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Abstract. In this paper, we study the uniform Hölder continuity of the generalized Riemann function $R_{\alpha,\beta}$ (with $\alpha > 1$ and $\beta > 0$) defined by

$$R_{\alpha,\beta}(x) = \sum_{n=1}^{+\infty} \frac{\sin(\pi n^{\beta} x)}{n^{\alpha}}, \quad x \in \mathbb{R},$$

using its continuous wavelet transform. In particular, we show that the exponent we find is optimal. We also analyse the behaviour of $R_{\alpha,\beta}$ as β tends to infinity.

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Keywords. Hölder continuity, continuous wavelet transform, Riemann function.

1. Introduction

In the 19th century, Riemann introduced the function R defined by

$$R(x) = \sum_{n=1}^{+\infty} \frac{\sin(\pi n^{2} x)}{n^{2}}, \quad x \in \mathbb{R},$$

to construct a continuous but nowhere differentiable function (see [6] for some historical informations). The regularity of this function has been extensively studied by many authors. In 1916, Hardy [9] showed that R is not differentiable at irrational numbers and at some rational numbers. Decades later, Gerver [8] and other people [12, 13, 21, 22, 24] proved that R is only differentiable at the rational numbers $(2p+1)/(2q+1)$ (with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$) with a derivative equal to $-1/2$.

The Hölder spaces allow to define a notion of smoothness or regularity for a function. Let us recall the general definition of Hölder spaces (see [5, 15, 17, 26]).

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Definition 1. Let $\alpha > 0$, $f \in L^{\infty}_{\text{loc}}(\mathbb{R})$ and $x_0 \in \mathbb{R}$.

(1) The function f belongs to $C^{\alpha}(x_0)$ if there exist $C > 0$, $\varepsilon > 0$ and a polynomial P of degree strictly less than α such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^{\alpha}$$

for all $x \in (x_0-\varepsilon, x_0+\varepsilon)$. In this case, we say that f is Hölder continuous with exponent α at x_0.

(2) The function f belongs to $C^{\alpha}(\mathbb{R})$ if there exist $C > 0$ and a polynomial P of degree strictly less than α such that

$$|f(x) - P(x - y)| \leq C|x - y|^{\alpha}$$

for all $x, y \in \mathbb{R}$. In this case, we say that f is uniformly Hölder continuous with exponent α (on $\mathbb{R}$).

In particular, these spaces provide an "intermediate level" between continuity and differentiability. This fact is more intuitive with the following remark which gives another way to define them for $\alpha \in (0, 1)$.

Remark 2. In the case $\alpha \in (0, 1)$, pointwise and uniform Hölder spaces can be expressed more simply in terms of r-oscillations ($r > 0$) of f, i.e. in terms of the function $\text{osc}_{r,f}$ defined by

$$\text{osc}_{r,f}(x) = \text{diam}(f(B(x, r))), \quad x \in \mathbb{R},$$

where diam denotes the diameter and $B(x, r)$ the open ball of centre x and radius r (see [18,27]). On the one hand, we indeed have $f \in C^{\alpha}(x_0)$ if and only if there exist $C > 0$ and $R > 0$ such that

$$\text{osc}_{r,f}(x_0) \leq Cr^{\alpha}$$

for all $r \in (0, R)$. On the other hand, $f \in C^{\alpha}(\mathbb{R})$ if and only if there exist $C > 0$ and $R > 0$ such that

$$\text{osc}_{r,f}(x) \leq Cr^{\alpha}$$

for all $x \in \mathbb{R}$ and $r \in (0, R)$.

The pointwise and uniform Hölder spaces are embedded: if $\beta > \alpha > 0$, then $C^{\beta}(x_0) \subset C^{\alpha}(x_0)$ for any $x_0 \in \mathbb{R}$ and $C^{\beta}(\mathbb{R}) \subset C^{\alpha}(\mathbb{R})$. This property allows to define a notion of regularity, known as Hölder exponent.

Definition 3. Let $f \in L^{\infty}_{\text{loc}}(\mathbb{R})$ and let $x_0 \in \mathbb{R}$.

(1) The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha > 0 : f \in C^{\alpha}(x_0)\}.$$

(2) The uniform Hölder exponent of f (on $\mathbb{R}$) is

$$h_f(\mathbb{R}) = \sup\{\alpha > 0 : f \in C^{\alpha}(\mathbb{R})\}.$$

Following this definition, if f is differentiable, then $h_f(\mathbb{R}) \geq 1$. Moreover, $h_f(\mathbb{R}) < 1$ implies that f is not differentiable. However, there exist non-differentiable functions with a uniform Hölder exponent equal to 1; the Takagi function (see [23,25]) is a famous example.

Based on a work with Littlewood [10], Hardy [9] showed that R is not Hölder continuous with exponent $3/4$ at irrational numbers and at some rational numbers. Using the continuous wavelet transform (of R), Holschneider and Tchamitchian [12] established that R is uniformly Hölder continuous with exponent $1/2$ and gave some results about its Hölder continuity at some particular points. With some similar techniques, Jaffard and Meyer [15,16] determined the Hölder exponent of R at each point and proved that R is a multifractal function, i.e. that the function $x \mapsto h_R(x)$ is not constant.

A generalization of R is given by the function $R_{\alpha,\beta}$ defined by

$$R_{\alpha,\beta}(x) = \sum_{n=1}^{+\infty} \frac{\sin(\pi n^\beta x)}{n^\alpha}, \quad x \in \mathbb{R}, \tag{1}$$

with $\alpha > 1$ and $\beta > 0$. Other generalizations of R are possible; for example, one can replace the element n^β in the definition of $R_{\alpha,\beta}$ by a polynomial with integer coefficients (see [3,22]).

The function $R_{\alpha,\beta}$ defined in (1) is clearly continuous and bounded on $\mathbb{R}$. If $\beta \in (0, \alpha - 1)$, it is easy to check that $R_{\alpha,\beta}$ is continuously differentiable on $\mathbb{R}$ (because the series of derivatives uniformly converges on $\mathbb{R}$). If $\beta \geq \alpha + 1$, Luther [20] proved that $R_{\alpha,\beta}$ is nowhere differentiable. If $\beta \in [\alpha - 1, \alpha + 1)$, several partial results about the differentiability of $R_{\alpha,\beta}$ are known (see [20,22]). Moreover, some results are also known for the cases $\beta = 2$ (see [9,15]), $\beta = 3$ (see [7]) and $\beta \in \mathbb{N} \setminus \{0\}$ (see [4]). Concerning the Hölder continuity and also the Hölder exponent of $R_{\alpha,\beta}$, several particular cases have been studied (see [2,4,15,16,19,28]).

In this paper, we study the uniform Hölder continuity of $R_{\alpha,\beta}$ with $\beta \geq \alpha - 1$ in complete and generalize a result of Johnsen [19] in 2010 which claims that, if $\beta > \alpha - 1$, then $R_{\alpha,\beta}$ is uniformly Hölder continuous with an exponent greater or equal to $(\alpha - 1)/\beta$. To achieve this, we use some techniques different from the ones of Johnsen. Our approach is based on the continuous wavelet transform of $R_{\alpha,\beta}$ related to the Lusin wavelet, and is similar to the ones used to obtain the Hölder continuity of R in [12,15,17]. This method has two advantages: we can consider both the cases $\beta = \alpha - 1$ and $\beta > \alpha - 1$ to study the uniform Hölder continuity of $R_{\alpha,\beta}$ and then show the optimality of the so obtained exponent. In other words, we calculate the uniform Hölder exponent of $R_{\alpha,\beta}$ for $\beta \geq \alpha - 1$. These results are summarized in the following theorem.

Theorem 4. *For $\beta \geq \alpha - 1$, we have*

$$h_{R_{\alpha,\beta}}(\mathbb{R}) = \frac{\alpha - 1}{\beta}.$$

If we fix $\alpha > 1$, the uniform Hölder exponent of $R_{\alpha,\beta}$ decreases to 0 as β increases to infinity. To illustrate this phenomenon, we give the graphical representation of $R_{\alpha,\beta}$ for some β. For β large enough, we can observe that $R_{\alpha,\beta}$ looks like the function $x \mapsto \sin(\pi x)$ with some "noise" or "oscillations" all around. In fact, this function is simply the first term of the series defining $R_{\alpha,\beta}$. We show that $R_{\alpha,\beta}$ can be, on average, compared to the function $x \mapsto \sin(\pi x)$ and we measure the amplitude of these fluctuations.

The paper is organized as follows. In Sect. 2, we recall some helpful properties about the continuous wavelet transform and the tool it provides to study the Hölder continuity of a function. We will extensively take advantage of the properties of the Lusin wavelet. The proof of Theorem 4 is given in Sect. 3. We analyse in Sect. 4 the behaviour of $R_{\alpha,\beta}$ as β increases. We present the graphical representation of $R_{2,\beta}$ for some particular values of β. In Sect. 5, we give some additional comments about the more general case of nonharmonic Fourier series. We also show the limitations of the Lusin wavelet to investigate the research of the maximal possible Hölder exponent of $R_{\alpha,\beta}$ at a point.

2. Hölder Continuity and Continuous Wavelet Transform

Let us recall some notions about the continuous wavelet transform and the Hölder continuity of a function (see [5,11,12,15,17,26]). The natural space associated with the continuous wavelet transform is the Hilbert space $L^2(\mathbb{R})$. Such a setting is of no interest for the function $R_{\alpha,\beta}$, since it does not belong to $L^2(\mathbb{R})$. As $R_{\alpha,\beta}$ is a continuous and bounded function on $\mathbb{R}$, the continuous wavelet transform of a function of $L^\infty(\mathbb{R})$ is more appropriate.

Definition 5. The function ψ is a wavelet if $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\hat{\psi}(0) = 0$, where $\hat{\psi}$ denotes the Fourier transform of ψ:

$$\hat{\psi}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-ix\xi} \psi(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}\,.$$

Using the wavelet ψ, the continuous wavelet transform of a function $f \in L^\infty(\mathbb{R})$ is the function $\mathcal{W}_\psi f$ defined by

$$\mathcal{W}_\psi f(a,b) = \int_{\mathbb{R}} f(x)\, \frac{1}{a}\, \overline{\psi}\left(\frac{x-b}{a}\right) \mathrm{d}x, \quad a > 0,\, b \in \mathbb{R},$$

where $\overline{\psi}$ denotes the complex conjugate of ψ.

To study the uniform Hölder continuity of $R_{\alpha,\beta}$, we will use a peculiar wavelet, known as the Lusin wavelet:

$$\psi(x) = \frac{1}{\pi(x+i)^2}, \quad x \in \mathbb{R}\,. \tag{2}$$

Since

$$\hat{\psi}(\xi) = \begin{cases} -2\xi \mathrm{e}^{-\xi} & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases},$$

this wavelet belongs to the second Hardy space

$$H^2(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \hat{f} = 0 \text{ a.e. on } (-\infty, 0) \right\}.$$

Such a property will be useful to obtain a simple explicit expression of $\mathcal{W}_\psi R_{\alpha,\beta}$ (in comparison with the derivatives of a gaussian function for example).

An exact reconstruction formula exists in such a situation: if ψ belongs to $H^2(\mathbb{R})$ and if f belongs to a certain class of continuous and bounded functions on $\mathbb{R}$, we can recover f from $\mathcal{W}_\psi f$ using a second wavelet satisfying some additional properties (see [12] with some adaptations to the case $H^2(\mathbb{R})$).

Theorem 6. *Let ψ be a wavelet which belongs to $H^2(\mathbb{R})$. Let φ be a differentiable wavelet such that $x \mapsto x\varphi(x)$ is integrable on $\mathbb{R}$, such that $D\varphi$ is square integrable on $\mathbb{R}$ and such that*

$$\int_0^{+\infty} \overline{\hat{\psi}}(\xi)\hat{\varphi}(\xi)\,\frac{\mathrm{d}\xi}{\xi} = 1. \tag{3}$$

If f is a continuous and bounded function on $\mathbb{R}$ and is weakly oscillating around the origin, such that

$$\lim_{r\to+\infty} \sup_{x\in\mathbb{R}} \left| \frac{1}{2r} \int_{x-r}^{x+r} f(t)\,\mathrm{d}t \right| = 0,$$

then we have

$$f(x) = \lim_{\substack{\varepsilon\to0^+\\ r\to+\infty}} 2\int_\varepsilon^r \left(\int_{-\infty}^{+\infty} \mathcal{W}_\psi f(a,b)\,\frac{1}{a}\varphi\left(\frac{x-b}{a}\right)\mathrm{d}b \right)\frac{\mathrm{d}a}{a}$$

for all $x \in \mathbb{R}$.

Thanks to this reconstruction formula, the Hölder continuity of a function can be characterized with its continuous wavelet transform, provided that the wavelet satisfies some additional conditions. We will use the following result to study the Hölder continuity of the generalized Riemann function (see [12,15,17]).

Theorem 7. *Let $\alpha \in (0,1)$, let ψ be a wavelet such that $x \mapsto x^\alpha\psi(x)$ is integrable on $\mathbb{R}$ and let f be a function as in Theorem 6.*

(1) *We have $f \in C^\alpha(\mathbb{R})$ if and only if there exists $C > 0$ such that*

$$|\mathcal{W}_\psi f(a,b)| \le C\,a^\alpha$$

for all $a > 0$ and $b \in \mathbb{R}$.

(2) *Let $x_0 \in \mathbb{R}$. If $f \in C^\alpha(x_0)$, then there exist $C > 0$ and $\eta > 0$ such that*

$$|\mathcal{W}_\psi f(a,b)| \le C\,a^\alpha \left(1 + \left(\frac{|b-x_0|}{a}\right)^\alpha\right)$$

for all $a \in (0,\eta)$ and $b \in (x_0-\eta, x_0+\eta)$. Conversely, if there exist $\alpha' \in (0,\alpha)$, $C > 0$ and $\eta > 0$ such that

$$|\mathcal{W}_\psi f(a,b)| \le C\,a^\alpha \left(1 + \left(\frac{|b-x_0|}{a}\right)^{\alpha'}\right)$$

for all $a \in (0,\eta)$ and $b \in (x_0-\eta, x_0+\eta)$, then $f \in C^\alpha(x_0)$.

Remark 8. Let us note that the proofs of the necessary conditions in Theorem 7 do not use all the hypotheses on the function f: the continuity and the weak oscillation around the origin of f are not useful for these implications.

The generalized Riemann function and the Lusin wavelet satisfy the conditions of the two previous theorems. Indeed, we know that $R_{\alpha,\beta}$ is continuous and bounded and that the Lusin wavelet ψ belongs to $H^2(\mathbb{R})$. Moreover, $R_{\alpha,\beta}$ is weakly oscillating around the origin because

$$\left|\frac{1}{2r}\int_{x-r}^{x+r} R_{\alpha,\beta}(t)\,\mathrm{d}t\right| \leq \left|\frac{1}{2r}\sum_{n=1}^{+\infty}\frac{\cos((x-r)\pi n^{\beta})-\cos((x+r)\pi n^{\beta})}{\pi n^{\alpha+\beta}}\right| \leq \frac{\zeta(\alpha+\beta)}{\pi r}$$

for all $x \in \mathbb{R}$ and $r > 0$, and $x \mapsto x^{\alpha}\psi(x)$ is clearly integrable for $\alpha \in (0,1)$. Besides, it is easy to find a differentiable wavelet φ such that $x \mapsto x\varphi(x)$ is integrable on $\mathbb{R}$, such that $D\varphi$ is square integrable on $\mathbb{R}$ and such that

$$\int_0^{+\infty} \hat{\varphi}(\xi)\mathrm{e}^{-\xi}\,\mathrm{d}\xi = -\frac{1}{2}.$$

In the following, ψ will systematically denote the Lusin wavelet [see (2)].

3. Hölder Continuity of Generalized Riemann Function

Since we know that the function $R_{\alpha,\beta}$ is continuously differentiable on $\mathbb{R}$ if $\alpha > 1$ and $\beta \in (0, \alpha-1)$, we can assume $\beta \geq \alpha - 1$ in the study of the uniform Hölder continuity of $R_{\alpha,\beta}$. To prove Theorem 4, we first need to determine the continuous wavelet transform of $R_{\alpha,\beta}$ related to the Lusin wavelet, as in [12,15,17] where the case $\alpha = \beta = 2$ is treated.

Proposition 9. *We have*

$$\mathcal{W}_{\psi}R_{\alpha,\beta}(a,b) = ia\pi\sum_{n=1}^{+\infty}\frac{\mathrm{e}^{i\pi n^{\beta}(b+ia)}}{n^{\alpha-\beta}} \tag{4}$$

for all $a > 0$ *and* $b \in \mathbb{R}$.

Proof. We can write

$$R_{\alpha,\beta}(x) = \frac{1}{2}\left(T_{\alpha,\beta}(x) - \widetilde{T}_{\alpha,\beta}(x)\right)$$

for $x \in \mathbb{R}$ with

$$T_{\alpha,\beta}(x) = -i\sum_{n=1}^{+\infty}\frac{\mathrm{e}^{i\pi n^{\beta}x}}{n^{\alpha}} \qquad \text{and} \qquad \widetilde{T}_{\alpha,\beta}(x) = T_{\alpha,\beta}(-x).$$

In other words, $R_{\alpha,\beta}$ is the odd part of $T_{\alpha,\beta}$.

Let us fix $a > 0$ and $b \in \mathbb{R}$. We have

$$\mathcal{W}_{\psi}T_{\alpha,\beta}(a,b) = \int_{\mathbb{R}} T_{\alpha,\beta}(x)\,\frac{1}{a}\overline{\psi}\left(\frac{x-b}{a}\right)\mathrm{d}x = \frac{a}{\pi}\int_{\mathbb{R}}\frac{T_{\alpha,\beta}(x)}{(x-(b+ia))^2}\,\mathrm{d}x.$$

For $\eta > 0$ and $r > 0$, let us denote by $\gamma_{\eta,r}$ the closed path formed by the juxtaposition of the two following ones: the first path describes the segment $[-r+i\eta, r+i\eta]$ and the second one the half-circle of centre $i\eta$ and radius r included in $H = \{z \in \mathbb{C} : \Im z > 0\}$. The function $T_{\alpha,\beta}$ is holomorphic on H because the series uniformly converges on every compact set of H. As the point $b+ia$ is situated inside the curve described by $\gamma_{\eta,r}$ for $\eta \in (0,a)$ and $r>a$, we obtain

$$\begin{aligned}\mathcal{W}_\psi T_{\alpha,\beta}(a,b) &= \frac{a}{\pi}\lim_{r\to+\infty}\lim_{\eta\to 0^+}\int_{\gamma_{\eta,r}} \frac{T_{\alpha,\beta}(z)}{(z-(b+ia))^2}\,\mathrm{d}z\\ &= 2ia\,(DT_{\alpha,\beta})(b+ia)\\ &= 2ia\pi\sum_{n=1}^{+\infty}\frac{\mathrm{e}^{i\pi n^\beta(b+ia)}}{n^{\alpha-\beta}},\end{aligned}$$

thanks to Cauchy's integral formula. Similarly, the continuous wavelet transform of $\widetilde{T}_{\alpha,\beta}$ is given by

$$\begin{aligned}\mathcal{W}_\psi \widetilde{T}_{\alpha,\beta}(a,b) &= \int_{\mathbb{R}} T_{\alpha,\beta}(-x)\,\frac{1}{a}\overline{\psi}\left(\frac{x-b}{a}\right)\mathrm{d}x\\ &= \frac{a}{\pi}\lim_{r\to+\infty}\lim_{\eta\to 0^+}\int_{\gamma_{\eta,r}}\frac{T_{\alpha,\beta}(z)}{(z-(-b-ia))^2}\,\mathrm{d}z = 0\end{aligned}$$

by homotopy invariance, because the point $-b-ia$ does not belong to H. We thus have the conclusion. $\square$

Let us now analyse $\mathcal{W}_\psi R_{\alpha,\beta}$ to study the uniform Hölder continuity of $R_{\alpha,\beta}$ with Theorem 7. We have

$$|\mathcal{W}_\psi R_{\alpha,\beta}(a,b)| \le a\pi\sum_{n=1}^{+\infty}\frac{\mathrm{e}^{-a\pi n^\beta}}{n^{\alpha-\beta}} = |\mathcal{W}_\psi R_{\alpha,\beta}(a,0)| \tag{5}$$

for $a>0$ and $b\in\mathbb{R}$. The function $f_{\alpha,\beta}: x\mapsto x^{\beta-\alpha}\,\mathrm{e}^{-a\pi x^\beta}$ is differentiable on $(0,+\infty)$ and

$$Df_{\alpha,\beta}(x) = \mathrm{e}^{-a\pi x^\beta}\,x^{\beta-\alpha-1}\left((\beta-\alpha)-a\pi\beta x^\beta\right),\quad x>0.$$

Then, $f_{\alpha,\beta}$ is decreasing on $(0,+\infty)$ if $\beta\in[\alpha-1,\alpha)$ and on $(((\beta-\alpha)/a\pi\beta)^{1/\beta},+\infty)$ if $\beta\ge\alpha$.

We note that $f_{\alpha,\beta}$ is integrable on $(0,+\infty)$ only if $\beta>\alpha-1$. We therefore split the study of the uniform Hölder continuity and the calculus of the uniform Hölder exponent of $R_{\alpha,\beta}$ into two cases: $\beta>\alpha-1$ and $\beta=\alpha-1$.

Proposition 10. *If $\beta>\alpha-1$, then*

$$h_{R_{\alpha,\beta}}(\mathbb{R}) = \frac{\alpha-1}{\beta}.$$

Proof. 1. Let us first consider the case $\beta\in(\alpha-1,\alpha)$. The function $f_{\alpha,\beta}$ is decreasing on $[1,+\infty)$ and we have

$$|\mathcal{W}_\psi R_{\alpha,\beta}(a,b)| \le a\pi\left(\mathrm{e}^{-a\pi}+\sum_{n=2}^{+\infty}\frac{\mathrm{e}^{-a\pi n^\beta}}{n^{\alpha-\beta}}\right) \le a\pi\left(\mathrm{e}^{-a\pi}+\int_1^{+\infty}\frac{\mathrm{e}^{-a\pi x^\beta}}{x^{\alpha-\beta}}\,\mathrm{d}x\right)$$

for $a > 0$ and $b \in \mathbb{R}$. For the second term of the right-hand side of the last inequality, we obtain

$$\int_1^{+\infty} \frac{\mathrm{e}^{-a\pi x^\beta}}{x^{\alpha-\beta}}\,\mathrm{d}x \le \int_0^{+\infty} \frac{\mathrm{e}^{-a\pi x^\beta}}{x^{\alpha-\beta}}\,\mathrm{d}x = \frac{1}{\beta}\pi^{\frac{\alpha-1}{\beta}-1}\,\Gamma\left(\frac{1+\beta-\alpha}{\beta}\right) a^{\frac{\alpha-1}{\beta}-1} \tag{6}$$

for $a > 0$, where Γ is defined by

$$\Gamma(x) = \int_0^{+\infty} \mathrm{e}^{-t}\,t^{x-1}\,\mathrm{d}t, \quad x > 0,$$

as usual. For the first term, we note that the function $a \mapsto \mathrm{e}^{-a\pi} a^{1-\frac{\alpha-1}{\beta}}$ is bounded on $(0, +\infty)$ because $\alpha - 1 < \beta$. Then, there exists $C_{\alpha,\beta} > 0$ such that

$$|\mathcal{W}_\psi R_{\alpha,\beta}(a, b)| \le C_{\alpha,\beta}\, a^{\frac{\alpha-1}{\beta}}$$

for all $a > 0$ and $b \in \mathbb{R}$, which implies $R_{\alpha,\beta} \in C^{\frac{\alpha-1}{\beta}}(\mathbb{R})$ using Theorem 7.

Let us show the optimality of this exponent $(\alpha - 1)/\beta$ related to the uniform Hölder continuity. Let $C > 0$ and $\eta > 0$; we have

$$\begin{aligned}|\mathcal{W}_\psi R_{\alpha,\beta}(a, 0)| &= a\pi \sum_{n=1}^{+\infty} \frac{\mathrm{e}^{-\pi n^\beta a}}{n^{\alpha-\beta}} \ge a\pi \int_1^{+\infty} \frac{\mathrm{e}^{-a\pi x^\beta}}{x^{\alpha-\beta}}\,\mathrm{d}x \\ &= \frac{1}{\beta}\,(a\pi)^{\frac{\alpha-1}{\beta}}\,\Gamma\left(\frac{\beta-\alpha+1}{\beta}, a\pi\right)\end{aligned}$$

for $a > 0$, where Γ is the incomplete Gamma function defined by

$$\Gamma(x, y) = \int_y^{+\infty} \mathrm{e}^{-t} t^{x-1}\,\mathrm{d}t, \quad (x, y) \in (0, +\infty) \times [0, +\infty).$$

Let us recall that $\Gamma(x, 0) = \Gamma(x)$ and $\Gamma(x, y)$ converges to $\Gamma(x)$ as $y \to 0^+$ for all $x > 0$. Since $\Gamma((\beta - \alpha + 1)/\beta, a\pi) \to \Gamma((\beta - \alpha + 1)/\beta)$ and $a^\eta \to 0$ as $a \to 0^+$, there exists $A > 0$ such that, for all $a \in (0, A)$, we have

$$|\mathcal{W}_\psi R_{\alpha,\beta}(a, 0)| > C\, a^{\frac{\alpha-1}{\beta}+\eta}.$$

Hence the conclusion using Theorem 7.

2. Let us now consider the case $\beta \ge \alpha$ and let us write $N_a = \lfloor((\beta - \alpha)/a\pi\beta)^{1/\beta}\rfloor + 1$, where $\lfloor x\rfloor$ denotes the largest integer smaller than or equal to the real x. If $a > 1$, then $N_a = 1$ and we can proceed as in the previous

case. Let us therefore suppose that $a \in (0,1]$. We have

$$\begin{aligned}
|\mathcal{W}_\psi R_{\alpha,\beta}(a,b)| &\le a\pi \left(\sum_{n=1}^{N_a} \frac{\mathrm{e}^{-a\pi n^\beta}}{n^{\alpha-\beta}} + \sum_{n=N_a+1}^{+\infty} \frac{\mathrm{e}^{-a\pi n^\beta}}{n^{\alpha-\beta}} \right) \\
&\le a\pi \left(N_a\, N_a^{\beta-\alpha} + \int_{N_a}^{+\infty} \frac{\mathrm{e}^{-a\pi x^\beta}}{x^{\alpha-\beta}}\, \mathrm{d}x \right) \\
&\le a\pi \left(\left(\left(\frac{\beta-\alpha}{\pi\beta} \right)^{\frac{1}{\beta}} + a^{\frac{1}{\beta}} \right)^{\beta-\alpha+1} a^{\frac{\alpha-1}{\beta}-1} + \int_0^{+\infty} \frac{\mathrm{e}^{-a\pi x^\beta}}{x^{\alpha-\beta}}\, \mathrm{d}x \right) \\
&\le a^{\frac{\alpha-1}{\beta}} \pi \left(\left(\left(\frac{\beta-\alpha}{\pi\beta} \right)^{\frac{1}{\beta}} + 1 \right)^{\beta-\alpha+1} \right. \\
&\quad \left. + \frac{1}{\beta} \pi^{\frac{\alpha-1}{\beta}-1}\, \Gamma\left(\frac{1+\beta-\alpha}{\beta} \right) \right),
\end{aligned}$$

where we have used relation (6) to obtain the last inequality. We then have $R_{\alpha,\beta} \in C^{\frac{\alpha-1}{\beta}}(\mathbb{R})$ using Theorem 7.

Let us show the optimality of the exponent related to the uniform Hölder continuity. Let $C > 0$ and $\eta > 0$; we have

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{\mathrm{e}^{-\pi n^\beta a}}{n^{\alpha-\beta}} &\ge \sum_{n=N_a}^{+\infty} \frac{\mathrm{e}^{-\pi n^\beta a}}{n^{\alpha-\beta}} \\
&\ge \int_{N_a}^{+\infty} \frac{\mathrm{e}^{-a\pi x^\beta}}{x^{\alpha-\beta}}\, \mathrm{d}x \\
&= \frac{1}{\beta} (a\pi)^{\frac{\alpha-1}{\beta}-1} \int_{a\pi N_a^\beta}^{+\infty} \mathrm{e}^{-u}\, u^{\frac{\beta-\alpha+1}{\beta}-1}\, \mathrm{d}u \\
&\ge \frac{1}{\beta} (a\pi)^{\frac{\alpha-1}{\beta}-1}\, \Gamma\left(\frac{\beta-\alpha+1}{\beta}, \left(\left(\frac{\beta-\alpha}{\beta} \right)^{1/\beta} + (a\pi)^{1/\beta} \right)^\beta \right)
\end{aligned}$$

for $a > 0$. As in the case $\beta \in (\alpha-1, \alpha)$, there exists $A > 0$ such that, for all $a \in (0, A)$, we have

$$|\mathcal{W}_\psi R_{\alpha,\beta}(a,0)| > C\, a^{\frac{\alpha-1}{\beta}+\eta},$$

hence the conclusion using once again Theorem 7. □

Remark 11. In fact, by taking $b = 2k$ with $k \in \mathbb{Z}$, we can show that $R_{\alpha,\beta} \in C^{\frac{\alpha-1}{\beta}}(2k)$ and that the exponent cannot be improved because $\mathcal{W}_\psi R_{\alpha,\beta}(a, 2k) = \mathcal{W}_\psi R_{\alpha,\beta}(a,0)$ for all $a > 0$. In other words, we have

$$h_{R_{\alpha,\beta}}(2k) = \frac{\alpha-1}{\beta}.$$

Since this quantity is strictly smaller than 1, $R_{\alpha,\beta}$ is consequently not differentiable at $2k$.

Proposition 12. *We have* $h_{R_{\alpha,\alpha-1}}(\mathbb{R}) = 1$.

Proof. We have

$$\begin{aligned}|\mathcal{W}_\psi R_{\alpha,\alpha-1}(a,b)| &\leq a\pi \left(\mathrm{e}^{-a\pi} + \int_1^{+\infty} \frac{\mathrm{e}^{-a\pi x^{\alpha-1}}}{x} \,\mathrm{d}x \right) \\ &= a\pi \left(\mathrm{e}^{-a\pi} + \frac{1}{\alpha-1} E_1(a\pi) \right)\end{aligned}$$

for $a > 0$ and $b \in \mathbb{R}$, where E_1 is the exponential integral defined by

$$E_1(x) = \int_1^{+\infty} \frac{\mathrm{e}^{-xt}}{t} \,\mathrm{d}t, \quad x > 0.$$

Since we have

$$\frac{1}{2}\,\mathrm{e}^{-x} \ln\left(1 + \frac{2}{x}\right) < E_1(x) < \mathrm{e}^{-x} \ln\left(1 + \frac{1}{x}\right) \tag{7}$$

for all $x > 0$ (see [1] p. 229), we obtain

$$|\mathcal{W}_\psi R_{\alpha,\alpha-1}(a,b)| \leq a\pi\,\mathrm{e}^{-a\pi} \left(1 + \frac{1}{\alpha-1} \ln\left(1 + \frac{1}{a\pi}\right)\right)$$

for $a > 0$ and $b \in \mathbb{R}$. Let us fix $\delta \in (0,1)$. There exists $A > 0$ such that, for all $a \in (0,A)$, we have

$$\frac{1}{\alpha-1} \frac{\ln\left(1 + \frac{1}{a\pi}\right)}{\left(1 + \frac{1}{a\pi}\right)^\delta} < 1$$

and then

$$\begin{aligned}|\mathcal{W}_\psi R_{\alpha,\alpha-1}(a,b)| &\leq a\pi\,\mathrm{e}^{-a\pi} \left(1 + \left(1 + \frac{1}{a\pi}\right)^\delta\right) \\ &\leq a\pi \left(1 + 2^\delta \left(1 + \left(\frac{1}{a\pi}\right)^\delta\right)\right).\end{aligned}$$

There also exists $A' \in (0,A)$ such that, for all $a \in (0,A')$, we have

$$|\mathcal{W}_\psi R_{\alpha,\alpha-1}(a,b)| \leq C'_\delta a^{1-\delta},$$

where C'_δ is a positive constant (depending only on δ). Since the function

$$a \mapsto a^\delta \mathrm{e}^{-a\pi} \left(1 + \frac{1}{\alpha-1} \ln\left(1 + \frac{1}{a\pi}\right)\right)$$

is bounded on $[A', +\infty)$, we also have

$$|\mathcal{W}_\psi R_{\alpha,\alpha-1}(a,b)| \leq C''_\delta a^{1-\delta}$$

for $a \in [A', +\infty)$, where C''_δ is a positive constant. We thus obtain

$$|\mathcal{W}_\psi R_{\alpha,\alpha-1}(a,b)| \leq C_\delta\, a^{1-\delta}$$

for all $a > 0$ and $b \in \mathbb{R}$ where $C_\delta = \max\{C'_\delta, C''_\delta\}$, which implies $R_{\alpha,\alpha-1} \in C^{1-\delta}(\mathbb{R})$ using Theorem 7.

Let us now show that this exponent of uniform Hölder continuity is optimal. Let $C > 0$; we have

$$|\mathcal{W}_\psi R_{\alpha,\alpha-1}(a,0)| \\ \geq a\pi \int_1^{+\infty} \frac{\mathrm{e}^{-a\pi x^{\alpha-1}}}{x}\,\mathrm{d}x = \frac{a\pi}{\alpha-1} E_1(a\pi) \geq a\,\frac{\pi}{2(\alpha-1)}\,\mathrm{e}^{-a\pi} \ln\left(1+\frac{2}{a\pi}\right)$$

for all $a > 0$ thanks to (7) and so, there exists $A > 0$ such that, for all $a \in (0, A)$, we have

$$|\mathcal{W}_\psi R_{\alpha,\alpha-1}(a,0)| > Ca,$$

hence the conclusion using one last time Theorem 7. □

4. Behaviour of $R_{\alpha,\beta}$ as β Increases

If we fix $\alpha > 1$, we know that the uniform Hölder exponent of $R_{\alpha,\beta}$ decreases as β increases, thanks to Theorem 4. Moreover, we know that this exponent is exactly the Hölder exponent of $R_{\alpha,\beta}$ at the origin. This phenomenon is clearly illustrated in Fig. 1 in the case $\alpha = 2$.

As β tends to infinity, we note that the graphical representation of $R_{\alpha,\beta}$ looks like the one of the function $s : x \mapsto \sin(\pi x)$ with some noise or oscillations all around (in some sense to establish). In the next two propositions, we give a convergence result and show that the fluctuations have a constant amplitude (i.e. independent of β). To do so, let us recall the usual definition of the mean of an integrable function over a bounded interval.

Definition 13. Let $a, b \in \mathbb{R}$ be such that $a < b$ and let f be an integrable function on (a, b). The mean of the function f over the interval (a, b) is defined by

$$m_f^{a,b} = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x.$$

Proposition 14. *Let $\alpha > 1$. For all $a, b \in \mathbb{R}$ such that $a < b$, we have*

$$\lim_{\beta \to +\infty} m_{R_{\alpha,\beta}}^{a,b} = m_s^{a,b}.$$

Proof. We have

$$\left|\int_a^b (R_{\alpha,\beta}(x) - \sin(\pi x))\,\mathrm{d}x\right| = \left|\sum_{n=2}^{+\infty} \frac{\cos(\pi n^\beta a) - \cos(\pi n^\beta b)}{\pi n^{\alpha+\beta}}\right| \leq \frac{2}{\pi}(\zeta(\alpha+\beta) - 1)$$

and we know that $\zeta(x) \to 1$ as $x \to +\infty$, hence the conclusion. □

Proposition 15. *Let $\alpha > 1$ and let $\beta \in \mathbb{N} \setminus \{0\}$. The function $R_{\alpha,\beta}$ is periodic of period 2 and we have*

$$\int_{-1}^1 (R_{\alpha,\beta}(x) - \sin(\pi x))^2\,\mathrm{d}x = \zeta(2\alpha) - 1.$$

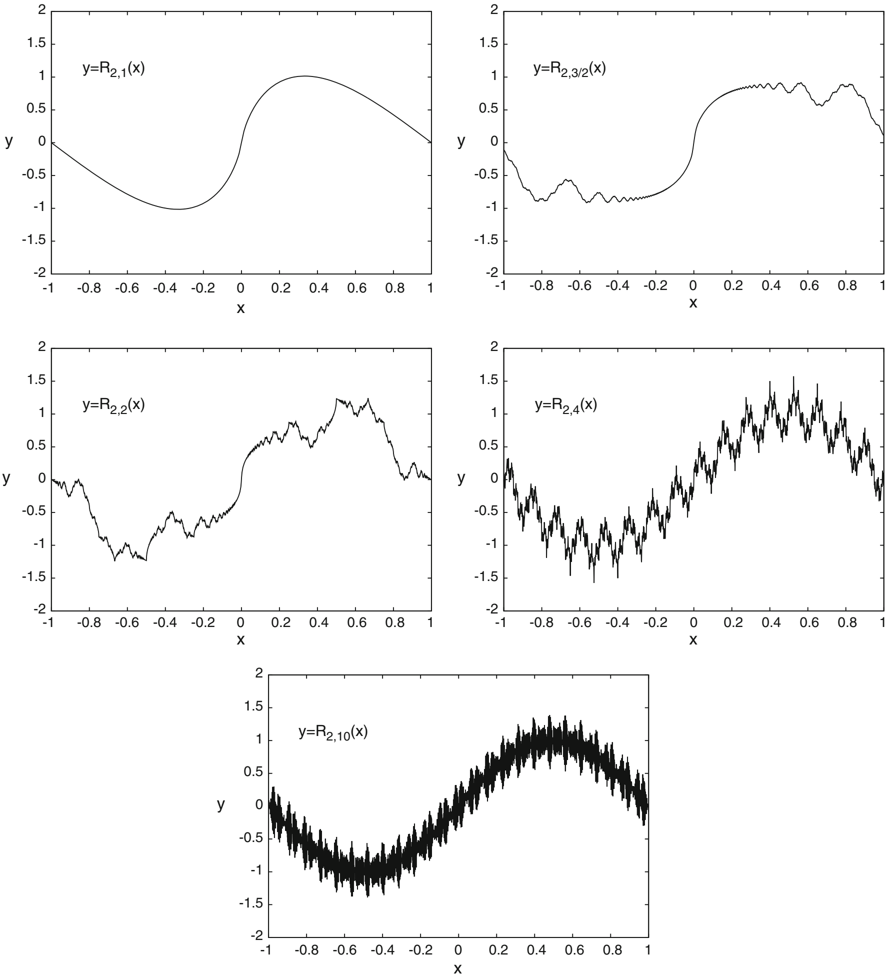

Figure 1. Graphical representation of $R_{2,1}$, $R_{2,3/2}$, $R_{2,2}$, $R_{2,4}$ and $R_{2,10}$

Proof. The periodicity of $R_{\alpha,\beta}$ is easy to check. Let us calculate the integral. By developing $x \mapsto R_{\alpha,\beta}(x) - \sin(\pi x)$ in Fourier series, we have

$$R_{\alpha,\beta}(x) - \sin(\pi x) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} \left(a_m \cos(\pi m x) + b_m \sin(\pi m x)\right)$$

in $L^2([-1,1])$ where $a_0 = a_m = 0$ and

$$\begin{aligned} b_m &= 2\int_0^1 (R_{\alpha,\beta}(x) - \sin(\pi x))\,\mathrm{d}x \\ &= \sum_{n=2}^{+\infty} \frac{1}{n^\alpha} \int_0^1 \left(\cos(x\pi(n^\beta - m)) - \cos(x\pi(n^\beta + m))\right)\,\mathrm{d}x \end{aligned}$$

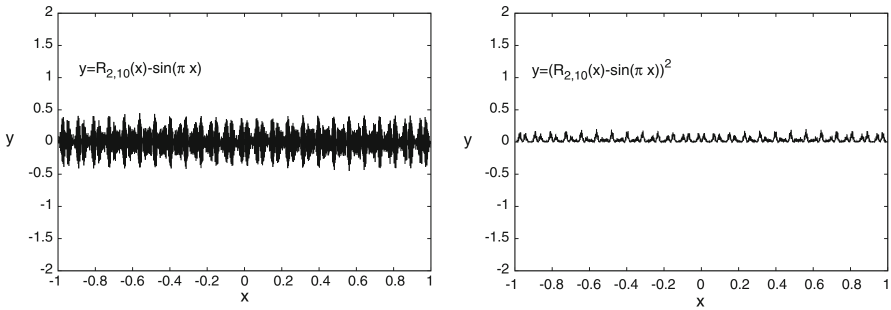

Figure 2. Mean value and amplitude of oscillations of $x \mapsto R_{2,10}(x) - \sin(\pi x)$

$$= \begin{cases} \dfrac{1}{m^{\alpha/\beta}} & \text{if } m = k^\beta \text{ for one } k \in \mathbb{N}\setminus\{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

for all $m \in \mathbb{N}\setminus\{0\}$. Consequently, by Parseval formula, we obtain

$$\int_{-1}^{1} \left(R_{\alpha,\beta}(x) - \sin(\pi x)\right)^2 \, \mathrm{d}x = \sum_{m=1}^{+\infty} b_m^2 = \sum_{k=2}^{+\infty} \frac{1}{k^{2\alpha}} = \zeta(2\alpha) - 1.$$

□

The two previous propositions are illustrated in Fig. 2. Let us end this section with a simple remark about the behaviour of $R_{\alpha,\beta}$ as α tends to infinity.

Remark 16. Proposition 14 is also "satisfied" for α, we have

$$\lim_{\alpha\to+\infty} m^{a,b}_{R_{\alpha,\beta}} = m^{a,b}_s$$

for all $\beta > 0$ and all $a, b \in \mathbb{R}$ such that $a < b$. Moreover, by Proposition 15, we have

$$\lim_{\alpha\to+\infty} \int_{-1}^{1} \left(R_{\alpha,\beta}(x) - \sin(\pi x)\right)^2 \, \mathrm{d}x = 0$$

for all $\beta \in \mathbb{N}\setminus\{0\}$. In fact, a stronger result holds: for any fixed $\beta > 0$, $R_{\alpha,\beta}$ uniformly converges on $\mathbb{R}$ to s as α tends to infinity because we have

$$|R_{\alpha,\beta}(x) - \sin(\pi x)| \leq \sum_{n=2}^{+\infty} \frac{1}{n^\alpha} = \zeta(\alpha) - 1$$

for all $x \in \mathbb{R}$.

5. Final Remarks

5.1. About Nonharmonic Fourier Series

A part of Theorem 4 can be adapted for particular nonharmonic Fourier series. Let us first recall the notion of nonharmonic Fourier series (see [14, 20, 29]).

Definition 17. Let $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}\setminus\{0\}}$ be a sequence of complex numbers and let $\boldsymbol{\lambda} = (\lambda_n)_{n\in\mathbb{N}\setminus\{0\}}$ be an increasing sequence of positive numbers which converges to infinity. A nonharmonic Fourier series (related to the sequences $\boldsymbol{a}$ and $\boldsymbol{\lambda}$) is a function S defined by

$$S(x) = \sum_{n=1}^{+\infty} a_n \, \mathrm{e}^{i\lambda_n x}, \quad x \in \mathbb{R},$$

if the series converges.

If the series $\sum_{n=1}^{+\infty} a_n$ is absolutely convergent, then the above series (related to S) uniformly converges on $\mathbb{R}$. We will assume that this is the case in the remainder of this discussion. Such a function S is then continuous and bounded on $\mathbb{R}$. As for $R_{\alpha,\beta}$, we can calculate the continuous wavelet transform of S (related to the Lusin wavelet).

Since $\lambda_n > 0$ for all $n \in \mathbb{N}\setminus\{0\}$, S is a holomorphic function on H and we have

$$\mathcal{W}_\psi S(a,b) = -2a \sum_{n=1}^{+\infty} a_n \lambda_n \, \mathrm{e}^{i\lambda_n(b+ia)}$$

for $a > 0$ and $b \in \mathbb{R}$, similarly to (4). If we assume that there exist positive constants C_1, C_2 and C_3, $\alpha > 1$ and $\beta > 0$ such that

$$|a_n| \leq \frac{C_1}{n^\alpha} \quad \text{and} \quad C_2 n^\beta \leq \lambda_n \leq C_3 n^\beta$$

for all $n \in \mathbb{N}\setminus\{0\}$, we then obtain

$$|\mathcal{W}_\psi S(a,b)| \leq 2aC_1C_3 \sum_{n=1}^{+\infty} \frac{\mathrm{e}^{-C_2 a n^\beta}}{n^{\alpha-\beta}}$$

for $a > 0$ and $b \in \mathbb{R}$ and we recover an expression similar to the one obtained for $|\mathcal{W}_\psi R_{\alpha,\beta}(a,b)|$ in (5). Using the same reasoning as in the study of the uniform Hölder continuity of $R_{\alpha,\beta}$ with $\alpha > 1$ and $\beta \geq \alpha - 1$, we can formulate the following result.

Corollary 18. *With the previous assumptions on* $\boldsymbol{a}$ *and* $\boldsymbol{\lambda}$, *we have* $S \in C^{\frac{\alpha-1}{\beta}}(\mathbb{R})$ *if* $\beta > \alpha - 1$ *and* $S \in C^{1-\delta}(\mathbb{R})$ *for all* $\delta \in (0,1)$ *if* $\beta = \alpha - 1$.

5.2. About the Lusin Wavelet

If $\alpha = \beta = 2$, we know that the largest Hölder exponent of $R = R_{2,2}$ at a point is $3/2$ and that it is attained at the rational numbers $(2p+1)/(2q+1)$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ (see [16]). The continuous wavelet transform related to the Lusin wavelet of R does not allow to find this exponent.

Indeed, for $a > 0$, we have

$$\begin{aligned}\mathcal{W}_\psi R(a,1) = ia\pi \sum_{n=1}^{+\infty} \mathrm{e}^{i\pi n^2(1+ia)} &= ia\pi \sum_{n=1}^{+\infty} (-1)^n \mathrm{e}^{-a\pi n^2} \\ &= \frac{ia\pi}{2}\left(\sum_{n\in\mathbb{Z}} \mathrm{e}^{i\pi n}\, \mathrm{e}^{-a\pi n^2} - 1\right)\end{aligned}$$

and, by the Poisson summation formula,

$$|\mathcal{W}_\psi R(a,1)| = \frac{a\pi}{2}\left|\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{a}}\mathrm{e}^{-\frac{(\pi+n)^2}{4a\pi}} - 1\right|$$
$$= \frac{a\pi}{2}\left|\frac{\mathrm{e}^{-\frac{\pi}{4a}}}{\sqrt{a}}\left(1+2\sum_{n=1}^{+\infty}\mathrm{e}^{-\frac{n^2}{4a\pi}}\cosh\left(\frac{n}{2a}\right)\right)-1\right|.$$

Let $C>0$ and $\eta>0$. We have

$$\lim_{a\to0^+}\frac{\mathrm{e}^{-\frac{\pi}{4a}}}{\sqrt{a}}\left(1+2\sum_{n=1}^{+\infty}\mathrm{e}^{-\frac{n^2}{4a\pi}}\cosh\left(\frac{n}{2a}\right)\right)=0$$

since we have

$$2\sum_{n=8}^{+\infty}\mathrm{e}^{-\frac{n^2}{4a\pi}}\cosh\left(\frac{n}{2a}\right)\le\int_7^{+\infty}\mathrm{e}^{-\frac{x^2}{4a\pi}}\left(1+\mathrm{e}^{\frac{x}{2a}}\right)\mathrm{d}x\le\pi\sqrt{a}+\int_7^{+\infty}\mathrm{e}^{-\frac{1}{2}\left(\frac{x^2}{2\pi}-x\right)}\mathrm{d}x$$

for all $a\in(0,1)$. The sum begins with the term related to $n=8$ for two reasons. On the one hand, the function $g:x\mapsto\mathrm{e}^{-\frac{x^2}{4a\pi}}\cosh\left(\frac{x}{2a}\right)$ is differentiable on $\mathbb{R}$ and

$$Dg(x)=\frac{\mathrm{e}^{-\frac{x^2}{4a\pi}}}{2a}\left(-\frac{x}{\pi}\cosh\left(\frac{x}{2a}\right)+\sinh\left(\frac{x}{2a}\right)\right)\le0\quad\Leftrightarrow\quad x\ge\pi\tanh\left(\frac{x}{2a}\right),$$

which implies that g is decreasing on $[4,+\infty)$. On the other hand, the function $x\mapsto\frac{x^2}{2\pi}-x$ is positive on $[7,+\infty)$. Consequently, there exists $A\in(0,1)$ such that, for all $a\in(0,A)$, we have

$$\frac{\pi}{2C}\left|\sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{a}}\mathrm{e}^{-\frac{(\pi+n)^2}{4a\pi}}-1\right|>a^\eta$$

and then

$$|\mathcal{W}_\psi R(a,1)|>Ca^{1+\eta}. \tag{8}$$

In fact, the Lusin wavelet has only one vanishing moment since $\hat{\psi}(0)=0$ and $(D\hat{\psi})(0)\neq0$, because the function $x\mapsto x\psi(x)$ is not integrable on $\mathbb{R}$. Inequality (8) thus shows that the second vanishing moment is essential for the study of the Hölder continuity of R when the exponent is (strictly) greater than 1. We could otherwise find $D>0$ and $\delta>0$ such that

$$|\mathcal{W}_\psi R(a,b)|\le D\,a^{3/2}\left(1+\left(\frac{|b-1|}{a}\right)^{3/2}\right)$$

for all $a\in(0,\delta)$ and $b\in(1-\delta,1+\delta)$ and then $|\mathcal{W}_\psi R(a,1)|\le D\,a^{3/2}$ for all $a\in(0,\delta)$, which is in contradiction with (8) by taking $C=D$, $\eta=1/2$ and $a\in(0,\min\{\delta,A\})$.

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