A CLASSIFICATION OF BARYCENTRICALLY ASSOCIATIVE POLYNOMIAL FUNCTIONS

JEAN-LUC MARICHAL, PIERRE MATHONET, AND JÖRG TOMASCHEK

Abstract. We describe the class of polynomial functions which are barycentrically associative over an infinite commutative integral domain.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. Let also X be an arbitrary nonempty set and let $X^* = \bigcup_{n\in\mathbb{N}} X^n$ be the set of all tuples on X, with the convention that $X^0 = \{\varepsilon\}$ (i.e., ε denotes the unique 0-tuple on X). As usual, a function $F\colon X^n \to X$ is said to be *n-ary*. Similarly, we say that a function $F\colon X^* \to X$ is *$*$-ary*. With a slight abuse of notation we may assume that every $*$-ary function $F\colon X^* \to X$ satisfies $F(\varepsilon) = \varepsilon$. The *$n$-ary part* F_n of a function $F\colon X^* \to X$ is the restriction of F to X^n, that is, $F_n = F|_{X^n}$. For tuples $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_m)$, the notation $F(\mathbf{x},\mathbf{y})$ stands for $F(x_1,\ldots,x_n,y_1,\ldots,y_m)$, and similarly for more than two tuples.

A function $F\colon X^* \to X$ is said to be *barycentrically associative*, or *B-associative* for short, if

$$F(\mathbf{x},\mathbf{y},\mathbf{z}) = F(\mathbf{x},k\cdot F(\mathbf{y}),\mathbf{z}), \tag{1}$$

for every integer $k \in \mathbb{N}$ and every $\mathbf{x},\mathbf{z} \in X^*$ and $\mathbf{y} \in X^k$, where the notation $k\cdot x$ means that the argument x is repeated k times. For instance, $F(x, 2\cdot y) = F(x,y,y)$.

Barycentric associativity was introduced in Schimmack [7] as a natural and suitable variant of associativity to characterize the arithmetic mean. Contrary to associativity, this property is satisfied by various means, including the geometric mean and the harmonic mean. It was also used by Kolmogoroff [5] and Nagumo [6] to characterize the class of quasi-arithmetic means.

Since its introduction this property was used under at least three different names: *associativity of means* [2], *decomposability* [3, Sect. 5.3], and *barycentric associativity* [1]. Here we have chosen the third one, which naturally recalls the associativity property of the barycenter as defined in affine geometry. For general background on barycentric associativity and its links with associativity, see [4, Sect. 2.3].

Date: October 31, 2014.

2010 *Mathematics Subject Classification.* Primary 39B72; Secondary 13B25, 26B35.

Key words and phrases. Barycentric associativity, decomposability, polynomial function, integral domain.

Corresponding author: Jean-Luc Marichal is with the Mathematics Research Unit, University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg.
Email: jean-luc.marichal[at]uni.lu.

Let $\mathcal{R}$ be an infinite commutative integral domain (with identity). We say that a function $F\colon\mathcal{R}^*\to\mathcal{R}$ is a *$*$-ary polynomial function*, or simply a *polynomial function*, if $F_n = F|_{\mathcal{R}^n}$ is a polynomial function for every integer $n\geqslant 1$.

In this note we provide a complete description of those polynomial functions $F\colon\mathcal{R}^*\to\mathcal{R}$ which are B-associative. This description is given in the Main Theorem below and the proof is given in the next section.

Any polynomial function $F\colon\mathcal{R}^*\to\mathcal{R}$ such that F_n is constant for every $n\geqslant 1$ is clearly B-associative. It is straightforward to see that nontrivial instances of B-associative polynomial functions include

- the first projection, defined by $F_n(x_1,\ldots,x_n) = x_1$ for every $n\geqslant 1$,
- the last projection, defined by $F_n(x_1,\ldots,x_n) = x_n$ for every $n\geqslant 1$,
- the arithmetic mean, defined by $F_n(x_1,\ldots,x_n) = n^{-1}\sum_{i=1}^n x_i$ for every $n\geqslant 1$ (assuming that every integer $n\geqslant 1$ is invertible in $\mathcal{R}$).

These examples are special cases of the following one-parameter family of polynomial functions. For every integer $n\geqslant 1$ and every $z\in\mathcal{R}$ such that

$$\Delta_n^z ~=~ \sum_{i=1}^n z^{n-i}(1-z)^{i-1} ~=~ \Delta_n^{1-z}$$

is invertible, define the weighted arithmetic mean function $M_n^z\colon\mathcal{R}^n\to\mathcal{R}$ by

$$M_n^z(\mathbf{x}) ~=~ (\Delta_n^z)^{-1}\sum_{i=1}^n z^{n-i}(1-z)^{i-1}\, x_i\, .$$

For every $z\in\mathcal{R}$ we define

$$n(z) ~=~ \inf\{n\geqslant 1 : \Delta_n^z \text{ is not invertible}\}.$$

Clearly, we have $n(z)\geqslant 3$. If Δ_n^z is invertible for every integer $n\geqslant 1$, then we set $n(z)=\infty$.

For every $z\in\mathcal{R}$, consider the function $M^z\colon\mathcal{R}^*\to\mathcal{R}$ whose restriction to $\mathcal{R}^n$ is M_n^z if $n<n(z)$, and 0, otherwise. The Main Theorem states that, up to special cases and constant functions, the typical B-associative polynomial functions are the functions M^z, where $z\in\mathcal{R}$. Note that the special functions M^1, M^0, and $M^{1/2}$ are precisely the three above-mentioned instances of B-associative polynomial functions.

Given a function $F\colon X^*\to X$ and an integer $k\geqslant 1$ or $k=\infty$, we denote by $[F]_k$ the class of functions $G\colon X^*\to X$ obtained from F by replacing F_n with a constant function for every $n\geqslant k$. In particular, we have $[F]_\infty = \{F\}$.

Main Theorem. *A polynomial function $F\colon\mathcal{R}^*\to\mathcal{R}$ is B-associative if and only if one of the following two conditions holds.*

(i) *There exist $z\in\mathcal{R}$ and an integer $k\geqslant 1$ or $k=\infty$, with $k\leqslant n(z)$, such that $F\in[M^z]_k$.*

(ii) *There exists a polynomial function $Q\colon\mathcal{R}^2\to\mathcal{R}$ of degree $\geqslant 1$ such that $F_1(x)=x$, $F_2(x,y) = Q(x,y)\,x + (1-Q(x,y))\,y$, and F_n is constant for every $n\geqslant 3$.*

Remark. By the very definition of function M^z, we see that the condition $k\leqslant n(z)$ is not really needed to describe the set of possible functions F in case (i) of the Main Theorem. However, we have added this condition to stress on the fact that F_n can be any constant function for every $n\geqslant n(z)$.

Example 1. Suppose that $\mathcal{R}$ is a field of characteristic zero. One can readily see that $\Delta_n^z = 0$ if and only if $(1-z)^n = z^n$ and $2z-1 \neq 0$, that is, if and only if $z = 1/(1+\omega_n)$, where $\omega_n \in \mathcal{R} \smallsetminus \{-1,1\}$ is an n-th root of unity. For instance, if $\mathcal{R}$ is the field $\mathbb{C}$ of complex numbers and $F\colon \mathbb{C}^* \to \mathbb{C}$ is a B-associative polynomial function such that $F_3 = M_3^z$, with $z = 1/(1+i)$, then necessarily F_n is constant for every $n \geqslant 4$.

Example 2. If $\mathcal{R}$ is the ring $\mathbb{Z}$ of integers, then $n(0) = n(1) = \infty$ and $n(z) = 3$ for every $z \in \mathbb{Z} \smallsetminus \{0,1\}$. Thus, if $F\colon \mathbb{Z}^* \to \mathbb{Z}$ is a B-associative polynomial function of type (i), then $F \in [M^0]_k$ or $F \in [M^1]_k$ for some integer $k \geqslant 1$ or $k = \infty$, or $F \in [M^z]_k$ for some $z \in \mathbb{Z} \smallsetminus \{0,1\}$ and some $k \in \{1,2,3\}$.

The following straightforward corollary concerns the special case when F_n is symmetric (i.e., invariant under any permutation of the arguments) for every $n \geqslant 1$.

Corollary 3. *Let $F\colon \mathcal{R}^* \to \mathcal{R}$ be a polynomial function such that F_n is symmetric for every $n \geqslant 1$. Then F is B-associative if and only if either F_n is constant for every $n \geqslant 1$ or $1/2 \in \mathcal{R}$ and one of the following two conditions holds.*

- (i) *There exists an integer $k \geqslant 2$ or $k = \infty$, with $k \leqslant n(1/2)$, such that $F \in [M^{1/2}]_k$.*
- (ii) *There exists a nonzero antisymmetric polynomial function $Q\colon \mathcal{R}^2 \to \mathcal{R}$ such that $F_1(x) = x$, $F_2(x,y) = \frac{x+y}{2} + (x-y)\,Q(x,y)$, and F_n is constant for every $n \geqslant 3$.*

2. Technicalities and proof of the Main Theorem

We observe that the definition of $\mathcal{R}$ enables us to identify the ring $\mathcal{R}[x_1,\ldots,x_n]$ of polynomials of n indeterminates over $\mathcal{R}$ with the ring of polynomial functions of n variables from $\mathcal{R}^n$ to $\mathcal{R}$.

It is a straightforward exercise to show that the $*$-ary polynomial functions given in the Main Theorem are B-associative.

We now show that no other $*$-ary polynomial function is B-associative. We first consider the special case when $\mathcal{R}$ is a field. We will then prove the Main Theorem in the general case (i.e., when $\mathcal{R}$ is an infinite commutative integral domain).

From the definition of B-associative functions, we immediately derive the following interesting fact.

Fact 4. *Let $F\colon X^* \to X$ be a B-associative function.*

- (i) *If F_n is constant for some $n \geqslant 1$, then so is F_{n+1}.*
- (ii) *Any $G \in \bigcup_{k \geqslant 1} [F]_k$ is B-associative.*

A function $F\colon X^n \to X$ is said to be *idempotent* if $F(n \cdot x) = x$ for every $x \in X$. It is said to be *range-idempotent* if $F(n \cdot x) = x$ for every x in the range of F. Equivalently, F is range-idempotent if $\delta_F \circ F = F$, where δ_F is the diagonal section of F, defined by $\delta_F(x) = F(n \cdot x)$. In this case we clearly have $\delta_F \circ \delta_F = \delta_F$.

Now let $F\colon \mathcal{R}^* \to \mathcal{R}$ be a B-associative polynomial function, where $\mathcal{R}$ is a field. Since F is B-associative, F_n is clearly range-idempotent for every $n \geqslant 1$ (just take $\mathbf{x} = \mathbf{z} = \varepsilon$ in Eq. (1)). The following lemma then shows that F_n is either constant or idempotent.

Lemma 5. *A polynomial function $F\colon \mathcal{R}^n \to \mathcal{R}$ is range-idempotent if and only if it is either constant or idempotent.*

Proof. The condition is trivially sufficient. To see that it is also necessary, we let $F:\mathcal{R}^n \to \mathcal{R}$ be a range-idempotent polynomial function and show that its diagonal section δ_F is either constant or the identity function. Clearly, if δ_F is constant, then so is $F = \delta_F \circ F$.

Suppose that δ_F is nonconstant and let us write $\delta_F(x) = \sum_{i=0}^{d} a_i x^i$, with $d \geqslant 1$ and $a_d \neq 0$. By equating the leading (i.e., highest degree) terms in both sides of the identity $\delta_F \circ \delta_F = \delta_F$, we obtain $a_d^2 x^{d^2} = a_d x^d$. Therefore, we must have $d = 1$ and $a_1 = 1$, that is, $\delta_F(x) = x + a_0$. Substituting again in $\delta_F \circ \delta_F = \delta_F$, we obtain $a_0 = 0$. □

Let us write F_n is the following standard form

$$F_n(\mathbf{x}) = \sum_{j=0}^{d} \sum_{|\boldsymbol{\alpha}|=j} a_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}, \quad \text{with } \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where the inner sum is taken over all $\boldsymbol{\alpha} \in \mathbb{N}^n$ such that $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n = j$. This polynomial function is said to be of *degree* d if there exists $\boldsymbol{\alpha} \in \mathbb{N}^n$, with $|\boldsymbol{\alpha}| = d$, such that $a_{\boldsymbol{\alpha}} \neq 0$.

Due to Fact 4, we may always assume that F_n is nonconstant. By Lemma 5, it is therefore idempotent, which means that

$$\sum_{j=0}^{d} \left(\sum_{|\boldsymbol{\alpha}|=j} a_{\boldsymbol{\alpha}} \right) x^j = x, \qquad x \in \mathcal{R},$$

or equivalently,

$$\sum_{|\boldsymbol{\alpha}|=1} a_{\boldsymbol{\alpha}} = 1 \quad \text{and} \quad \sum_{|\boldsymbol{\alpha}|=j} a_{\boldsymbol{\alpha}} = 0 \quad \text{for } j \neq 1.$$

We then have the following results.

Lemma 6. *Let $F:\mathcal{R}^* \to \mathcal{R}$ be a B-associative polynomial function and assume that F_{n+1} is nonconstant for some $n \geqslant 2$. Then there exists an idempotent binary polynomial function $P:\mathcal{R}^2 \to \mathcal{R}$ such that*

$$F_{n+1}(x_1,\ldots,x_{n+1}) = P(F_n(x_1,\ldots,x_n),x_{n+1}), \tag{2}$$
$$= P(F_n(x_1,(n-1)\cdot F_n(x_2,\ldots,x_{n+1})),F_n(x_2,\ldots,x_{n+1})) \tag{3}$$

and

$$P(F_n(F_n(x_2,\ldots,x_{n+1}),x_2,\ldots,x_n),x_{n+1}) = F_n(x_2,\ldots,x_{n+1}). \tag{4}$$

Proof. Consider the binary polynomial functions $P:\mathcal{R}^2 \to \mathcal{R}$ and $Q:\mathcal{R}^2 \to \mathcal{R}$ defined by $P(x,y) = F_{n+1}(n \cdot x, y)$ and $Q(x,y) = F_{n+1}(x, n \cdot y)$, respectively. Since F_{n+1} is nonconstant, by Lemma 5 it must be idempotent and therefore so are P and Q. By B-associativity of F, we then obtain Eq. (2) and

$$P(F_n(x_1,\ldots,x_n),x_{n+1}) = Q(x_1,F_n(x_2,\ldots,x_{n+1})). \tag{5}$$

Clearly, F_n is nonconstant by Fact 4. Setting $x_{n+1} = x_n = \cdots = x_2$ in Eq. (5) and then using idempotence, we obtain

$$P(F_n(x_1,(n-1)\cdot x_2),x_2) = Q(x_1,x_2).$$

Then, substituting for Q in Eq. (5) from the latter equation, we obtain Eq. (3). Finally, setting $x_1 = F_n(x_2,\ldots,x_{n+1})$ in either Eq. (3) or Eq. (5) and then using idempotence, we obtain Eq. (4). □

Proposition 7. *Let $F:\mathcal{R}^* \to \mathcal{R}$ be a B-associative polynomial function. If F_3 is nonconstant, then F_2 must be of degree 1.*

Proof. Let us particularize Lemma 6 to the case $n = 2$. There exists an idempotent binary polynomial function $P:\mathcal{R}^2 \to \mathcal{R}$ such that

$$P(F_2(x_1,x_2),x_3) \;=\; P(F_2(x_1,F_2(x_2,x_3)),F_2(x_2,x_3)) \tag{6}$$

and

$$P(F_2(F_2(x_2,x_3),x_2),x_3) - F_2(x_2,x_3) \;=\; 0. \tag{7}$$

Clearly, F_2 is nonconstant by Fact 4. Let us express F_2 and P in the following convenient ways. Let p (resp. q) be the degree of P (resp. F_2) in the first variable. Then there are polynomial functions $P_i:\mathcal{R} \to \mathcal{R}$ ($i = 0,\dots,p$) and $Q_j:\mathcal{R} \to \mathcal{R}$ ($j = 0,\dots,q$), with $P_p \neq 0$ and $Q_q \neq 0$, such that

$$P(x,y) \;=\; \sum_{i=0}^{p} x^i\, P_i(y) \quad\text{and}\quad F_2(x,y) \;=\; \sum_{j=0}^{q} x^j\, Q_j(y). \tag{8}$$

Considering the standard form of F_2, we can also write

$$F_2(x,y) \;=\; \sum_{k+\ell\leqslant d} a_{k,\ell}\, x^k y^\ell \;=\; \sum_{m=0}^{d} R_m(x,y),$$

where d is the degree of F_2 and

$$R_m(x,y) \;=\; \sum_{k+\ell=m} a_{k,\ell}\, x^k y^\ell, \quad \text{with } R_d \neq 0.$$

Claim. If $p > 0$ and $q > 0$, then the polynomial functions P_p and Q_q are constant.

Proof. Substituting for P and F_2 from Eq. (8) in Eq. (6) and then equating the leading terms in x_1 in the resulting equation, we obtain

$$(x_1^q\, Q_q(x_2))^p\, P_p(x_3) \;=\; (x_1^q\, Q_q(F_2(x_2,x_3)))^p\, P_p(F_2(x_2,x_3)),$$

or, equivalently, $G(x_2,x_3) - H(x_2,x_3) = 0$, where

$$G(x_2,x_3) \;=\; Q_q^p(F_2(x_2,x_3))\, P_p(F_2(x_2,x_3)) \quad\text{and}\quad H(x_2,x_3) \;=\; Q_q^p(x_2)\, P_p(x_3).$$

Denote by ax^α (resp. bx^β) the leading term of P_p (resp. Q_q); hence $ab \neq 0$. Clearly, the leading term in x_2 of G is

$$(b(x_2^q\, Q_q(x_3))^\beta)^p\, a(x_2^q\, Q_q(x_3))^\alpha \tag{9}$$

and is therefore of degree $pq\beta + q\alpha$. Similarly, the leading term in x_2 of H is

$$(bx_2^\beta)^p\, P_p(x_3)$$

and is of degree $p\beta$.

If $pq\beta + q\alpha > p\beta$, then the expression in Eq. (9) must be the zero polynomial function, which is impossible since $Q_q \neq 0$. Therefore we must have $pq\beta + q\alpha = p\beta$, that is $\alpha = 0$ (i.e., P_p is the constant a) and $(q-1)\beta = 0$. If $q = 1$, then the leading term in x_2 of $G(x_2,x_3) - H(x_2,x_3)$ is

$$(b(x_2\, Q_q(x_3))^\beta)^p\, a - (bx_2^\beta)^p\, a \;=\; (bx_2^\beta)^p\, a\, (Q_q(x_3)^{p\beta} - 1)\,,$$

and hence Q_q must be constant. □

Let us now prove that F_2 is of degree 1. We consider the following cases, which cover all the possibilities.

Case $q = 0$: We have $F_2(x,y) = Q_0(y)$ and therefore $y = F_2(y,y) = Q_0(y) = F_2(x,y)$, which shows that F_2 is of degree 1.

Case $p = 0$: We have $P(x,y) = P_0(y)$. Using idempotence, we obtain $y = P(y,y) = P_0(y)$ and therefore $P(x,y) = y$. Substituting for P in Eq. (6), we obtain $x_3 = F_2(x_2,x_3)$ and therefore F_2 is of degree 1.

Case $p > 0$ and $q = 1$: We have $F_2(x_1,x_2) = x_1Q_1(x_2) + Q_0(x_2)$ with $Q_1 \neq 0$. Since F_2 is idempotent, we also have $x = F_2(x,x) = xQ_1(x) + Q_0(x)$. But Q_1 is constant by the claim. It follows that Q_0 is of degree 1 and therefore so is F_2.

Case $p > 0$ and $q > 1$: By definition of q we must have $d \geqslant 2$. Let us compute the leading terms (i.e., homogeneous terms of highest degree) of the left-hand side of Eq. (7). On the one hand, we have

$$F_2(F_2(x_2,x_3),x_2) \;=\; \sum_{k+\ell\leqslant d} a_{k,\ell} \underbrace{\left(\sum_{m=0}^{d} R_m(x_2,x_3)\right)^k x_2^\ell}_{(*)},$$

where the expression $(*)$ is of degree $kd+\ell$, with $R_d^k(x_2,x_3)\,x_2^\ell$ as leading terms. We also have

$$\max\{kd+\ell : k+\ell \leqslant d,\ a_{k,\ell}\neq 0\} \;=\; qd.$$

Indeed, if $k > q$, then $a_{k,\ell} = 0$ by definition of q. If $k = q$ and $\ell \neq 0$, then $a_{k,\ell} = 0$ by the claim. If $k = q$ and $\ell = 0$, then $a_{k,\ell} \neq 0$ and $kd + \ell = qd$. Finally, if $k \leqslant q-1$, then

$$\begin{aligned} kd+\ell &\leqslant kd+d-k = k(d-1)+d \leqslant (q-1)(d-1)+d \\ &= qd-q+1 < qd \quad (\text{since } q>1). \end{aligned}$$

This shows that the leading terms of $F_2(F_2(x_2,x_3),x_2)$ are of degree qd and consist of $a_{q,0}\,R_d^q(x_2,x_3)$, where $a_{q,0}\neq 0$.

Now, to compute the leading terms of $P(F_2(F_2(x_2,x_3),x_2),x_3)$, it is convenient to express P as

$$P(x,y) \;=\; \sum_{rqd+s\leqslant e} b_{r,s}\,x^r y^s \;=\; \sum_{m=0}^{e} S_m(x,y),$$

where $e = \max\{rqd+s : b_{r,s}\neq 0\}$ and

$$S_m(x,y) \;=\; \sum_{rqd+s=m} b_{r,s}\,x^r y^s, \quad \text{with } S_e \neq 0.$$

It follows that the leading terms of $P(F_2(F_2(x_2,x_3),x_2),x_3)$ are of degree e and consist of $S_e(a_{q,0}\,R_d^q(x_2,x_3),x_3)$. On the other hand, the leading terms of $F_2(x_2,x_3)$ are of degree d and consist of $R_d(x_2,x_3)$.

We observe that there exists $r > 0$ such that $b_{r,s}\neq 0$ (otherwise, if $b_{r,s} = 0$ for every $r > 0$, then $p = 0$, a contradiction). By definition of e, we then have $e \geqslant rdq > d$. By Eq. (7), we then have $S_e(a_{q,0}\,R_d^q(x_2,x_3),x_3) = 0$, or equivalently,

$$\sum_{rqd+s=e} b_{r,s}\,\left(a_{q,0}\,R_d^q(x_2,x_3)\right)^r x_3^s \;=\; 0. \tag{10}$$

Since $R_d(x_2,x_3)$ is of degree $\geqslant 1$ in x_2 (otherwise, we would have $R_d(x,y) = T(y)$ and therefore $0 = R_d(y,y) = T(y) = R_d(x,y)$, a contradiction), we can write

$$R_d(x_2,x_3) \;=\; \sum_{k=0}^{f} x_2^k\, T_k(x_3), \quad \text{with } f > 0 \text{ and } T_f \neq 0.$$

Equating the leading terms in x_2 in Eq. (10), we obtain

$$b_{r_0,e-r_0qd}\,\Big(a_{q,0}\,x_2^{fq}\,T_f^q(x_3)\Big)^{r_0}\,x_3^{e-r_0qd} \;=\; 0,$$

where $r_0 = \max\{r : rqd + s = e,\ b_{r,s} \neq 0\}$. This is a contradiction.

This completes the proof of the proposition. □

Proposition 8. *Let $F{:}\mathcal{R}^* \to \mathcal{R}$ be a B-associative polynomial function. If $F_n = M_n^z$ for some $n \geqslant 2$ and some $z \in \mathcal{R}$ such that $\Delta_n^z \neq 0$, then either $F_{n+1} = M_{n+1}^z$ or F_{n+1} is constant. Moreover, if $\Delta_{n+1}^z = 0$, then F_{n+1} is constant.*

Proof. Assume that $F_n = M_n^z$ for some $n \geqslant 2$ and some $z \in \mathcal{R}$ such that $\Delta_n^z \neq 0$ and assume that F_{n+1} is nonconstant. Substituting in Eq. (4) and observing that $(1-z)\,\Delta_n^z + z^n = \Delta_{n+1}^z$, we obtain

(11)

$$P\left(\Delta_{n+1}^z \sum_{i=2}^{n} \frac{z^{n-i}\,(1-z)^{i-2}}{(\Delta_n^z)^2}\,x_i + \frac{z^{n-1}(1-z)^{n-1}}{(\Delta_n^z)^2}\,x_{n+1}, x_{n+1}\right) \;=\; \sum_{i=1}^{n} \frac{z^{n-i}\,(1-z)^{i-1}}{\Delta_n^z}\,x_{i+1}.$$

If $z = 0$, then Eq. (11) reduces to $P(x_n, x_{n+1}) = x_{n+1}$. By Eq. (2), we obtain $F_{n+1}(x_1,\ldots,x_{n+1}) = x_{n+1}$, that is, $F_{n+1} = M_{n+1}^z$. We can henceforth assume that $z \neq 0$.

If $\Delta_{n+1}^z = 0$, then we obtain a contradiction; indeed, the left-hand side of Eq. (11) is independent of x_2 whereas the coefficient of x_2 in the right-hand side is z^{n-1}/Δ_n^z. In this case F_{n+1} must be constant.

We can now assume that $\Delta_{n+1}^z \neq 0$. Using the expression of P given in Eq. (8) and equating the leading terms in x_2 in Eq. (11), we obtain

$$\left(\frac{\Delta_{n+1}^z}{(\Delta_n^z)^2}\,z^{n-2}\,x_2\right)^p P_p(x_{n+1}) \;=\; \frac{z^{n-1}}{\Delta_n^z}\,x_2.$$

It follows that $p = 1$ and that P_1 is constant, say $P_1 = c$, where $c = z\,\Delta_n^z/\Delta_{n+1}^z$. We then have $P(x,y) = cx + P_0(y)$ and, by idempotence of P, we also have $cx + P_0(x) = x$. Therefore, $P(x,y) = cx + (1-c)y$. Finally, by Eq. (2) we obtain

$$F_{n+1}(x_1,\ldots,x_{n+1}) \;=\; c\,F_n(x_1,\ldots,x_n) + (1-c)\,x_{n+1} \;=\; M_{n+1}^z\,.$$

This completes the proof of the proposition. □

Let us now show that any B-associative polynomial function $F{:}\mathcal{R}^* \to \mathcal{R}$, where $\mathcal{R}$ is a field, falls into one of the two cases given in the Main Theorem.

Suppose first that F_1 or F_2 is constant. In the latter case, F_1 is either constant or the identity function by Lemma 5. By Fact 4, F_n is constant for every $n \geqslant 2$ and therefore F falls into case (i) with $k = 1$ or $k = 2$.

Suppose now that F_1 and F_2 are nonconstant. These functions are idempotent by Lemma 5 and therefore F_1 is the identity function. If F_2 is of degree 1, then by Lemma 5 we have $F_2(x,y) = zx + (1-z)y$ for some $z \in \mathcal{R}$ and therefore F falls into case (i) by Propositions 8 and Fact 4. Otherwise if F_2 is of degree $\geqslant 2$, then by Proposition 7 and Fact 4 we have $F_1(x) = x$, $F_2(x,y) = zx + (1-z)y + R(x,y)$

for some $z \in \mathcal{R}$ and some polynomial function $R\colon \mathcal{R}^2 \to \mathcal{R}$ of degree $\geqslant 2$ such that $R(x,x) = 0$ for all $x \in \mathcal{R}$, and F_n is constant for every $n \geqslant 3$. It is easy to see that a polynomial function $R\colon \mathcal{R}^2 \to \mathcal{R}$ satisfies $R(x,x) = 0$ for all $x \in \mathcal{R}$ if and only if we have $R(x,y) = (x-y)\, Q'(x,y)$ for some polynomial function $Q'\colon \mathcal{R}^2 \to \mathcal{R}$. Indeed, if we write the homogeneous terms of degree k of $R(x,y)$ in the form

$$\sum_{j=0}^{k} c_j\, x^j y^{k-j} \;=\; (x-y)\sum_{j=1}^{k}\left(\sum_{i=0}^{k-j} c_{k-i}\right) x^{j-1} y^{k-j} + \left(\sum_{j=0}^{k} c_j\right) y^k\,,$$

then we see that $R(x,x) = 0$ if and only if $\sum_{j=0}^{k} c_j = 0$. Thus, we have $F_2(x,y) = y + (x-y)\, Q(x,y)$ for some polynomial function $Q\colon \mathcal{R}^2 \to \mathcal{R}$ of degree $\geqslant 1$. Therefore, F falls into case (ii). This completes the proof of the Main Theorem when $\mathcal{R}$ is a field.

Let us now prove the Main Theorem when $\mathcal{R}$ is an infinite integral domain. Using the identification of polynomials and polynomial functions, we can extend every B-associative $\ast$-ary polynomial function over an infinite integral domain $\mathcal{R}$ to a $\ast$-ary polynomial function on the fraction field $\mathrm{Frac}(\mathcal{R})$ of $\mathcal{R}$. The latter function is still B-associative since the B-associativity property for $\ast$-ary polynomial functions is defined by a set of polynomial equations on the coefficients of the polynomial functions. Therefore, every B-associative $\ast$-ary polynomial function F over $\mathcal{R}$ is the restriction to $\mathcal{R}$ of a B-associative $\ast$-ary polynomial function $\overline{F}$ over $\mathrm{Frac}(\mathcal{R})$. The possible expressions for such a polynomial function $\overline{F}$ are given by the Main Theorem over $\mathrm{Frac}(\mathcal{R})$. Clearly, if $\overline{F}$ falls into case (ii), then so does F. If $\overline{F}$ falls into case (i), then there exist $z \in \mathrm{Frac}(\mathcal{R})$ and an integer $k \geqslant 1$ or $k = \infty$, with $k \leqslant \inf\{n \geqslant 1 : \Delta_n^z = 0\}$, such that $\overline{F} \in [M^z]_k$. If $k = 1$, then $\overline{F}_n$ is constant for every $n \geqslant 1$. Therefore F_n is also a constant (in $\mathcal{R}$) for every $n \geqslant 1$ and hence F falls into case (i). If $k \geqslant 2$, then $\overline{F} \in [M^z]_k$, where $z = \overline{F}_2(1,0) = F_2(1,0) \in \mathcal{R}$. For every integer $n < k$, we have

$$\overline{F}_n(\mathbf{x}) \;=\; M_n^z(\mathbf{x}) \;=\; \sum_{i=1}^{n} (\Delta_n^z)^{-1}\, z^{n-i} (1-z)^{i-1}\, x_i\,.$$

Since $\overline{F}_n$ is the extension of F_n, the coefficient $(\Delta_n^z)^{-1}\, z^{n-i}(1-z)^{i-1}$ of x_i in $\overline{F}_n(\mathbf{x})$ is in $\mathcal{R}$ for $i = 1,\ldots,n$. A straightforward induction shows that $(\Delta_n^z)^{-1}\, z^{n-j} \in \mathcal{R}$ for $j = 1,\ldots,n$. Therefore Δ_n^z is invertible in $\mathcal{R}$ for every $n < k$ and hence $k \leqslant n(z)$. This shows that F falls into case (i). The proof is now complete.

Acknowledgments

This research is partly supported by the internal research project F1R-MTH-PUL-12RDO2 of the University of Luxembourg. J. Tomaschek is supported by the National Research Fund, Luxembourg (AFR 3979497), and cofunded under the Marie Curie Actions of the European Commission (FP7-COFUND).

References

[1] C. Antoine. Les Moyennes. Volume 3383 of *Que sais-je?* Presse Universitaires de France, Paris, 1998.

[2] B. de Finetti. Sul concetto di media. *Giornale dell' Instituto Italiano degli Attari* 2(3):369–396, 1931.

[3] J. Fodor and M. Roubens. *Fuzzy preference modelling and multicriteria decision support.* Kluwer, Dordrecht, 1994.

[4] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Encyclopedia of Mathematics and its Applications, vol. 127. Cambridge University Press, Cambridge, 2009.

[5] A. N. Kolmogoroff. Sur la notion de la moyenne. (French). *Atti Accad. Naz. Lincei*, 12(6):388–391, 1930.

[6] M. Nagumo. Über eine klasse der mittelwerte. (German). *Japanese Journ. of Math.*, 7:71–79, 1930.

[7] R. Schimmack. Der Satz vom arithmetischen Mittel in axiomatischer Begründung. *Math. Ann.* 68:125–132, 1909.

Mathematics Research Unit, FSTC, University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg
E-mail address: jean-luc.marichal[at]uni.lu

University of Liège, Department of Mathematics, Grande Traverse, 12 - B37, B-4000 Liège, Belgium
E-mail address: p.mathonet[at]ulg.ac.be

Mathematics Research Unit, FSTC, University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg
E-mail address: research[at]jtomaschek.eu