An analogue of Cobham's theorem for graph directed iterated function systems

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Abstract

Feng and Wang showed that two homogeneous iterated function systems in $\mathbb{R}$ with multiplicatively independent contraction ratios necessarily have different attractors. In this paper, we extend this result to graph directed iterated function systems in $\mathbb{R}^n$ with contraction ratios that are of the form $\frac{1}{\beta}$, for integers β. By using a result of Boigelot *et al.*, this allows us to give a proof of a conjecture of Adamczewski and Bell. In doing so, we link the graph directed iterated function systems to Büchi automata. In particular, this link extends to real numbers β. We introduce a logical formalism that permits to characterize sets of $\mathbb{R}^n$ whose representations in base β are recognized by some Büchi automata. This result depends on the algebraic properties of the base: β being a Pisot or a Parry number. The main motivation of this work is to draw a general picture representing the different frameworks where an analogue of Cobham's theorem is known.

Keywords: Iterated function systems, Graph-directed iterated function systems, Self-similar set, Cobham's theorem, β-numeration, Büchi automaton

1. Introduction

According to Hutchinson [1], any iterated function system (IFS) $\Phi = \{\phi_i\}_{i=1}^t$ admits a unique compact set K (called the *attractor*) such that $K = \phi_1(K) \cup \cdots \cup \phi_t(K)$. Such a set is said to be *self-similar* or *self-affine* when the contraction mappings are affine functions. For instance, it is well known that the usual Triadic Cantor set is self-affine for $\phi_1(x) = x/3$ and $\phi_2(x) = x/3 + 2/3$.

IFSs play an important role in fractal geometry, notably thanks to the so-called Collage theorem stating that any subset of a complete metric space can be approximated (with arbitrary precision) by a self-similar set [2].

Given an IFS, a natural question is whether its attractor can be obtained as the attractor of another IFS. This question has recently been completely answered by Feng and Wang [3] in the case of homogeneous self-affine sets in $\mathbb{R}$ (all contraction mappings have the same contraction ratio) that satisfy the *open set condition* (OSC).

Theorem (Feng and Wang [3])**.** Let X be the attractor of a homogeneous IFS $\Phi = \{\phi_i\}_{i=1}^N$ in the Euclidean space $\mathbb{R}$ with contraction ratio r_Φ and satisfying the OSC. Consider $\psi(x) = \lambda x + d$.

1. Suppose that the Hausdorff dimension of X, $\dim_H(X)$, is less than 1 and that $\lambda \neq 0$. If $\psi(X) \subset X$, then $\frac{\log |\lambda|}{\log |r_\Phi|} \in \mathbb{Q}$;
2. Suppose $\dim_H(X) = 1$ and X is not a finite union of intervals and $\lambda > 0$. If $\min(X) \in \psi(X) \subset X$, then $\frac{\log \lambda}{\log |r_\Phi|} \in \mathbb{Q}$.

Under a more restrictive separation condition (the so-called *strong separation condition*), Elekes, Keleti and Máthé [4] proved an extension of this result for self-similar sets in higher dimension and without the homogeneous hypothesis. In their setting, if $r_1, \ldots, r_t$ denote the different contraction ratios of the IFS and if ψ is a similarity with contraction ratio r_ψ such that $\psi(X) \subset X$, then $\log |r_\psi|$ is a linear combination of $\log |r_1|$, ..., $\log |r_t|$ with rational coefficients.

A way to understand these results is the following: If a "non-simple" set $X \subset \mathbb{R}$ (meaning X is not a finite union of intervals) is the attractor of two different IFSs, then the contraction ratios of these two IFSs have to be strongly linked.

Surprisingly enough, a rather similar result has recently been proved by Adamczewski and Bell [5]. If X is a compact set in $[0, 1]$ and if $b \geq 2$ is an integer, the *b-kernel* of X is the collection of sets $(b^k X - a) \cap [0, 1]$ for $a, k \in \mathbb{N}$, $0 \leq a < b^k$. Such a set X is said to be *b-self-similar* if its *b*-kernel is finite.

Theorem (Adamczewski and Bell [5]). Let $b, b' \geq 2$ be two integers such that $\frac{\log b}{\log b'} \notin \mathbb{Q}$. A compact set $X \subset [0, 1]$ is simultaneously *b*- and *b'*-self-similar if and only if it is a finite union of intervals with rational endpoints.

They conjectured an equivalent result in higher dimension.

Conjecture (Adamczewski and Bell [5]). Let $b, b' \geq 2$ be two integers such that $\frac{\log b}{\log b'} \notin \mathbb{Q}$. A compact set $X \subset [0, 1]^n$ is simultaneously *b*- and *b'*-self-similar if and only if it is a finite union of polyhedra whose vertices have rational coordinates.

Even more surprisingly, a third similar result having an important meaning in the framework of IFSs has recently been proved, but in a theoretical computer science setting [6, 7, 8]: It concerns sets of points in $\mathbb{R}^n$ whose representations in some integer base b are accepted by some particular Büchi automaton. In this paper we provide a bridge between these three results. This can be achieved by use of graph directed iterated functions systems (GDIFS) and this allows us to provide extensions of the results in the three frameworks, for instance by proving Adamcweski and Bell's conjecture (which has also independently been proved by Chan and Hare [9]) and by extending the IFS results to a large class of GDIFS. Furthermore, we also extend a logical characterization of recognizable sets in some non integer bases, thus providing a fourth framework to study them.

Let us give a few more details about the result of Boigelot *et al.*. A *Büchi automaton* is a finite state automaton with an accepting procedure adapted to infinite words [10]. Given an integer base $b \geq 2$, one can therefore study subsets X of $\mathbb{R}^n$ such that the base b representations of the elements of X are accepted by some Büchi automaton. Such a set is said to be *b-recognizable*. These sets have been well studied and in particular, they can be characterized by some first order formula in the structure $\langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle$ extended with a special predicate related to the chosen base [11]. This logical characterization has for instance found applications in verification of systems with unbounded mixed variables taking integer or real values. In that setting, timed automata or hybrid systems are considered [12, 13].

Weak Büchi automata are a subclass of Büchi automata that mainly behaves like automata accepting finite words [14]. Sets recognized by weak automata are said to be *weakly recognizable*.

Theorem (Boigelot, Brusten, Bruyère, Jodogne, Leroux and Wolper [15, 6, 7]). Let $b, b' \geq 2$ be two integers such that $\frac{\log b}{\log b'} \notin \mathbb{Q}$. A set $X \subset \mathbb{R}^n$ is simultaneously weakly *b*- and *b'*-recognizable if and only if it is definable by a first order formula in the structure $\langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle$.

Observe that subsets of $\mathbb{R}^n$ that are definable by a first order formula in the structure $\langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle$ are exactly the periodic repetitions of finite unions of polyhedra whose vertices have rational coordinates.

As already announced, in this paper, we obtain a better and complete understanding of the links and interactions existing between those results. The different notions under consideration are not exactly equivalent but very closely related. By means of GDIFSs, we are able to determine exactly what are the similarities and the differences between them.

GDIFSs generalize IFSs in the sense that the similarities can be applied accordingly to a labeled directed graph (V, E) [16]. As for IFSs, such a system admits a unique *attractor* which is a list of non-empty compact sets $(K_v)_{v \in V}$ such that for all $u \in V$,

$$K_u = \bigcup_{e: u \to v} S_e(K_v).$$

Using these GDIFS, we obtain the following picture of inclusions, in the extended framework of some real base $\beta > 1$.

β-self-similar

$\Updownarrow$ Theorems 62 and 64

particular GDIFS

$\Updownarrow$ Theorem 57

particular weakly
β-recognizable

IFS

GDIFS

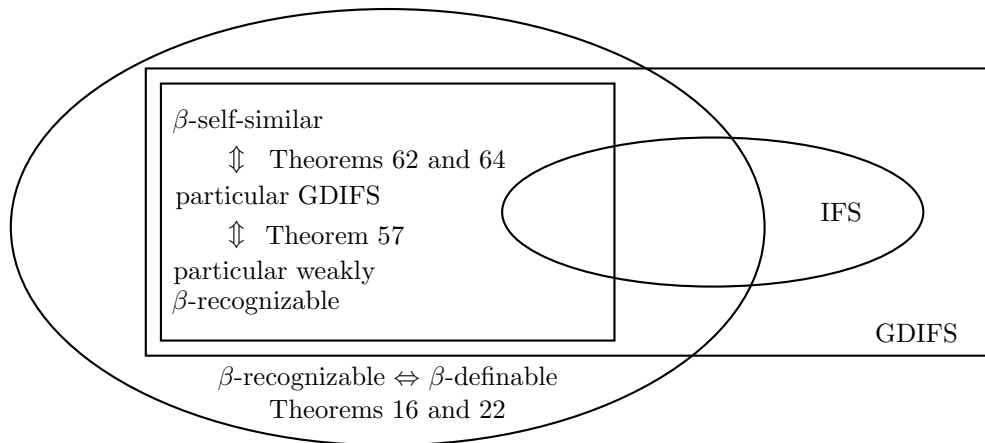β-recognizable $\Leftrightarrow$ β-definable
Theorems 16 and 22

Figure 1: General picture of our contributions.

It is interesting to determine the strengths and weaknesses of the different approaches. On the one hand, the IFS method is very powerful from the point of view of contraction ratios: these can be any number smaller than 1 although we are restricted to reciprocal of positive integers in the automata framework. Furthermore we can simultaneously work with different contraction ratios [4]. From an automata point of view, this would correspond to represent numbers by simultaneously using several bases. On the other hand, the homogeneous case of IFS corresponds to recognizable sets whose underlying automaton has a unique state. In this direction, Büchi automata provide a more general result.

Observe that both the result of Adamwzewski and Bell and the one of Boigelot *et al.* belong to a large class of results that are generalisations of a famous theorem in discrete mathematics and theoretical computer science: Cobham's theorem. As in the real case described above, one can consider sets of integers whose base b expansions are accepted by some finite automaton. These sets are also said to be *b-recognizable*. In 1969, Cobham obtained a fundamental result initiating the systematic study of b-recognizable sets and showing that the b-recognizability depends on the chosen base [17]. Let $b, b' \geq 2$ be two integers such that $\frac{\log b}{\log b'} \notin \mathbb{Q}$. If a subset X of $\mathbb{N}$ is simultaneously b-recognizable and b'-recognizable, then it is a finite union of arithmetic progressions.

Cobham's theorem led to a deep study, for instance by considering non-standard numeration systems: It is meaningful to consider other numeration systems to handle new sets of integers recognizable by finite automata. Making use of numeration systems like the Zeckendorf system based on the Fibonacci sequence, other kinds of sets of integers can be recognized by means of some finite automaton. The bibliography in [18] provides many pointers to various extensions of Cobham's theorem.

In the setting of recognizable subsets of $\mathbb{R}^n$, as for the representation of integers where one has the opportunity to consider non-standard numeration systems, we are not limited to base b

expansions. We can consider other ways to represent real numbers. A large class of extensively studied numeration systems is made of the so-called β-numeration systems [19, Chap. 7] and [20], where the idea is to replace the integer base with a real base $\beta > 1$. In such a system, a real number usually can have more than one representation as an infinite sequence and this redundancy can be exploited in information theory with the use of beta encoders for analogues to digital conversion, see for instance [21, 22]. The results mentioned in Figure 1 depend on the algebraic properties of β and Pisot numbers play a special role.

The paper is organized as follows. In Section 2 we recall the notions of β-expansions and of Büchi automata. In Section 3 and Section 4 we prove the equivalence between β-recognizable sets and β-definable sets (Theorems 16 and 22). In Section 5, we prove the equivalence between β-self-similar sets, attractors of particular GDIFS and particular weakly β-recognizable sets (Theorems 57, 62 and 64).

2. Büchi automata and β-expansions

In this section, we describe the conventions that we will use to represent real numbers or n-tuples of reals numbers using a real base β. We recap some well-known results on Büchi automata.

Let A be a finite alphabet. We denote the set of nonempty finite words over A by A^+ and the set of infinite words by A^ω.

2.1. β-expansions

Let $\beta > 1$ be a real number and let $C \subset \mathbb{Z}$ be an alphabet. For a real number x, any infinite word $u = u_k \cdots u_1 u_0 \star u_{-1} u_{-2} \cdots$ over $C \cup \{\star\}$ such that

$$\mathrm{val}_\beta(u) := \sum_{-\infty < i \leq k} u_i \beta^i = x$$

is a β-representation of x. In general, the representation is not unique. For $x \geq 0$, among all such β-representations of x, we distinguish the β-expansion $d_\beta(x) = x_k \cdots x_1 x_0 \star x_{-1} x_{-2} \cdots$ which is an infinite word over the canonical alphabet $A_\beta = \{0, \ldots, \lceil \beta \rceil - 1\}$ containing one symbol $\star$ and obtained by a greedy algorithm, i.e., we fix the minimal $k \geq 0$ such that

$$x = \sum_{-\infty < i \leq k} x_i \beta^i \text{ and, for all } i \leq k, \ x_i \in A_\beta$$

and, for all $\ell \leq k$,

$$\sum_{-\infty < i \leq \ell} x_i \beta^i < \beta^{\ell+1}.$$

The choice of k implies that $x_k \neq 0$ if and only if $x \geq 1$ and that reals in $[0, 1)$ have a β-expansion of the form $0 \star u$ with $u \in A_\beta^\omega$. In particular $d_\beta(0) = 0 \star 0^\omega$.

For an example, take $\varphi = (1 + \sqrt{5})/2$ be the golden mean. Let x be the real number defined as

$$x = \sum_{i \geq 1} \varphi^{-2i}.$$

The words $u = 0 \star 001111 \cdots$ and $v = 0 \star 0101010 \cdots$ are φ-representations of x. But one can check that v is the φ-expansion of x.

To deal with negative real numbers and negative digits, we make use of the notation $\overline{a}$ to denote the integer $-a$ for all $a \in \mathbb{Z}$. This notation is extended to a morphism acting on finite or infinite words over $\mathbb{Z}$: $\overline{uv} = \overline{u}\,\overline{v}$, $\overline{u \star v} = \overline{u} \star \overline{v}$ and $\overline{\overline{u}} = u$. Let $\bar{A}_\beta = \{\bar{0}, \bar{1}, \ldots, \overline{\lceil \beta \rceil - 1}\}$. For $x < 0$, the β-expansion of x is defined as the word $d_\beta(x) = u \star v$ over $\bar{A}_\beta \cup \{\star\}$ such that $\overline{d_\beta(-x)} = u \star v$. With the identification $\bar{0} = 0$, we let $\tilde{A}_\beta$ denote the alphabet $A_\beta \cup \bar{A}_\beta$.

Definition 1. A real number x is a β-integer, if $d_\beta(x)$ is of the kind $u \star 0^\omega$. The set of β-integers is denoted by $\mathbb{Z}_\beta$.

Let $\beta > 1$. A particular β-representation of 1 plays a special role. We let $d_\beta^*(1)$ denote the greatest word $w \in A_\beta^\omega$ (in the lexicographic order) not ending in 0^ω such that $0 \star w$ is a β-representation of 1. The next result characterizes the β-expansions of real numbers in $[0,1)$ in terms of their shifted words. Recall that the *shift map* is the function $\sigma : A^\omega \to A^\omega$ defined by $\sigma(u_1 u_2 u_3 \cdots) = u_2 u_3 u_4 \cdots$.

Theorem 2 (Parry [20]). *Let $\beta > 1$ be a real number. An infinite word u is such that $0 \star u$ is the β-expansion of a real number in $[0,1)$ if and only if for all $k \geq 0$, $\sigma^k(u) < d_\beta^*(1)$.*

Example 3. For all $\beta > 1$ one has $d_\beta(1) = 1 \star 0^\omega$. The definition of $d_\beta^*(1)$ indeed depends on β. Let $\varphi = (1 + \sqrt{5})/2$ be the golden mean. We have $d_\varphi^*(1) = (10)^\omega$. Theorem 2 shows that the φ-expansions of real numbers in $[0,1)$ are of the form $0 \star u$, where u is an infinite word over $\{0,1\}$ not containing 11 as a factor.

2.2. Büchi automata and ω-regular languages

Definition 4. Given an alphabet A, a *Büchi automaton* over A is a labeled directed graph given by a 5-tuple $\mathcal{A} = (Q, A, E, I, T)$ where Q is the finite set of states, $E \subseteq Q \times A \times Q$ is the set of transitions, I is the set of initial states and T is the set of terminal states. An infinite word $u \in A^\omega$ is said to be *accepted* by $\mathcal{A}$ if there exists an infinite path in $\mathcal{A}$ whose label is u starting in an initial state and visiting infinitely often the set T. The language $L \subseteq A^\omega$ accepted by $\mathcal{A}$ is the set of all infinite words over A that are accepted by $\mathcal{A}$; such a language is said to be ω-*regular*. A Büchi automaton is said to be *deterministic* if I is a singleton and for all states $q \in Q$ and all letters $a \in A$, there is at most one transition in $\{q\} \times \{a\} \times Q$. As usual when representing an automaton, initial states are depicted with an incoming arrow and terminal states with an outgoing arrow.

If $u = u_0 \cdots u_{\ell-1}, v = v_0 \cdots v_{\ell-1}$ are two finite words of the same length ℓ (resp. $u = u_0 u_1 \cdots, v = v_0 v_1 \cdots$ are two infinite words), then one can define the *direct product* $u \times v$ of u and v where the ith symbol is $(u \times v)_i = (u_i, v_i)$ for all $i < \ell$ (resp. for all $i \geq 0$). Let $L \subseteq A^\omega$ and $M \subseteq B^\omega$ be two languages of infinite words, we define the *direct product* of L and M as $L \times M = \{u \times v \mid u \in L, v \in M\}$. The morphisms of projections are denoted by π_1 and π_2, satisfy $\pi_1(u \times v) = u$, $\pi_2(u \times v) = v$ and are naturally extended to the product of languages. These notions of product and projection can obviously be extended to the direct product of n words.

Facts 5 ([10]). *We collect some well-known result on ω-regular languages.*

1. *The class of ω-regular languages is closed under complementation, finite union, finite intersection, morphic image and inverse image under a morphism.*
2. *If L and M are ω-regular languages over the alphabets A and B respectively, then $L \times M$ is an ω-regular language over $A \times B$.*
3. *If L is an ω-regular language over an alphabet $A \times B$, then the projections $\pi_1(L)$ and $\pi_2(L)$ are ω-regular languages.*
4. *The class of languages accepted by deterministic Büchi automaton is strictly included in the class of ω-regular languages.*

Example 6. Let us illustrate Fact 5 (4). The non-deterministic Büchi automaton in Figure 2 accepts the language over $\{a,b\}$ of the words containing finitely many a's. However it can easily be shown that no deterministic Büchi automaton accepts the same language.
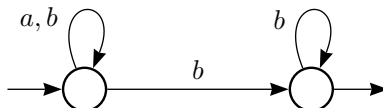

Figure 2: A non-deterministic Büchi automaton

There is no obstacle to consider a multi-dimensional framework. Let $n \geq 1$ be an integer. Let us define the following three alphabets of n-tuples:

$$\mathbf{A}_\beta = \underbrace{A_\beta \times \cdots \times A_\beta}_{n \text{ times}}, \quad \bar{\mathbf{A}}_\beta = \underbrace{\bar{A}_\beta \times \cdots \times \bar{A}_\beta}_{n \text{ times}} \quad \text{and} \quad \tilde{\mathbf{A}}_\beta = \underbrace{\tilde{A}_\beta \times \cdots \times \tilde{A}_\beta}_{n \text{ times}}.$$

We also set

$$\star = \underbrace{(\star, \ldots, \star)}_{n \text{ times}} \quad \text{and} \quad \mathbf{0} = \underbrace{(0, \ldots, 0)}_{n \text{ times}}.$$

Definition 7. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a point in $\mathbb{R}^n$. We define the β-expansion of $\mathbf{x}$ as being the word $d_\beta(\mathbf{x})$ over the alphabet $\tilde{\mathbf{A}}_\beta \cup \{\star\}$ that belongs to $0^* d_\beta(x_1) \times 0^* d_\beta(x_2) \times \cdots \times 0^* d_\beta(x_n)$ and that does not start with $\mathbf{0}$ except if $|x_i| < 1$ for all i, in which case we consider the word starting with $\mathbf{0}\star$. Otherwise stated, the n β-expansions are synchronized by possibly using some leading zeroes in such a way that all the $\star$ symbols occur at the same position in every β-expansions.

Example 8. Take $\varphi = (1 + \sqrt{5})/2$. Consider $\mathbf{x} = (x_1, x_2) = ((1 + \sqrt{5})/4, 2 + \sqrt{5})$. We have

$$d_\varphi(\mathbf{x}) = \begin{array}{cccccccccccc} 0 & 0 & 0 & \star & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & \star & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \end{array}$$

Where the first φ-expansion is padded with some leading zeroes. Consider an example where all the components have moduli less than one. With $\mathbf{y} = (x_1, x_2) = ((1 + \sqrt{5})/4, -1/2)$, we get

$$d_\varphi(\mathbf{y}) = \begin{array}{ccccccccc} 0 & \star & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & \star & 0 & \bar{1} & 0 & 0 & \bar{1} & 0 & \cdots \end{array}$$

where the two φ-expansions start with one symbol 0 followed by $\star$.

Definition 9. A set $X \subseteq \mathbb{R}^n$ is β-*recognizable* if there is some Büchi automaton over the alphabet $\tilde{\mathbf{A}}_\beta \cup \{\star\}$ accepting the language $d_\beta(X)$. We let $\mathrm{Rec}_\beta(\mathbb{R}^n)$ denote the set of β-recognizable sets of $\mathbb{R}^n$.

In this paper, we will sometimes obtain automata accepting words of the form $0^k w$ with $w \in d_\beta(X)$. The following result shows that it does change the notion of β-recognizability.

Fact 10. *Let $X \subseteq \mathbb{R}^n$. The following three assertions are equivalent:*

1. *X is β-recognizable;*
2. *there exists some Büchi automaton over the alphabet $\tilde{\mathbf{A}}_\beta \cup \{\star\}$ accepting the language $\mathbf{0}^* d_\beta(X)$;*
3. *there exists a Büchi automaton over the alphabet $\tilde{\mathbf{A}}_\beta \cup \{\star\}$ accepting a language of the form $\{\mathbf{0}^{m(\mathbf{x})} d_\beta(\mathbf{x}) \mid \mathbf{x} \in X\}$ for some map $m : \mathbf{x} \to \mathbb{N}$.*

A *Parry number* is a real number β for which $d_\beta^*(1)$ is ultimately periodic. The next result is a direct consequence of Theorem 2. The following fact provides a justification to restrict ourselves to Parry numbers. It is indeed desirable that the set of all β-expansions is accepted by some (deterministic) Büchi automaton.

Fact 11. *[23, 20] If $\beta > 1$ is a Parry number, then $d_\beta([0,1)^n)$ is accepted by a deterministic Büchi automaton.*

Example 12. The following Büchi automaton in Figure 3 accepts $d_\varphi([0,1))$ where $\varphi = (1+\sqrt{5})/2$.

The next definition introduces a predicate describing that some fixed digit occurs in a specific position of the β-expansion of a given real number (assuming that β-expansions are extended to the left with infinitely many zeroes).
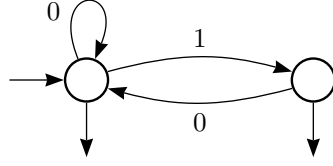
Figure 3: A deterministic Büchi automaton accepting $d_\varphi([0,1))$.

Definition 13. Let x be a real number and let $d_\beta(x) = x_k \cdots x_0 \star x_{-1} \cdots$ be its β-expansion. For each $a \in \tilde{A}_\beta$, the binary predicate $X_{\beta,a}(x,y)$ over $\mathbb{R}^2$ holds true whenever y is an integral power of β and either

- $|x| < y$ and $a = 0$, or

- $|x| \geq y$ and in the β-expansion of x, the coefficient corresponding to y is a, i.e., $y = \beta^i$ for some $i \leq k$ and $x_i = a$.

Let X_β denote the finite collection of predicates $\{X_{\beta,a} \mid a \in \tilde{A}_\beta\}$. A set $X \subseteq \mathbb{R}^n$ is β-*definable* if it can be defined by a first order formula in the structure

$$\langle \mathbb{R}, 1, \leq, +, X_\beta \rangle.$$

We let $\mathrm{Def}_\beta(\mathbb{R}^n)$ denote the set of β-definable sets of $\mathbb{R}^n$.

Remark 14. The property of being a power of β is definable in the structure $\langle \mathbb{R}, 1, \leq, X_\beta \rangle$ by the formula

$$x \text{ is a power of } \beta \Leftrightarrow (\exists y)\,(X_{\beta,1}(x,y) \wedge x = y).$$

We can also define the properties of being a positive or negative power of β by adding $x > 1$ or $x < 1$ respectively. Let b be a power of β. One can easily define the next (or the previous) power of β as follows:

$$b' = \beta b \Leftrightarrow (b' \text{ is a power of } \beta) \wedge (b' > b) \wedge (\forall c)((c \text{ is a power of } \beta \wedge c > b) \implies c \geq b').$$

Consequently, any constant (positive or negative) power of β is definable in the structure.

Remark 15. The two structures $\langle \mathbb{R}, 1, \leq, +, X_\beta \rangle$ and $\langle \mathbb{R}, \mathbb{Z}_\beta, \leq, +, X_\beta \rangle$ are equivalent. Indeed, the set $\mathbb{Z}_\beta$ can be defined in $\langle \mathbb{R}, 1, \leq, X_\beta \rangle$ by the formula

$$z \in \mathbb{Z}_\beta \Leftrightarrow (\forall y)\big[(y \text{ is a negative power of } \beta) \implies X_{\beta,0}(z,y)\big].$$

Conversely, 1 can be defined in $\langle \mathbb{R}, \mathbb{Z}_\beta, \leq, + \rangle$ by the formula

$$z = 1 \Leftrightarrow (z \in \mathbb{Z}_\beta) \wedge \big[(\forall x)\big(((x \in \mathbb{Z}_\beta) \wedge (x > 0)) \implies (x \geq z)\big)\big]$$

where 0 is defined by $(\forall x)(x + 0 = x)$.

3. If the base is a Parry number, recognizability implies definability

The main result of this section is the following one. Compared with the next section, note that the only assumption is that β is a Parry number.

Theorem 16. *Let $\beta > 1$ be a Parry number. If $X \subseteq \mathbb{R}^n$ is β-recognizable, then X is β-definable.*

The aim of the following technical lemma is to ensure the construction of some valid β-expansions.

Lemma 17. *Let $\beta > 1$ be a Parry number. There exists an integer k such that for any infinite word $u \in \{0,1\}^\omega$ of the form $0^{n_1} 1 0^{n_2} 1 0^{n_3} \cdots$ with $n_i \geq k$ for all $i \geq 1$, there exists some real x such that $d_\beta(x) = 0 \star u$.*

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Proof. Since β is a Parry number, there exist two finite words u and v such that $d_\beta^*(1) = uv^\omega$ where v is nonempty and contains at least one non-zero digit. Taking $k = |u| + |v|$, the result follows from Theorem 2. $\square$

In the proof of Theorem 16, we will make use of the following construction. We can replace one automaton by M copies of this automaton in such a way that any path cyclically visits these copies. The reason of this construction is to use a number of copies corresponding to the constant k obtained in the previous lemma. Let $M \geq 1$ be an integer. If $\mathcal{A} = (Q, A, E, I, T)$ is a Büchi automaton, we define the automaton $\mathcal{A}^{(M)} = (Q', A, E', I', T')$ as follows. Its set of states is made of M distinct copies of the states of $\mathcal{A}$: If $Q = \{q_1, \ldots, q_t\}$ then $Q' = \cup_{n=1}^{M}\{q_{1,n}, \ldots, q_{t,n}\}$. If (q_i, a, q_j) belongs to E, then $(q_{i,n}, a, q_{j,n+1}) \in E'$ for all $n \in \{1, \ldots, M-1\}$ and $(q_{i,M}, a, q_{j,1}) \in E'$. The set of initial states I' is made of the sates $q_{i,1}$ such that $q_i \in I$. A state $q_{i,n}$, $1 \leq n \leq M$, is terminal whenever $q_i \in T$.

Example 18. Consider the automaton $\mathcal{A}$ depicted in Figure 3. We consider the automaton $\mathcal{A}^{(3)}$ depicted in Figure 4. All states are terminal but for the sake of readability, outgoing arrows have been omitted. The three copies have been drawn consecutively from left to right.
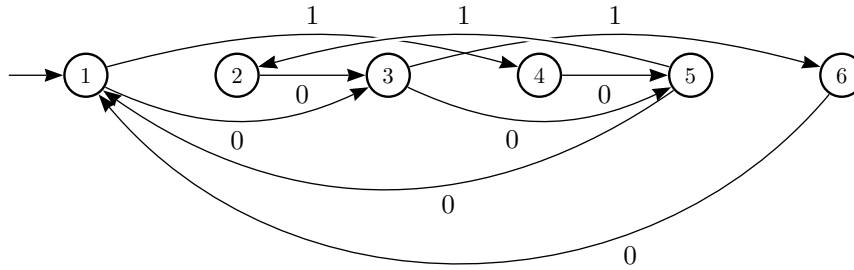

Figure 4: An automaton of the kind $\mathcal{A}^{(3)}$.

Lemma 19. *Let $\mathcal{A}$ be a Büchi automaton. The automaton $\mathcal{A}^{(M)}$ accepts the same ω-regular language.*

Proof. Clear from the definition of $\mathcal{A}^{(M)}$. $\square$

For the proof of Theorem 16, we follow essentially the same lines as [11]. We code runs (i.e., infinite sequences of states) in a Büchi automaton $\mathcal{A}$ using a finite number of infinite words over $\mathbf{A}_\beta \cup \{\star\}$. These infinite words in turn can be viewed as β-expansions of some real vectors. Hence, this coding permits to replace runs with real vectors. Successful computations in $\mathcal{A}$ can therefore be translated into a formula in $\langle \mathbb{R}, 1, \leq, +, X_\beta \rangle$. The main technical difference when dealing with β-expansions instead of classical base b expansions is that we have to ensure that the infinite words that are built are valid β-expansions (cf. Lemma 17).

We will make often use of the following fact.

Remark 20. Multiplication (or division) by β and thus by a constant power of β is definable:

$$y = \beta x \Leftrightarrow (\forall b)\Big[\bigwedge_{a \in \tilde{A}_\beta} (X_{\beta,a}(x,b) \implies X_{\beta,a}(y, \beta b))\Big].$$

This formula expresses that we get the β-expansion of y simply by shifting the one of x.

Proof of Theorem 16. To keep notations as simple as possible, without loss of generality, we may assume that $n = 1$. By assumptions, there exists a Büchi automaton $\mathcal{A}$ with t states accepting the language $d_\beta(X)$. Moreover, we can also assume that it has a single initial state by possibly adding a new initial state with convenient transitions.

Let k be given by Lemma 17. We replace $\mathcal{A}$ with the automaton $\mathcal{A}^{(k+1)} = (Q, \tilde{A}_\beta \cup \{\star\}, E, I, T)$ having $s := (k+1)t$ states. We can enumerate its states as $Q = \{q_1, \ldots, q_s\}$ and assume that q_1

is the initial state of $\mathcal{A}^{(k+1)}$. Each state q_j will be coded by a unit column vector in $\{0,1\}^s$ where each component is equal to zero except for the jth component. This vector is denoted by $c(q_j)$.

To any infinite run $r = (q_{i_j})_{j \geq 0} \in Q^\omega$ in $\mathcal{A}^{(k+1)}$, it corresponds an infinite sequence of vectors $\{0,1\}^s$ coding the sequence of visited states. If we concatenate these column vectors, for each row, we get s infinite words $w_1, \ldots, w_s$ over $\{0,1\}$ with the property that each symbol 1 is followed by at least k zeroes. Indeed, the shortest cycles in $\mathcal{A}^{(k+1)}$ have length at least $k+1$ and therefore, the same unit vector cannot be encountered more than once every $k+1$ times. As an example, consider the automaton $\mathcal{A}^{(3)}$ depicted in Figure 4. One of the shortest loops is given by $1 \xrightarrow{0} 3 \xrightarrow{0} 5 \xrightarrow{0} 1$ and it is derived from the loop with label 0 in the original automaton depicted in Figure 3. The periodic run $(1,3,5)^\omega$ corresponds to the 6 infinite words

$$
\begin{array}{rcccccccc}
w_1 & = & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\
w_2 & = & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
w_3 & = & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\
w_4 & = & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
w_5 & = & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\
w_6 & = & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}
$$

Thanks to Lemma 17, the s infinite words $w_1, \ldots, w_s$ over $\{0,1\}$ are such that $0 \star w_1, \ldots, 0 \star w_s$ are valid β-expansions. We define the map

$$
f : Q^\omega \to \mathbb{R}^s, \; r = (q_{i_j})_{j \geq 0} \mapsto (\mathrm{val}_\beta(0 \star w_1), \ldots, \mathrm{val}_\beta(0 \star w_s)).
$$

In our running example, $f(r) = ((1 + \sqrt{5})/4, 0, 1/2, 0, 1/(1 + \sqrt{5}), 0)$.

We now define a $(s+1)$-ary predicate $R_\mathcal{A}(x, y_1, \ldots, y_s)$ over $\mathbb{R}^{s+1}$ which holds true if and only if there exists an execution of $\mathcal{A}^{(k+1)}$ with the infinite word $d_\beta(x)$ producing a successful run r such that $f(r) = (y_1, \ldots, y_s)$.

Recall that a successful run must start in the initial state, be compatible with the transitions of $\mathcal{A}^{(k+1)}$ and visit infinitely often the set of terminal states. Our final task is to show that $R_\mathcal{A}(x, y_1, \ldots, y_s)$ can be expressed in $\langle \mathbb{R}, 1, \leq, +, X_\beta \rangle$. Therefore the set X of reals accepted by $\mathcal{A}$ is β-definable:

$$
\{x \in \mathbb{R} \mid (\exists y_1) \ldots (\exists y_s)(R_\mathcal{A}(x, y_1, \ldots, y_s))\}.
$$

We first overcome some technicalities to make some kind of synchronization between the expansion of x and the expansions of $y_1, \ldots, y_s$. Indeed, we have access through the predicates $X_{\beta, \cdot}(\cdot, b)$ to the digits corresponding to a same power b of β in the β-expansions of x and $y_1, \ldots, y_s$. The sequence of visited states is encoded in $y_1, \ldots, y_s$ and we have to synchronize the state reached at step n with the symbol read at the same step n and the state at step $n + 1$. We assume in all what follows that x is positive (the final formula should involve a disjunction of the two possible cases to take into account the sign of x). In the next formula which is a part of the definition of $R_\mathcal{A}(x, y_1, \ldots, y_s)$, we define s new intermediate variables $z_1, \ldots, z_s$ that are roughly shifted versions of $y_1, \ldots, y_s$:

$$
(\exists b)[(b \text{ is a power of } \beta) \wedge (b \leq x < \beta b) \wedge ([(x \geq 1) \wedge \bigwedge_{i=1}^{s}(z_i = \beta b y_i)] \vee [(x < 1) \wedge \bigwedge_{i=1}^{s}(z_i = y_i)])].
$$

More precisely, if $x \geq 1$, then $d_\beta(x) = u \star v$ where u is a finite nonempty word not starting with 0. We set $d_\beta(z_i) = u_i \star v_i$ for all i. Then at least one of the u_i's is non reduced to 0 and the longest u_i as the same length as u. If $x < 1$, then $d_\beta(z_i) = 0 \star v_i$ for all i and one of the v_i has 1 as prefix.

To give the reader an idea about this construction, we continue our running example with $(y_1, \ldots, y_s) = ((1 + \sqrt{5})/4, 0, 1/2, 0, 1/(1 + \sqrt{5}), 0)$ coding a sequence of states and taking $x = \varphi^3$.

In such a case, we get $z_i = \varphi^4 y_i$, for all i, and the following φ-expansions:

$$
\begin{array}{rcccccccccc}
d_\varphi(x) & = & 1 & 0 & 0 & 0 & \star & 0 & 0 & \cdots \\
d_\varphi(z_1) & = & 1 & 0 & 0 & 1 & \star & 0 & 0 & \cdots \\
d_\varphi(z_2) & = & 0 & 0 & 0 & 0 & \star & 0 & 0 & \cdots \\
d_\varphi(z_3) & = & 0 & 1 & 0 & 0 & \star & 1 & 0 & \cdots \\
d_\varphi(z_4) & = & 0 & 0 & 0 & 0 & \star & 0 & 0 & \cdots \\
d_\varphi(z_5) & = & 0 & 0 & 1 & 0 & \star & 0 & 1 & \cdots \\
d_\varphi(z_6) & = & 0 & 0 & 0 & 0 & \star & 0 & 0 & \cdots
\end{array}
$$

The leftmost column contains the coding of the first state of the sequence of states but also the first symbol that is read. Generally, the nth column contains the coding of the nth state and the nth read symbol. Note that if $|x| < 1$, then the leftmost column is $(0, \dots, 0)$ directly followed with $(\star, \dots, \star)$.

The predicate $R_{\mathcal{A}}(x, y_1, \dots, y_s)$ is true if and only if

1. the run starts in the initial state:

$$
(\exists b_0)\big[X_{\beta,1}(z_1, b_0) \wedge \bigwedge_{i=2}^{s} X_{\beta,0}(z_i, b_0) \wedge (\forall c)[c > b_0 \implies \bigwedge_{i=1}^{s} X_{\beta,0}(z_i, c))]\big].
$$

Note that b_0 is defined once and for all and corresponds to the largest power of β occurring with a non-zero coefficients in one of the β-developments of $z_1, \dots, z_s$.

2. The run follows the transitions: for all powers c of β less or equal to b_0, being in the state coded by the digits corresponding to c occurring in $d_\beta(z_1), \dots, d_\beta(z_s)$ and reading the digit corresponding to c occurring in $d_\beta(x)$, the reached state must correspond to the digits corresponding to c/β (we make use of Remark 20) occurring in $d_\beta(z_1), \dots, d_\beta(z_s)$. Such a finite transition relation can be coded by a formula:

$$
(\forall c)\big[(c \leq b_0) \implies \{ \bigwedge_{\substack{(q,d,q') \in E \\ c(q)=(a_1,\dots,a_s) \\ c(q')=(a'_1,\dots,a'_s)}} X_{\beta,d}(x, c) \wedge \bigwedge_{i=1}^{s} X_{\beta,a_i}(z_i, c) \implies \bigwedge_{i=1}^{s} X_{\beta,a'_i}(z_i, c/\beta)\}\big].
$$

3. Finally, the run must visit infinitely often a terminal state: for all powers c of β less or equal to b_0, there exists a power $d < c$ of β such that the state coded by the digits corresponding to d occurring in $d_\beta(z_1), \dots, d_\beta(z_s)$ is terminal:

$$
(\forall c)\big[(c \text{ is a power of } \beta) \wedge (c \leq b_0) \implies (\exists d)(d < c) \wedge \bigvee_{\substack{q \in T \\ c(q)=(a_1,\dots,a_s)}} \bigwedge_{i=1}^{s} X_{\beta,a_i}(z_i, d)\big].
$$

$\square$

4. If the base is a Pisot number, definability implies recognizability

A real algebraic integer greater than 1 whose Galois conjugates have modulus less than 1 is called a *Pisot number*. Under this stronger assumption, we get the converse of Theorem 16.

Fact 21. *[19, Chapter 7] If β is a Pisot number, then it is a Parry number. But the converse does not hold.*

Theorem 22. *Let $\beta > 1$ be a Pisot number. If $X \subseteq \mathbb{R}^n$ is β-definable, then X is β-recognizable.*

To prove this result, we use the classical method consisting in proving the recognizability of any formula by induction, i.e., we prove that sets defined by atomic formulae are recognizable and that adding connectors and quantifiers does not alter recognizability. We first need a few results.

Lemma 23. *If $\beta > 1$ is a Parry number, then $\mathbb{R}^n$ is β-recognizable.*

Proof. Let $\mathcal{A}_{\beta,\text{frac}} = (Q_{\text{frac}}, \mathbf{A}_\beta \cup \{\star\}, E_{\text{frac}}, I_{\text{frac}}, T_{\text{frac}})$ denote a deterministic Büchi automaton accepting the language $d_\beta([0,1)^n)$ (see Fact 11). Thus the language accepted by $\mathcal{A}_{\beta,\text{frac}}$ is included in $\mathbf{0}\star\mathbf{A}_\beta^\omega$. Let $\mathcal{A}'$ be a deterministic Büchi automaton over the alphabet $\mathbf{A}_\beta$ such that $L(\mathcal{A}_{\beta,\text{frac}}) = \mathbf{0} \star L(\mathcal{A}')$. Starting from $\mathcal{A}'$, we will show how to build an automaton $\mathcal{A}_\beta$ that accepts $\mathbf{0}^* d_\beta(\mathbb{R}^n)$. We then conclude by using Fact 10.

If $r \in \mathbb{R}$ we define $\text{sign}(r)$ to be $+$ if $r \geq 0$ and $-$ else. If $\mathbf{x} = (x_1, \ldots, x_n)$ is a point in $\mathbb{R}^n$, then $\text{sign}(\mathbf{x}) = (\text{sign}(x_1), \ldots, \text{sign}(x_n))$. Given an n-tuple $\mathbf{z} = (z_1, \ldots, z_n)$ whose components belong to $\{+, -\}$, we will build a Büchi automaton $\mathcal{A}_{\beta,\mathbf{z}}$ accepting $\mathbf{0}^* d_\beta(\{\mathbf{x} \in \mathbb{R}^n \mid \text{sign}(\mathbf{x}) = \mathbf{z}\})$. Then the Büchi automaton $\mathcal{A}_\beta$ accepting $\mathbf{0}^* d_\beta(\mathbb{R}^n)$ will be the (disjoint) union of these 2^n automata.

We construct such an automaton $\mathcal{A}_{\beta,\mathbf{z}}$ by considering two copies of $\mathcal{A}'$, one for the β-integer part and one for the β-fractional part of the representations. In all labels of transitions of both copies of $\mathcal{A}'$, we replace the i-th component by its "opposite value" if $\text{sign}(z_i) = -$ and we leave it unchanged otherwise.

W.l.o.g. we can pick the automaton $\mathcal{A}'$ so that its initial state has a loop labeled by $\mathbf{0}$. The unique initial state of $\mathcal{A}_{\beta,\mathbf{z}}$ is a new additional state i and, for each transition $(p, \mathbf{a}, q)$ with $\mathbf{a} \in \tilde{\mathbf{A}}_\beta$ and p, q states of the β-integer part copy of $\mathcal{A}'$ with p initial, there is a transition $(i, \mathbf{a}, q)$ in $\mathcal{A}_{\beta,\mathbf{z}}$. We add a loop on the initial state i with label $\mathbf{0}$.

The terminal states are the terminal states of the β-fractional part copy. We complete $\mathcal{A}_{\beta,\mathbf{z}}$ by adding, for each state q of $\mathcal{A}'$, a transition from (q, int) to (q, frac) labeled by $\star$, where (q, int) (resp. (q, frac)) is the state of $\mathcal{A}_{\beta,\mathbf{z}}$ that corresponds to q in the β-integer part copy (resp. β-fractional part copy). $\qquad\square$

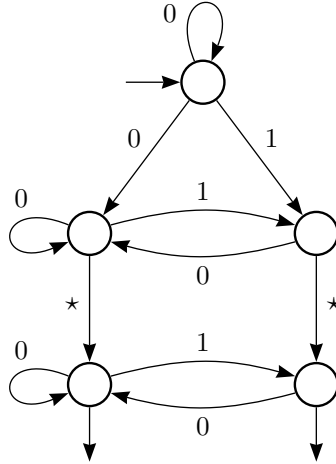**Remark 24.** We illustrate the proof of the previous lemma. The automaton $\mathcal{A}_{\varphi,+}$ is depicted in Figure 5.

Figure 5: The automaton $\mathcal{A}_{\varphi,+}$.

Lemma 25. *If $\beta > 1$ is a Parry number, then $\mathbb{Z}_\beta^n$ is β-recognizable.*

Proof. Indeed, the automaton recognizing $\mathbb{Z}_\beta^n$ is simply the intersection of the one recognizing $\mathbb{R}^n$ with the one accepting $\tilde{\mathbf{A}}_\beta^+ \star \mathbf{0}^\omega$. $\qquad\square$

Lemma 26. *If $\beta > 1$ is a Parry number, then the set $X_< = \{(x,y) \in \mathbb{R}^2 \mid x < y\}$ is β-recognizable.*

Proof. For all $x, y \in \mathbb{R}$, we have $x < y$ if and only if $d_\beta(x) \prec d_\beta(y)$ where $\prec$ is a natural generalization of the lexicographic order with negative digits: it is defined as the partial order

over $\tilde{A}_\beta^+ \star \tilde{A}_\beta^\omega$ by, for $u = u_k \cdots u_1 u_0 \star u_{-1} u_{-2} \cdots$ and $v = v_\ell \cdots v_1 v_0 \star v_{-1} v_{-2} \cdots$, $u \prec v$ if and only if

$$\left[\left(u \in \bar{A}_\beta^+ \star \bar{A}_\beta^\omega\right) \wedge \left(v \in A_\beta^+ \star A_\beta^\omega\right) \wedge \left(u, v \notin 0^+ \star 0^\omega\right)\right]$$
$$\vee \quad \left[\left(u, v \in A_\beta^+ \star A_\beta^\omega\right) \wedge (k < \ell)\right]$$
$$\vee \quad \left[\left(u, v \in \bar{A}_\beta^+ \star \bar{A}_\beta^\omega\right) \wedge (k > \ell)\right]$$
$$\vee \quad \left[\left((u, v \in A_\beta^+ \star A_\beta^\omega) \vee (u, v \in \bar{A}_\beta^+ \star \bar{A}_\beta^\omega)\right) \wedge (k = \ell)\right.$$
$$\left. \wedge \left((\exists i \leq k)[(\forall j \in \{i+1, \ldots, k\})(u_j = v_j) \wedge (u_i < v_i)]\right)\right].$$

Thus, the set $X_<$ is recognized by the intersection of the automaton accepting $d_\beta\left(\mathbb{R}^2\right)$ with the automaton represented in Figure 6. $\qquad\blacksquare$
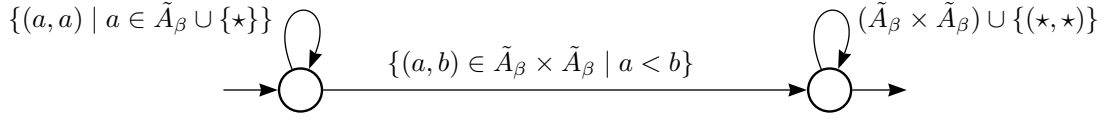

Figure 6: Automaton for the order.

Definition 27. Let $\beta > 1$ be a real number and let $C \subset \mathbb{Z}$ be an alphabet. The *normalization function* is the function

$$\nu_{\beta,C} : C^+ \star C^\omega \to \tilde{A}_\beta^+ \star \tilde{A}_\beta^\omega$$

that maps any β-representation of a real number x onto its β-expansion $d_\beta(x)$.

Definition 28. Given two alphabets A and B, a *transducer* is a Büchi automaton given by a 6-tuple $\mathcal{T} = (Q, A, B, E, I, T)$, where the edges are labeled by elements in $A^* \times B^*$ instead of considering the unique alphabet A. Thus a transducer defines a relation $R_\mathcal{T} \subseteq A^\omega \times B^\omega$ given by $(u, v) \in R_\mathcal{T}$ if and only if (u, v) is accepted by $\mathcal{T}$. A transducer is said to be *letter-to-letter* if its edges are labeled by elements of $A \times B$. If $L \subseteq A^\omega$ is a language and $\mathcal{T}$ is a transducer, we let $\mathcal{T}(L)$ denote the language over B such that $L \times \mathcal{T}(L) \subseteq R_\mathcal{T}$ and which is maximal (with respect to the inclusion) for this property.

Frougny [24] studied the normalization $\nu_{\beta,C,\mathrm{frac}} : C^\omega \to A_\beta^\omega$ that maps any word $u \in C^\omega$ such that $\mathrm{val}_\beta(0 \star u) \in [0, 1)$ onto the word $v \in A_\beta^\omega$ such that $d_\beta(\mathrm{val}_\beta(0 \star u)) = 0 \star v$. She proved the following result.

Theorem 29 (Frougny [24]). *Let β be a Pisot number and $C \subset \mathbb{Z}$ be an alphabet. The normalization $\nu_{\beta,C,\mathrm{frac}}$ is realizable by a (non-deterministic) letter-to-letter transducer $\mathcal{T}_{\beta,C,\mathrm{frac}} = (Q, C, A_\beta, E, \{i\}, T)$: For every infinite word $u \in C^\omega$ such that $\mathrm{val}_\beta(0 \star u) \in [0, 1)$, there exists a unique infinite word $v \in A_\beta^\omega$ such that $(u, v) \in R_{\mathcal{T}_{\beta,C,\mathrm{frac}}}$ and, moreover, $d_\beta(\mathrm{val}_\beta(0 \star u)) = 0 \star v$.*

Example 30. Let φ be the Golden ratio $(1 + \sqrt{5})/2$. Frougny's transducer $\mathcal{T}_{\varphi,C,\mathrm{frac}}$ for the alphabet $C = \{\bar{1}, 0, 1\}$ is the composition of the transducer $\mathcal{T}$ depicted in Figure 7 and the one realizing the intersection with the set $D(\varphi) = \{w \in A_\varphi^\omega \mid \exists x \in [0, 1) \ d_\varphi(x) = 0 \star w\} = \{w \in \{0, 1\}^\omega \mid w \text{ does not contain } 11 \text{ as a factor}\}$. In Figure 7 all states are considered terminal. To obtain the transducer $\mathcal{T}_{\varphi,C,\mathrm{frac}}$, we make the product of $\mathcal{T}$ and the automaton in Figure 3: each state q of the transducer $\mathcal{T}$ is split into 2 states $(q, 0)$ and $(q, 1)$ in order to maintain the information whether the last output letter is 0 or 1. If it was 1, then it is forbidden to output another 1 at the next step. More precisely, $(p, i) \xrightarrow{a|b} (q, j)$ is an edge if and only if $p \xrightarrow{a|b} q$ is an edge in $\mathcal{T}$ and (i, b, j) equals $(0, 0, 0)$, $(0, 1, 1)$ or $(1, 0, 0)$. Note that there is a transition in $\mathcal{T}$ of the form $r \xrightarrow{a|b} s$ whenever $\varphi r + a - b = s$. In [25, Proposition 2.3.38] this transducer is called the converter $\mathcal{C}_\varphi$.
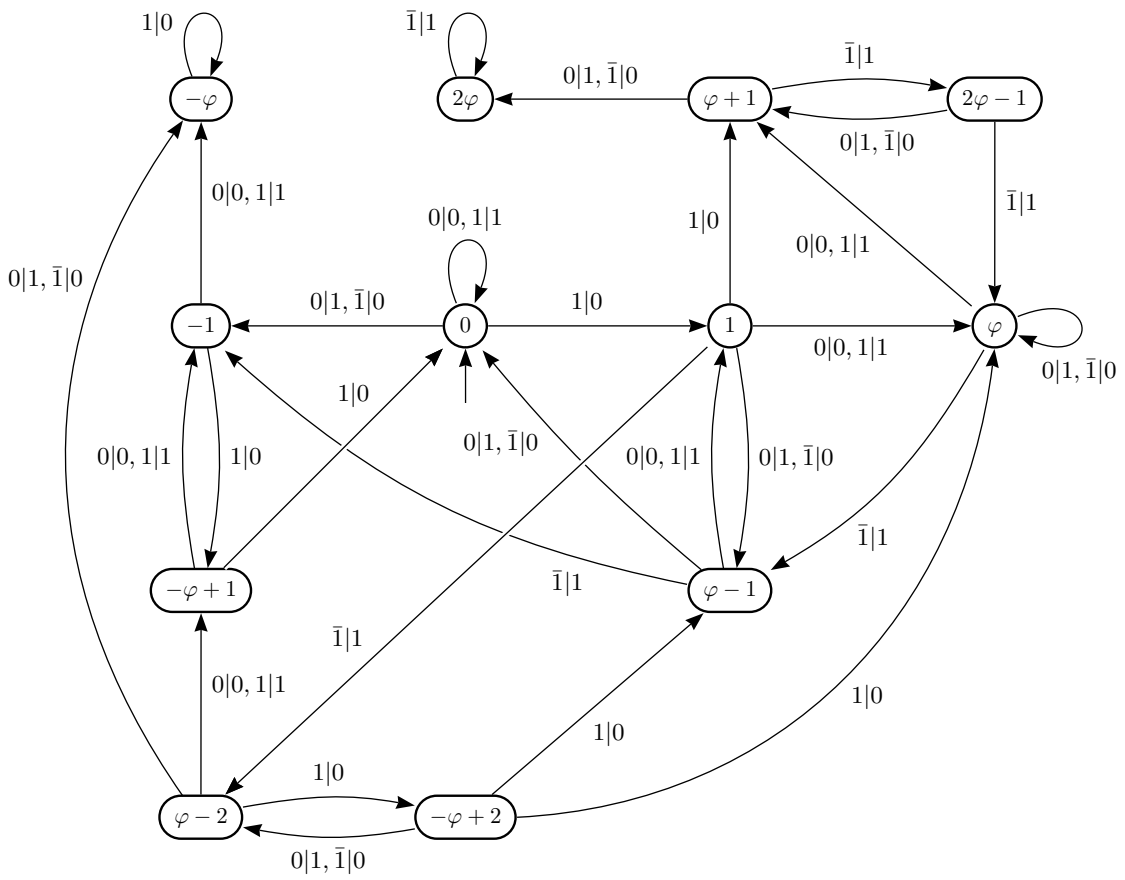
Figure 7: The transducer $\mathcal{T}$.

Remark 31. The transducer $\mathcal{T}_{\beta,C,\text{frac}}$ is built in such a way that for each state q in Q, there is at most one pair of words $(0^k, u)$ such that u does not begin with 0 and that labels a path from i to q.

The next result extends Theorem 29 to $\nu_{\beta,C}$. We first need to allow a transducer to have an initial function instead of initial states.

Definition 32. A *transducer with an initial function* is a transducer given by a 6-tuple $\mathcal{B} = (Q, A, B, E, \alpha, T)$, where α is a partial function over Q with values in $A^* \times B^*$. A pair of infinite words $(u, v) \in A^\omega \times B^\omega$ is accepted by such a transducer if there is a state $q \in \text{dom}(\alpha)$ and a pair of infinite words $(u', v') \in A^\omega \times B^\omega$ such that $(u, v) = \alpha(q)(u', v')^1$ and (u', v') labels an infinite path in $\mathcal{B}$ starting in q and going infinitely often through terminal states.

Proposition 33. *Let $\beta > 1$ be a Pisot number and let $C \subset \mathbb{Z}$ be an alphabet. The normalization $\nu_{\beta,C}$ is realizable by a (non-deterministic) letter-to-letter transducer $\mathcal{T}_{\beta,C} = (Q_{\beta,C}, C \cup \{\star\}, \tilde{A}_\beta \cup \{\star\}, E, \alpha_{\beta,C}, T_{\beta,C})$ with an initial function $\alpha_{\beta,C} : Q_{\beta,C} \to \{\varepsilon\} \times \tilde{A}_\beta^*$: For every infinite word $u \in C^+ \star C^\omega$, there exists a unique infinite word $v \in (\tilde{A}_\beta \cup \{\star\})^\omega$ such that $(u, v) \in R_{\mathcal{T}_{\beta,C}}$ and, moreover, $v \in 0^* d_\beta(\text{val}_\beta(u))$.*

Proof. Let us construct such a transducer $\mathcal{T}_{\beta,C}^+$ but only for words $u \in C^+ \star C^\omega$ such that $\text{val}_\beta(u) \geq 0$. The case where $\text{val}_\beta(u) < 0$ is obtained by considering two copies of $\mathcal{T}_{\beta,C}^+$ where, in one of them, we have exchanged all labels by their respective opposite value.

As in the proof of Lemma 23, we consider the transducer $\mathcal{T}$ that consists of two copies of $\mathcal{T}_{\beta,C,\text{frac}}$, one for the β-integer part and one for the β-fractional part. The initial state of $\mathcal{T}$ is the initial state of the β-integer part copy and the terminal states of $\mathcal{T}$ are the terminal states of the β-fractional part copy. For any state q of $\mathcal{T}_{\beta,C,\text{frac}}$, we add a transition labeled by $(\star, \star)$ from (q, int) to (q, frac) where (q, int) (resp. (q, frac)) is the state of $\mathcal{T}$ that corresponds to q in the β-integer part copy (resp. β-fractional part copy).

By construction of $\mathcal{T}$, a pair of words $(u, v) \in (C^+ \star C^\omega) \times (\tilde{A}_\beta \cup \{\star\})^\omega$ is accepted by $\mathcal{T}$ if and only if $(u, v) \in (C \times A_\beta)^+ (\star, \star)(C \times A_\beta)^\omega$ and $v \in 0^* d_\beta(\text{val}_\beta(u))$. Thus, what remains to consider is the case where $u = u_k \cdots u_0 \star u_{-1} u_{-2} \cdots \in C^+ \star C^\omega$ is such that $d_\beta(\text{val}_\beta(u)) = v_\ell \cdots v_0 \star v_{-1} v_{-2} \in A_\beta^+ \star A_\beta^\omega$ with $\ell > k$. The purpose of the initial function is to deal with this situation.

To be able to define such an initial function, it is first easily seen that in that case, the difference between k and ℓ is bounded. Indeed, if $c = \max C$, we have

$$\text{val}_\beta(u) \leq \frac{c\beta^k}{\beta - 1}.$$

Note that we have $c \geq 1$ because $\text{val}_\beta(u) \geq 0$ and $\ell > k$. The integer ℓ is the greatest one for which $\beta^\ell \leq \frac{c\beta^k}{\beta-1}$, which is equivalent to

$$\ell - k \leq \left\lfloor \log_\beta \left(\frac{c}{\beta - 1} \right) \right\rfloor =: K.$$

Now let $u = u_k \cdots u_0 \star u_{-1} u_{-2} \cdots \in C^+ \star C^\omega$ be such that $d_\beta(\text{val}_\beta(u)) = v_\ell \cdots v_0 \star v_{-1} v_{-2} \in A_\beta^+ \star A_\beta^\omega$ with $\ell > k$. By construction of $\mathcal{T}$, the pair of words $(0^{\ell-k}u, d_\beta(\text{val}_\beta(u)))$ is accepted by $\mathcal{T}$. Thus, if q is a state reached after reading $(0^{\ell-k}, d_\beta(\text{val}_\beta(u))[1\colon \ell - k])$, we consider an initial function α defined by $\alpha(q) = (\varepsilon, d_\beta(\text{val}_\beta(u))[1\colon \ell - k])$. Notice that Remark 31 implies that we can always define α in such a way.

Finally $\mathcal{T}_{\beta,C}^+$ is the transducer $\mathcal{T}$ to which we add the initial function α. □

The following result is folklore. The proof is not very difficult but quite long and technical. Therefore we will only sketch the proof on an example.

---

[1] Concatenation of letters in $A \times B$ is as follows: $(a, b)(c, d) = (ac, bd)$.

Lemma 34. *If $\mathcal{T} = (Q, A, B, E, \alpha, T)$ is a letter-to-letter transducer with partial initial function $\alpha : Q \to A^{\leq k} \times B^{\leq \ell}$ for some positive constants k and ℓ, then the relation $R_{\mathcal{T}}$ realized by $\mathcal{T}$ is an ω-regular language over the alphabet $A \times B$. In particular, if L is an ω-regular language over an alphabet A then $\mathcal{T}(L)$ is an ω-regular language over B.*

Sketch of the proof. Consider the transducer $\mathcal{T}$ of Figure 8. We build a new transducer consisting
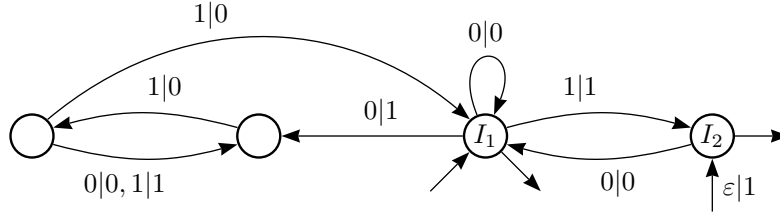
Figure 8: Transducer with partial initial function.

of two modified copies of $\mathcal{T}$, one for each initial state. The copy associated with I_1 is unchanged, except that I_2 is no longer initial. In the copy associated with I_2, we remove the initial function from I_2 and then we must take into account the generated shift caused by the incoming digit 1. From I_2 we must write 1 instead of 0. Each state is duplicated if one can write 0 or 1 when entering this state and we record this information by labeling each new state by 0 or 1. Else we label the state by the unique digit that one can write when entering the state. Each new state has the same incoming and outgoing transitions as the state it comes from. The first component of the labels of the transitions are unchanged while the second components are the labels of the outgoing corresponding state. The transducer that we obtain (see Figure 9) and the transducer
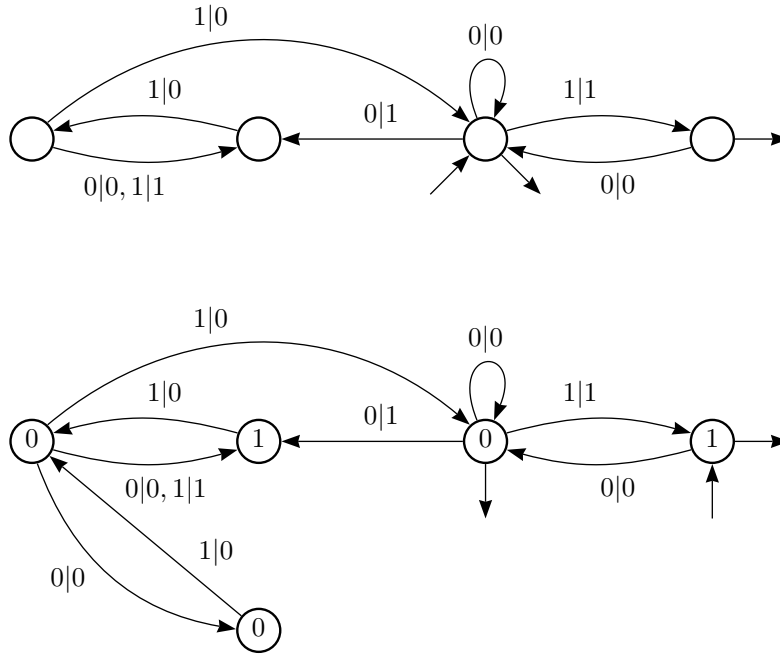

Figure 9: Modified transducer without a partial initial function.

$\mathcal{T}$ both recognize the same language.

If, for a state q, the initial function α is such that $\alpha(q) = (\varepsilon, u_1 \cdots u_n)$ where $n \geq 2$ and u_i are digits, then in the copy corresponding to q, the new initial state q' is such that $\alpha'(q') =$

$(\varepsilon, u_1 \cdots u_{n-1})$. Then we iterate this process until the second component of the initial function is empty. $\qquad\square$

Corollary 35. *Let $X, Y \subset \mathbb{R}$. If X and Y are β-recognizable, then so is $X + Y$.*

Proof. Let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be the Büchi automata that respectively accept $d_\beta(X)$ and $d_\beta(Y)$. We build another automaton $\mathcal{A}_{X \times Y}$ that accepts $d_\beta(X \times Y)$ by intersecting an automaton accepting $0^* d_\beta(X) \times 0^* d_\beta(Y)$ (see Fact 5) with an automaton accepting the words that contain exactly one occurrence of the letter $(\star, \star)$. Now, let us consider the automaton $\mathcal{A}_+$ obtained from $\mathcal{A}_{X \times Y}$ by replacing each label (a, b) by $a + b$ and replacing $(\star, \star)$ by $\star$. By construction, the automaton $\mathcal{A}_+$ accepts a language L over the alphabet $B_\beta = \{0, 1, 2, \ldots, 2(\lceil \beta \rceil - 1), \bar{1}, \bar{2}, \ldots, \overline{2(\lceil \beta \rceil - 1)}\}$ such that $\text{val}_\beta(L) = X + Y$. Finally, using the transducer $\mathcal{T}_{\beta, B_\beta}$ of Proposition 33, Lemma 34 and Fact 10 imply that $X + Y$ is β-recognizable. $\qquad\square$

Lemma 36. *For all $a \in \tilde{A}_\beta$, the set $X_a = \{(x, y) \in \mathbb{R}^2 \mid X_{\beta, a}(x, y) \text{ is true}\}$ is β-recognizable.*

Proof. Indeed, $d_\beta(X_a)$ is accepted by the intersection of the automaton recognizing $d_\beta(\mathbb{R}^2)$ with the one represented in Figure 10. $\qquad\square$
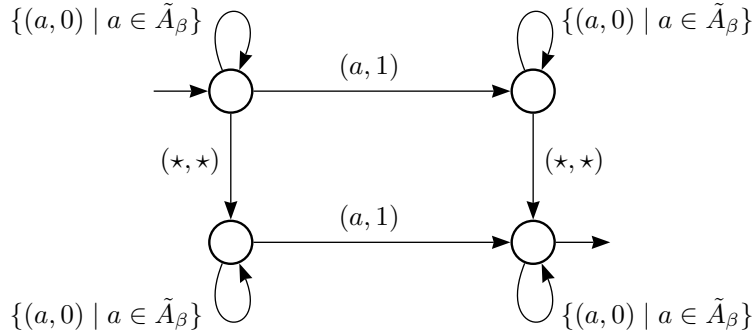

Figure 10: Automaton for X_a.

We are now ready to prove Theorem 22.

Proof of Theorem 22. We follow the lines of [26] by proving the result by induction on the length of the formula. If $\phi(x_1, \ldots, x_n)$ is a formula written in the structure $\langle \mathbb{R}, 1, \leq, +, X_\beta \rangle$, we let X_ϕ denote the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \phi(x_1, \ldots, x_n) \text{ is true}\}$ and we let $\mathcal{A}_\phi$ denote a Büchi automaton (when it exists) accepting $d_\beta(X_\phi)$.

The existence of $\mathcal{A}_\phi$ for atomic formulae ϕ comes from Lemma 23, Lemma 25, Corollary 35 and Lemma 36. We have to prove that β-recognizability is preserved under disjunction, negation and existence, i.e., we prove that if X_ϕ and X_ψ are β-recognizable, then so are $X_{\phi \vee \psi}$, $X_{\neg \phi}$ and $X_{\exists x \phi}$. As a consequence, we obtain that $X_{\phi \wedge \psi} = X_{\neg(\neg \phi \vee \neg \psi)}$, $X_{\phi \implies \psi} = X_{(\neg \phi) \vee (\phi \wedge \psi)}$ and $X_{\forall x \phi} = X_{\neg(\exists x \neg \phi)}$ are also β-recognizable. This is a consequence of Fact 5 and Lemma 23:

1. Given a formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_\ell)$ defined by

$$\phi(x_1, \ldots, x_n, y_1, \ldots, y_m) \vee \psi(y_1, \ldots, y_m, z_1, \ldots, z_\ell).$$

The language $d_\beta(X_\varphi)$ is equal to $(d_\beta(X_\phi) \times d_\beta(\mathbb{R}^\ell)) \cap (d_\beta(\mathbb{R}^n) \times d_\beta(X_\psi))$, which is ω-regular if so are $d_\beta(X_\phi)$ and $d_\beta(X_\psi)$.
2. Given a formula $\phi(x_1, \ldots, x_n)$, the language $d_\beta(X_{\neg \phi})$ is equal to $d_\beta(\mathbb{R}^n) \setminus d_\beta(X_\phi)$, which is ω-regular if so is $d_\beta(X_\phi)$.
3. Given a formula $\phi(x, x_1, \ldots, x_n)$, the language $d_\beta(X_{\exists x \phi})$ is equal to $(\pi_{[2, n+1]}(d_\beta(X_\phi))$, where $\pi_{[2, n+1]}$ is the projection over the last n components, and $(\pi_{[2, n+1]}(d_\beta(X_\phi))$ is ω-regular if so is $d_\beta(X_\phi)$.

$\qquad\square$

5. Towards a Cobham-like theorem for β-numeration systems

The famous theorem of Cobham from 1969 states that recognizability in integer bases of sets of integers strongly depends on the chosen base b [17]. If b and b' are such that $\log(b)/\log(b')$ is irrational, then the only sets $X \subset \mathbb{N}$ that are simultaneously b- and b'-recognizable are the finite unions of arithmetic progressions. In [6, 7], the authors studied particular β-recognizable sets of real numbers and obtained a Cobham-like theorem about these sets. The authors of [5] independently obtained almost the same result with other techniques. By means of graph directed iterated function systems, we provide here a translation between the two results. In doing so we also answer a conjecture of [5] and improve a result of [3]. Prior to this, we prove that when β is a Pisot number, the sets $X \subset \mathbb{R}^n$ that are β-recognizable or β^k-recognizable are the same.

5.1. Taking powers of the base does not change recognizability

Proposition 37. *Let β be a Pisot number. For all positive integers k, a set $X \subset \mathbb{R}^n$ is β-recognizable if and only if it is β^k-recognizable.*

Proof. Let k be an integer greater than 1 (the case $k = 1$ is obvious). The number β being a Pisot number, β^k is also Pisot [27]. We define the morphism ζ_k by $\zeta_k(i) = 0^{k-1}i$ for all $i \in \tilde{A}_{\beta^k}$ and $\zeta_k(\star) = \star$. To alleviate the proof, we only prove the result for $n = 1$. The general case can be handled by considering multidimensional versions of the transducers we use.

If $X \subset \mathbb{R}$ is β^k-recognizable, the language $d_{\beta^k}(X)$ is ω-regular so the language $\zeta_k\left(d_{\beta^k}(X)\right)$ is also ω-regular, see Fact 5. Moreover, any element u in $\zeta_k\left(d_{\beta^k}(X)\right)$ is a β-representation of an element of X: there exists $x \in X$ such that $\mathrm{val}_\beta(u) = x$. More precisely, if $x \in d_{\beta^k}(X)$, then $\mathrm{val}_\beta(\zeta_k(x)) = \mathrm{val}_{\beta^k}(x)$. If $\mathcal{T}_{\beta,\tilde{A}_{\beta^k}}$ is the transducer given by Proposition 33, then the language $\mathcal{T}_{\beta,\tilde{A}_{\beta^k}}\left(\zeta_k\left(d_{\beta^k}(X)\right)\right)$ is ω-regular so X is β-recognizable.

Now suppose that $X \subset \mathbb{R}$ is β-recognizable and let $\mathcal{A}_X$ denote the Büchi automaton accepting $d_\beta(X)$. Let $R_\beta \subset (\tilde{A}_{\beta^k}^+ \star \tilde{A}_{\beta^k}^\omega) \times (\tilde{A}_\beta^+ \star \tilde{A}_\beta^\omega)$ be the relation defined by $\mathcal{T}_{\beta,\tilde{A}_{\beta^k}}$. It is ω-regular by Lemma 34. Let L_{ζ_k} be the ω-regular language $(0^{k-1}\tilde{A}_{\beta^k})^+ \star (0^{k-1}\tilde{A}_{\beta^k})^\omega$ accepted by the Büchi automaton $\mathcal{A}_{\zeta_k}$ depicted in Figure 11. Since $d_\beta(X)$ is ω-regular, if π_1 is the projection on the first component, the language $L_X := \pi_1\left(R_\beta \cap (L_{\zeta_k} \times 0^*d_\beta(X))\right) \subset \tilde{A}_{\beta^k}^+ \star \tilde{A}_{\beta^k}^\omega$ is ω-regular. Furthermore, due to the intersection with R_β, any word u in L_X is such that $\mathrm{val}_\beta(u) \in X$ and, conversely, for all $x \in X$, the word $\zeta_k(d_{\beta^k}(x))$ belongs to $(0^{k-1}\tilde{A}_{\beta^k})^+ \star (0^{k-1}\tilde{A}_{\beta^k})^\omega$ and is such that $\left(\zeta_k(d_{\beta^k}(x)), 0^{m(x)}d_\beta(x)\right)$ belongs to R_β for some integer $m(x)$. Thus we have $\mathrm{val}_\beta(L_X) = X$.

Let $M_X := \zeta_k^{-1}(L_X)$. This language is ω-regular from Fact 5 and $\mathrm{val}_{\beta^k}(M_X) = X$. Thanks to the transducer $\mathcal{T}_{\beta^k,\tilde{A}_{\beta^k}}$, the language $d_{\beta^k}(X)$ is ω-regular. $\qquad\square$

5.2. Background on Cobham theorem for real numbers in integer bases

Definition 38. *Two real numbers β and γ greater than 1 are multiplicatively independent if $\frac{\log \beta}{\log \gamma} \notin \mathbb{Q}$.*

Definition 39. *Let $b \geq 2$ be an integer. A compact set $X \subset [0,1]^n$ is b-self-similar if its b-kernel is finite where the b-kernel of X is the collection of sets*

$$\left\{(b^k X - \mathbf{a}) \cap [0,1]^n \mid k \geq 0, \mathbf{a} = (a_1,\ldots,a_n) \in \mathbb{Z}^n \text{ and } (\forall i)\, 0 \leq a_i < b^k\right\}.$$

Examples of b-self-similar sets are the Pascal's triangle modulo 2 (it is 2-self-similar) that consists in the adherence of the set

$$\left\{(\mathrm{val}_2(0 \star \mathrm{rep}_2(n)), \mathrm{val}_2(0 \star \mathrm{rep}_2(m))) \mid \binom{m}{n} \equiv 1 \mod 2\right\}$$
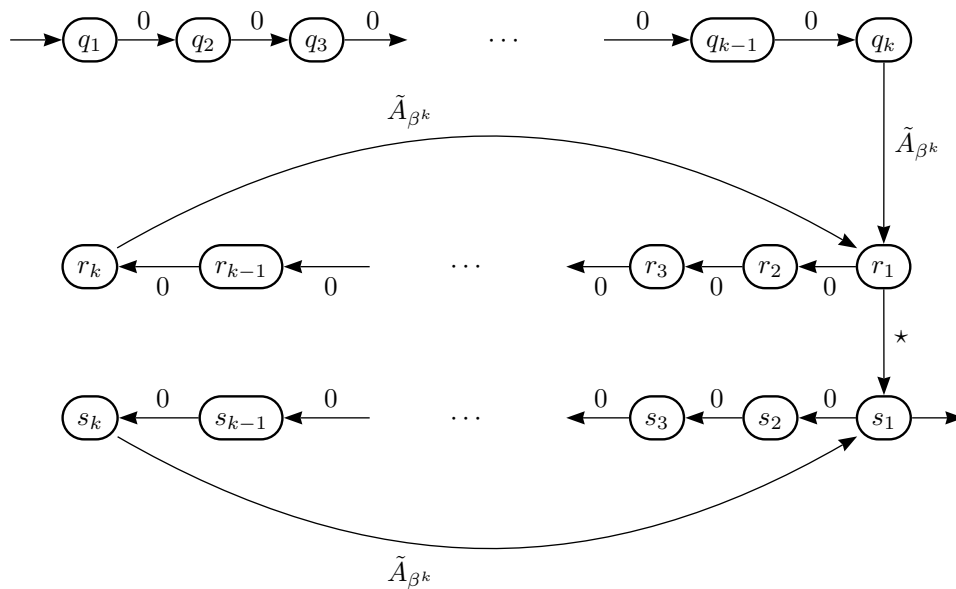
Figure 11: Automaton for L_{ζ_k}

depicted in Figure 12 and the Menger sponge (it is 3-self-similar) that consists in the adherence of the set of points $\mathbf{x} \in [0,1]^3$ such that $\mathrm{rep}_3(\mathbf{x})$ does not contain occurrences[2] of digits in $\{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$; it is depicted in Figure 13.
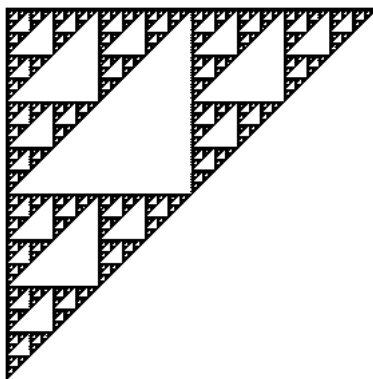

Figure 12: Pascal's triangle modulo 2 is 2-self-similar.

Theorem 40 (Adamczewski and Bell [5]). *Let b and b' be two multiplicatively independent integers greater than 1. A compact set $X \subset [0,1]$ is simultaneously b- and b'-self-similar if and only if it is a finite union of closed intervals with rational endpoints.*

Conjecture 41 (Adamczewski and Bell [5]). *Let b and b' be two multiplicatively independent integers greater than 1. A compact set $X \subset [0,1]^n$ is simultaneously b- and b'-self-similar if and only if it is a finite union of polyhedra whose vertices have rational coordinates.*

The aim of this section is to prove that this conjecture holds true: this is a consequence of Theorem 43, Theorem 57, Theorem 62 and Theorem 64 below.

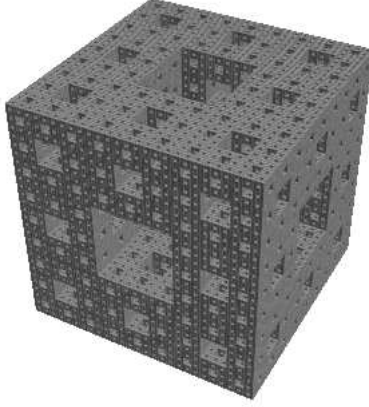[2]Each face of the Menger sponge is a Sierpinski carpet.

Figure 13: Menger sponge is 3-self-similar.

Definition 42. A Büchi automaton $\mathcal{A} = (Q, A, E, I, T)$ is said to be *weak* if all its strongly connected components are subsets either of T, or of $Q \setminus T$. A set X such that $d_\beta(X)$ is accepted by a weak automaton is said to by *weakly β-recognizable*.

The use of weak automata finds its motivation in [15] where the authors prove that any set definable by a first order formula in the structure $\langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle$ is weakly b-recognizable for any integer base $b \geq 2$. Moreover, the behaviour of such automata are comparable with finite automata: minimal weak Büchi automata can be defined, the class of weak Büchi automata is closed under complementation, etc. The next result shows that the converse also holds true.

Theorem 43 (Boigelot, Brusten, Bruyère, Jodogne, Leroux and Wolper [15, 6, 7])**.** *Let b and b' be two multiplicatively independent integers greater than 1. A set $X \subset \mathbb{R}^n$ is simultaneously weakly b- and b'-recognizable if and only if it is definable by a first order formula in the structure $\langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle$.*

5.3. Background on graph directed iterated function systems

By means of graph directed iterated function systems, we are able to determine the link between the notions of weakly b-recognizable and b-self-similar. As in the first part of the paper, we extend these notions to β-numeration systems for real numbers $\beta > 1$.

Definition 44. Let $G = (V, E)$ be a directed graph. For all $u, v \in V$, we let E_{uv} denote the set of edges from u to v. An *iterated function system realizing G* is given by a collection of metric spaces (X_v, ρ_v), $v \in V$, and of similarities[3] $S_e : X_v \to X_u$, $e \in E_{uv}$, of ratio r_e. An *attractor* (or *invariant list*) for such an iterated function system is a list of nonempty compact sets $K_u \subset X_u$ such that for all $u \in V$,

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} S_e(K_v).$$

A *graph-directed iterated function system* (*GDIFS* for short) is given by a 4-tuple

$$(V, E, (X_v, \rho_v)_{v \in V}, (S_e)_{e \in E})$$

with (V, E) the underlying directed graph, $(X_v, \rho_v)_{v \in V}$ the collection of metric spaces and $(S_e)_{e \in E}$ the collection of similarities.

Remark 45. In the directed graph of a GDIFS, it is always assumed that each vertex has at least one outgoing edge.

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[3]Recall that a *similarity* of ratio r is a function $f : X_v \to X_u$ such that $\rho_u(f(\mathbf{x}), f(\mathbf{y})) = r\rho_v(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X_v$.

Example 46 (Rauzy fractal). A classical example of attractor of GDIFS is the *Rauzy fractal* [28] represented in Figure 14. It can obtained as follows. Let σ be the tribonacci substitution defined over the alphabet $A = \{1, 2, 3\}$ by $\sigma(1) = 12$, $\sigma(2) = 13$ and $\sigma(3) = 1$. We define the abelianization map $P : A^+ \to \mathbb{R}^3$ by, for $u_0, \ldots, u_n \in A$, $P(u_0 \cdots u_n) = \sum_{i=0}^{n} \mathbf{e}_{u_i}$ where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. The incidence matrix M_σ of σ is primitive and its dominating eigenvalue β is a unit Pisot number (real root of $x^3 - x^2 - x - 1$). Thus, the space $\mathbb{R}^3$ can be decomposed into the *expanding line* $\mathbb{H}_e$ (eigenspace of M_σ associated with β) and the *contracting plane* $\mathbb{H}_c$ (eigenspace associated with the conjugates of β). Then, if $\sigma^\omega(1) = u_0 u_1 u_2 \cdots$, the sequence $(P(u_0 u_1 \cdots u_n))_{n \in \mathbb{N}}$ describes a broken line whose distance to $\mathbb{H}_e$ is bounded. If $\pi : \mathbb{R}^3 \to \mathbb{H}_c$ is the projection along $\mathbb{H}_e$, then the Rauzy fractal is defined by

$$\mathcal{R} = \overline{\{\pi\left(P(u_0 u_1 \cdots u_n)\right) \mid n \in \mathbb{N}\}}$$

and its subtiles are defined by

$$\mathcal{T}(i) = \overline{\{\pi\left(P(u_0 u_1 \cdots u_n)\right) \mid n \in \mathbb{N}, u_n = i\}}.$$
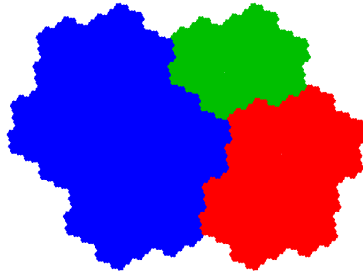

Figure 14: The Rauzy fractal is the attractor of a GDIFS.

Then, the family $\{\mathcal{T}(1), \mathcal{T}(2), \mathcal{T}(3)\}$ satisfies the equations [29, 30]

$$
\begin{aligned}
\mathcal{T}(1) &= h(\mathcal{T}(1)) \cup h(\mathcal{T}(2)) \cup h(\mathcal{T}(3)) \\
\mathcal{T}(2) &= h(\mathcal{T}(1)) + \pi(P(1)) \\
\mathcal{T}(3) &= h(\mathcal{T}(2)) + \pi(P(1))
\end{aligned}
$$

where h is the similarity πM_σ. So $(\mathcal{T}(1), \mathcal{T}(2), \mathcal{T}(3))$ is an attractor of the GDIFS represented in Figure 15.
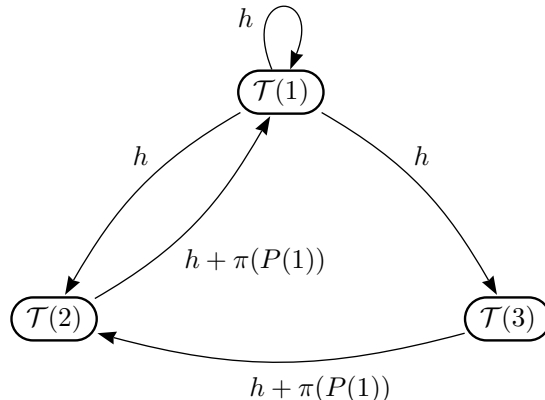

Figure 15: GDIFS for the Rauzy fractal.

Example 47. The Pascal triangle modulo 2 P (Figure 12) can be easily obtained by a GDIFS construction: we have

$$P = \frac{P}{2} \cup \frac{P + (0,1)}{2} \cup \frac{P + (1,1)}{2}$$

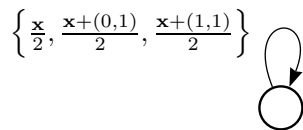so P is an attractor of the GDIFS represented in Figure 16.

Figure 16: GDIFS associated with the Pascal triangle.

Similarly, the Menger sponge MS (Figure 13) can be obtained by the following GDIFS construction: if $E = \{0,1,2\}^3 \setminus \{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$, we have

$$MS = \bigcup_{\mathbf{e} \in E} \frac{MS + \mathbf{e}}{3}$$

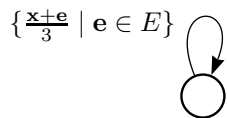so MS is an attractor of the GDIFS represented in Figure 17.

Figure 17: GDIFS associated with the Menger sponge.

The previous two examples are actually obtained by an *iterated function system* (*IFS* for short): the underlying directed graph has a unique vertex. Furthermore, they are obtained by a so called *homogeneous* IFS because all similarities are affine maps with the same contraction ratio. Finally, these homogeneous IFS satisfy the *open set condition*: there exists an open set V such that $S_i(V) \cap S_j(V) = \emptyset$ for any two distinct similarities S_i and S_j. In that setting, Feng and Wang proved, among other things, the following result which is closely linked to our work.

Theorem 48 (Feng and Wang [3]). *Let* $\Phi = \{\phi_i\}_{i=1}^{N}$ *and* $\Psi = \{\psi_j\}_{j=1}^{M}$ *be two homogeneous IFS with metric spaces* $\mathbb{R}$ *embedded with the Euclidean distance, with contraction ratios* r_Φ *and* r_Ψ *respectively, and that satisfy the open set condition. Suppose that* $X \subset \mathbb{R}$ *is an attractor of both* Φ *and* Ψ.

1. *If* [4] $\dim_H(X) < 1$, *then* $\frac{\log |r_\Phi|}{\log |r_\Psi|} \in \mathbb{Q}$;

2. *If* $\dim_H(X) = 1$ *and* X *is not a finite union of intervals, then* $\frac{\log |r_\Phi|}{\log |r_\Psi|} \in \mathbb{Q}$.

The result we obtain with our translation is therefore stronger than Feng and Wang's one in two directions: it is not limited to dimension 1 and it concerns GDIFSs instead of IFSs. It is also weaker in the sense that the similarities are affine functions with contraction ratios of the form $1/b$ for $b \in \mathbb{Z}_{\geq}2$.

Let us provide an example of GDIFS with a non-trivial directed graph.

Example 49. Let $\mathcal{G}$ be the GDIFS whose directed graph is represented in Figure 18, whose metric spaces are $\mathbb{R}$ with the Euclidean distance and whose similarities are $S_i : x \mapsto \frac{x+i}{3}$ for $i \in \{-2,0,2\}$. If X_T is the usual triadic Cantor set that consist of all real numbers in $[0,1]$ whose ternary

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[4]$\dim_H$ stands for Hausdorff dimension.

expansion does not contain the digit 1, then $\mathcal{G}$ admits the attractor $(X_T, -X_T, X_T \cup (-X_T))$. Indeed, X_T corresponds to the vertex C, $-X_T$ to the vertex B and $X_T \cup (-X_T)$ to the vertex A since

$$
\begin{aligned}
X_T &= S_0(X_T) \cup S_2(X_T) \\
-X_T &= S_0(-X_T) \cup S_{-2}(-X_T) \\
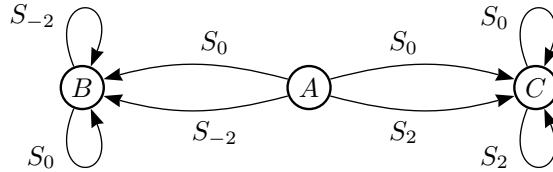X_T \cup (-X_T) &= S_0(X_T) \cup S_2(X_T) \cup S_0(-X_T) \cup S_{-2}(-X_T)
\end{aligned}
$$

Figure 18: Underlying directed graph of the GDIFS representing $X_T \cup (-X_T)$.

The following result is well known.

Theorem 50 ([16]). *If $\mathcal{G} = (V, E, (X_v, \rho_v)_{v \in V}, (S_e)_{e \in E})$ is a GDIFS such that all (X_v, ρ_v) are nonempty complete metric spaces and all similarities S_e have a ratio $r_e < 1$, then $\mathcal{G}$ admits a unique attractor.*

In what follows, the complete metric spaces (X_v, ρ_v) are always $\mathbb{R}^n$ with the Euclidean distance.

5.4. Link between β-recognizable sets and sets obtained by some GDIFS construction

From the GDIFS point of view, we do not need to take care about properties of β such as being Parry or Pisot, neither about limiting the coefficients of the similarities to canonical alphabets such as A_β. Thus, we consider another notion of recognizability as follows.

Definition 51. Let β be a real number greater than 1 and let C be the alphabet $\{-c, -c + 1, \ldots, 0, \ldots, c\}$ for some $c \in \mathbb{Z}_{\geq 1}$. As in Section 2.3, we defined the alphabet

$$
\mathbf{C} = \underbrace{C \times \cdots \times C}_{n \text{ times}}.
$$

A set $X \subset \mathbb{R}^n$ is (β, C)-*acceptable* if there exists a Büchi automaton $\mathcal{A}$ over $\mathbf{C}$ such that $\mathrm{val}_\beta(\mathbf{0} \star L(\mathcal{A})) = X$. We say that $\mathcal{A}$ is an underlying automaton for X.

Remark 52. Let $C = \{-c, \ldots, c\}$ for some $c \in \mathbb{Z}_{\geq 1}$. A (β, C)-acceptable set is always a subset of $\left[\frac{-c}{\beta - 1}, \frac{c}{\beta - 1} \right]^n$.

Remark 53. Thanks to Proposition 33, Fact 10 and Lemma 34, if β is a Pisot number, then for all alphabets $C \subset \mathbb{Z}$, any (β, C)-acceptable set is β-recognizable.

Definition 54. Let $\mathcal{A} = (Q, A, E, I, T)$ be an automaton. A state $q \in Q$ is said to be *accessible* (resp. *co-accessible*) if there is a path from I to q (resp. from q to T). An automaton is said to be *trim* if all its states are accessible and co-accessible. Given an automaton $\mathcal{A}$, we can build a trim automaton $\mathcal{A}'$ that accepts the same language (just by removing non accessible and non co-accessible states). We say that an automaton is *closed* if for all states q such that there is a nonempty path from q to q, q is final.

Remark 55. If an accessible Büchi automaton $\mathcal{A}$ is closed then, for all infinite words w labeling a path in $\mathcal{A}$ not necessarily starting from an initial state, w is an accepting tail, i.e., there is an infinite word v accepted by $\mathcal{A}$ that admits w as a suffix.

Remark 56. Any closed trim Büchi automaton is weak: all its strongly connected components are subsets of T.

Theorem 57. *Let β be a real number greater than 1 and let C be the alphabet $\{-c, -c + 1, \ldots, 0, \ldots, c\}$ for $c \in \mathbb{Z}_{\geq 1}$. A set $X \subset \mathbb{R}^n$ is (β, C)-acceptable set whose underlying trim Büchi automaton is closed if and only if X is a finite union of compact sets belonging to the attractor of a GDIFS whose similarities are of the form $S_\mathbf{a}(\mathbf{x}) = \frac{\mathbf{x}+\mathbf{a}}{\beta}$ for $\mathbf{a} \in \mathbf{C}$*

We first need the following lemma.

Lemma 58. *If the underlying trim Büchi automaton of a (β, C)-acceptable set X is closed, then X is a (topologically) closed set.*

Proof. Consider a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ of elements of X that converges to some $\mathbf{x} \in \mathbb{R}^n$ and let us prove that $\mathbf{x}$ belongs to X. Let $(\mathbf{w}_n)_{n \in \mathbb{N}}$ be a sequence of infinite words over $\mathbf{C}$ accepted by $\mathcal{A}$ such that $\mathrm{val}_\beta(\mathbf{0} \star \mathbf{w}_n) = \mathbf{x}_n$ for all n. There is no reason for which $(\mathbf{w}_n)_{n \in \mathbb{N}}$ would converge to a limit word. However, by the pigeonhole principle, $(\mathbf{w}_n)_{n \in \mathbb{N}}$ admits a subsequence $(\mathbf{w}_{k(n)})_{n \in \mathbb{N}}$ such that $\mathbf{w}_{k(n)}$ and $\mathbf{w}_{k(n+1)}$ share a prefix of length at least n. Thus, $(\mathbf{w}_{k(n)})_{n \in \mathbb{N}}$ converges to some word $\mathbf{w}$ over $\mathbf{C}$ that labels an infinite path in $\mathcal{A}$ starting from an initial state and such that $\mathrm{val}_\beta(\mathbf{0} \star \mathbf{w}) = \mathbf{x}$. Since $\mathcal{A}$ is closed, $\mathbf{w}$ is accepted by $\mathcal{A}$ so we have $\mathbf{x} \in X$. $\square$

Remark 59. If a set X is (β, C)-acceptable with underlying trim automaton $\mathcal{A}$, then its adherence is also (β, C)-acceptable and an underlying automaton for it is $\mathcal{A}$ where all states are final; this automaton is closed. Thus, for β Pisot, any (β, C)-acceptable set $X \subset [-1, 1]^n$ is β-recognizable and its adherence belongs to the small rectangle in Figure 1 (page 3).

Proof of Theorem 57. Let $\mathcal{A} = (Q, \mathbf{C}, E, I, T)$ be a trim Büchi automaton over $\mathbf{C}$ which is closed and such that $\mathrm{val}_\beta(\mathbf{0} \star L(\mathcal{A})) = X$. The GDIFS that we build is obtained from $\mathcal{A}$ by considering, for all states $q \in Q$, the complete metric space $X_q = \mathbb{R}^n$ (with the Euclidean distance) and by replacing the label $\mathbf{a} \in \mathbf{C}$ or each transition by the similarity $S_\mathbf{a} \colon \mathbf{x} \mapsto \frac{\mathbf{x}+\mathbf{a}}{\beta}$ of ratio $1/\beta < 1$. By Theorem 50, the GDIFS admits a unique attractor $(K_q, q \in Q)$: for all $q \in Q$,

$$K_q = \bigcup_{p \in Q} \bigcup_{q \xrightarrow{\mathbf{a}} p} S_\mathbf{a}(K_p).$$

Let us prove that

$$X = \bigcup_{q \in I} K_q.$$

For all states $q \in Q$, we let W_q denote the set of infinite words that, starting from q, label an infinite path in $\mathcal{A}$ (hence the tail of an accepted run since $\mathcal{A}$ is closed). We also let Y_q denote the set $\{\mathrm{val}_\beta(\mathbf{0} \star w) \mid w \in W_q\}$. The automaton $\mathcal{A}$ being closed, all sets Y_q are closed by Lemma 58 and so are compact sets of $\mathbb{R}^n$ that satisfy

$$X = \bigcup_{q \in I} Y_q.$$

Now let us show that $Y_q = K_q$ for all $q \in Q$. By uniqueness of the attractor of the GDIFS, it suffices to show that the list $(Y_q, q \in Q)$ satisfies

$$\forall q \in Q, \quad Y_q = \bigcup_{p \in Q} \bigcup_{q \xrightarrow{\mathbf{a}} p} S_\mathbf{a}(Y_p).$$

This is clear from the following two observations:

$$\forall q \in Q, \quad W_q = \bigcup_{p \in Q} \bigcup_{q \xrightarrow{\mathbf{a}} p} \mathbf{a} W_p$$

$$\forall \mathbf{w} \in \mathbf{C}^\omega, \forall \mathbf{a} \in \mathbf{C}, \quad \mathrm{val}_\beta(\mathbf{0} \star \mathbf{a}\mathbf{w}) = S_\mathbf{a}\left(\mathrm{val}_\beta(\mathbf{0} \star \mathbf{w})\right).$$

Now let $\mathcal{G}$ be a GDIFS with underlying directed graph $G = (V, E)$, $V = \{K_1, \ldots, K_m\}$, complete metric spaces $\mathbb{R}^n$ with Euclidean distance and similarities of the form $S_{\mathbf{a}} \colon \mathbf{x} \mapsto \frac{\mathbf{x}+\mathbf{a}}{\beta}$ for $\mathbf{a} \in \mathbf{C}$. Suppose that X is the union of the compact sets $K_1, \ldots, K_l$, $l \leq m$.

Let $\mathcal{A}$ be the closed automaton $(V, \mathbf{C}, E, \{K_1, \ldots, K_l\}, V)$ where the transitions correspond to the edges of G in which we have replaced the label $S_{\mathbf{a}}$ by $\mathbf{a}$. Remark 45 ensures that, starting from any state q, we can read infinite words so this automaton can be viewed as a Büchi automaton. Let Y be the set $\mathrm{val}_\beta(\mathbf{0} \star L(\mathcal{A}))$. By Lemma 58 Y is closed since so is $\mathcal{A}$. We also have $Y = X$ because if we consider the GDIFS built from $\mathcal{A}$ as we did in the first part of the proof, we would obtain the initial GDIFS whose attractor is $\{K_i \mid 1 \leq i \leq m\}$ and we would get $Y = \bigcup_{1 \leq i \leq l} K_i = X$. $\quad\square$

Now let us extend the definition of b-kernel and of b-self-similar set to real numbers β and arbitrary alphabets C. Let β be a real number greater than 1 and let C be the alphabet $\{-c, -c+1, \ldots, 0, \ldots, c\}$ for $c \in \mathbb{Z}_{\geq 1}$. We let $C[\beta]$ denote the set of polynomials in β with coefficients in C, i.e., $C[\beta] = \{\mathrm{val}_\beta(u \star 0^\omega) \mid u \in C^+\}$ and for $k \in \mathbb{Z}_{\geq 0}$, we let $C^{<k}[\beta]$ denote the set of polynomials of degree less than k in β with coefficients in C, i.e., $C^{<k}[\beta] = \{\mathrm{val}_\beta(u \star 0^\omega) \mid u \in C^{\leq k}\}$. We extend these notation to the multidimensional case, i.e.,

$$\mathbf{C}[\beta] = \underbrace{C[\beta] \times \cdots \times C[\beta]}_{n \text{ times}} \quad \text{and} \quad \mathbf{C}^{<k}[\beta] = \underbrace{C^{<k}[\beta] \times \cdots \times C^{<k}[\beta]}_{n \text{ times}}.$$

Definition 60. A compact set $X \subset \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1}\right]^n$ is (β, C)-*self-similar* if its (β, C)-kernel is finite, where the (β, C)-*kernel* of X is the family of sets

$$N_{k,\mathbf{b}}(X) = \beta^k X - \mathbf{b} \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1}\right]^n,$$

with $k \in \mathbb{Z}_{\geq 0}$ and $\mathbf{b} \in \mathbf{C}^{<k}[\beta]$.

Example 61. Let us consider the notation of Example 49. If $\tilde{A}_3$ is the alphabet $\{-2, -1, 0, 1, 2\}$, then the set $X_T \cup (-X_T)$ is $(3, \tilde{A}_3)$-self-similar since its $(3, \tilde{A}_3)$-kernel is $\{X_T \cup (-X_T), X_T \cup \{-1\}, -X_T \cup \{1\}, X_T, -X_T, \{1\}, \{-1\}, \emptyset\}$.

Theorem 62. *Let β be a Pisot number and C be the alphabet $\{-c, -c+1, \ldots, 0, \ldots, c\}$ for $c \in \mathbb{Z}_{\geq 1}$. If $X \subset \mathbb{R}^n$ is a finite union of compact sets belonging to the attractor of a GDIFS whose similarities are of the form $S_{\mathbf{a}}(\mathbf{x}) = \frac{\mathbf{x}+\mathbf{a}}{\beta}$ for $\mathbf{a} \in \mathbf{C}$, then X is (β, C)-self-similar.*

We will need the following result where, again, algebraic properties of β enter the picture.

Theorem 63 (Berend and Frougny [31, 25]). *Let $\beta > 1$ be a real number. The set $C[\beta] \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1}\right]$ is finite for all alphabets $C = \{-c, -c+1, \ldots, 0, \ldots, c\}$, $c \in \mathbb{Z}_{\geq 1}$, if and only if β is a Pisot number.*

Proof of Theorem 62. By Theorem 50, the GDIFS has a unique attractor $\{K_i \mid 1 \leq i \leq m\}$. Suppose that $X = K_1 \cup K_2 \cup \cdots \cup K_r$, $r \leq m$. For all vertices p and q of the directed graph, we let E_{pq}^ℓ denote the set of words $\mathbf{a}_1 \cdots \mathbf{a}_\ell$ of length ℓ over $\mathbf{C}$ for which there is a sequence of vertices $q = q_0, \ldots, q_\ell = p$ of the directed graph such that for all i, $S_{\mathbf{a}_i}$ labels an edge from q_{i-1} to q_i. For all $i \in \{1, \ldots, m\}$ and all $\ell \geq 1$, it comes

$$K_i = \bigcup_{j=1}^{m} \bigcup_{\mathbf{a}_1 \cdots \mathbf{a}_\ell \in E_{ij}^\ell} S_{\mathbf{a}_1} \circ \cdots \circ S_{\mathbf{a}_\ell}(K_j)$$

and so

$$X = \bigcup_{i=1}^{r} \bigcup_{j=1}^{m} \bigcup_{\mathbf{a}_1 \cdots \mathbf{a}_\ell \in E_{ij}^\ell} S_{\mathbf{a}_1} \circ \cdots \circ S_{\mathbf{a}_\ell}(K_j). \tag{1}$$

Let us prove that the collection of sets

$$N_{\ell,\mathbf{b}}(X) = \beta^{\ell} X - \mathbf{b} \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1} \right]^n, \quad \ell \in \mathbb{Z}_{\geq 0}, \; \mathbf{b} \in \mathbf{C}^{\leq \ell}[\beta]$$

is finite. Using (1), we get, for some ℓ and $\mathbf{b}$,

$$
\begin{aligned}
N_{\ell,\mathbf{b}}(X) &= \left(\beta^{\ell} \left(\bigcup_{i=1}^{r} \bigcup_{j=1}^{m} \bigcup_{\mathbf{a}_1 \cdots \mathbf{a}_{\ell} \in E_{ij}^{\ell}} S_{\mathbf{a}_1} \circ \cdots \circ S_{\mathbf{a}_{\ell}}(K_j) \right) - \mathbf{b} \right) \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1} \right]^n \\
&= \bigcup_{i=1}^{r} \bigcup_{j=1}^{m} \bigcup_{\mathbf{a}_1 \cdots \mathbf{a}_{\ell} \in E_{ij}^{\ell}} \left(\left(\beta^{\ell} S_{\mathbf{a}_1} \circ \cdots \circ S_{\mathbf{a}_{\ell}}(K_j) - \mathbf{b} \right) \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1} \right]^n \right)
\end{aligned}
$$

Let D be the alphabet $\{-2c, -2c+1, \ldots, 0, \ldots, 2c\}$. Observe that we have

$$\beta^{\ell} \left(S_{\mathbf{a}_1} \circ \cdots \circ S_{\mathbf{a}_{\ell}}(K_j) \right) - \mathbf{b} = K_j + \left(\mathbf{a}_{\ell} + \beta \mathbf{a}_{\ell-1} + \beta^2 \mathbf{a}_{\ell-2} + \cdots + \beta^{\ell-1} \mathbf{a}_1 \right) - \mathbf{b}$$

and

$$\mathbf{a}_{\ell} + \beta \mathbf{a}_{\ell-1} + \beta^2 \mathbf{a}_{\ell-2} + \cdots + \beta^{\ell-1} \mathbf{a}_1 - \mathbf{b} \in \mathbf{D}[\beta],$$

where

$$\mathbf{D} = \underbrace{D \times \cdots \times D}_{n \text{ times}}.$$

Since K_j is included in $\left[\frac{-c}{\beta-1}, \frac{c}{\beta-1} \right]^n$, Theorem 63 implies that there is only a finite number of $\mathbf{x} \in \mathbf{D}[\beta]$ such that

$$(K_j + \mathbf{x}) \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1} \right]^n$$

is nonempty and this number does not depend on ℓ. Consequently, if $\mathcal{F}$ is the family of sets

$$(K_j + \mathbf{x}) \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1} \right]^n, \quad \mathbf{x} \in \mathbf{D}[\beta], \; j \in \{1, \ldots, m\},$$

then $\mathcal{F}$ is finite and any set $N_{\ell,\mathbf{b}}$ is a finite union of elements of $\mathcal{F}$. The (β, C)-kernel of X is thus finite. $\quad\square$

Theorem 64. *Let $\beta > 1$ be a real number and C be the alphabet $\{-c, -c+1, \ldots, 0, \ldots, c\}$ for $c \in \mathbb{Z}_{\geq 1}$. If $X \subset \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1} \right]^n$ is (β, C)-self-similar, then X belongs to the attractor of a GDIFS whose similarities are of the form $S_{\mathbf{a}}(\mathbf{x}) = \frac{\mathbf{x}+\mathbf{a}}{\beta}$ for $\mathbf{a} \in \mathbf{C}$.*

Proof. Let

$$\mathcal{N}(X) = \{ N_{k_i, \mathbf{b}_i}(X) \mid i \in \{1, \ldots, m\}, k_i \in \mathbb{Z}_{\geq 0}, \mathbf{b}_i \in \mathbf{C}^{<k_i}[\beta] \}$$

be the (β, C)-kernel of X. We will build a GDIFS with m vertices, whose attractor is $\mathcal{N}(X)$. This will prove the result since X belongs to $\mathcal{N}(X)$: $X = N_{0,\mathbf{0}}(X)$.

For all $k \in \mathbb{Z}_{\geq 0}$ and all $\mathbf{b} \in \mathbf{C}^{<k}[\beta]$, we let $f_{k,\mathbf{b}}$ denote the function $\mathbf{x} \mapsto \frac{\mathbf{x}+\mathbf{b}}{\beta^k}$. It comes

$$N_{k,\mathbf{b}}(X) = f_{k,\mathbf{b}}^{-1}(X) \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1} \right]^n$$

and, for all $k_1, k_2, \mathbf{b}_1, \mathbf{b}_2$,

$$f_{k_1+k_2, \beta^{k_2} \mathbf{b}_1 + \mathbf{b}_2}^{-1} = f_{k_2, \mathbf{b}_2}^{-1} \circ f_{k_1, \mathbf{b}_1}^{-1}$$

Thus, for all $k \in \mathbb{Z}_{\geq 0}$ and all $\mathbf{b} \in \mathbf{C}^{<k}[\beta]$, we have

$$
\begin{aligned}
\bigcup_{a \in \mathbf{C}} f_{1,\mathbf{a}}\big(N_{k+1,\beta\mathbf{b}+\mathbf{a}}(X)\big) &= \bigcup_{\mathbf{a} \in \mathbf{C}} f_{1,\mathbf{a}}\left(f_{k+1,\beta\mathbf{b}+\mathbf{a}}^{-1}(X) \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1}\right]^n\right) \\
&= \bigcup_{\mathbf{a} \in \mathbf{C}} f_{k,\mathbf{b}}^{-1}(X) \cap f_{1,\mathbf{a}}\left(\left[\frac{-c}{\beta-1}, \frac{c}{\beta-1}\right]^n\right) \\
&= f_{k,\mathbf{b}}^{-1}(X) \cap \bigcup_{\mathbf{a} \in \mathbf{C}} f_{1,\mathbf{a}}\left(\left[\frac{-c}{\beta-1}, \frac{c}{\beta-1}\right]^n\right) \\
&= f_{k,\mathbf{b}}^{-1}(X) \cap \left[\frac{-c}{\beta-1}, \frac{c}{\beta-1}\right]^n \\
&= N_{k,\mathbf{b}}(X)
\end{aligned}
$$

To complete the proof, we build the following GDIFS whose attractor is $\mathcal{N}(X)$:

1. the vertices of the directed graph are $v_1, \ldots, v_m$, each v_i corresponding to $N_{k_i, \mathbf{b}_i}(X)$;
2. for all vertices v_i and v_j, there is an edge from v_i to v_j if there is a letter $\mathbf{a} \in \mathbf{C}$ such that $N_{k_j, \mathbf{b}_j} = N_{k_i+1, \beta\mathbf{b}_i+\mathbf{a}}$. This edge is labeled by $f_{1,\mathbf{a}} = S_{\mathbf{a}}$.

$\square$

We show that Conjecture 41 follows from Theorem 43, Theorem 57 and Theorem 64.

First, notice that the theory $\langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle$ admits the elimination of quantifiers [32]. As a consequence of this result, a set X is definable by a first order formula in $\langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle$ if and only if there is a finite number of rational polyhedra (i.e., a region delimited by a finite number of hyperplanes whose equation has integer coefficients) and the intersection of X with each of these polyhedra is a rational periodic repetition of a finite number of rational polyhedra. In particular, if X is bounded, then X is a finite union of rational polyhedra.

Let $b \geq 2$ be an integer. It is clear that a finite union of polyhedra whose vertices have rational coordinates is b-self-similar.

Let $b, b' \geq 2$ be two multiplicatively independent integers. Let $X \subset [0,1]^n$ be a compact set that is both b- and b'-self-similar. Applying first Theorem 64 and then Theorem 57, the base b expansions of the elements in X are accepted by a trim Büchi automaton that is closed. From Remark 56 this automaton is also weak and we can therefore use Theorem 43.

6. Further work

To conclude the paper, let us describe what are the constructions for which an analogue of Cobham's theorem exists.

In Figure 1, the small ellipse concerns IFS and the analogue of Cobham's theorem is Theorem 48 for homogeneous IFSs in $\mathbb{R}$. As explained in the introduction of the paper, this result can been extended to non-homogeneous IFSs satisfying the strong separation condition in $\mathbb{R}^n$ [4].

In the small rectangle, an analogue of Cobham's theorem exists in $\mathbb{R}^n$ for $\beta \in \mathbb{Z}$: it is Theorem 43. Another one also exists in the big ellipse but the condition of independence of the integer bases is stronger: b and b' cannot share common prime factors [33]. In the latter paper, the authors provide an example of a set which is b- and b'-recognizable, where b and b' are multiplicatively independent and share a prime factor, but which is not definable in $\langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle$.

There exist sets that do not belong to any of these cases, for instance the Rauzy fractal (Example 46). Let α be a complex root of the polynomial $x^3 - x^2 - x - 1$. The Rauzy fractal is the attractor of a non-homogeneous IFS (contraction ratios are different powers of α) not satisfying the strong separation condition. It is also the attractor of a homogeneous GDIFS with contraction ratio α. Finally, it is the set of complex number $\left\{ \sum_{i=3}^{+\infty} u_i \alpha_i \mid \forall i \geq 3, \ u_i \in \{0,1\} \text{ and } u_i u_{i+1} u_{i+2} \neq 111 \right\}$; the α-representation of its elements are therefore accepted by some Büchi automaton. Since these

constructions do not meet the settings where an analogue of Cobham's theorem is known, there currently exists no result attesting that this set cannot be obtained in another way (through an IFS, a GDIFS or an automata for instance) with some contraction ratio "independent" from α.

7. Acknowledgement

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