\mathbf{w}$ is an abelian recurrent word such that $\mathcal{APR}_{\mathbf{w}}$ is finite, then the number of factors of length n of $\mathbf{w}$ up to abelian equivalence is bounded.

But the converse does not hold in general.

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Perspective:

bounded number of factors up to abelian equivalence + condition
$\Rightarrow$ abelian recurrent word with finite $\mathcal{APR}_{\mathbf{w}}$?

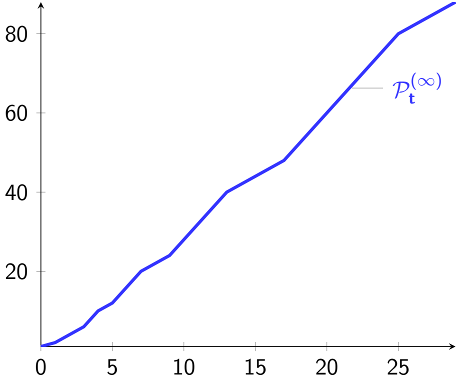
Problem 4

ℓ-abelian complexity

Thue–Morse word $\mathbf{t} = 0110100110010110\cdots$

Factor complexity $\mathcal{P}_{\mathbf{t}}^{(\infty)}$ [Brlek 1989, de Luca–Varricchio 1988]

$$\mathcal{P}_{\mathbf{t}}^{(\infty)}(n) = \begin{cases} 4n - 2\cdot 2^m - 4 & \text{if } 2\cdot 2^m < n \leq 3\cdot 2^m \\ 2n + 4\cdot 2^m - 2 & \text{if } 3\cdot 2^m < n \leq 4\cdot 2^m. \end{cases}$$



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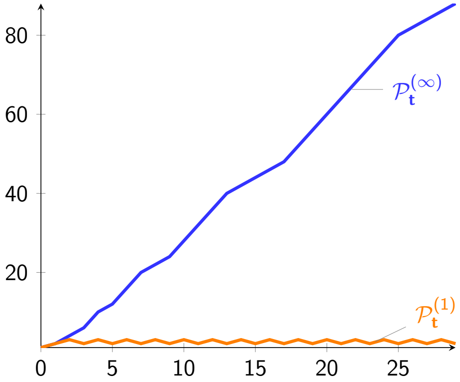

Abelian complexity $\mathcal{P}_{\mathbf{t}}^{(1)}$

$$\mathcal{P}_{\mathbf{t}}^{(1)}(2n) = 3 \text{ and } \mathcal{P}_{\mathbf{t}}^{(1)}(2n+1) = 2$$

ℓ-abelian complexity

Two words u, v are ℓ-abelian equivalent if

$$|u|_x = |v|_x \qquad \text{for any } x \text{ of length at most } \ell.$$

Example:

u	$\|u\|_0$	$\|u\|_1$	$\|u\|_{00}$	$\|u\|_{01}$	$\|u\|_{10}$	$\|u\|_{11}$
11010011	3	5	1	2	2	2
11101001	3	5	1	2	2	2

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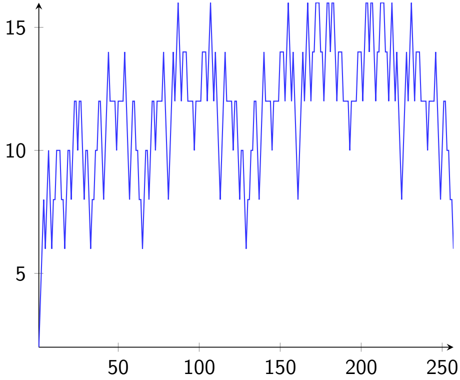
Number of factors of length n up to ℓ-abelian equivalence: $\mathcal{P}_{\mathbf{w}}^{(\ell)}(n)$

$$\mathcal{P}_{\mathbf{w}}^{(1)}(n) \leq \cdots \leq \mathcal{P}_{\mathbf{w}}^{(\ell)}(n) \leq \mathcal{P}_{\mathbf{w}}^{(\ell+1)}(n) \leq \cdots \leq \mathcal{P}_{\mathbf{w}}^{(\infty)}(n)$$

The ℓ-abelian complexity of a word $\mathbf{w}$ is the sequence $\mathcal{P}_{\mathbf{w}}^{(\ell)}(n)_{n\geq 0}$.

[Karhumäki–Saarela–Zamboni 2013]

2-abelian complexity of the Thue–Morse word



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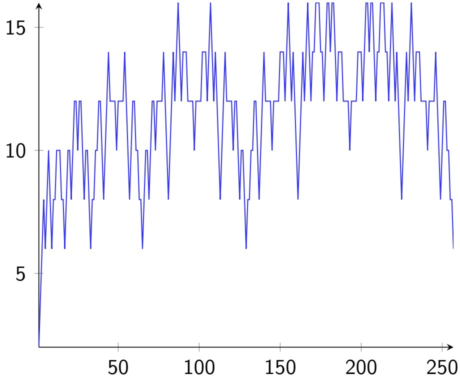

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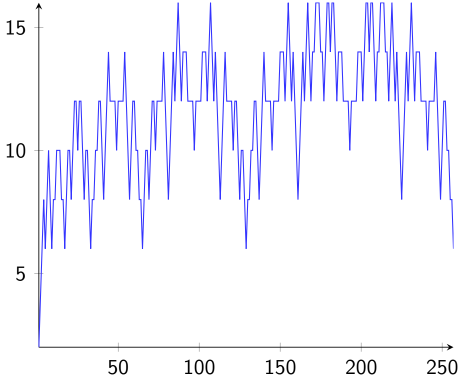

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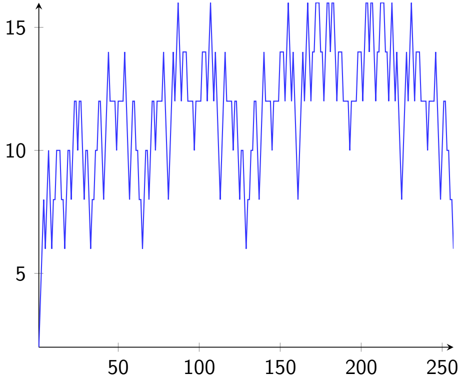

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- Regular?

A definition of regularity

A sequence $\mathbf{s} = s(n)_{n \geq 0}$ is *k-regular* if the $\mathbb{Z}$-module generated by its k-kernel

$$\mathcal{K}_k(\mathbf{s}) = \{s(k^e n + r)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq r < k^e\}$$

is finitely generated. [Allouche–Shallit 1992]

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$$\mathbf{t} = \quad 01101001100101101001011001101001 \cdots$$

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$$\begin{aligned} \mathbf{t} &= 01101001100101101001011001101001 \cdots \\ (\mathbf{t}_{2n})_{(n \geq 0)} &= 0110100110010110100101100110 1001 \cdots = \mathbf{t} \end{aligned}$$

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$$\mathcal{K}_2(\mathbf{t}) = \{\mathbf{t}, \overline{\mathbf{t}}\}$$

Complexity and regularity

- The factor complexity of a k-automatic sequence is k-regular. [Carpi–D'Alonzo 2010, Charlier–Rampersad–Shallit 2012]
- The abelian complexity of the Thue-Morse sequence is 2-regular.
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Question

Is the ℓ-abelian complexity of a k-automatic sequence always k-regular?

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Is the ℓ-abelian complexity of a k-automatic sequence always k-regular?

Is the 2-abelian complexity of the Thue–Morse word 2-regular?

Regularity via relations

`Mathematica` experiments

$$\mathbf{x}_{2^e+r} = \mathcal{P}_{\mathbf{t}}^{(2)}(2^e n + r)_{n \geq 0}$$

$$\begin{aligned}
\mathbf{x}_5 &= \mathbf{x}_3 \\
\mathbf{x}_9 &= \mathbf{x}_3 \\
\mathbf{x}_{12} &= -\mathbf{x}_6 + \mathbf{x}_7 + \mathbf{x}_{11} \\
\mathbf{x}_{13} &= \mathbf{x}_7 \\
\mathbf{x}_{16} &= \mathbf{x}_8 \\
\mathbf{x}_{17} &= \mathbf{x}_3 \\
\mathbf{x}_{18} &= \mathbf{x}_{10} \\
\mathbf{x}_{20} &= -\mathbf{x}_{10} + \mathbf{x}_{11} + \mathbf{x}_{19} \\
\mathbf{x}_{21} &= \mathbf{x}_{11} \\
\mathbf{x}_{22} &= -\mathbf{x}_3 - 2\mathbf{x}_6 + \mathbf{x}_7 + 3\mathbf{x}_{10} + \mathbf{x}_{11} - \mathbf{x}_{19} \\
\mathbf{x}_{23} &= -\mathbf{x}_3 - 3\mathbf{x}_6 + 2\mathbf{x}_7 + 3\mathbf{x}_{10} + \mathbf{x}_{11} - \mathbf{x}_{19} \\
\mathbf{x}_{24} &= -\mathbf{x}_3 + \mathbf{x}_7 + \mathbf{x}_{10} \\
\mathbf{x}_{25} &= \mathbf{x}_7 \\
\mathbf{x}_{26} &= -\mathbf{x}_3 + \mathbf{x}_7 + \mathbf{x}_{10} \\
\mathbf{x}_{27} &= -2\mathbf{x}_3 + \mathbf{x}_7 + 3\mathbf{x}_{10} - \mathbf{x}_{19} \\
\mathbf{x}_{28} &= -2\mathbf{x}_3 + \mathbf{x}_7 + 3\mathbf{x}_{10} - \mathbf{x}_{14} + \mathbf{x}_{15} - \mathbf{x}_{19} \\
\mathbf{x}_{29} &= \mathbf{x}_{15} \\
\mathbf{x}_{30} &= -\mathbf{x}_3 + 3\mathbf{x}_6 - \mathbf{x}_7 - \mathbf{x}_{10} - \mathbf{x}_{11} + \mathbf{x}_{15} + \mathbf{x}_{19} \\
\mathbf{x}_{31} &= -3\mathbf{x}_3 + 6\mathbf{x}_6 - 2\mathbf{x}_{11} - 3\mathbf{x}_{14} + 2\mathbf{x}_{15} + \mathbf{x}_{19}
\end{aligned}$$

Regularity via relations

`Mathematica` experiments $\quad \mathbf{x}_{2^e+r} = \mathcal{P}_{\mathbf{t}}^{(2)}(2^e n + r)_{n\geq 0}$

$$
\begin{aligned}
\mathbf{x}_{32} &= \mathbf{x}_{8}\\
\mathbf{x}_{33} &= \mathbf{x}_{3}\\
\mathbf{x}_{34} &= \mathbf{x}_{10}\\
\mathbf{x}_{35} &= \mathbf{x}_{11}\\
\mathbf{x}_{36} &= -\mathbf{x}_{10} + \mathbf{x}_{11} + \mathbf{x}_{19}\\
\mathbf{x}_{37} &= \mathbf{x}_{19}\\
\mathbf{x}_{38} &= -\mathbf{x}_{3} + \mathbf{x}_{10} + \mathbf{x}_{19}\\
\mathbf{x}_{39} &= -\mathbf{x}_{3} + \mathbf{x}_{11} + \mathbf{x}_{19}\\
\mathbf{x}_{40} &= -\mathbf{x}_{3} + \mathbf{x}_{10} + \mathbf{x}_{11}\\
\mathbf{x}_{41} &= \mathbf{x}_{11}\\
\mathbf{x}_{42} &= -\mathbf{x}_{3} + \mathbf{x}_{10} + \mathbf{x}_{11}\\
\mathbf{x}_{43} &= -2\mathbf{x}_{3} + 3\mathbf{x}_{10}\\
\mathbf{x}_{44} &= -2\mathbf{x}_{3} - \mathbf{x}_{6} + \mathbf{x}_{7} + 3\mathbf{x}_{10}\\
\mathbf{x}_{45} &= -\mathbf{x}_{3} - 3\mathbf{x}_{6} + 2\mathbf{x}_{7} + 3\mathbf{x}_{10} + \mathbf{x}_{11} - \mathbf{x}_{19}\\
\mathbf{x}_{46} &= -2\mathbf{x}_{3} - 3\mathbf{x}_{6} + 2\mathbf{x}_{7} + 5\mathbf{x}_{10} + \mathbf{x}_{11} - 2\mathbf{x}_{19}\\
\mathbf{x}_{47} &= -2\mathbf{x}_{3} + \mathbf{x}_{7} + 3\mathbf{x}_{10} - \mathbf{x}_{19}\\
\mathbf{x}_{48} &= -\mathbf{x}_{3} + \mathbf{x}_{7} + \mathbf{x}_{10}\\
\mathbf{x}_{49} &= \mathbf{x}_{7}\\
\mathbf{x}_{50} &= -\mathbf{x}_{3} + \mathbf{x}_{7} + \mathbf{x}_{10}\\
\mathbf{x}_{51} &= -\mathbf{x}_{3} - 3\mathbf{x}_{6} + 2\mathbf{x}_{7} + 3\mathbf{x}_{10} + \mathbf{x}_{11} - \mathbf{x}_{19}\\
\mathbf{x}_{52} &= -2\mathbf{x}_{3} - 3\mathbf{x}_{6} + 2\mathbf{x}_{7} + 5\mathbf{x}_{10} + \mathbf{x}_{11} - 2\mathbf{x}_{19}\\
\mathbf{x}_{53} &= -2\mathbf{x}_{3} + \mathbf{x}_{7} + 3\mathbf{x}_{10} - \mathbf{x}_{19}\\
\mathbf{x}_{54} &= -4\mathbf{x}_{3} + 3\mathbf{x}_{6} + \mathbf{x}_{7} + 3\mathbf{x}_{10} - \mathbf{x}_{11} - 2\mathbf{x}_{14} + \mathbf{x}_{15}\\
\mathbf{x}_{55} &= -4\mathbf{x}_{3} + 3\mathbf{x}_{6} + \mathbf{x}_{7} + 3\mathbf{x}_{10} - \mathbf{x}_{11} - 3\mathbf{x}_{14} + 2\mathbf{x}_{15}\\
\mathbf{x}_{56} &= -\mathbf{x}_{3} + \mathbf{x}_{10} + \mathbf{x}_{15}\\
\mathbf{x}_{57} &= \mathbf{x}_{15}\\
\mathbf{x}_{58} &= -\mathbf{x}_{3} + \mathbf{x}_{10} + \mathbf{x}_{15}\\
\mathbf{x}_{59} &= -2\mathbf{x}_{3} + 3\mathbf{x}_{6} - \mathbf{x}_{7} - \mathbf{x}_{11} + \mathbf{x}_{15} + \mathbf{x}_{19}\\
\mathbf{x}_{60} &= -4\mathbf{x}_{3} + 6\mathbf{x}_{6} + \mathbf{x}_{10} - 2\mathbf{x}_{11} - 3\mathbf{x}_{14} + 2\mathbf{x}_{15} + \mathbf{x}_{19}\\
\mathbf{x}_{61} &= -3\mathbf{x}_{3} + 6\mathbf{x}_{6} - 2\mathbf{x}_{11} - 3\mathbf{x}_{14} + 2\mathbf{x}_{15} + \mathbf{x}_{19}\\
\mathbf{x}_{62} &= -\mathbf{x}_{3} + 3\mathbf{x}_{6} - \mathbf{x}_{7} - \mathbf{x}_{10} - \mathbf{x}_{11} + \mathbf{x}_{15} + \mathbf{x}_{19}\\
\mathbf{x}_{63} &= \mathbf{x}_{15}
\end{aligned}
$$

Regularity via relations

If the relations hold, then any sequence $\mathbf{x}_n$ for $n \geq 32$ is a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_{19}$.

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Example: $\mathbf{x}_{154} = \mathcal{P}_{\mathbf{t}}^{(2)}(128n + 26)_{n \geq 0}$

Using $\mathbf{x}_{58} = -\mathbf{x}_3 + \mathbf{x}_{10} + \mathbf{x}_{15}$,

$$\begin{aligned}
\mathcal{P}_{\mathbf{t}}^{(2)}(128n + 26) &= \mathcal{P}_{\mathbf{t}}^{(2)}(32(4n) + 26) \\
&= -\mathcal{P}_{\mathbf{t}}^{(2)}(2(4n) + 1) + \mathcal{P}_{\mathbf{t}}^{(2)}(8(4n) + 2) + \mathcal{P}_{\mathbf{t}}^{(2)}(8(4n) + 7) \\
&= -\mathcal{P}_{\mathbf{t}}^{(2)}(8n + 1) + \mathcal{P}_{\mathbf{t}}^{(2)}(32n + 2) + \mathcal{P}_{\mathbf{t}}^{(2)}(32n + 7).
\end{aligned}$$

So

$$\mathbf{x}_{154} = -\mathbf{x}_9 + \mathbf{x}_{34} + \mathbf{x}_{39} = -2\mathbf{x}_3 + \mathbf{x}_{10} + \mathbf{x}_{11} + \mathbf{x}_{19}.$$

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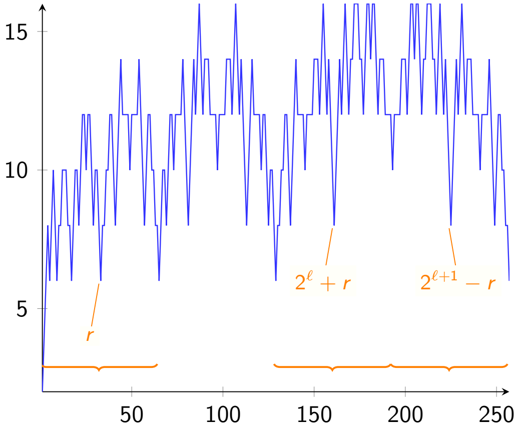
Theorem (Greinecker 2014)

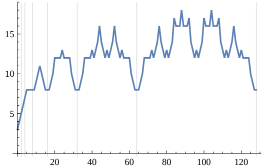
The relations hold and the 2-abelian complexity is 2-regular.

Regularity via another approach

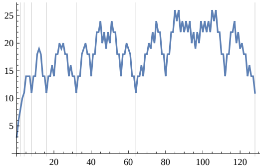


- Symmetry of the form $\mathcal{P}_{\mathbf{t}}^{(2)}(2^{\ell+1} - r) = \mathcal{P}_{\mathbf{t}}^{(2)}(2^\ell + r)$
- Some relation between $\mathcal{P}_{\mathbf{t}}^{(2)}(2^\ell + r)$ and $\mathcal{P}_{\mathbf{t}}^{(2)}(r)$

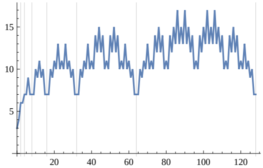
It is the case for many 2-abelian complexity functions.



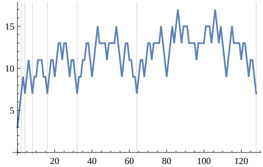
$0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 01$



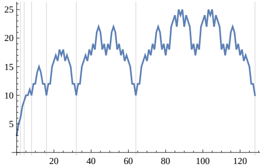
$0 \mapsto 01,\ 1 \mapsto 12,\ 2 \mapsto 01$



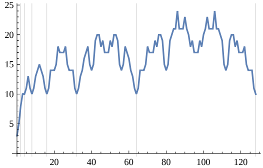
$0 \mapsto 01,\ 1 \mapsto 12,\ 2 \mapsto 11$



$0 \mapsto 01,\ 1 \mapsto 12,\ 2 \mapsto 21$



$0 \mapsto 01,\ 1 \mapsto 20,\ 2 \mapsto 01$



$0 \mapsto 01,\ 1 \mapsto 20,\ 2 \mapsto 10$

Symmetry and recurrence relation

Theorem (Parreau–Rigo–Rowland–V.)

If $s(n)_{n\geq 0}$ satisfies

$$s(2^\ell + r) = \begin{cases} s(r) + c & \text{if } r \leq 2^{\ell-1} \\ s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}, \end{cases}$$

then $s(n)_{n\geq 0}$ is 2-regular.

Abelian complexity of the fixed point of $0 \mapsto 12,\ 1 \mapsto 12,\ 2 \mapsto 00$

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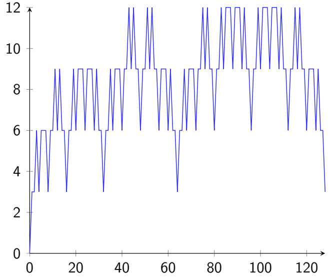
then $s(n)_{n\geq 0}$ is 2-regular.

Abelian complexity of the fixed point of $0 \mapsto 12,\ 1 \mapsto 12,\ 2 \mapsto 00$



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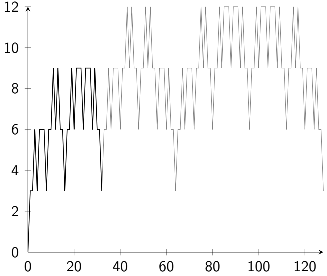

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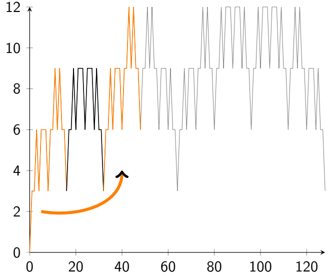

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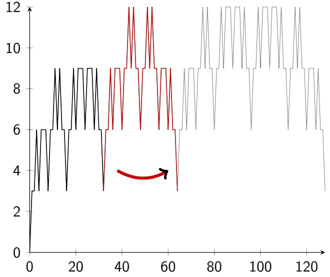

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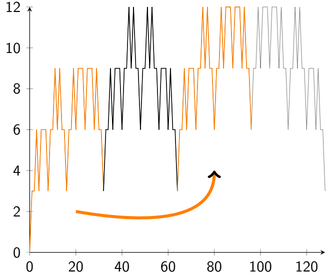

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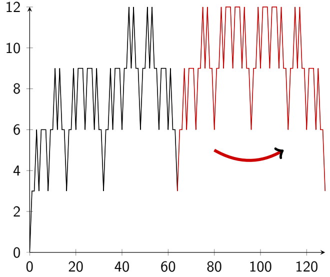

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How to prove the recurrence and reflection relations

For abelian complexity of the fixed point of $0 \mapsto 12, 1 \mapsto 12, 2 \mapsto 00$

$$\mathbf{x} = 1200121212001200120012121200121212 00\ldots$$

- Consider $\Delta_0(n) = \max_{|u|=n} |u|_0 - \min_{|u|=n} |u|_0$.
- It is closely related to the abelian complexity.
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- The 2-abelian complexity of the Thue–Morse word is 2-regular.

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- The 2-abelian complexity of the Thue–Morse word and of the period-doubling word is 2-regular.

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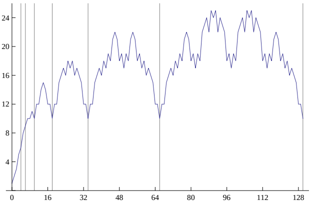
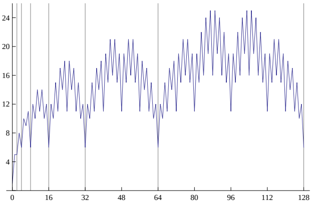
Theorem (Parreau–Rigo–Rowland–V.)

- The 2-abelian complexity of the Thue–Morse word and of the period-doubling word is 2-regular.
- The abelian complexity of some other 2-automatic words is 2-regular.

Perspectives

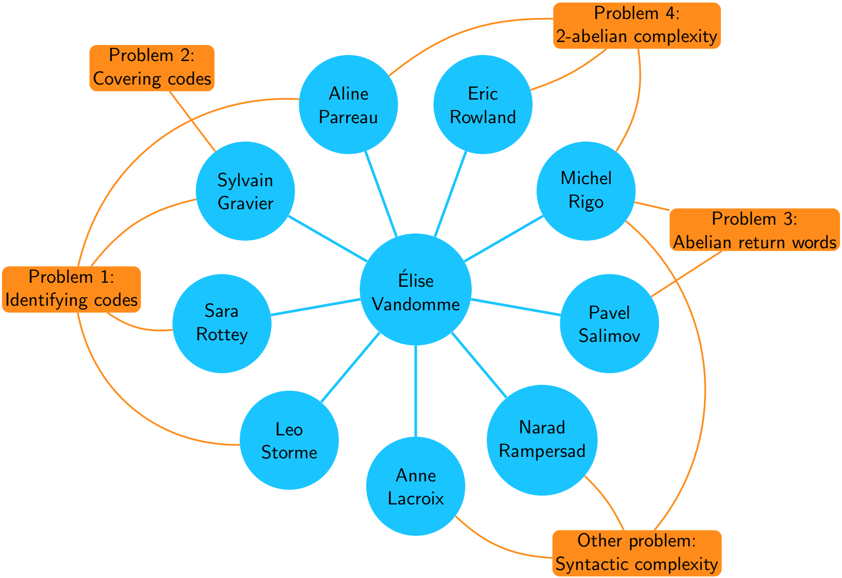
It seems that lots of (ℓ-)abelian complexity functions satisfy similar recurrence.





For the 3-abelian complexity of period-doubling word **p**, the abelian complexity of the 3-block coding **z** of **p** seems to satisfy:

$$\mathcal{P}_{\mathbf{z}}^{(1)}(2^\ell + r) = \begin{cases} \mathcal{P}_{\mathbf{z}}^{(1)}(r) + 5 & \text{if } r \leq 2^{\ell-1} \text{ and } r \text{ even} \\ \mathcal{P}_{\mathbf{z}}^{(1)}(r) + 7 & \text{if } r \leq 2^{\ell-1} \text{ and } r \text{ odd} \\ \mathcal{P}_{\mathbf{z}}^{(1)}(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}. \end{cases}$$


Problem 4:
2-abelian complexity
Problem 2:
Covering codes
Aline
Parreau
Eric
Rowland
Sylvain
Gravier
Michel
Rigo
Problem 3:
Abelian return words
Problem 1:
Identifying codes
Élise
Vandomme
Sara
Rottey
Pavel
Salimov
Leo
Storme
Narad
Rampersad
Anne
Lacroix
Other problem:
Syntactic complexity