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On the Use of Principal Component Analysis for Parameter Identification and Damage Detection in Structures

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Principal Component Analysis (PCA)

Principal component analysis is a multi-variate statistical method.

Aim: to obtain a compact representation of the data.

Principal Component Analysis = Proper Orthogonal Decomposition

PCA (or POD) is applied here for three purposes:

1. Damage detection
2. Structural health monitoring
3. Identification of nonlinear parameters

Outline

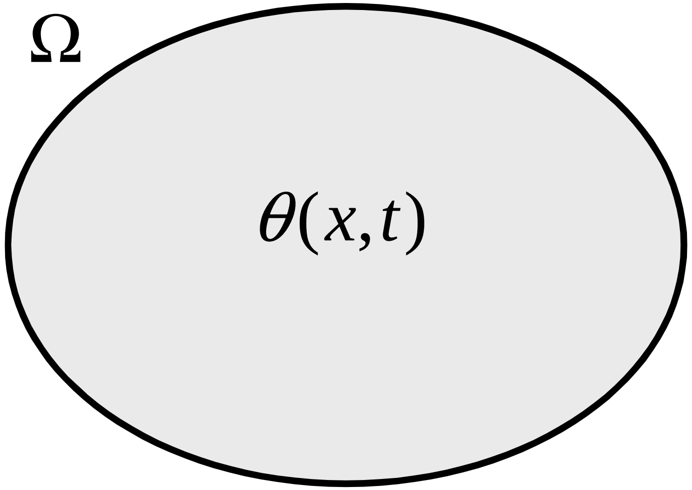
- **Principal Component Analysis**
- **Damage detection**
- **Structural Health Monitoring**
- **Identification of nonlinear parameters**
- **Conclusion**

Proper Orthogonal Decomposition (POD)

Mathematical formulation

Let $\theta(x,t)$ be a random field on a domain Ω

$$\boxed{\theta(x,t)=\overbrace{\mu(x)}^{\text{mean}}+\underbrace{\vartheta(x,t)}_{\text{time varying part}}}$$



At time t_k , the system displays a snapshot $\vartheta^k(x)=\vartheta(x,t_k)$

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Proper Orthogonal Decomposition (POD)

The POD aims at obtaining the most characteristic structure $\phi(x)$ of an ensemble of snapshots i.e.

$$\boxed{\text{Maximize } \left\langle \left| \left(\vartheta^k, \phi \right) \right|^2 \right\rangle \quad \text{with} \quad \left\| \phi \right\|^2 = 1}$$

where $\left(f, g \right) = \int_\Omega f(x)\, g(x)\, d\Omega$

$\langle \cdot \rangle$ denotes the averaging operation

$\| \cdot \|$ denotes the norm

It can be shown that the problem reduces to the following integral eigenvalue problem

$$\boxed{\int_\Omega \underbrace{\left\langle \vartheta^k(x)\, \vartheta^k(x') \right\rangle}_{\text{averaged auto-correlation function}} \phi(x')\, dx' = \lambda\, \phi(x)}$$

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Proper Orthogonal Decomposition (POD)

Thus the solution of the optimization problem

$$\text{Maximize } \left\langle \left| \left(\vartheta^k, \phi \right) \right|^2 \right\rangle \quad \text{with} \quad \left\| \phi \right\|^2 = 1$$

is given by the orthogonal eigenfunctions $\phi_i(x)$ of the integral equation

$$\int_\Omega \left\langle \vartheta^k(x)\, \vartheta^k(x') \right\rangle \phi(x')\, dx' = \lambda\, \phi(x)$$

$\phi_i(x)$ are called the proper orthogonal modes (POM)

λ_i are called the proper orthogonal values (POV)

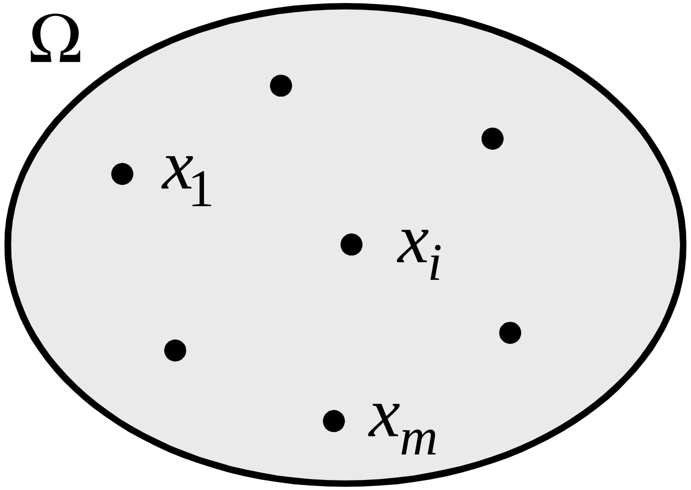
and we have $$\vartheta(x,t) = \sum_{i=1}^{\infty} a_i(t)\, \phi_i(x)$$ where $$a_i(t) = \left(\vartheta(x,t), \phi_i(x) \right)$$

→ uncorrelated coefficients

Proper Orthogonal Decomposition (POD)

In practice, the data are discretized in space and time.

Instrumented structure



Observation matrix:

$$\mathbf{X}_{m\times N} = \overbrace{\begin{bmatrix} x_1(t_1) & \cdots & x_1(t_N) \\ \vdots & \ddots & \vdots \\ x_m(t_1) & \cdots & x_m(t_N) \end{bmatrix}}^{N \text{ snapshots}} \left.\vphantom{\begin{bmatrix} x \\ x \\ x \end{bmatrix}}\right\} \begin{array}{c} m \\ \text{measurement} \\ \text{co-ordinates} \end{array}$$

Proper Orthogonal Decomposition (POD)

The m x m correlation matrix $\mathbf{R}$ is built

$$\boxed{\mathbf{R} = \frac{1}{m}\,\mathbf{X}\,\mathbf{X}^T}$$

The eigenvalue problem is solved

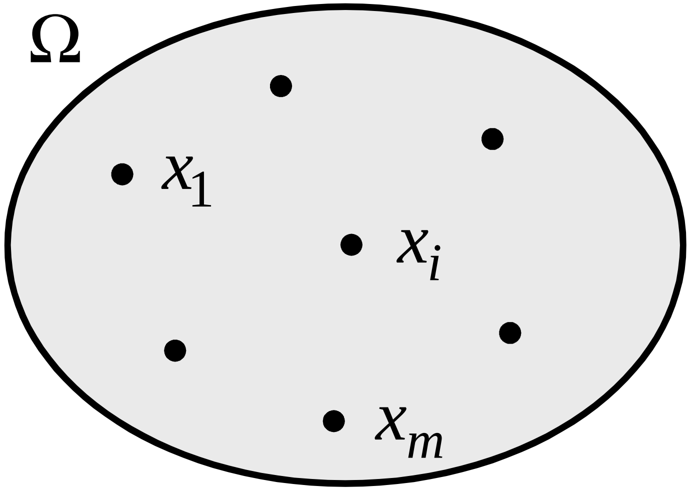
eigenvalues (POVs)

$$\boxed{\mathbf{R}\,\mathbf{u} = \lambda\,\mathbf{u}}$$

eigenvectors of $\mathbf{X}\,\mathbf{X}^T$ (POMs)

Principal Component Analysis (PCA)

Computation of the POMs using SVD



m measurement co-ordinates

N time samples

$$\mathbf{X}_{m\times N} = \begin{bmatrix} x_1(t_1) & \cdots & x_1(t_N) \\ \vdots & \ddots & \vdots \\ x_m(t_1) & \cdots & x_m(t_N) \end{bmatrix}$$

Using SVD

$$diag(\sqrt{\lambda_i}) \quad (\lambda_i \equiv \text{POV})$$

$$\boxed{\mathbf{X}_{m\times N} = \mathbf{U}_{m\times m}\,\Sigma_{m\times N}\,\mathbf{V}^T_{N\times N}}$$

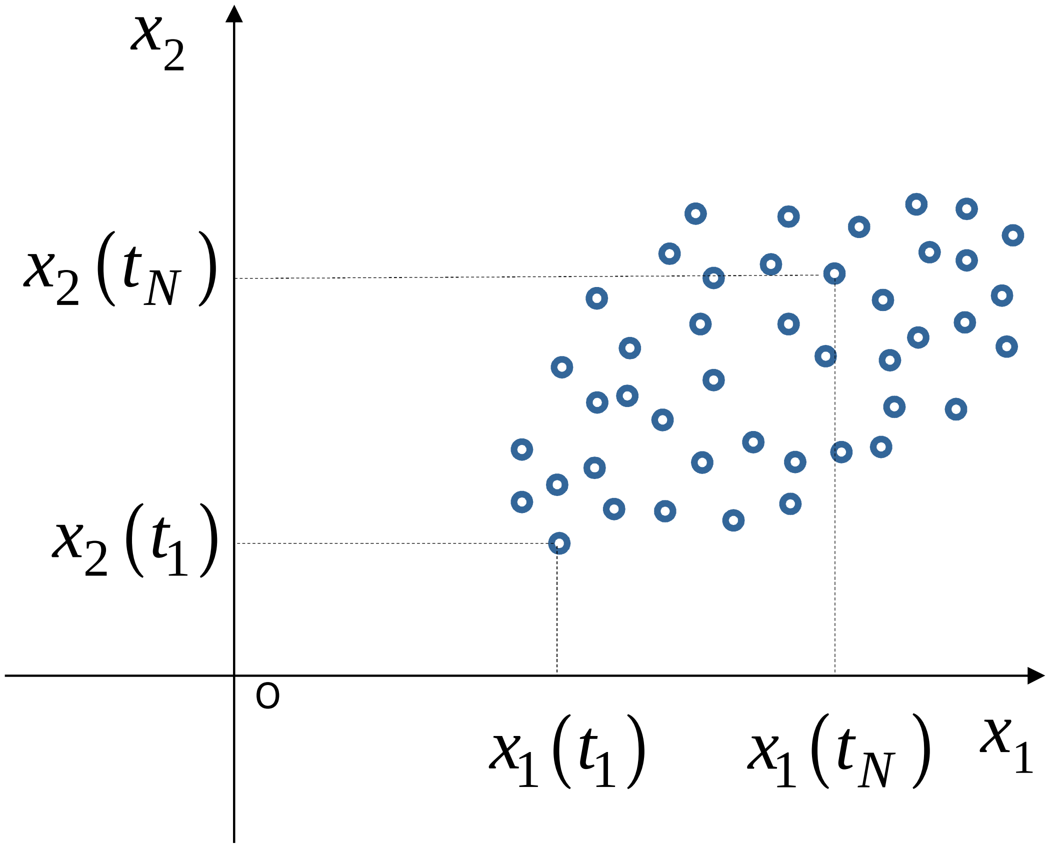
eigenvectors of $\mathbf{X}\,\mathbf{X}^T$ (POM)

Principal Component Analysis (PCA)

PCA in 2D-space

Geometric interpretation

$$\mathbf{X} = \begin{bmatrix} x_1(t_1) & x_1(t_2) & \cdots & x_1(t_N) \\ x_2(t_1) & x_2(t_2) & \cdots & x_2(t_N) \end{bmatrix}$$

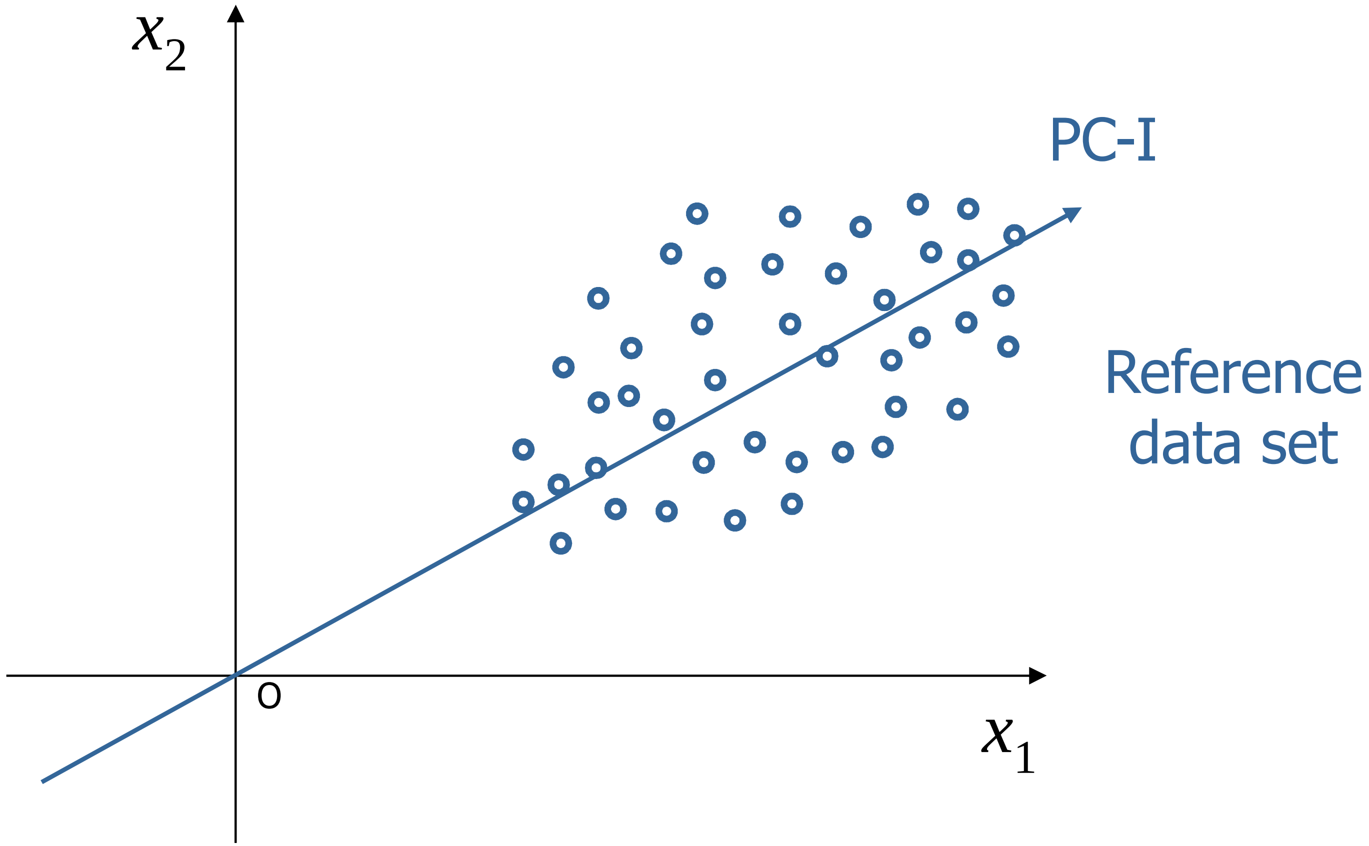


Reference data set

Principal Component Analysis (PCA)

Geometric interpretation

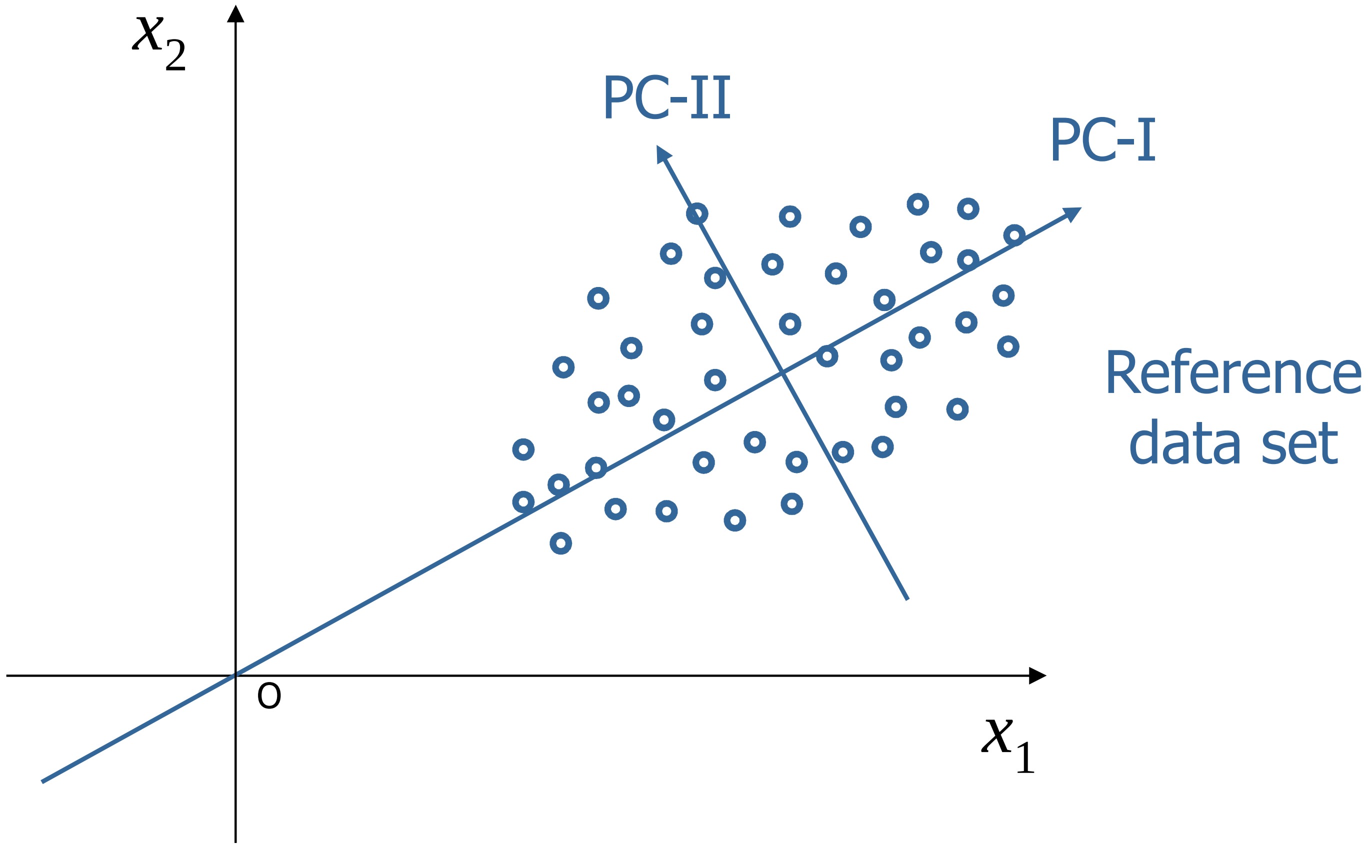
PCA in 2D-space



Principal Component Analysis (PCA)

Geometric interpretation

PCA in 2D-space



Outline

- Principal Component Analysis (PCA)
- **Damage detection**
- Structural Health Monitoring
- Identification of nonlinear parameters
- Conclusion

Principal Component Analysis (PCA)

$$\boxed{\mathbf{M}\,\ddot{\mathbf{x}}+\mathbf{C}\,\dot{\mathbf{x}}+\mathbf{K}\,\mathbf{x}=\mathbf{p}(t)}$$

Modal Analysis

Deterministic approach

Eigenvalue problem:

$$\boxed{(\mathbf{K}-\omega^2\mathbf{M})\,\boldsymbol{\Phi}=\mathbf{0}}$$

Response:

$$\boxed{\mathbf{x}(t)=\sum_{i=1}^{n}\eta_i(t)\,\boldsymbol{\Phi}_{(i)}}$$

→ **Spatial information**

where

$$\boxed{\eta_i=A_i\cos(\omega_i\,t)+B_i\sin(\omega_i\,t)}$$

→ **Natural frequencies**

Principal Component Analysis

Statistical approach

Eigenvalue problem:

$$\boxed{\mathbf{R}\,\mathbf{u}=\lambda\mathbf{u}}$$

$$\boxed{\mathbf{x}(t)=\sum_{j=1}^{n}a_j(t)\,\mathbf{u}_{(j)}}$$

→ **Spatial information**

→ **Time information**

→ **Instantaneous frequencies**

Damage detection problem

Concept of subspace angle (Golub-Van Loan)

Key idea

- Use PCA to extract the structural response subspace
- Use the concept of subspace angles to compare the hyperplanes associated with the reference (undamaged) state and with the current (possibly damaged?) state of the structure.

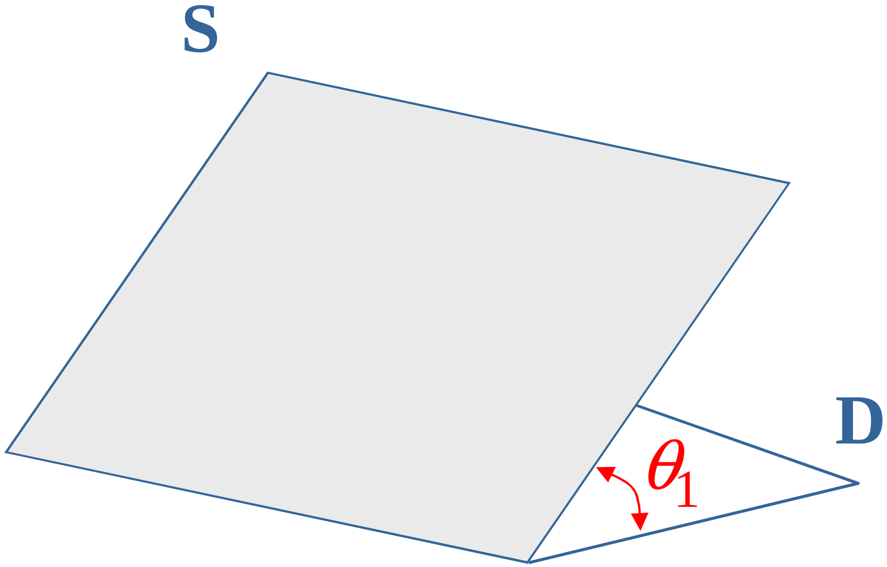
Damage detection problem

Concept of subspace angle (Golub-Van Loan)

Given two subspaces $\mathbf{S} \in \Re^{n \times p}$ and $\mathbf{D} \in \Re^{n \times q}$ $(p > q)$

Carry out the QR-factorizations $\mathbf{S} = \mathbf{Q}_S \, \mathbf{R}_S$ and $\mathbf{D} = \mathbf{Q}_D \, \mathbf{R}_D$

define orthonormal bases



The angles θ_i between subspaces $\mathbf{S}$ and $\mathbf{D}$ are defined through the singular values associated to

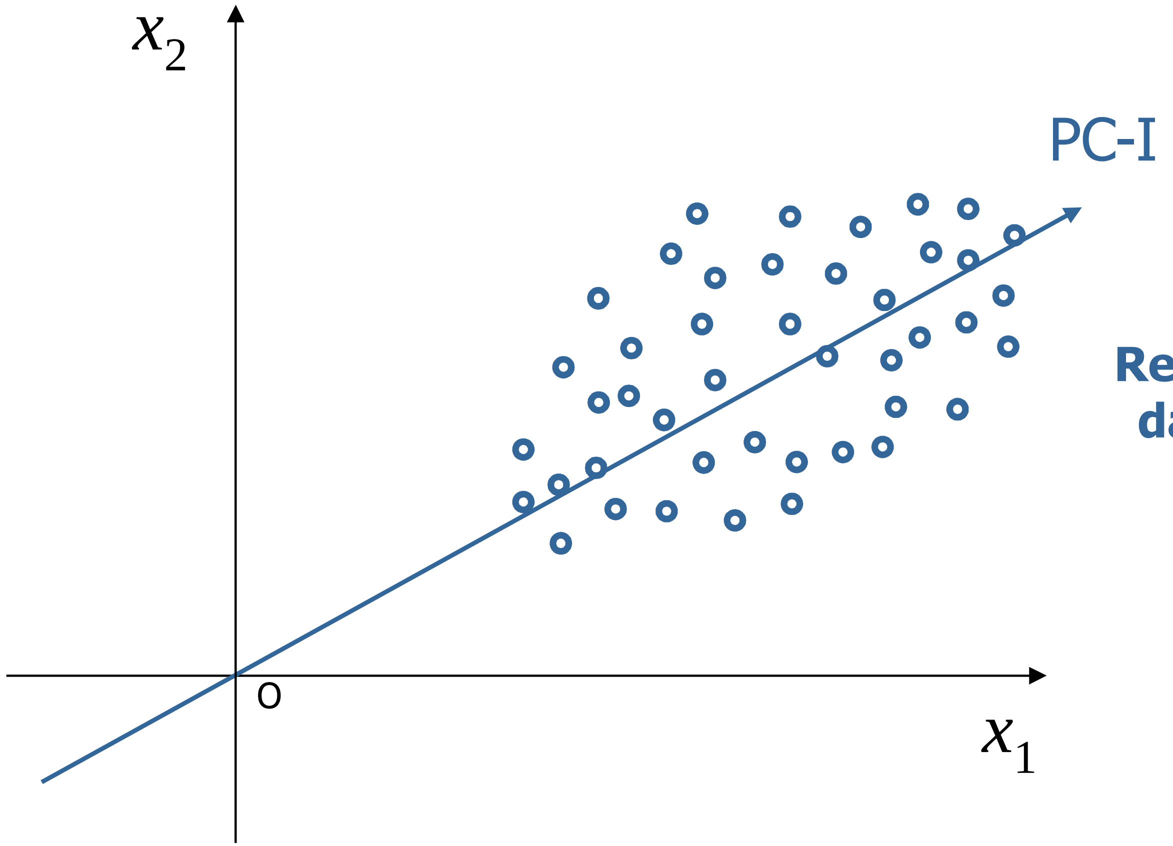
$$\mathbf{Q}_S^T \, \mathbf{Q}_D = \mathbf{U}_{SD} \, \mathbf{\Sigma}_{SD} \, \mathbf{V}_{SD}^T$$

$$\mathbf{\Sigma}_{SD} = diag(\cos\theta_i) \quad (i = 1, \ldots, q)$$

Damage detection problem

Geometric interpretation

PCA in 2D-space

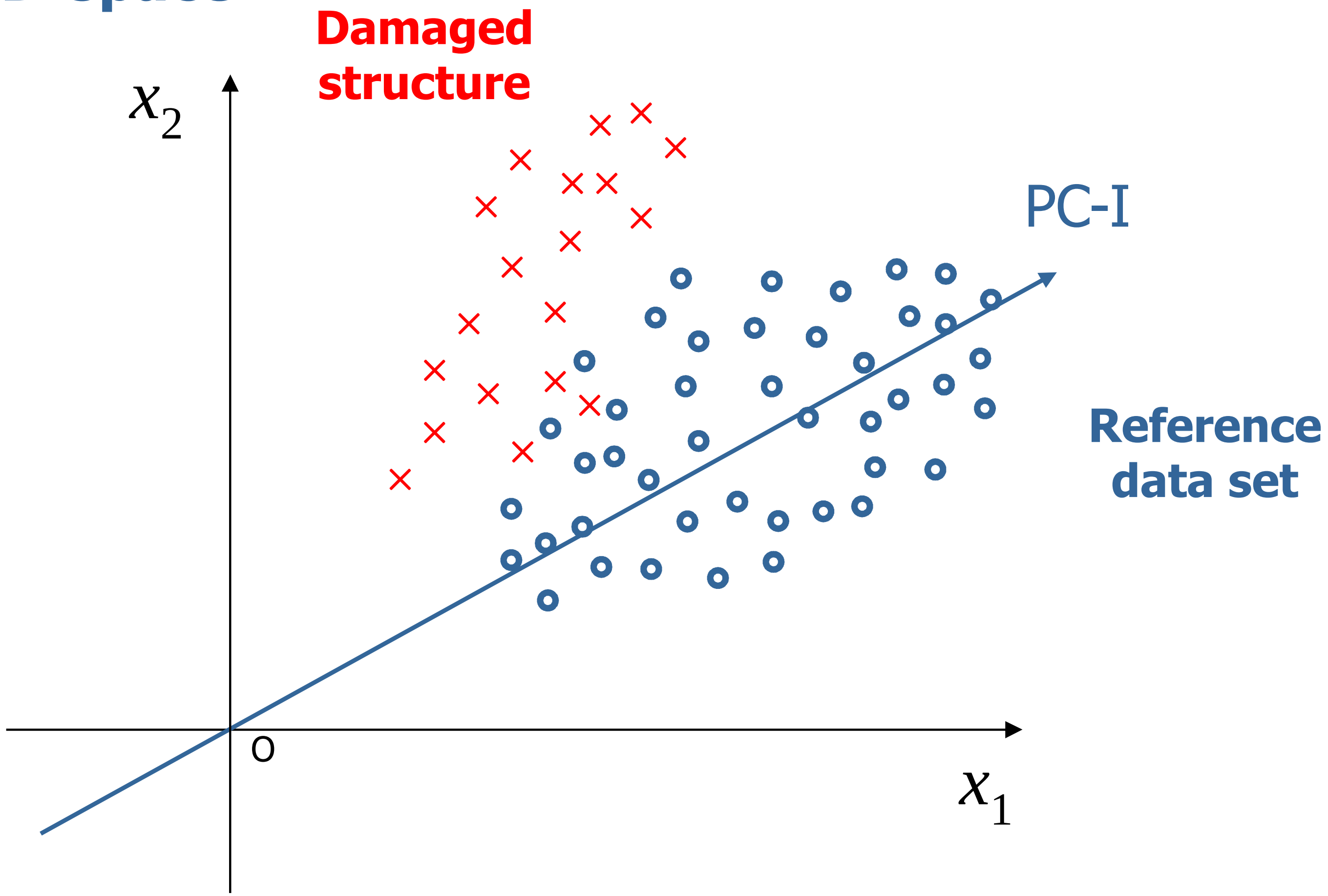


Reference data set

Damage detection problem

Geometric interpretation

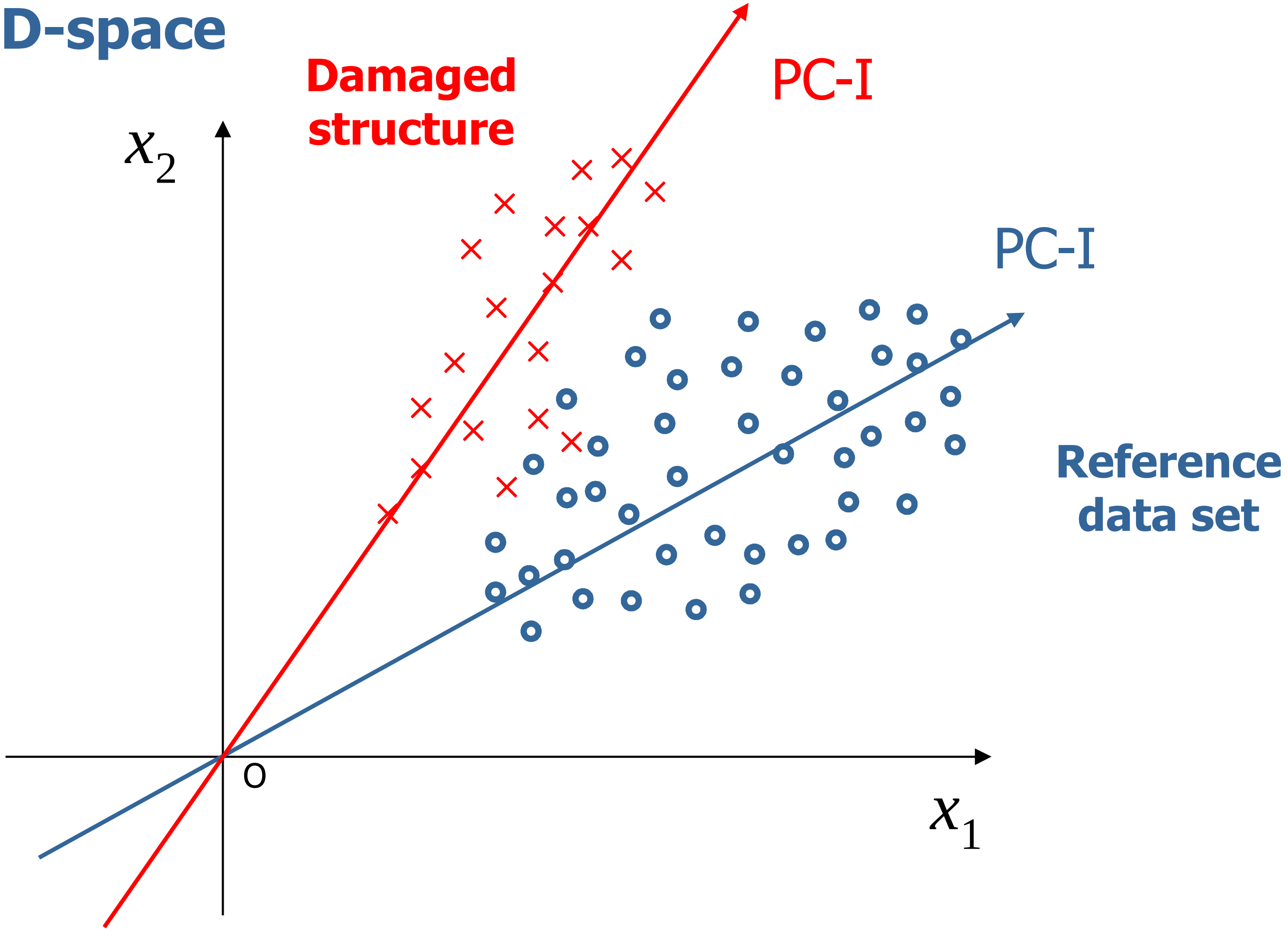
PCA in 2D-space



Damage detection problem

Geometric interpretation

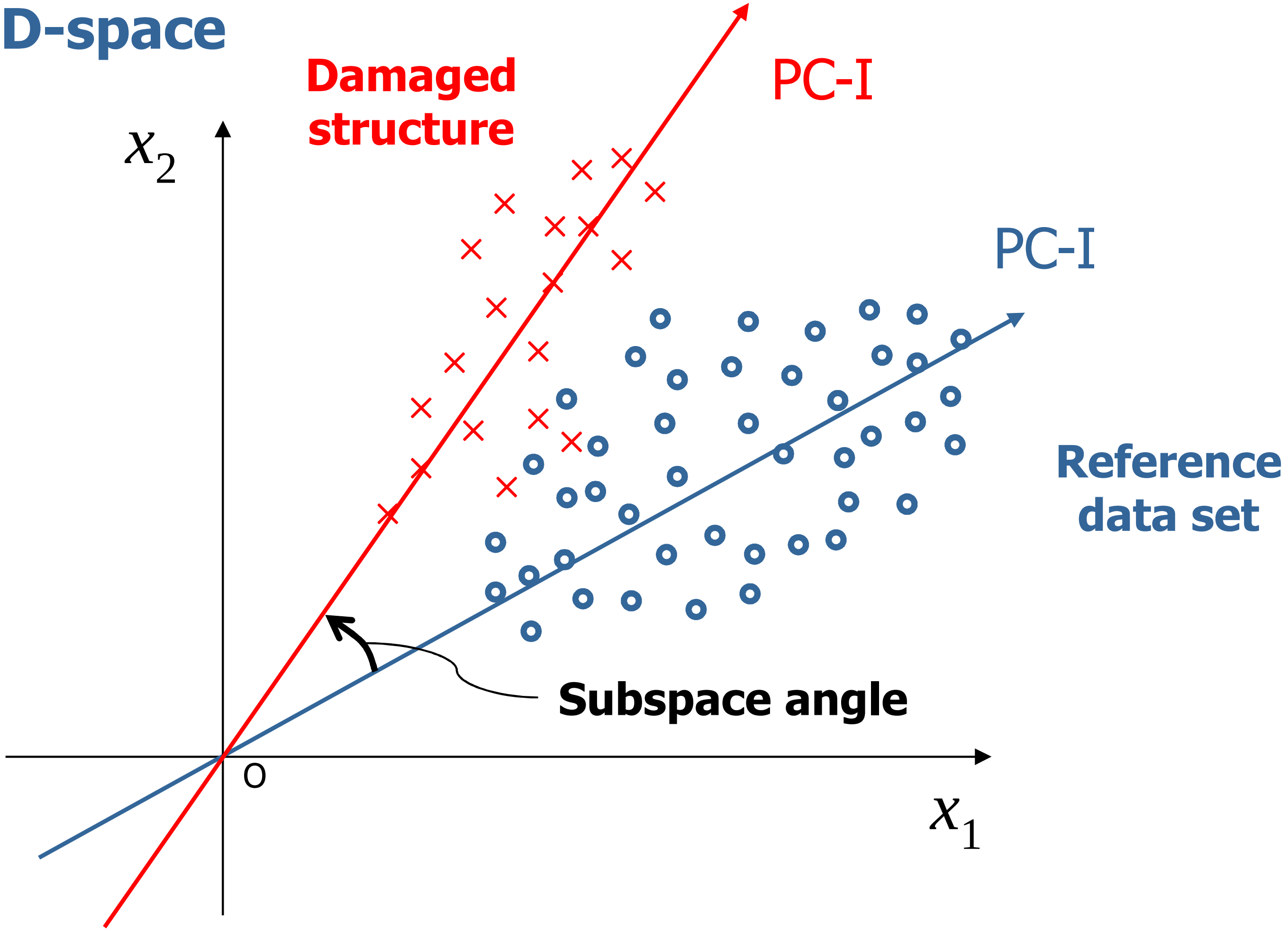
PCA in 2D-space



Damage detection problem

Geometric interpretation

PCA in 2D-space



Damage detection problem

Novelty analysis → the aim is to build a prediction model using the principal components of reference.

Consider the transformation matrix $\mathbf{T}$ containing the p first eigenvectors associated with the largest eigenvalues.

PCA provides a linear mapping of the measured data from the original dimension m to a lower dimension p

$$\boxed{\underset{\text{score matrix}}{\mathbf{Y}_{p\times N}} = \overbrace{\begin{bmatrix} \mathbf{u}_{(1)} & \cdots & \mathbf{u}_{(p)} \end{bmatrix}}^{\text{loading matrix } \mathbf{T} \text{ of dimension } m \text{ x } p}{}^{T} \; \mathbf{X}_{m\times N}}$$

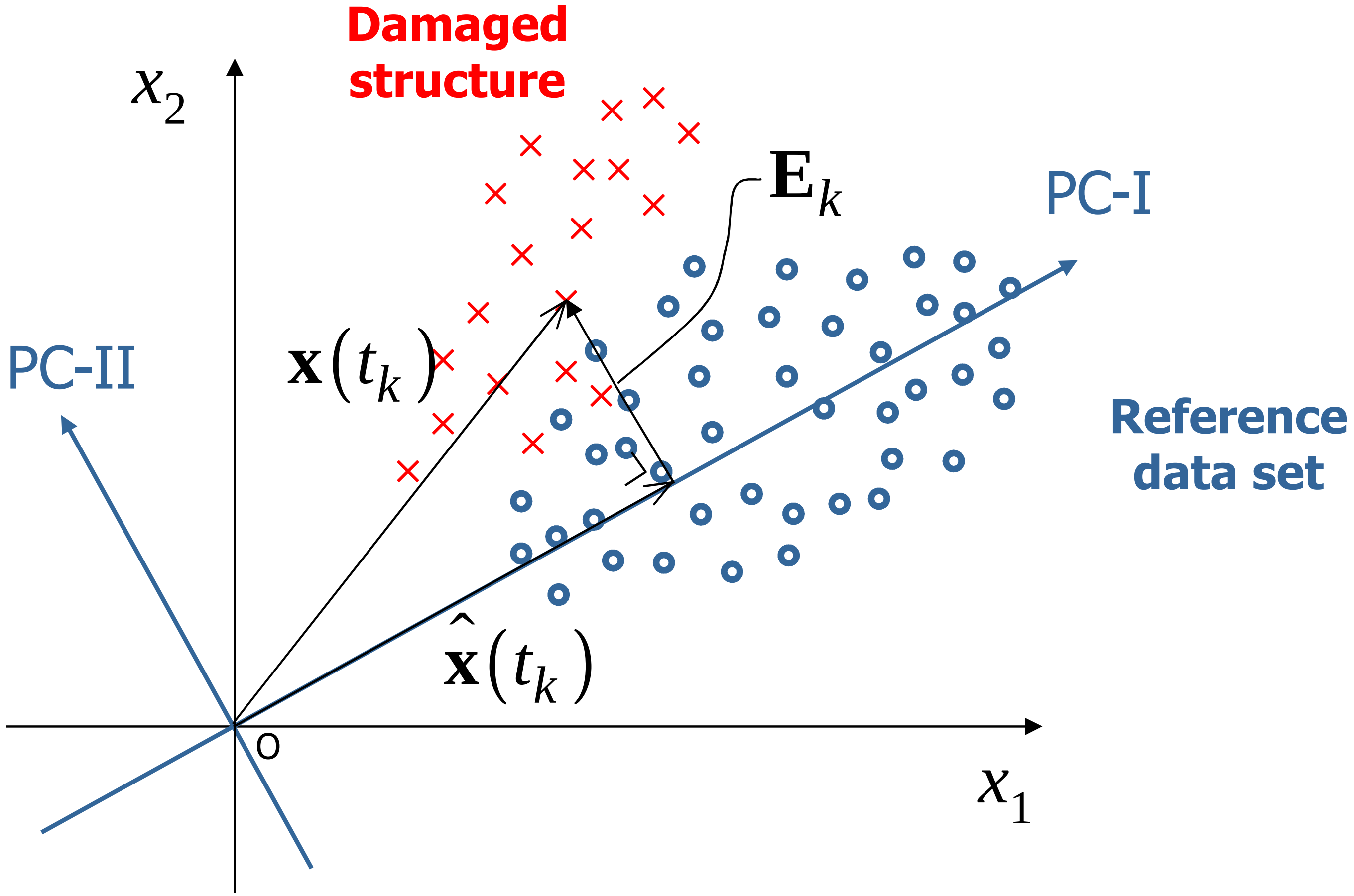
Re-mapping of the projected data back to the original subspace gives

$$\text{estimated} \leftarrow \boxed{\hat{\mathbf{X}} = \mathbf{T}^T \; \mathbf{Y} = \mathbf{T}^T \; \mathbf{T} \, \mathbf{X}}$$

The residual error matrix is defined as: $\boxed{\mathbf{E} = \mathbf{X} - \hat{\mathbf{X}}}$

Damage detection problem

Geometric interpretation



Damage detection problem

Definition of the Novelty Index

Residual error matrix : $\boxed{\mathbf{E} = \mathbf{X} - \hat{\mathbf{X}}}$

Euclidean norm : $\boxed{NI_k^E = \left\| \mathbf{E}_k \right\|}$

↳ prediction error vector at time t_k

Mahalanobis norm : $\boxed{NI_k^M = \sqrt{\mathbf{E}_k^T \ \mathbf{R}^{-1} \ \mathbf{E}_k}}$

↳ $\mathbf{R} = \dfrac{1}{N} \mathbf{X} \ \mathbf{X}^T$ (covariance matrix)

Damage detection problem

damaged state

The ratio $\boxed{\dfrac{NI_d}{NI_r}}$ may be used as a quantitative indicator of damage level.

reference state

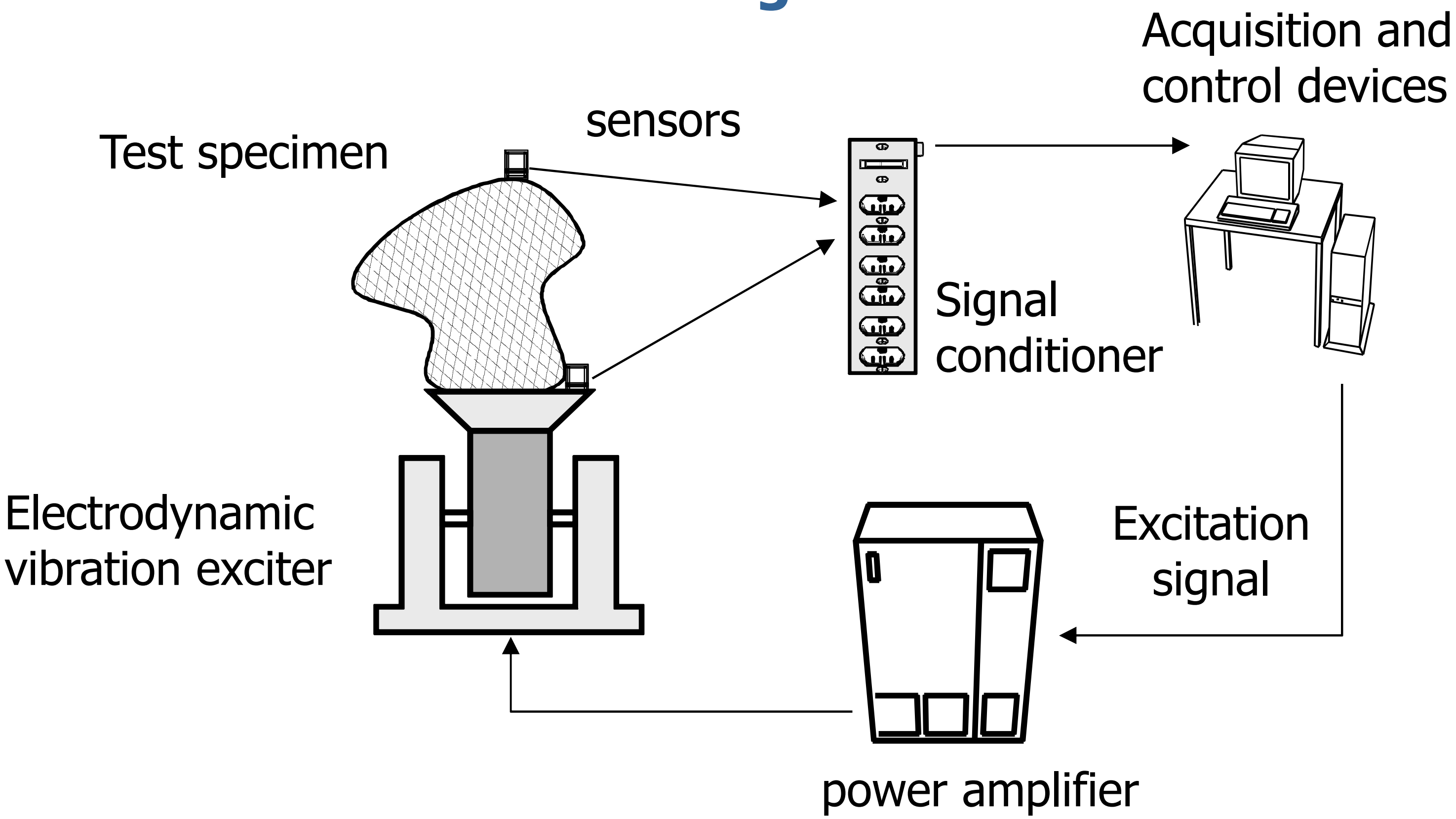
mean value

Statistical tool : $\boxed{CL = \overline{NI} + 3\,\sigma}$ (Upper Control Limit at 99.7 % confidence interval)

standard deviation

Example of damage detection

Environmental Vibration Testing



Detection of damages usually by visual inspection or by comparison of frequency spectra before and after the test.

Objective : to be able to detect damage as soon as it appears.

Example of damage detection

Fatigue testing of a street lighting device

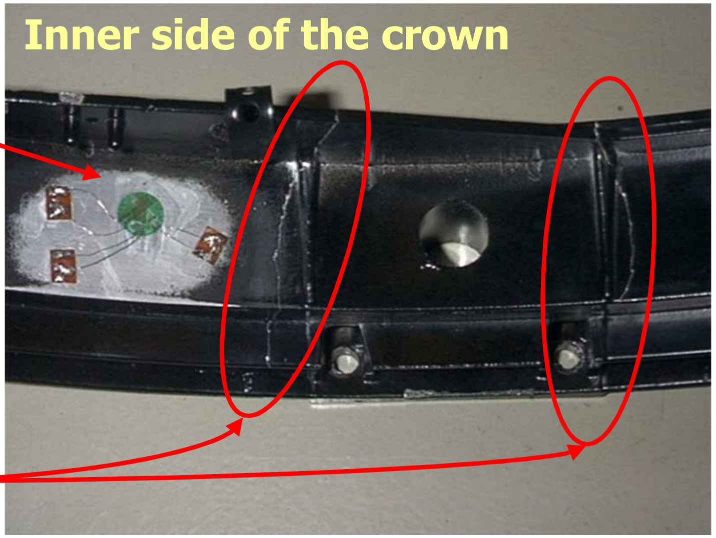
Control accelerometer

+ 10 measurement accelerometers

Vibration specifications

- Sine excitation at the first resonance frequency (~ 12,4 Hz) during 1 hour.
- Acceleration level of 0,5 g at the fixation.

Strain gauges

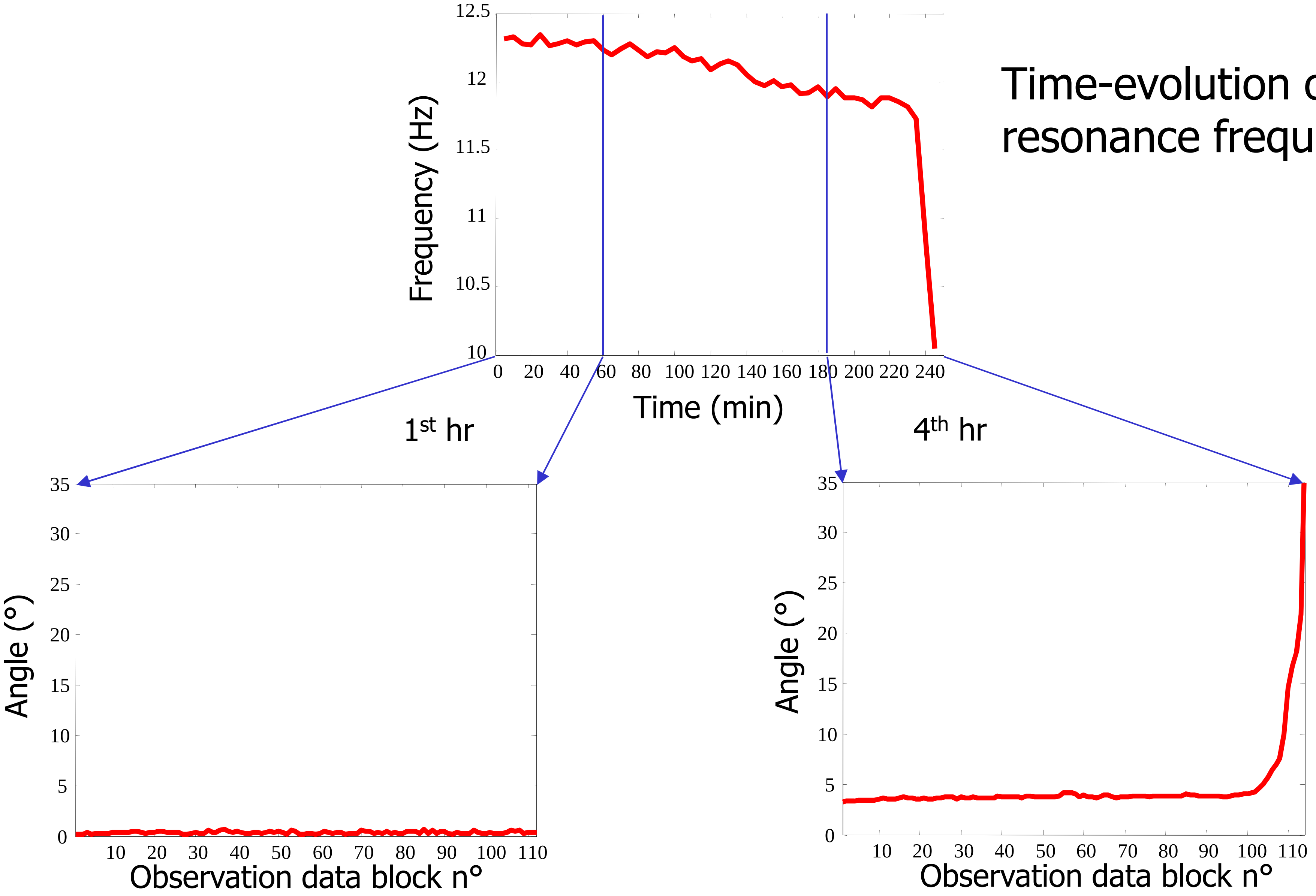


Total test duration: ~ 4 hours

Mode of failure of the crown

Crack initiation and propagation

Damage detection (Results)



Time-evolution of the resonance frequency

Time-evolution of the angle

Principal Component Analysis (PCA)

Limitation of the PCA-based method

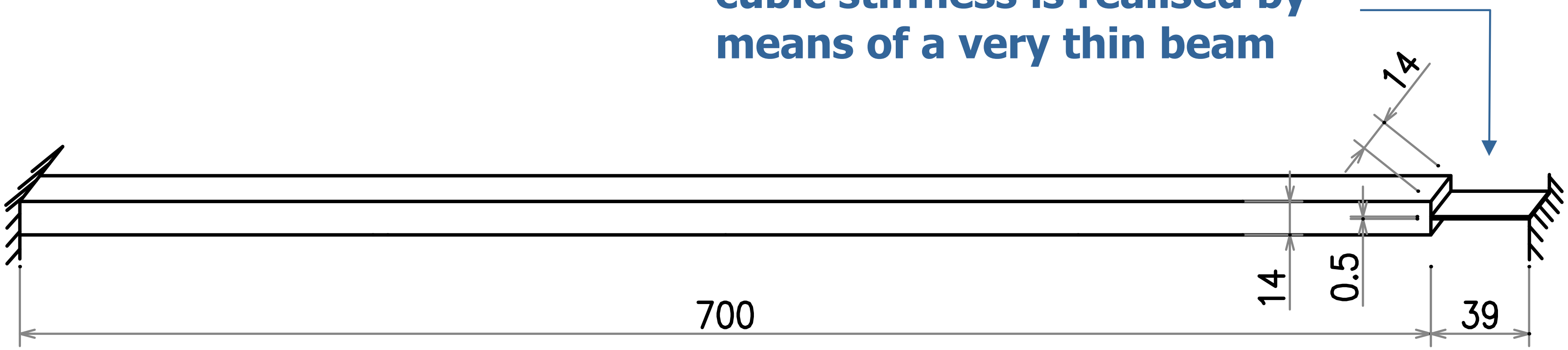
The number of sensors must be larger than the number of active modes → it can be solved using the concept of null-subspace of the Hankel matrices of responses.

Null Subspace Analysis (NSA)

Aim : to replace the observation matrix $\mathbf{X}$ by a "dynamic" response matrix (i.e. the Hankel matrix)

Detection of nonlinearity onset

Example: beam with nonlinear stiffness used as benchmark in the framework of the European action COST F3.


cubic stiffness is realised by means of a very thin beam

For weak excitation, the system behaviour may be considered as linear. When the excitation level increases, the thin beam exhibits large displacements and a nonlinear geometric effect is activated resulting in a stiffening effect at the end of the main beam.

Detection of nonlinearity onset

Impact excitation

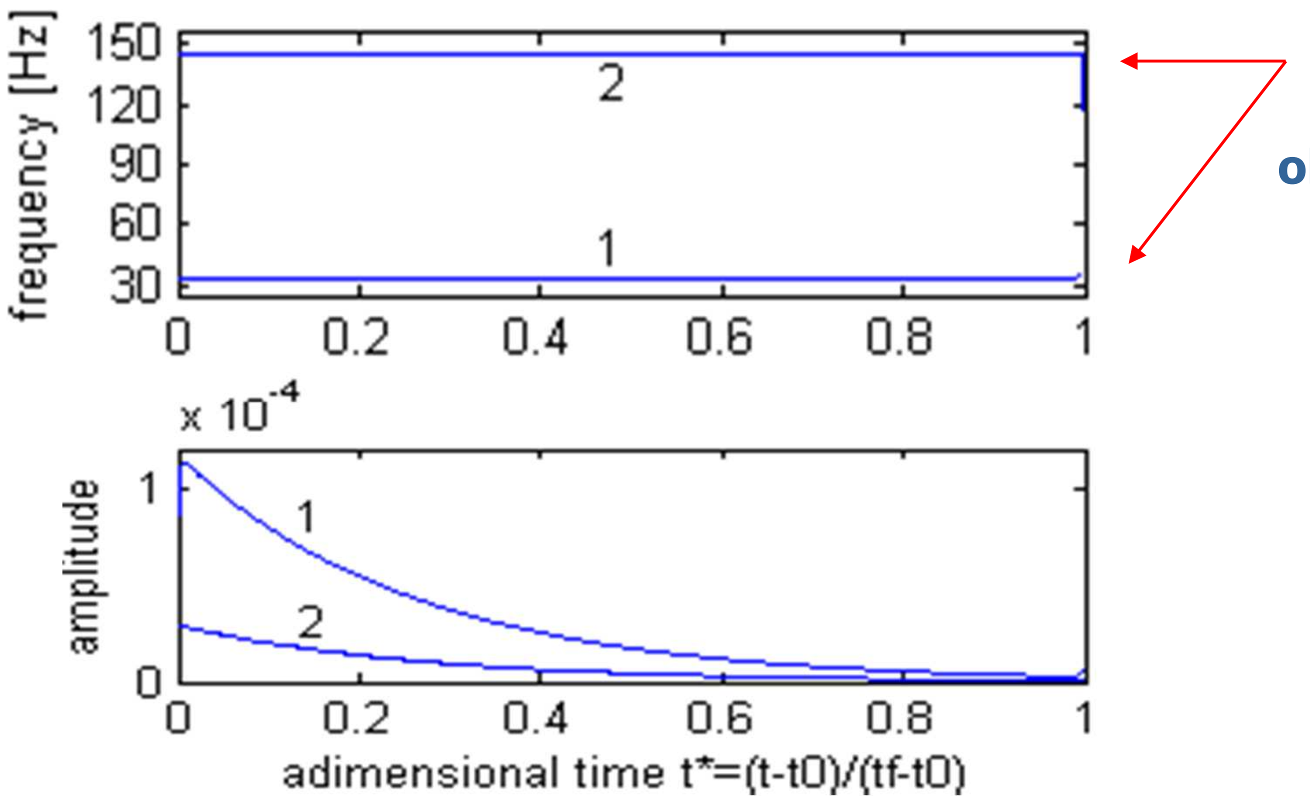
Test n°	1-7	8	9	10	11	12
Largest displacement (*mm*)	< 0.04	0.48	0.72	0.93	1.20	1.37

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{The nonlinearity is activated}}$$

Detection of nonlinearity onset

Detection based on the Wavelet Transform (WT)

WT of the displacement at coordinate n° 7



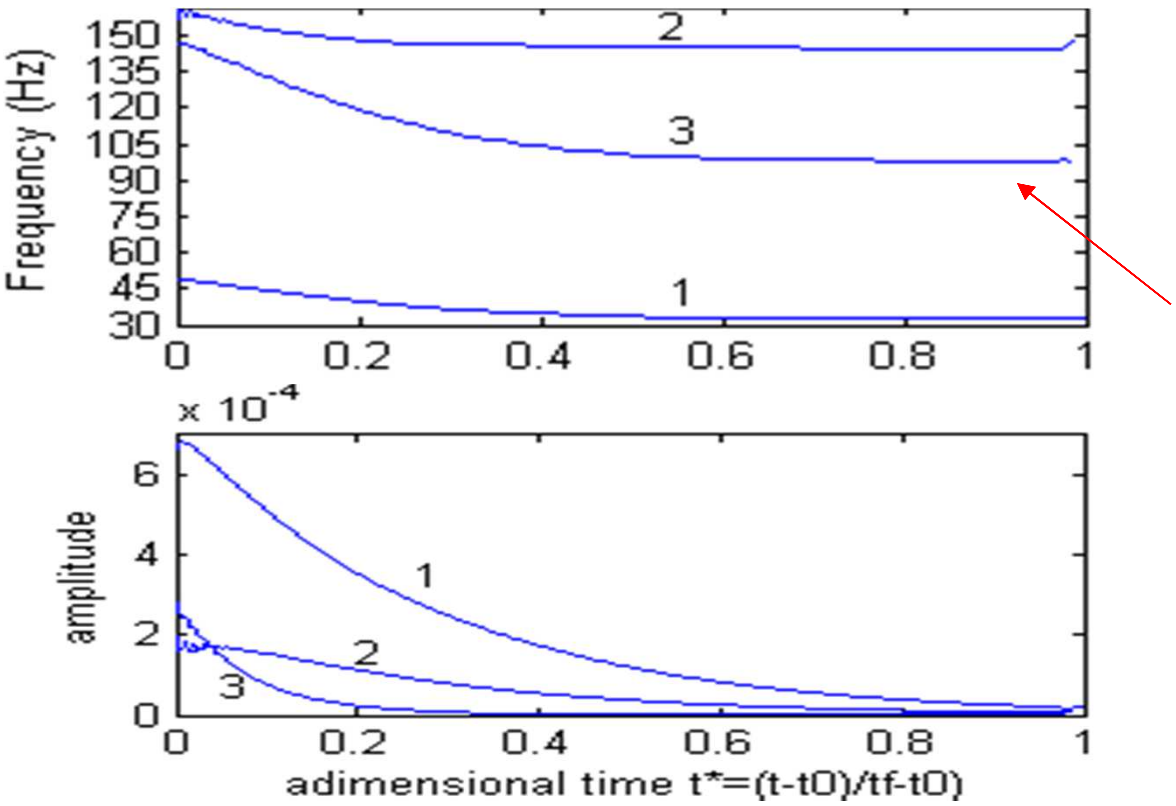
Two frequency lines (called 'ridges') are observed respectively at 32.6 Hz and 144.7 Hz

→ at low excitation level (impact of 70N), the behaviour of the beam appears as linear (largest displacement lower than 0.15 mm).

Detection of nonlinearity onset

Detection based on the Wavelet Transform (WT)

WT of the displacement at coordinate n° 7

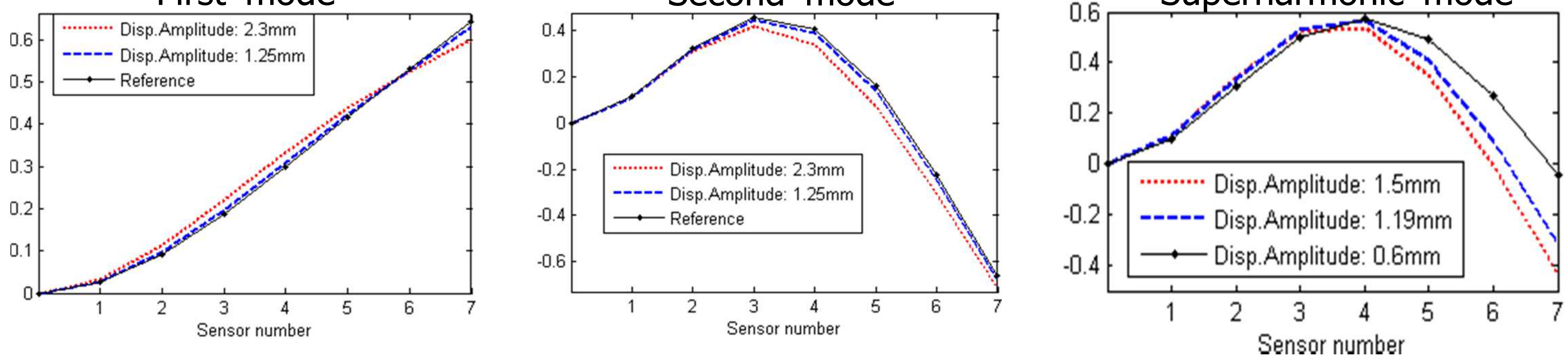


- **drop-off of the frequencies down to the linear system values as the nonlinear effect vanishes progressively**
- **presence of a third order superharmonic of the first frequency (curve n° 3)**

→ at high level of excitation (impact of 1500N), the behaviour is clearly nonlinear (maximum displacement at the right end of about 2.4 mm).

Detection of nonlinearity onset

Instantaneous deformation shapes



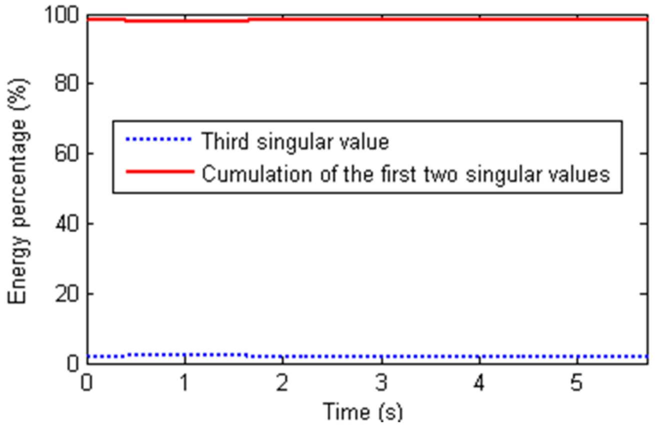
First 'mode'

Second 'mode'

Superharmonic 'mode'

Defining an instantaneous deformation matrix **A** = [$\mathbf{M}_1$ $\mathbf{M}_2$ $\mathbf{M}_3$].

The instantaneous singular values of the decomposition in terms of energy percentage reveals that the third singular value is negligible.

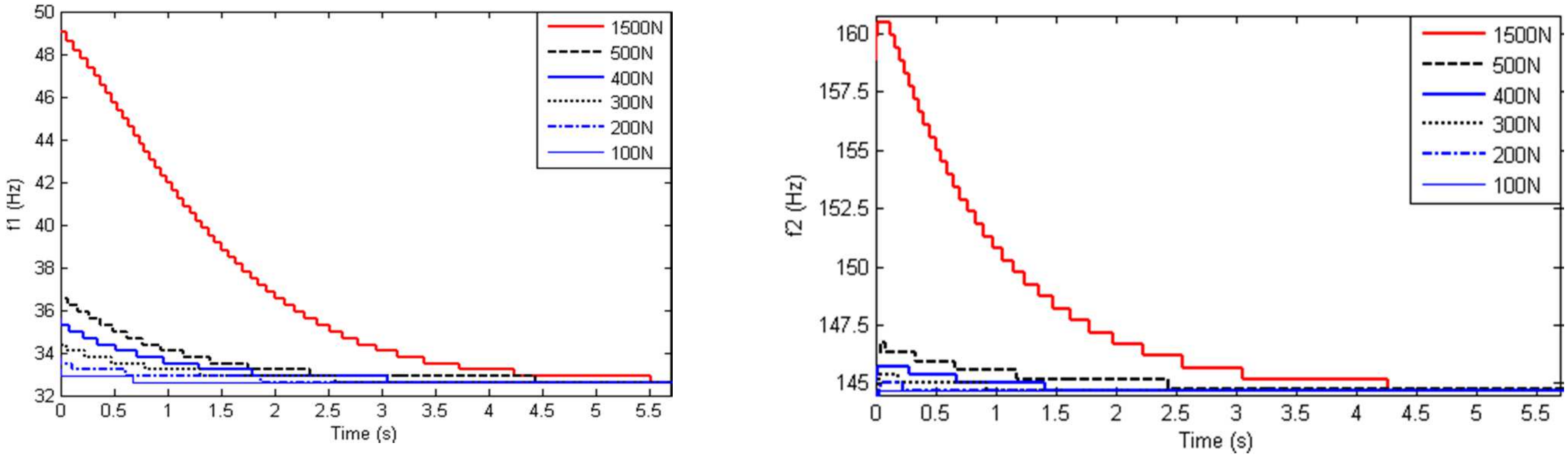


→ the third (superharmonic) deformation 'mode' ($\mathbf{M}_3$) is actually a linear combination of the two other 'modes'.

Detection of nonlinearity onset

Detection based on the concept of subspace angle

- the structure is now excited at increasing level of impact forces (amplitudes ranging from 100 N to 1500 N).

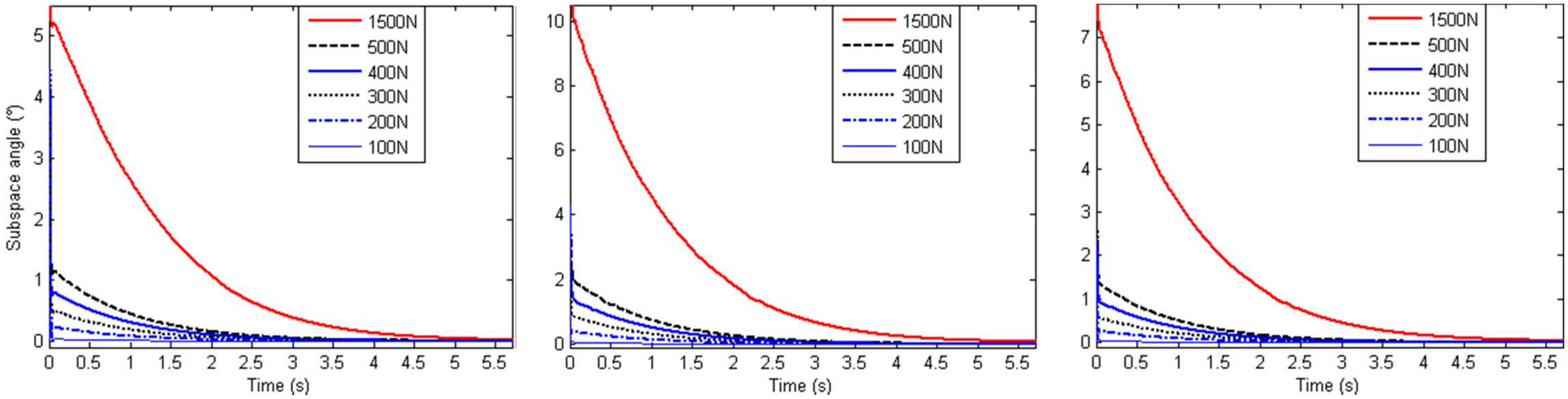


Evolution of the instantaneous frequencies

Detection of nonlinearity onset

The instantaneous deformation shapes associated to the two frequencies may be considered as instantaneous active modes to define a subspace which characterises the dynamic state of the structure. The comparison of subspace angles between the reference state (defined by the linear normal modes) and current states at different excitation levels reveals the range of activation of the nonlinearity.

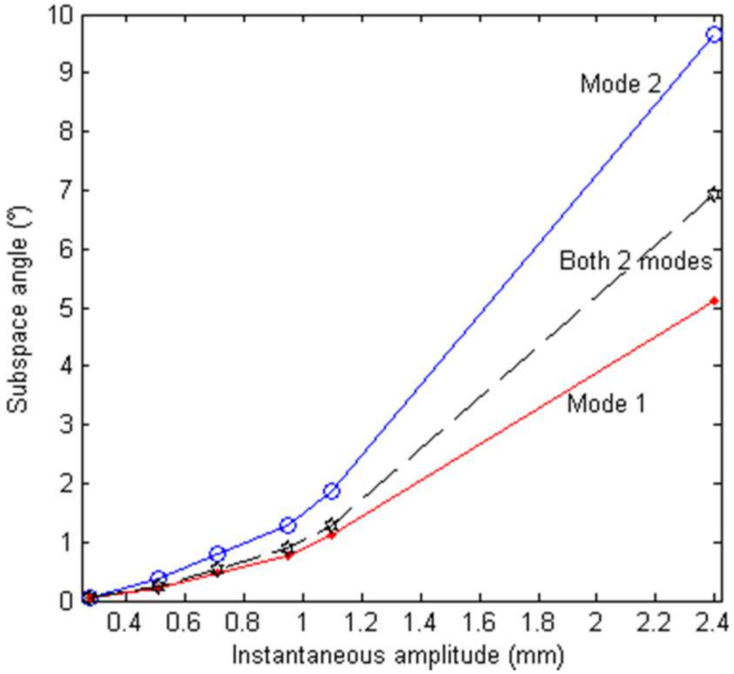
Time evolution of subspace angles for different excitation levels



a) Based on the 1st mode **b) Based on the 2nd mode** **c) Based on both modes**

Detection of nonlinearity onset

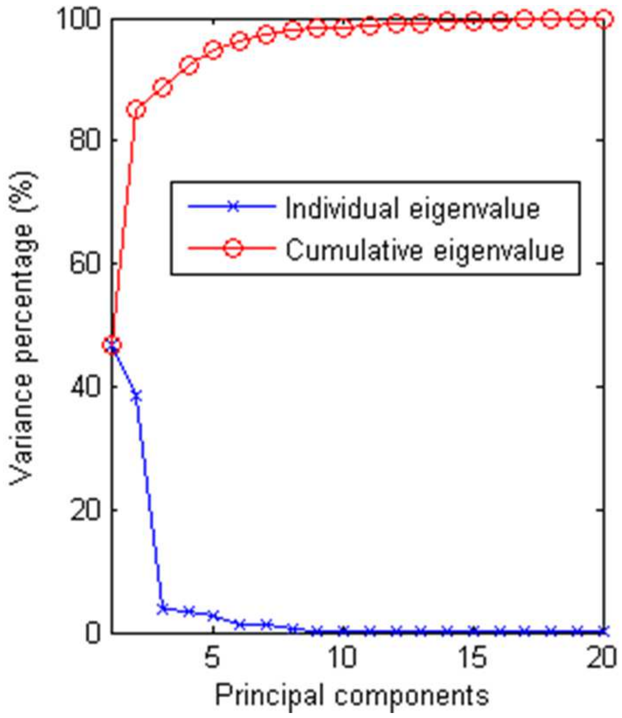
Angle–displacement amplitude at the end of the beam at $t = 0.1\ s$



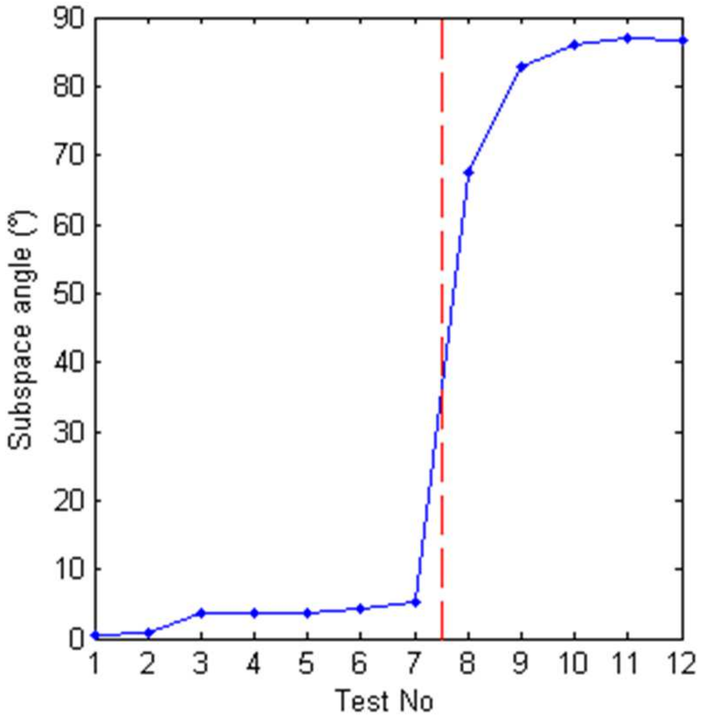
Detection of nonlinearity onset

EKPCA-based detection method

Eigenvalue diagram



EKPCA detection based on the subspace angle



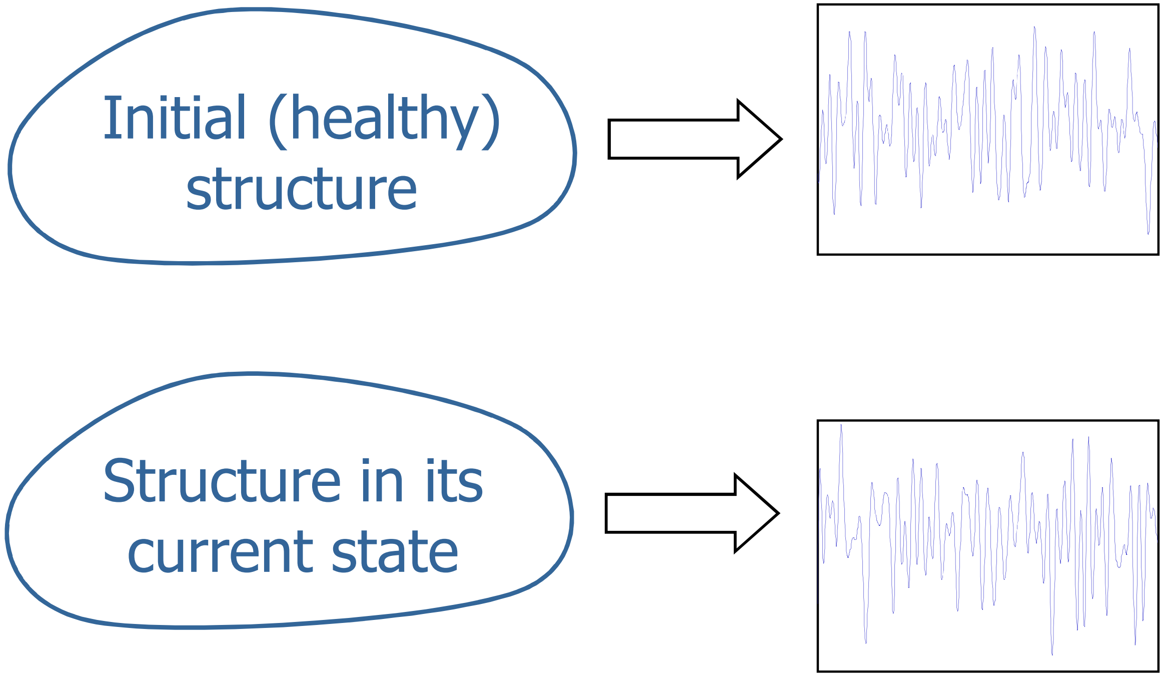
Outline

- Principal Component Analysis (PCA)
- Damage detection
- **Structural Health Monitoring**
- Identification of nonlinear parameters
- Conclusion

Structural Health Monitoring (SHM) process (1)

The SHM process involves three steps:

1) the observation of a system over time using periodically sampled dynamic measurements from an array of sensors,



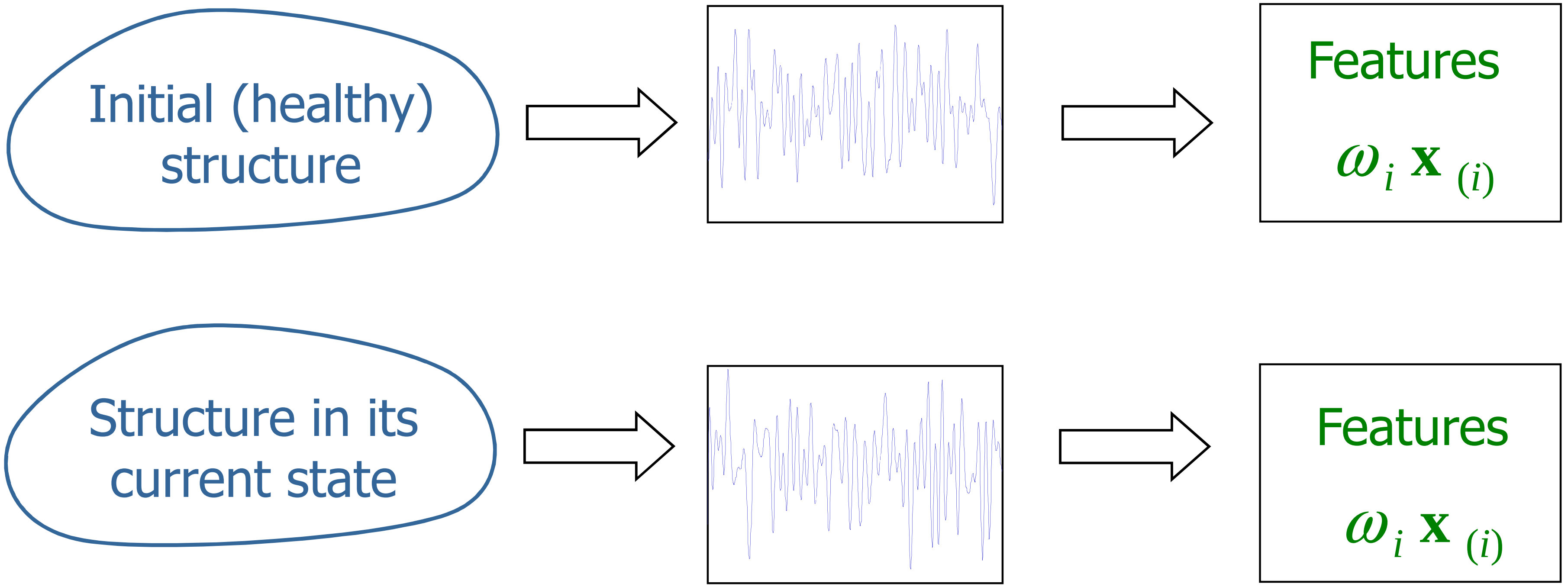
Initial (healthy) structure

Structure in its current state

Structural Health Monitoring (SHM) process (2)

The SHM process involves three steps:

1) the observation of a system over time using periodically sampled dynamic measurements from an array of sensors,
2) the extraction of damage-sensitive features from these measurements,



Initial (healthy) structure

Features

$\omega_i \; \mathbf{x}_{(i)}$

Structure in its current state

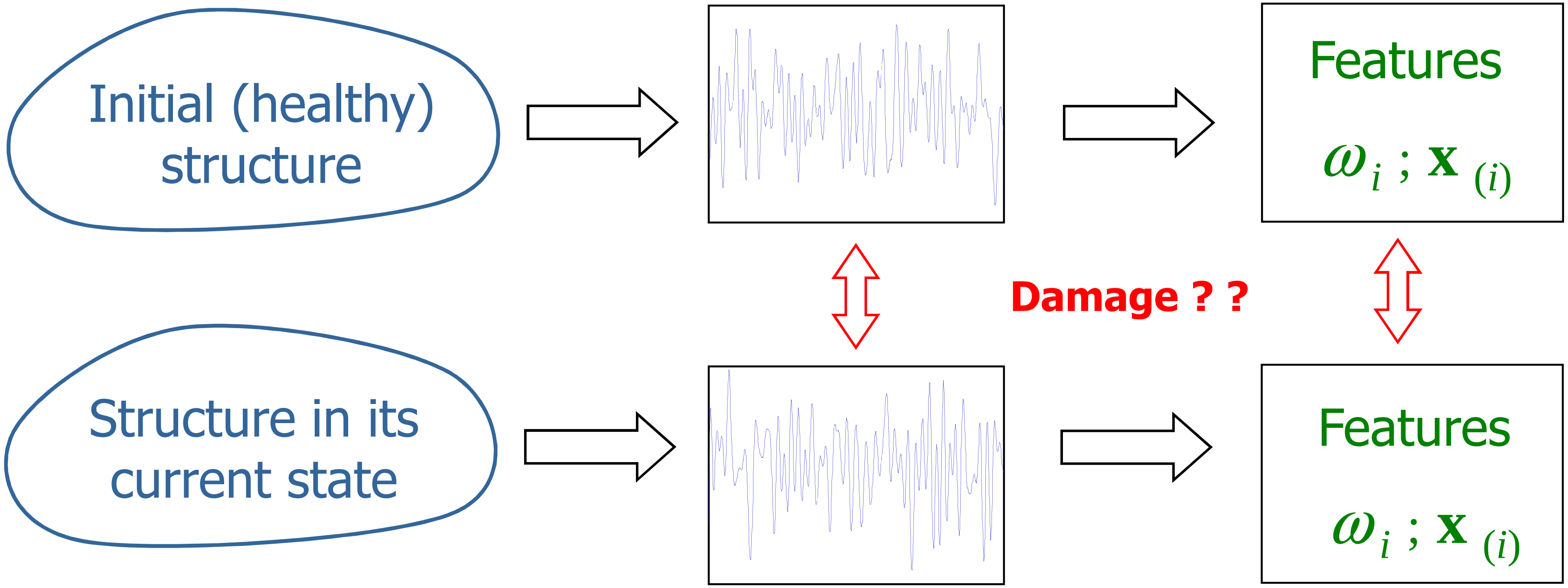
Features

$\omega_i \; \mathbf{x}_{(i)}$

Structural Health Monitoring (SHM) process (3)

The SHM process involves three steps:

1) the observation of a system over time using periodically sampled dynamic measurements from an array of sensors,
2) the extraction of damage-sensitive features from these measurements,
3) the statistical analysis of these features to determine if the structure is healthy or damaged.



Initial (healthy) structure

Features

$\omega_i \; ; \; \mathbf{x}_{(i)}$

Damage ? ?

Structure in its current state

Features

$\omega_i \; ; \; \mathbf{x}_{(i)}$

Damage problem

The damage state of a system can be described as a five-step process (Rytter, 1993) to answer the following questions.

- Level 1: **Existence**. Is there damage in the system?
- Level 2: **Location**. Where is the damage in the system?
- Level 3: **Type**. What kind of damage is present?
- Level 4: **Extent**. How severe is the damage?
- Level 5: **Prognosis**. How much useful life remains?

Some difficulties

Categories of false indications of damage

- False-positive damage indication = indication of damage when none is present.
- False-negative damage indication = no indication of damage when damage is present.

→ Use of statistical procedures to increase robustness

Influence of environmental conditions

- e.g. temperature variations in civil engineering structure

The Champangshiehl bridge (1)

- Located in Luxembourg
- Two spans (102m total) concrete box girder bridge built in 1966
- 112 pre-stressed steel cables
- Destruction for territory development purpose

Views of the Champangshiehl bridge (Luxembourg)

The Champangshiehl bridge (2)

Views of the Champangshiehl bridge (Luxembourg)

Context

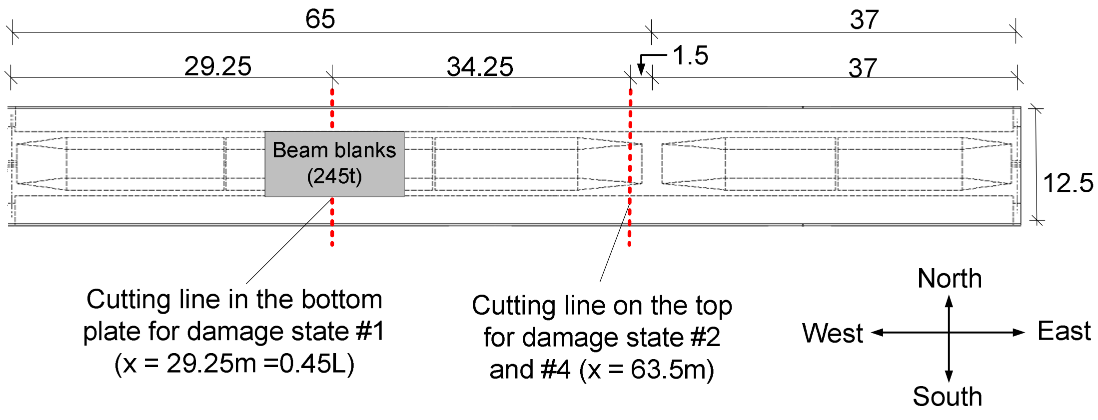
- Project: « Dynamic and Static evaluation of civil engineering structures » led by the University of Luxembourg
  - **Aim:** to assess the feasibility of non-destructive testing methods for condition monitoring of civil structures
  - **Mean:** introduce controlled damage in the Champangshiehl bridge

- Collaboration with the University of Liège to test damage detection techniques.

The Champangshiehl bridge (3)

Damage scenarios

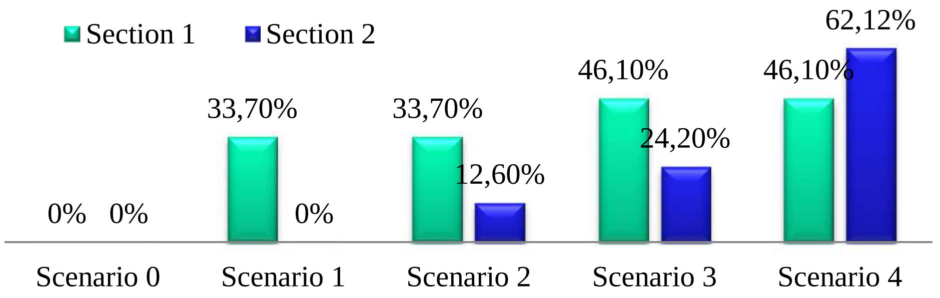
Bridge top-view – cutting sections



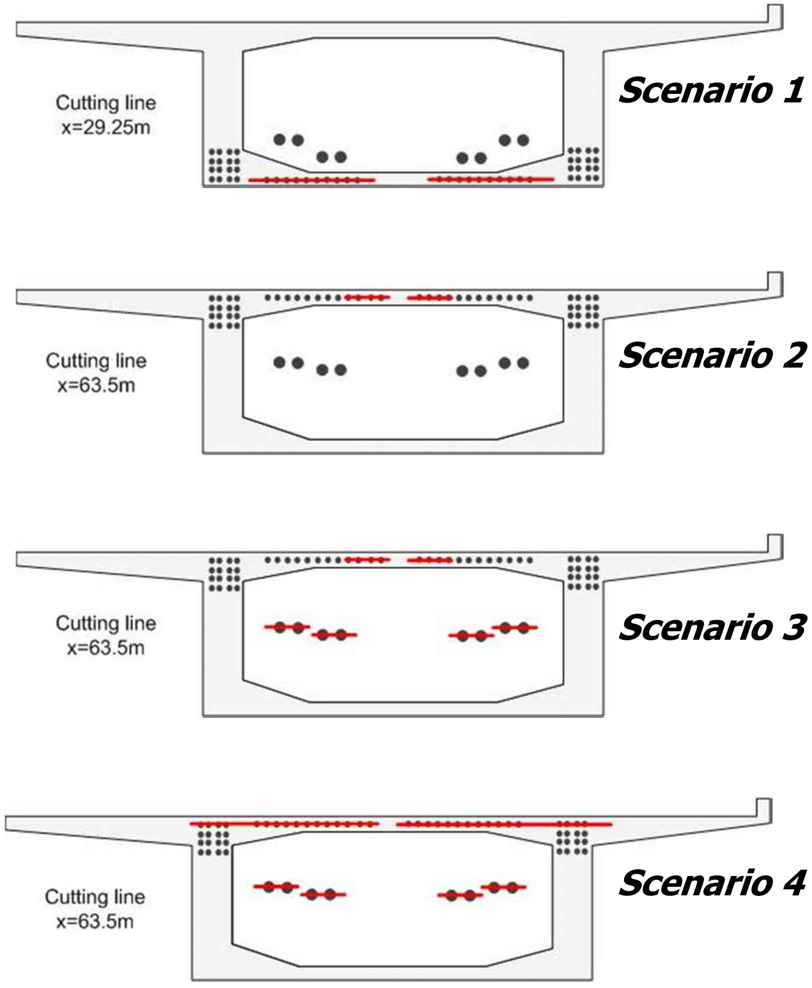
Cutting line in the bottom plate for damage state #1 (x = 29.25m =0.45L)

Cutting line on the top for damage state #2 and #4 (x = 63.5m)

Percentage of cut tendons in each section

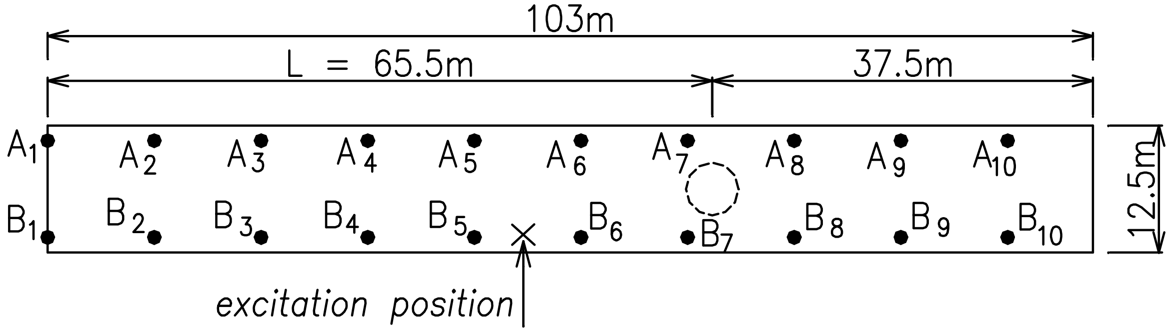


4 damage cases



The Champangshiehl bridge (4)

Measurement set-up



Location of the sensors on the bridge deck

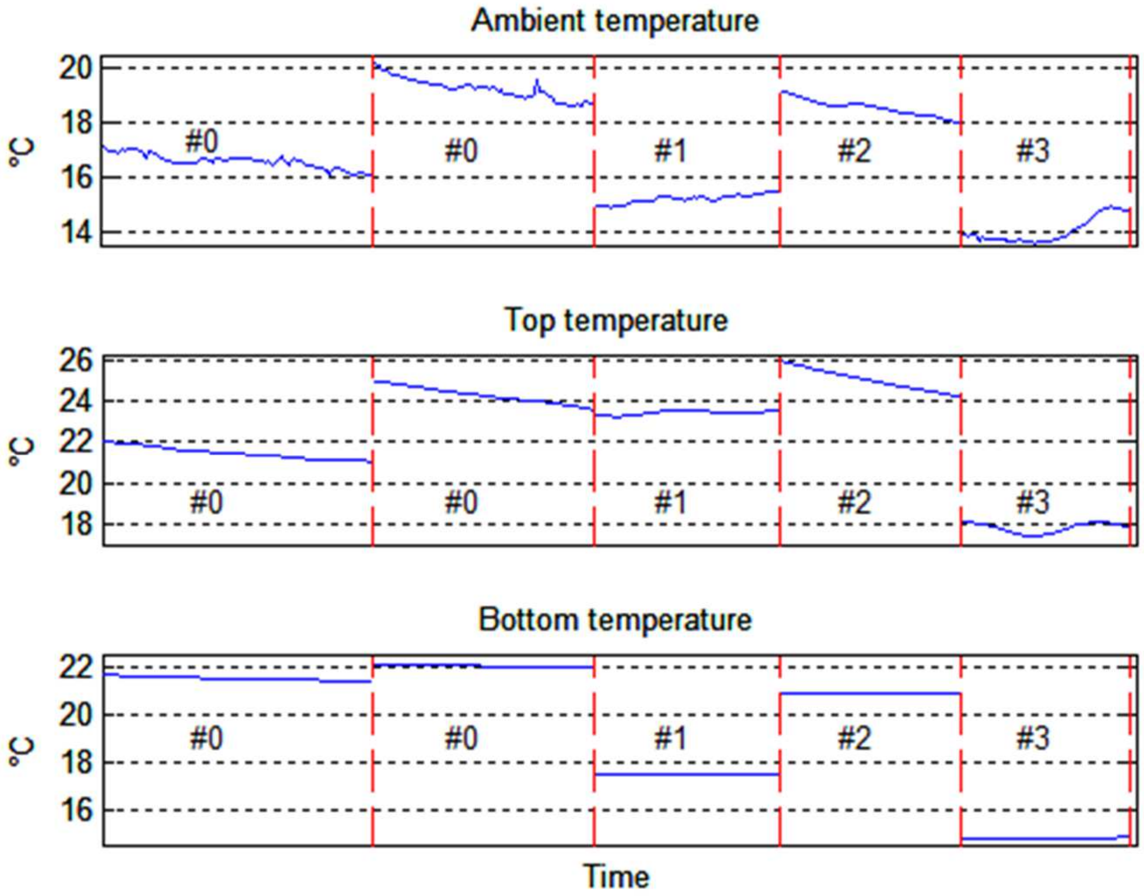
- Dynamic measurements
  - Tri-axial accelerometers
  - 20 measurement points (A1-A10, B1-B10).

- Excitation
  - Impact excitation between B5 and B6
  - Swept sine excitation by a reaction-type vibration machine using two rotating unbalances

The Champangshiehl bridge (4)

Measurement set-up

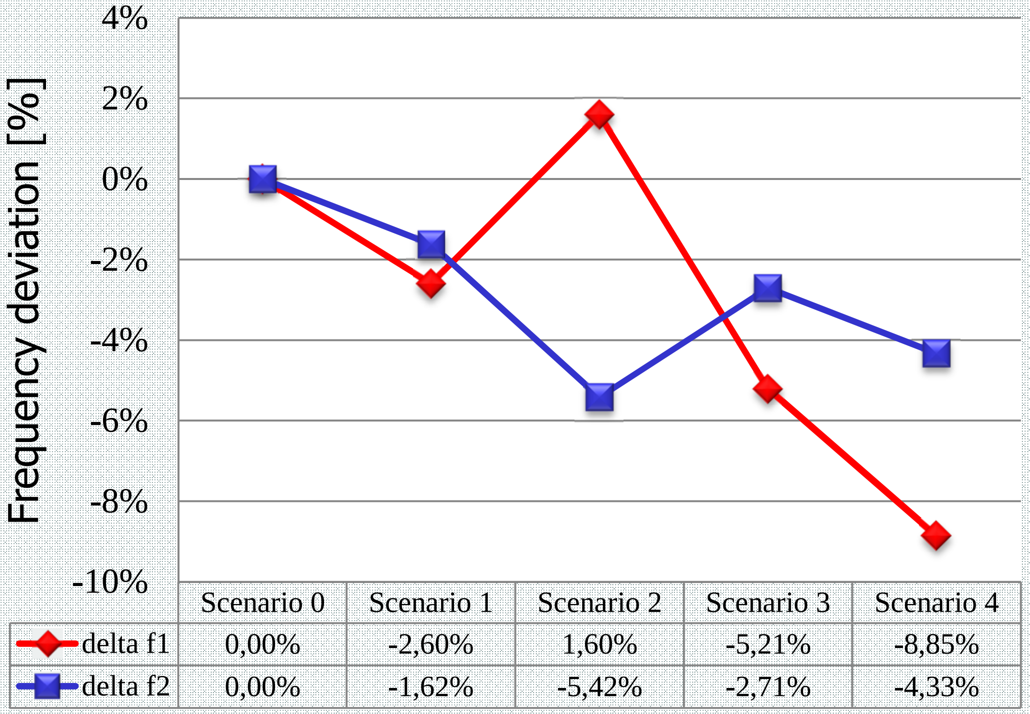
Evolution of the temperatures recorded
at different locations on the bridge



The Champangshiehl bridge (5)

Identification of natural frequencies using SSI

- Use of the free response after impact excitation
- Detection of the 2 first natural frequencies in all the scenarios
- In the healthy state (scenario #0): $f_1 = 1.92\ Hz$, $f_2 = 5.54\ Hz$

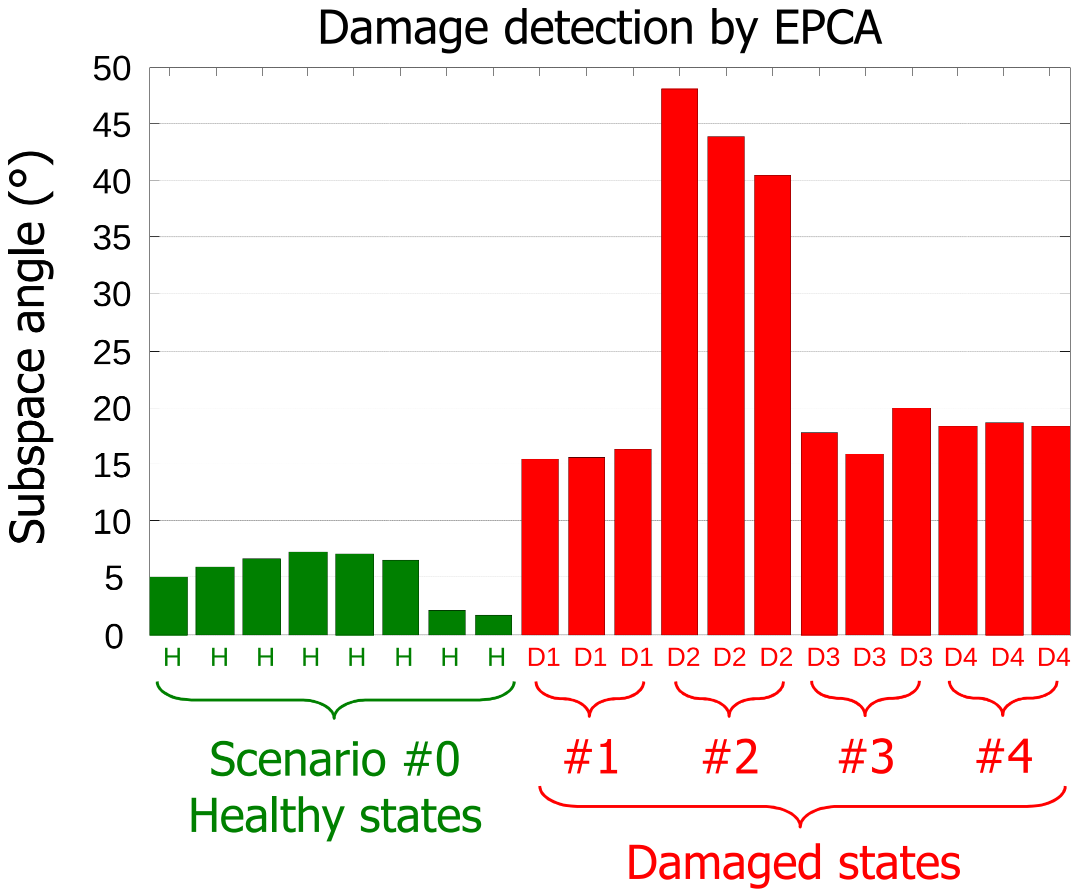


	Scenario 0	Scenario 1	Scenario 2	Scenario 3	Scenario 4
◆ delta f1	0,00%	-2,60%	1,60%	-5,21%	-8,85%
■ delta f2	0,00%	-1,62%	-5,42%	-2,71%	-4,33%

The natural frequencies decrease as more tendons are cut (except for f_1 in damage scenario 2)

The Champangshiehl bridge (6)

Damage detection results



Eliminating environmental effects (1)

Use of Principal Component Analysis (*)

In the following method, measurement of environmental variables is not required but their effects are merely observed from the variation of measured features (i.e. natural frequencies in this case).

Let us denote by $\mathbf{y}_k$ a vector of n vibration features identified at time t_k and let us collect all the samples ($k=1, \ldots, N$) in a (nxN) matrix $\mathbf{Y}$.

Performing SVD of the covariance matrix gives

$$\boxed{\mathbf{Y}\,\mathbf{Y}^T = \mathbf{U}\,\boldsymbol{\Sigma}^2\,\mathbf{U}^T} \quad \text{with} \quad \mathbf{U}\,\mathbf{U}^T = \mathbf{I}$$

and

$$\boxed{\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_1 & 0 \\ 0 & \boldsymbol{\Sigma}_2 \end{bmatrix}} \longrightarrow \text{negligible (due to noise)}$$

(*) A.-M. Yan et al., Structural damage diagnosis under varying environmental conditions, MSSP 19 (2005) 847-864

Eliminating environmental effects (2)

The first m columns of $\mathbf{U}$ are taken to build matrix $\mathbf{T}$ so that

score matrix ← → loading matrix

$$\boxed{\mathbf{X}_{m\times N} = \mathbf{T}_{m\times n}\ \mathbf{Y}_{n\times N}}$$

The dimension m may be thought as the physical order of the system which corresponds to the number of combined environmental factors that affect the features.

The loss of information can be assessed by re-mapping the projected data back to the original space

estimated ←

$$\boxed{\hat{\mathbf{Y}} = \mathbf{T}^T\ \mathbf{X} = \mathbf{T}^T\ \mathbf{T}\,\mathbf{Y}}$$

and the residual error matrix is defined as : $\boxed{\mathbf{E} = \mathbf{Y} - \hat{\mathbf{Y}}}$

Eliminating environmental effects (3)

Definition of the Novelty Index

Residual error matrix : $\boxed{\mathbf{E} = \mathbf{Y} - \hat{\mathbf{Y}}}$

Euclidean norm : $\boxed{NI_k^E = \left\| \mathbf{E}_k \right\|}$

prediction error vector at time t_k

Mahalanobis norm : $\boxed{NI_k^M = \sqrt{\mathbf{E}_k^T \, \mathbf{R}^{-1} \, \mathbf{E}_k}}$

$\mathbf{R} = \frac{1}{N} \mathbf{Y} \, \mathbf{Y}^T$ (covariance matrix)

mean value

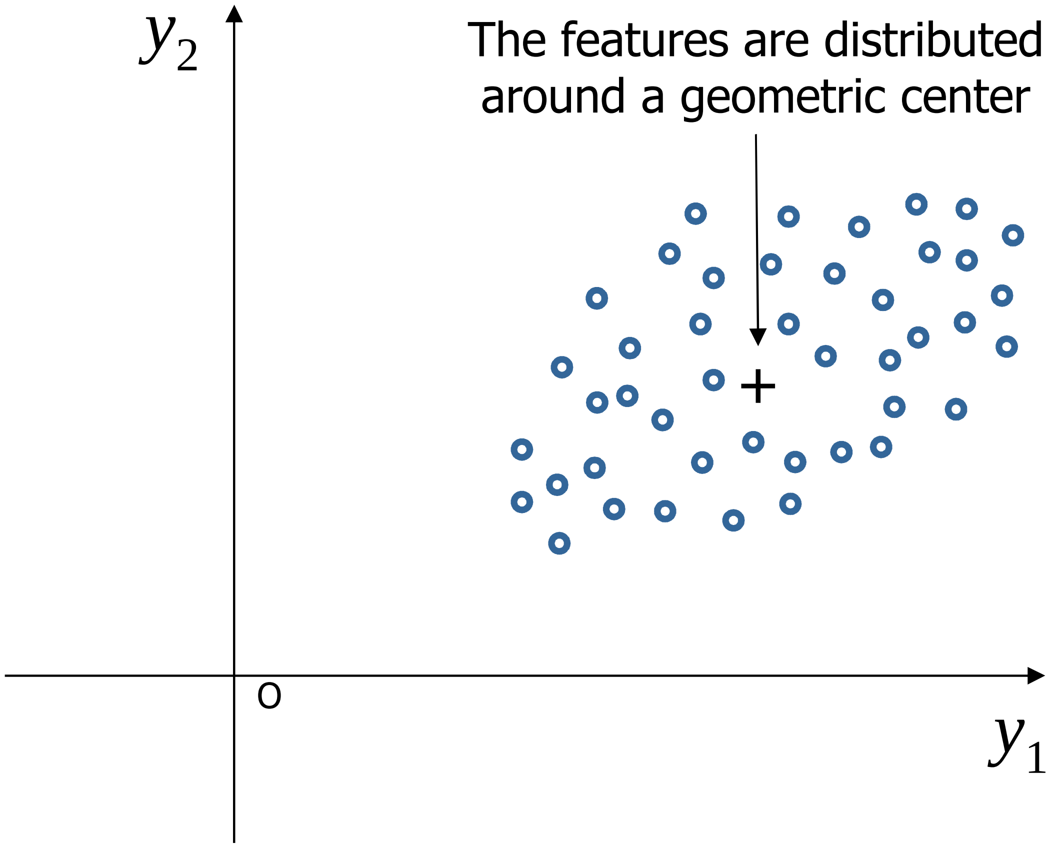
Statistical tool : $\boxed{CL = \overline{NI} + 3\,\sigma}$ (Upper Control Limit at 99.7 % confidence interval)

standard deviation

Eliminating environmental effects (4)

Principal component analysis

In the 2D-space



The features are distributed around a geometric center

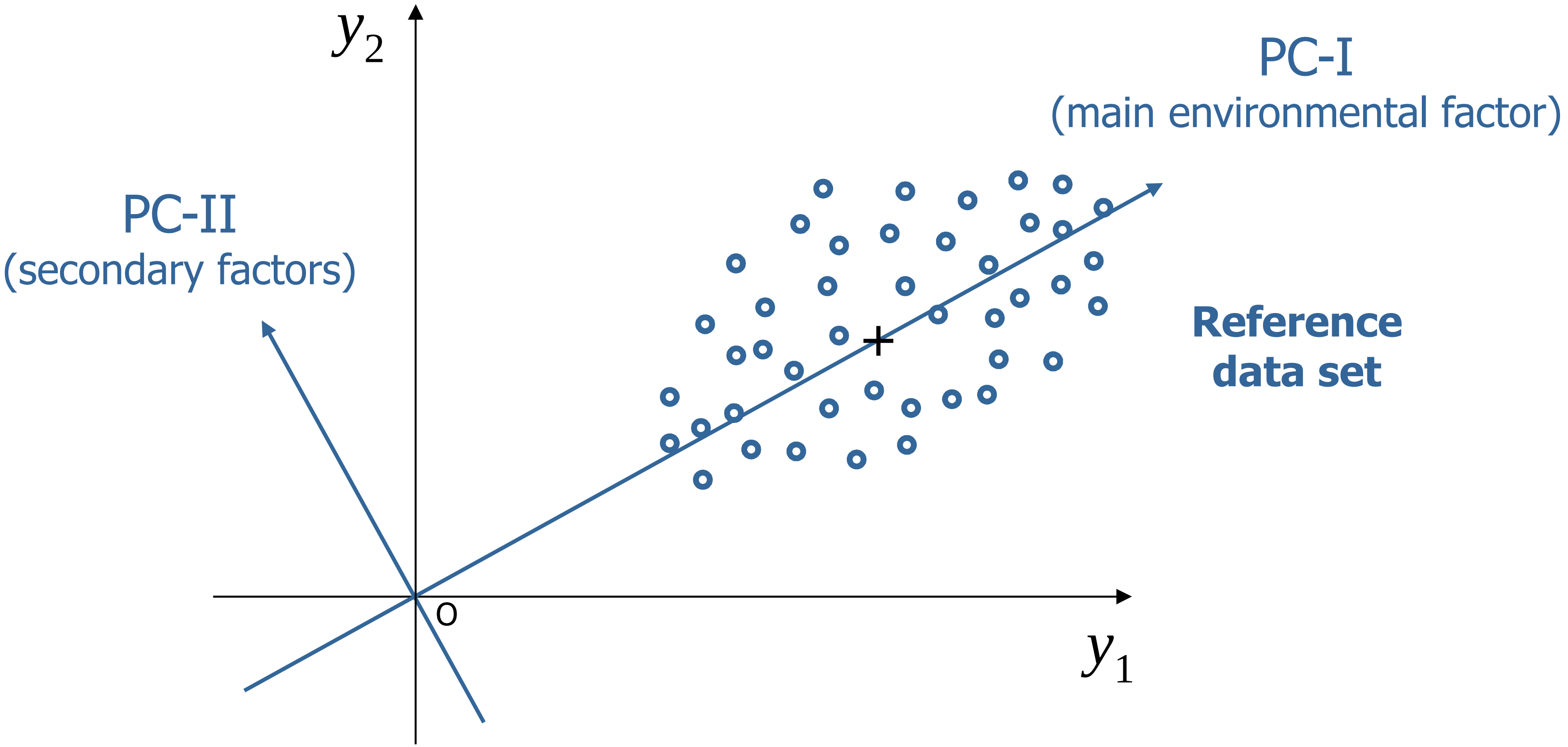
Reference data set

Environmental variations are responsible for the dispersion

Eliminating environmental effects (5)

Principal component analysis

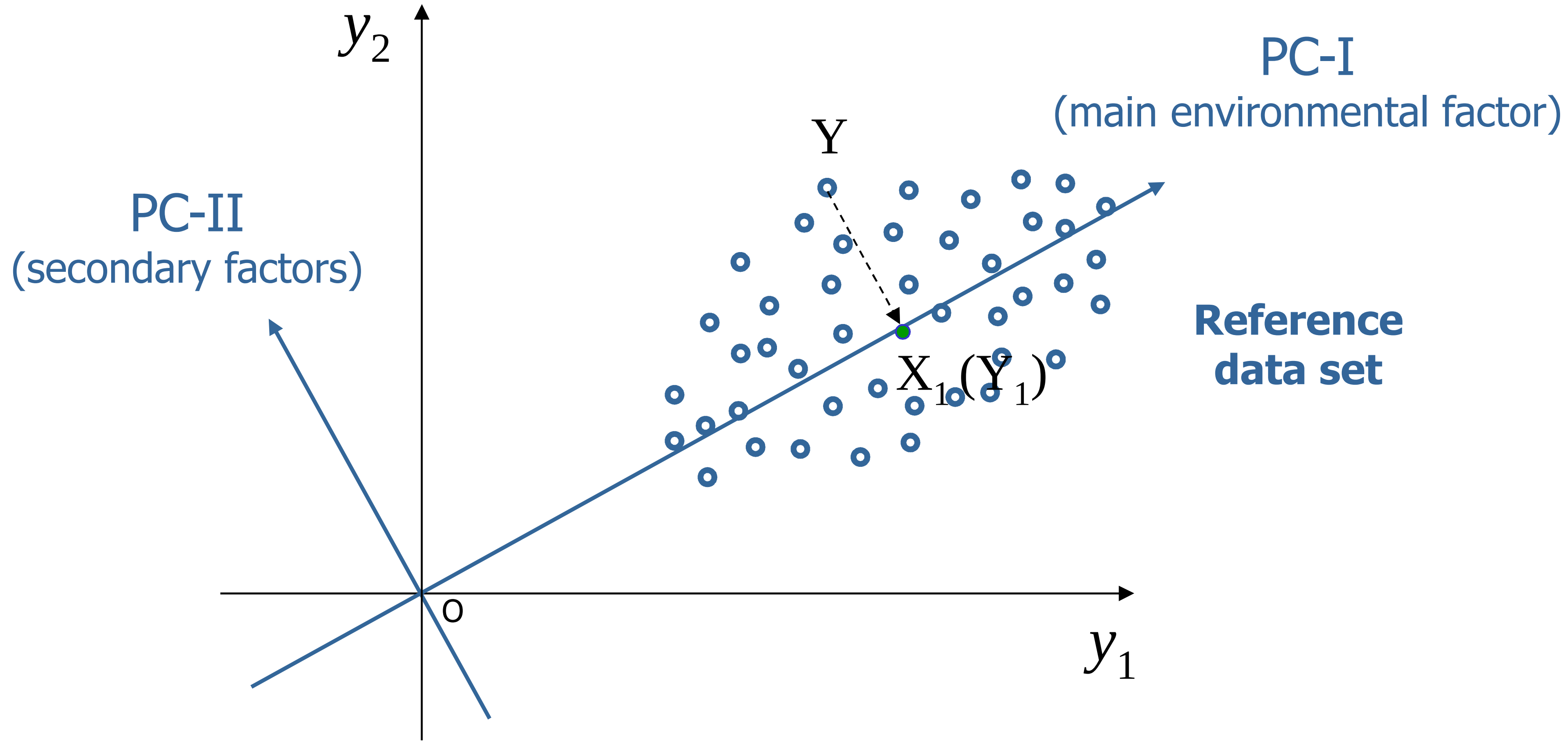
In the 2D-space



Eliminating environmental effects (6)

Principal component analysis

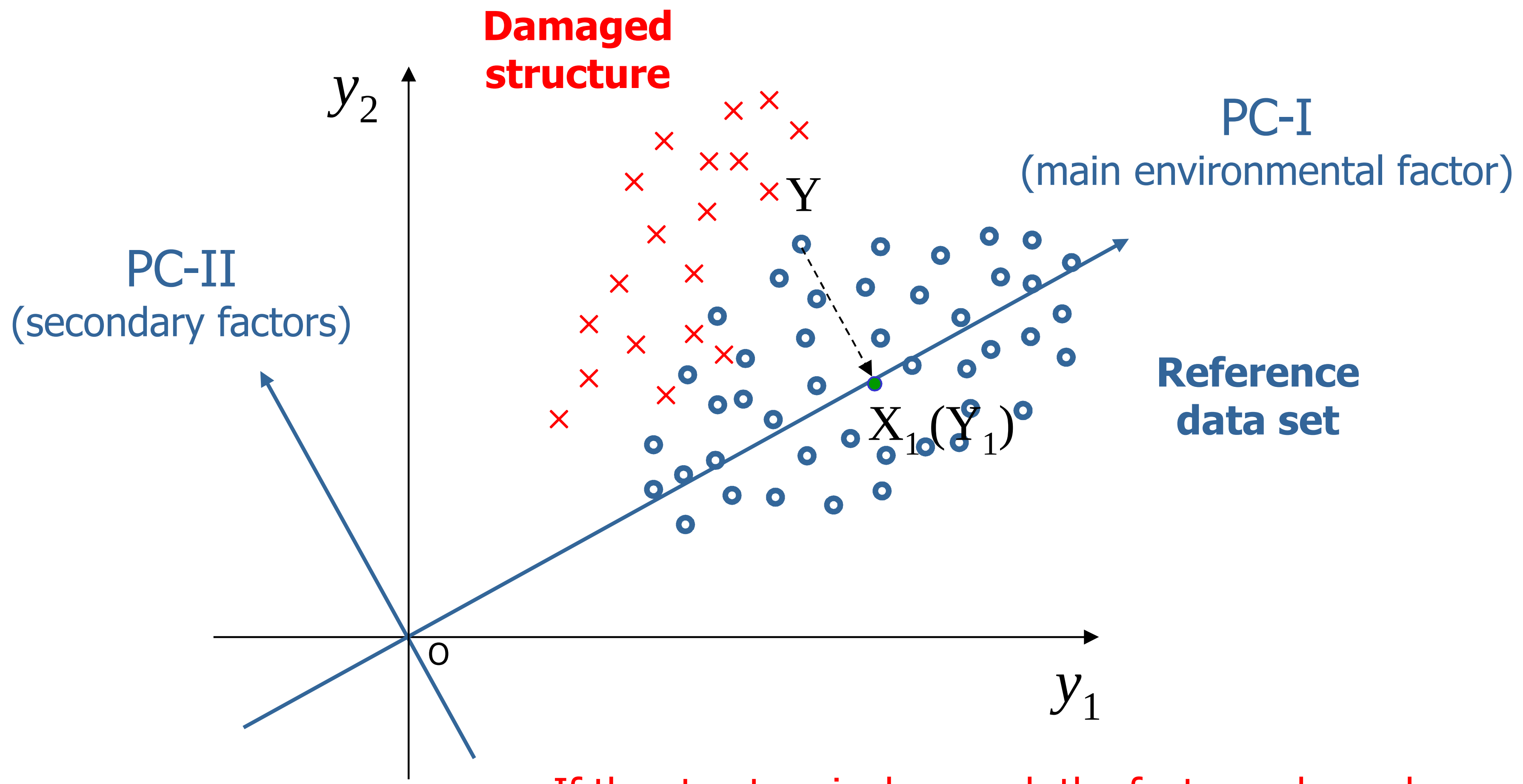
In the 2D-space



Eliminating environmental effects (7)

Principal component analysis

In the 2D-space

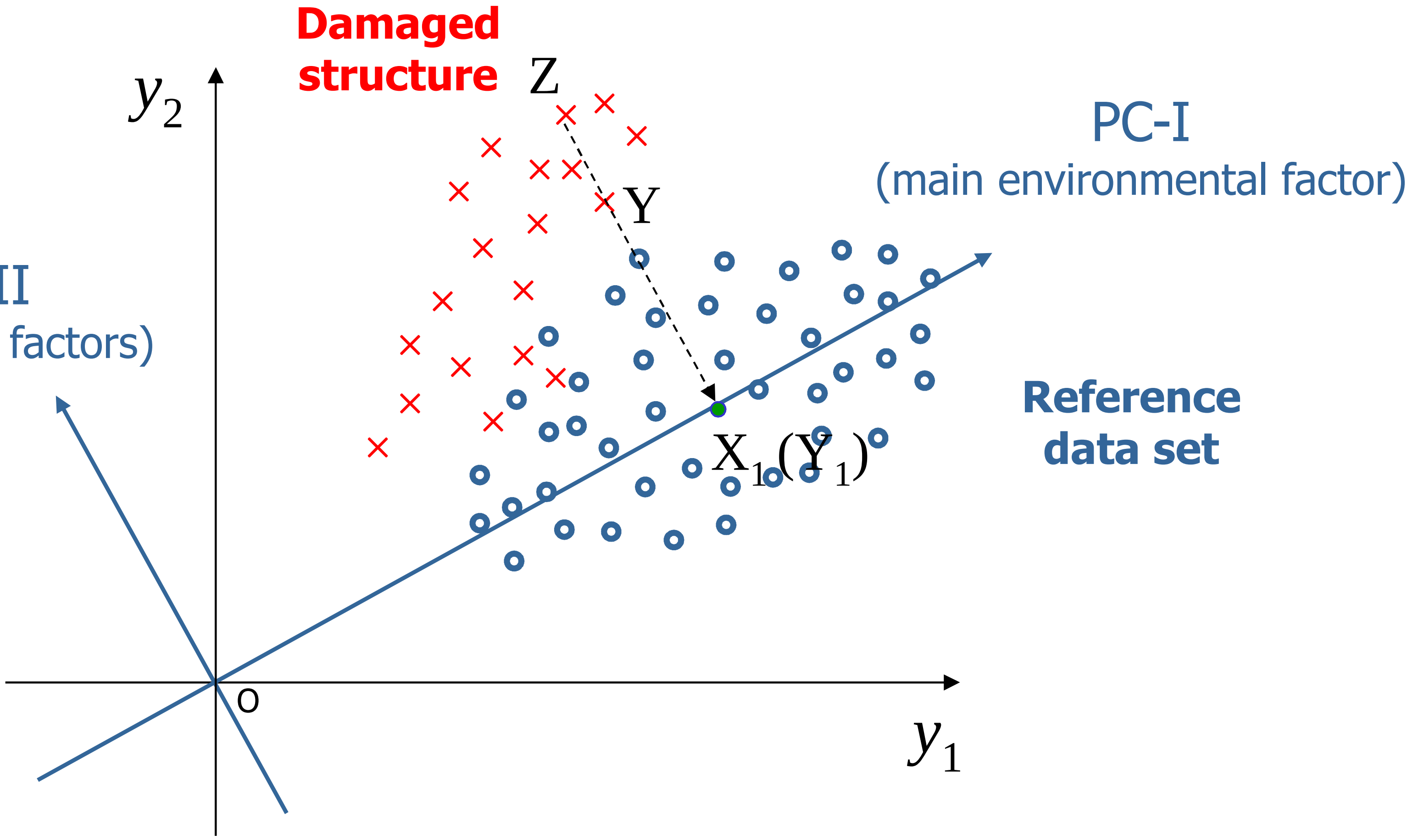


If the structure is damaged, the features depend in a different way on the environmental factors

Eliminating environmental effects (8)

Principal component analysis

In the 2D-space

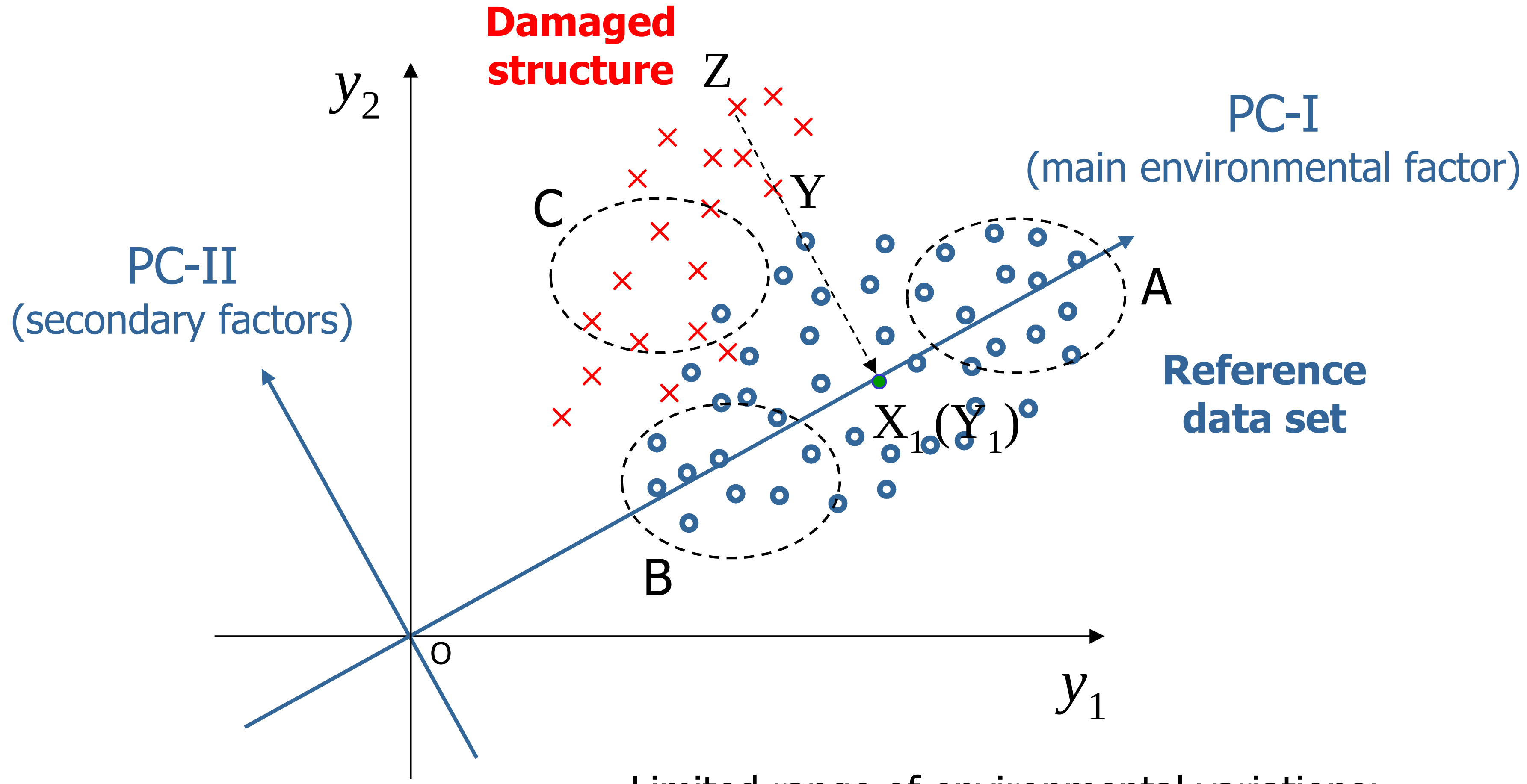


Remark: data normalization (zero-mean and unitary standard deviation) should be avoided in the present case !

Eliminating environmental effects (9)

Principal component analysis

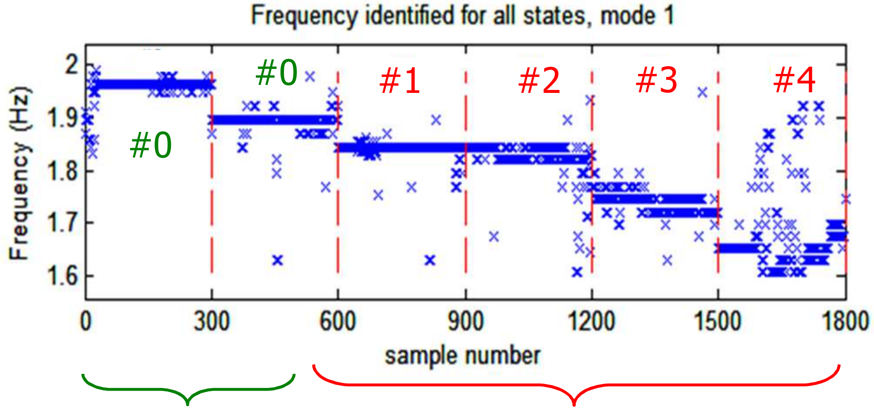
In the 2D-space

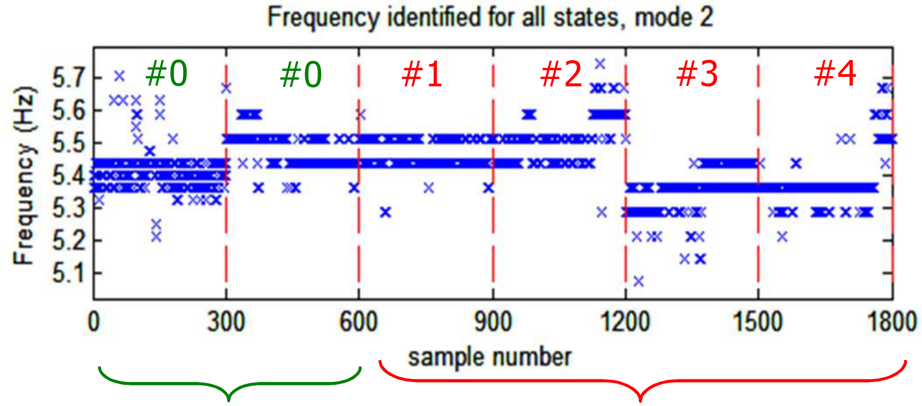


Limited range of environmental variations:
PCA-I from data set A ~ PCA-I from data set B

The Champangshiehl bridge (7)

Natural frequencies are identified using of the Wavelet Transform and are chosen as system features



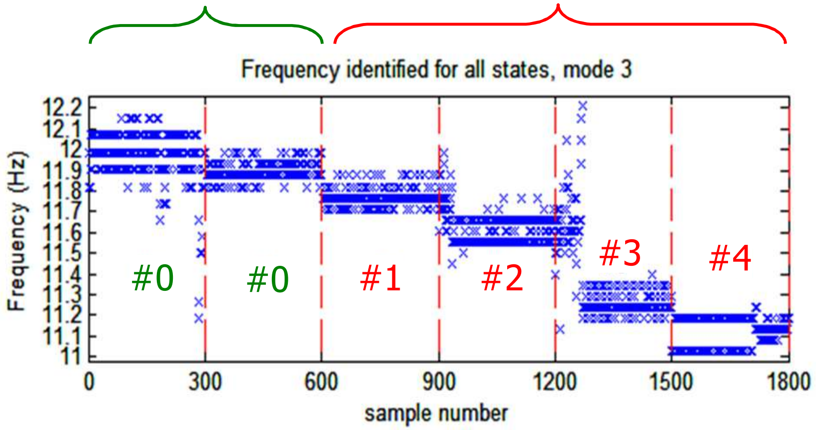


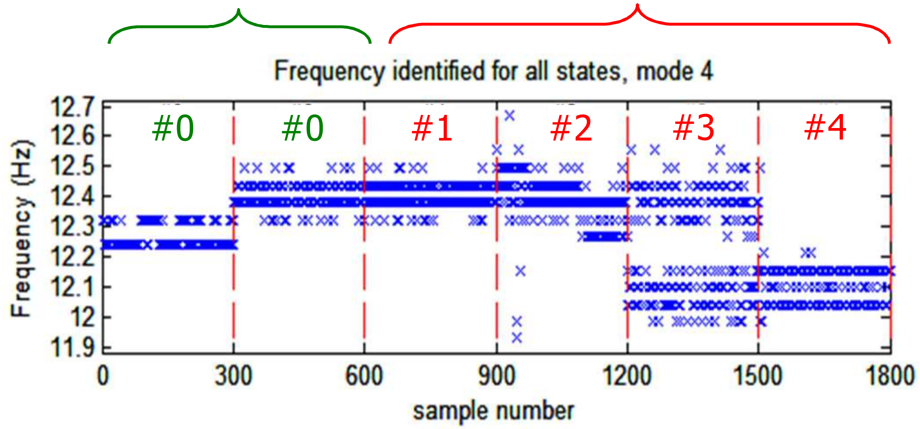
Healthy states

Damaged states

Healthy states

Damaged states

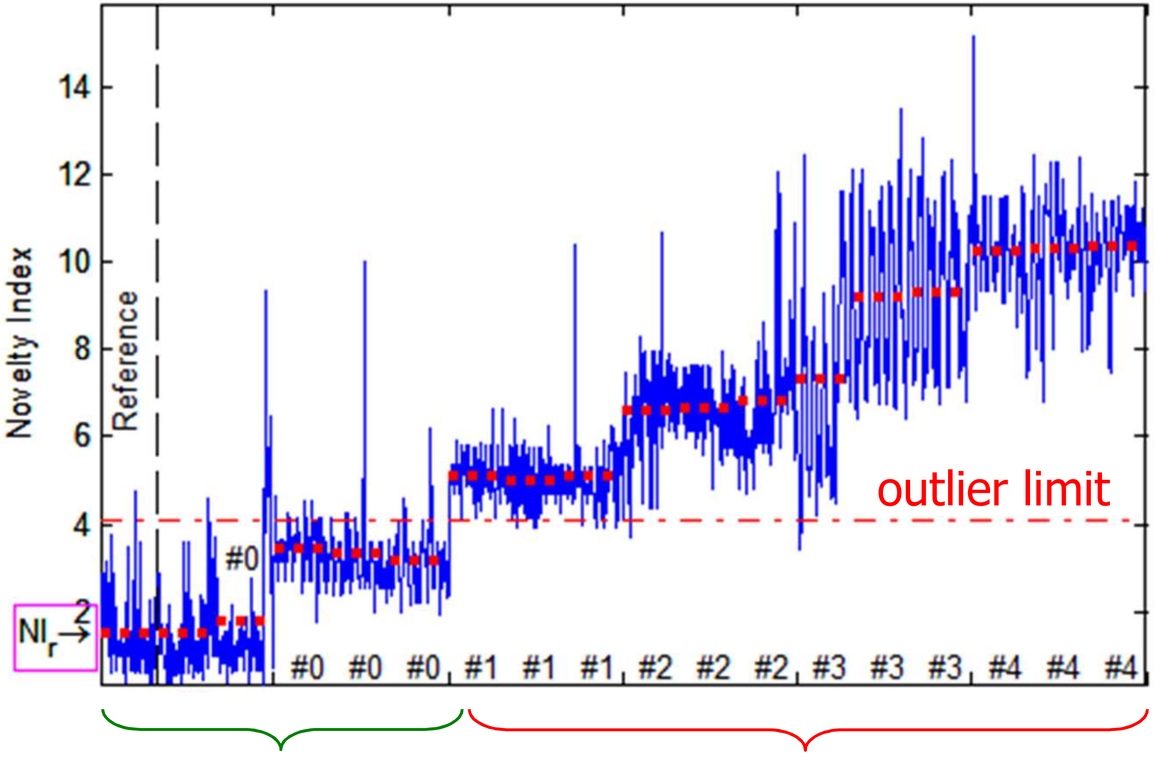




The Champangshiehl bridge (8)

PCA procedure applied on the matrix of collected features → one single environmental factor has an influence (temperature)

Evolution of the Novelty Index



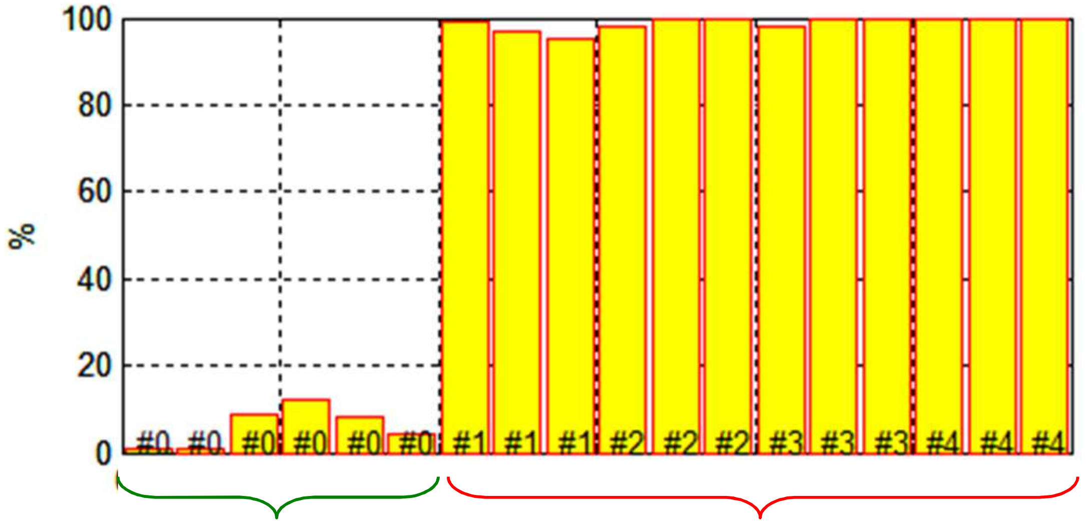
Mean value of NI for the reference data set →

Healthy states

Damaged states

The Champangshiehl bridge (9)

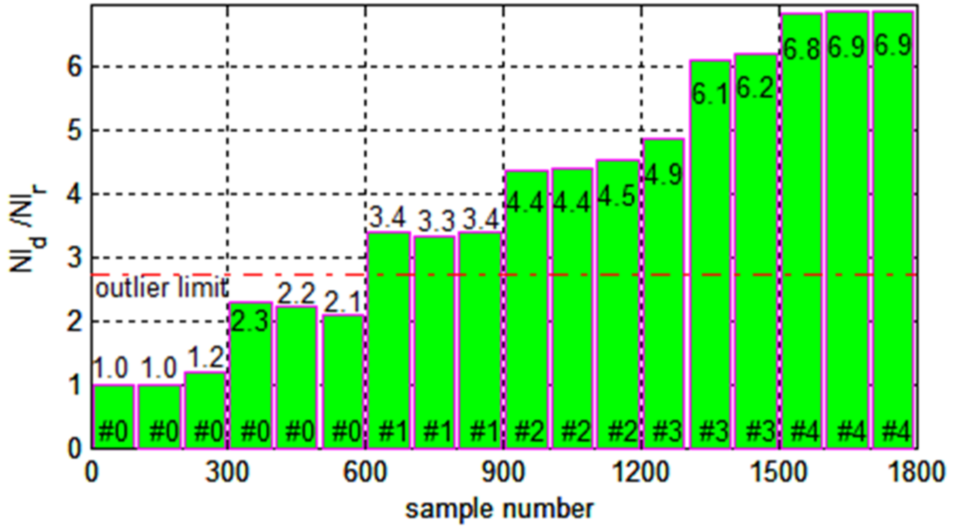
Percentage of samples exceeding the outlier limit



Healthy states

Damaged states

Ratio Nid/Nir between the damaged and the healthy states



Outline

- Principal Component Analysis (PCA)
- Damage detection
- Structural Health Monitoring
- **Identification of nonlinear parameters**
- Conclusion

Nonlinearity in engineering applications

backlash and friction in
control surfaces and joints

fluid-structure interaction

composite materials

hardening nonlinearities in
engine-to-pylon connections

Many works are reported in the literature on dynamic testing and identification of nonlinear systems but very few address nonlinear phenomena during modal survey tests.

Theoretical Modal Analysis

Linear systems

Finite Element model

$$\mathbf{M}\,\ddot{\mathbf{x}}+\mathbf{C}\,\dot{\mathbf{x}}+\mathbf{K}\,\mathbf{x}=0$$

Eigenvalue problem

$$\mathbf{K}\,\boldsymbol{\Phi}_j=\omega_j^2\,\mathbf{M}\,\boldsymbol{\Phi}_j$$

Natural frequencies (ω_j^2)

Mode shapes ($\boldsymbol{\Phi}_j$)

Nonlinear systems

Finite Element model

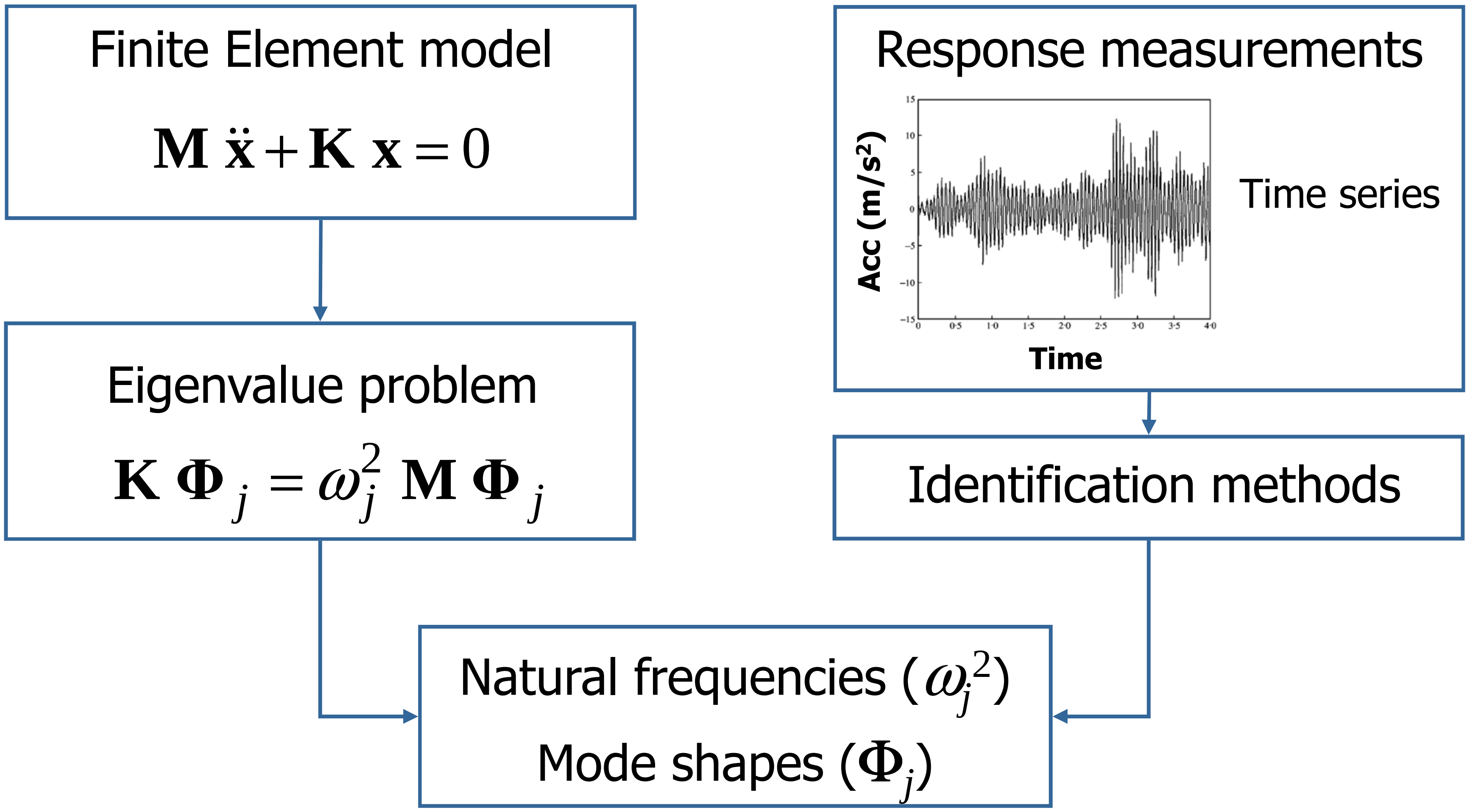
$$\mathbf{M}\,\ddot{\mathbf{x}}+\mathbf{f}_{NL}\left(\mathbf{x},\dot{\mathbf{x}}\right)=0$$

NNM computation

NNM frequencies

NNM modal curves

Experimental Modal Analysis (EMA)



Theoretical approach

Finite Element model

$$\mathbf{M}\,\ddot{\mathbf{x}}+\mathbf{K}\,\mathbf{x}=0$$

Eigenvalue problem

$$\mathbf{K}\,\mathbf{\Phi}_j=\omega_j^2\,\mathbf{M}\,\mathbf{\Phi}_j$$

Experimental approach

Response measurements

Identification methods

Natural frequencies (ω_j^2)

Mode shapes (Φ_j)

EMA for linear systems is now mature and widely used in structural engineering → well established techniques.

Normal mode testing (Phase resonance method)

EMA for nonlinear systems is still a challenge.

Theoretical approach

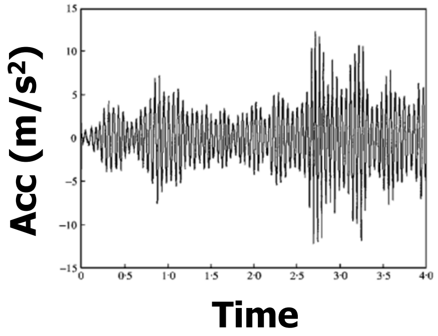
Finite Element model

$$\mathbf{M}\,\ddot{\mathbf{x}}+\mathbf{f}_{NL}\left(\mathbf{x},\dot{\mathbf{x}}\right)=0$$

Numerical NNM computation

Experimental approach

Response measurements



Time series

Experimental NNM extraction

NNM frequencies
NNM modal curves

Experimental Modal Analysis (EMA)

There are two main techniques for EMA.

1. Phase resonance methods (Normal mode testing)

One of the normal mode at a time is excited using multi-point sine excitation at the corresponding natural frequency. The modes are identified one by one.

→ can be extended to nonlinear structures according to the invariance property of NNMs:

« *If the motion is initiated on one specific NNM, the remaining NNMs remain quiescent for all time.* »

Remark

- Expensive and difficult.
- Extremely accurate mode shapes → a way to identify NNMs (but still a research topic).

Experimental Modal Analysis (EMA)

2. Phase separation methods

Several modes are excited at once using either broadband excitation (e.g., hammer impact and random excitation) or swept-sine excitation in the frequency range of interest.

→ in the nonlinear case, extraction of individual NNMs is not possible generally, because modal superposition is no longer valid.

→ use of the proper orthogonal decomposition (POD) method to extract features from the time series .

Remark

- All structures encountered in practice are nonlinear to some degree.
- If a nonlinear structure is excited with a broadband excitation signal (e.g. random force), then the results will appear linear → experimental modal analysis will lead to an updated linearized model !

Proper Orthogonal Decomposition (POD)

Linear systems

Finite Element model

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{p}(t)$$

Eigenvalue problem

$$\mathbf{K}\,\mathbf{\Phi}_j = \omega_j^2\,\mathbf{M}\,\mathbf{\Phi}_j$$

normal modes

$$\mathbf{x}(t) = \sum_{j=1}^{n} \eta_j(t)\,\mathbf{\Phi}_{(j)}$$

$$\eta_j = A_j \cos(\omega_j\,t) + B_j \sin(\omega_j\,t)$$

→ natural frequencies

Nonlinear systems

Finite Element model

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{f}_{NL}\left(\mathbf{x},\dot{\mathbf{x}}\right) = \mathbf{p}(t)$$

POD of the response

$$\mathbf{X} = \mathbf{U}\,\Sigma\,\mathbf{V}^{\mathbf{T}}$$

POM

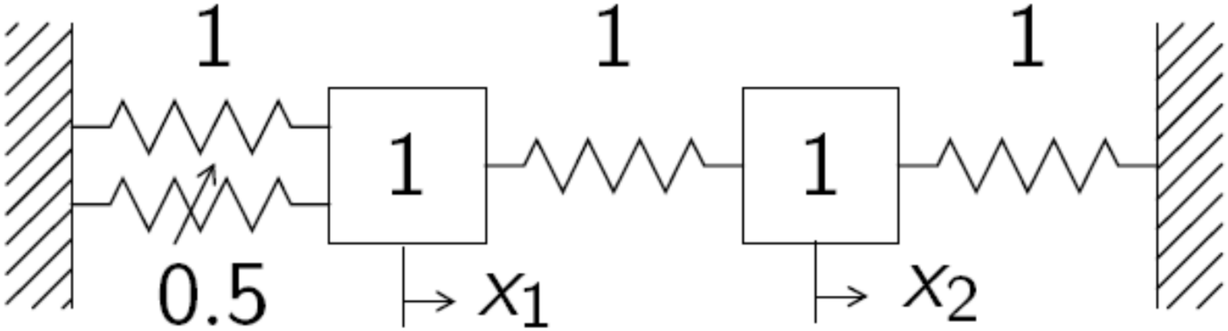
$$\mathbf{x}(t) = \sum_{j=1}^{n} a_j(t)\,\mathbf{u}_{(j)}$$

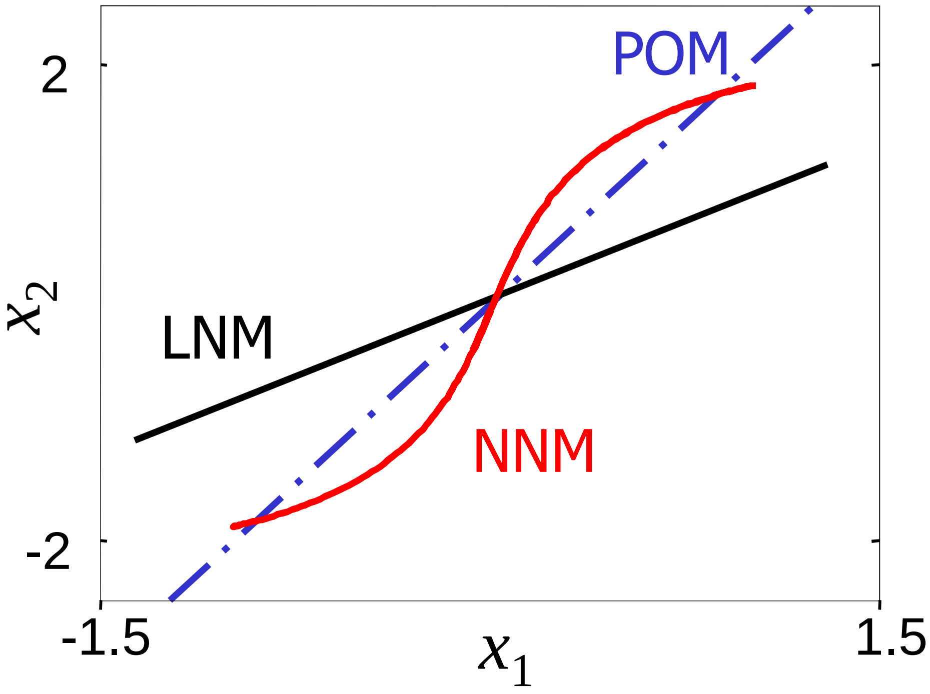
Time information

→ instantaneous frequencies

Geometric interpretation of the POMs

Comparison of LNM, NNM and POM on the 2 DOF example





First mode

The POM is the best linear representation of the nonlinear normal mode.

Model parameter estimation using POD

Assumption

- The linear counterpart of the structure is known (updated).

Methodology

- Estimation of nonlinear parameters only (which will be based on FE updating techniques).

Parameters for model updating (Crucial step!)

The number of parameters :

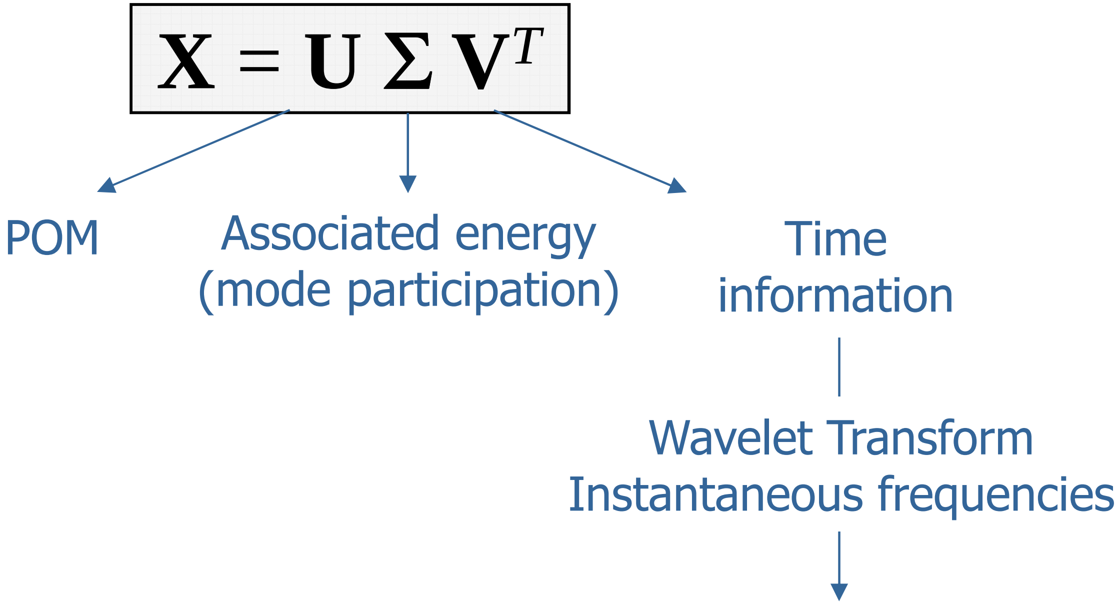
- should be kept small to avoid problems of ill-conditioning,
- should be chosen with the aim of correcting recognised features in the model.

→ requires physical insight → leads to knowledge-based models.

Parameter estimation using POD

Principle of the method

Minimise the residuals between the bi-orthogonal decompositions of the measured and simulated data.



$$\boxed{\mathbf{X} = \mathbf{U}\,\Sigma\,\mathbf{V}^T}$$

POM

Associated energy
(mode participation)

Time
information

Wavelet Transform
Instantaneous frequencies

Penalty function

$$\boxed{J = \sum_i \sum_j (\Delta U_{ij})^2 + \sum_j (\Delta \Sigma_{jj})^2 + \sum_j \sum_k (\Delta V_{jk})^2}$$

→ selection of the POMs with the highest POV

Model parameter estimation techniques

Definition of a measurement vector $\mathbf{v}$ containing the modal features.

- In the case of **linear** systems

ith mode shape vector

$$\mathbf{v}^T = \left(\omega_1, \boldsymbol{\Phi}_1^T, \ldots, \omega_i, \boldsymbol{\Phi}_i^T, \ldots, \omega_r, \boldsymbol{\Phi}_r^T\right)^T$$

ith eigenvalue

- In the case of **nonlinear** systems

ith POM

$$\mathbf{v}^T = \left(\boldsymbol{\omega}_1, \mathbf{U}_1^T, \ldots, \boldsymbol{\omega}_i, \mathbf{U}_i^T, \ldots, \boldsymbol{\omega}_r, \mathbf{U}_r^T\right)^T$$

ith set of instantaneous frequencies

Model parameter estimation techniques

The vector of modal features **v** depends on parameters **p**

$$\boxed{\mathbf{v} = \mathbf{v}(\mathbf{p})}$$

A residual between analytical results and measured data is defined as

$$\boxed{\boldsymbol{\varepsilon} = \bar{\mathbf{v}} - \mathbf{v}(\mathbf{p})}$$

Penalty function methods are based on the Taylor series expansion of the modal data in terms of the unknown parameters

$$\boxed{\mathbf{v} = \mathbf{v}(p_0) + \left[\frac{\partial \mathbf{v}}{\partial \mathbf{p}}\right]_{\mathbf{p}=\mathbf{p}_0} (\mathbf{p} - \mathbf{p}_0) + O(\mathbf{p}^2)}$$

sensitivity matrix

initial estimation
of the parameters

This expansion is often limited to the first two terms.

Model parameter estimation techniques

The weighted penalty function is defined as

weighting matrix

$$\boxed{J = \boldsymbol{\varepsilon}^T \ \mathbf{W} \ \boldsymbol{\varepsilon}}$$

where $\boldsymbol{\varepsilon} = \Delta\mathbf{v} - \mathbf{S} \ \Delta\mathbf{p}$ is the error in the predicted measurements.

$$\mathbf{S} = \left[\frac{\partial \mathbf{v}}{\partial \mathbf{p}} \right]_{\mathbf{p}=\mathbf{p}_0}$$ is the sensitivity matrix.

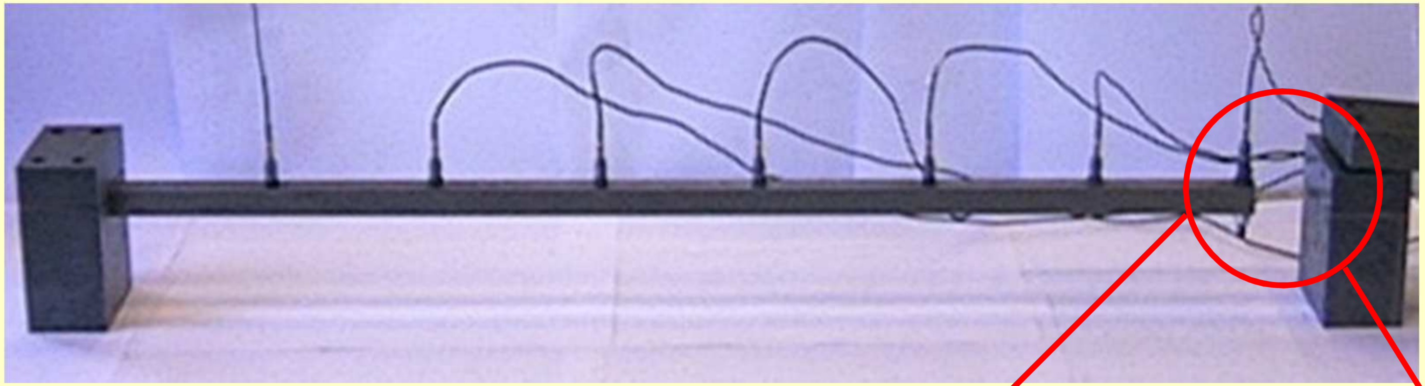
Minimising J with respect to $\Delta\mathbf{p}$ leads to

$$\boxed{\Delta\mathbf{p} = \left(\mathbf{S}^T \ \mathbf{W} \ \mathbf{S}\right)^{-1} \mathbf{S}^T \ \mathbf{W} \ \Delta\mathbf{v}}$$

With the assumption that the number of measurements is larger than the number of parameters, the matrix $\mathbf{S}^T \ \mathbf{W} \ \mathbf{S}$ is square and hopefully full rank.

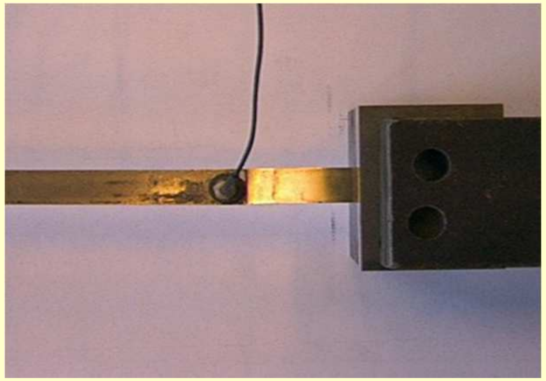
Example of a nonlinear beam

Benchmark of the European COST Action F3 « Structural Dynamics »

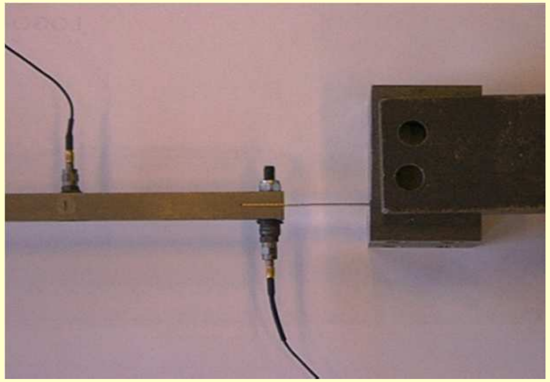


Experimental set-up

Horizontal view

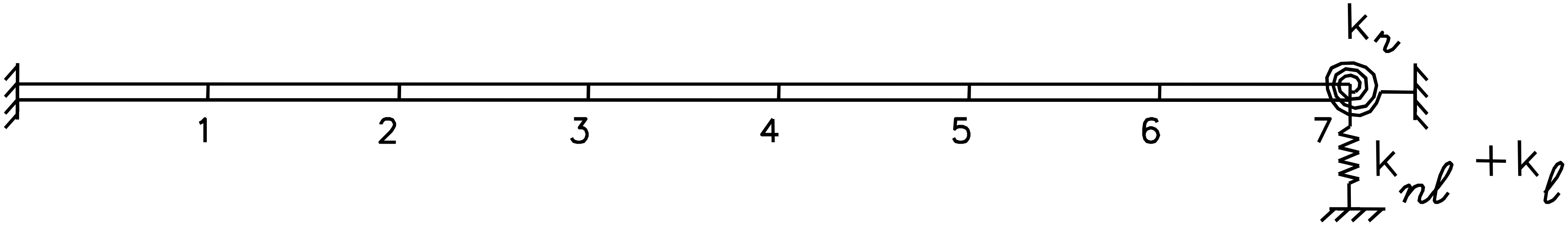


Vertical view



Example of a nonlinear beam

Finite Element model of the beam



The nonlinear stiffening effect of the thin beam is modelled by a nonlinear function in displacement of the form:

$$f_{nl} = A\left|x\right|^{\alpha}\ sign\left(x\right)$$

where A and α are nonlinear parameters to be identified.

Example of a nonlinear beam

Simulated results

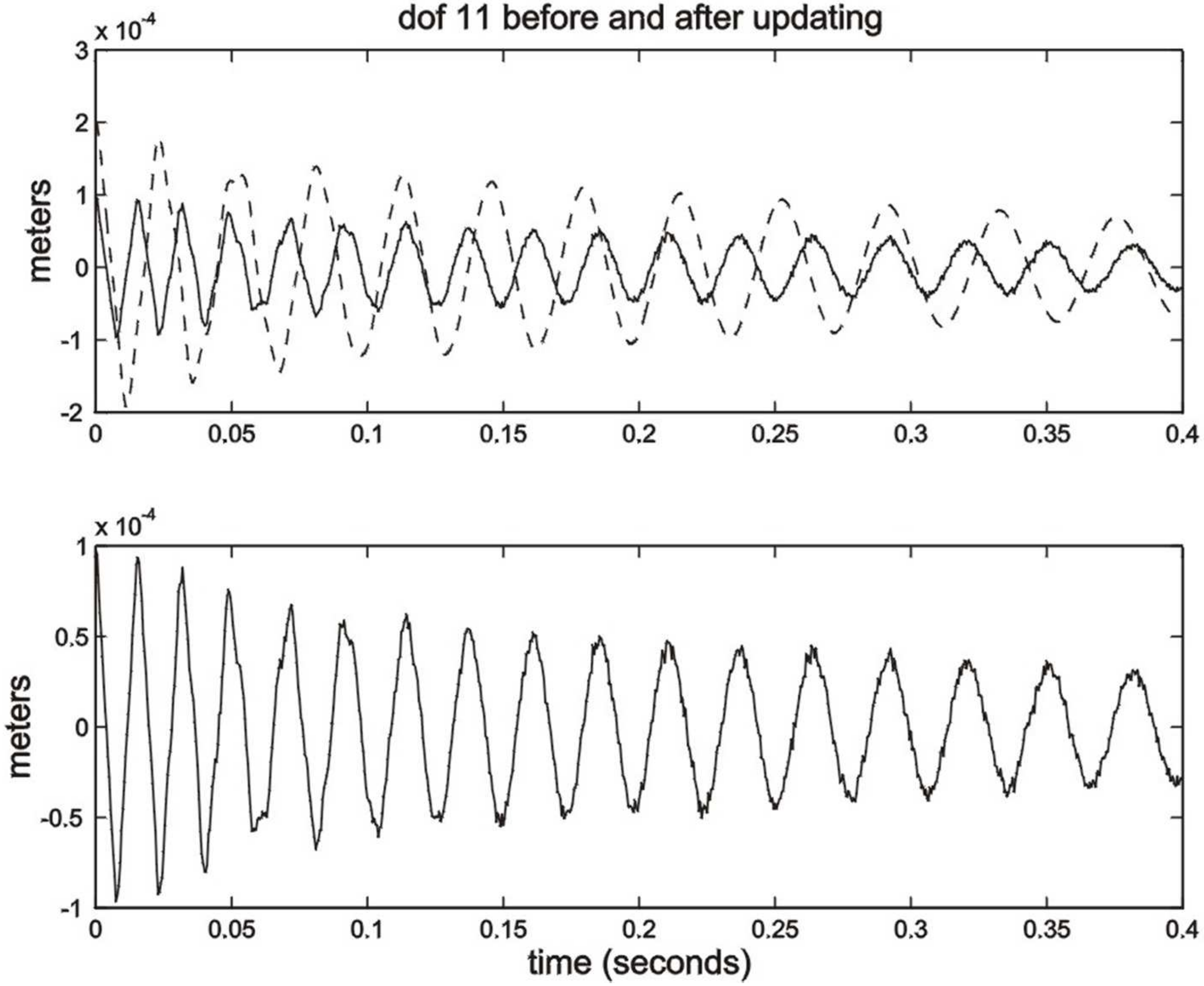
Identification of linear and nonlinear parameters

- 2 parameters : nonlinear stiffness + Young’s modulus
- Penalty function in terms of the first POM
- Simulation time = 0.4 sec
- Gaussian white noise of 1 %
- Nonlinear parameter correction < 10 %
- Linear parameter correction < 50 %

Example of a nonlinear beam

Simulated results

Comparison between the original (–) and the reconstructed (--) signals



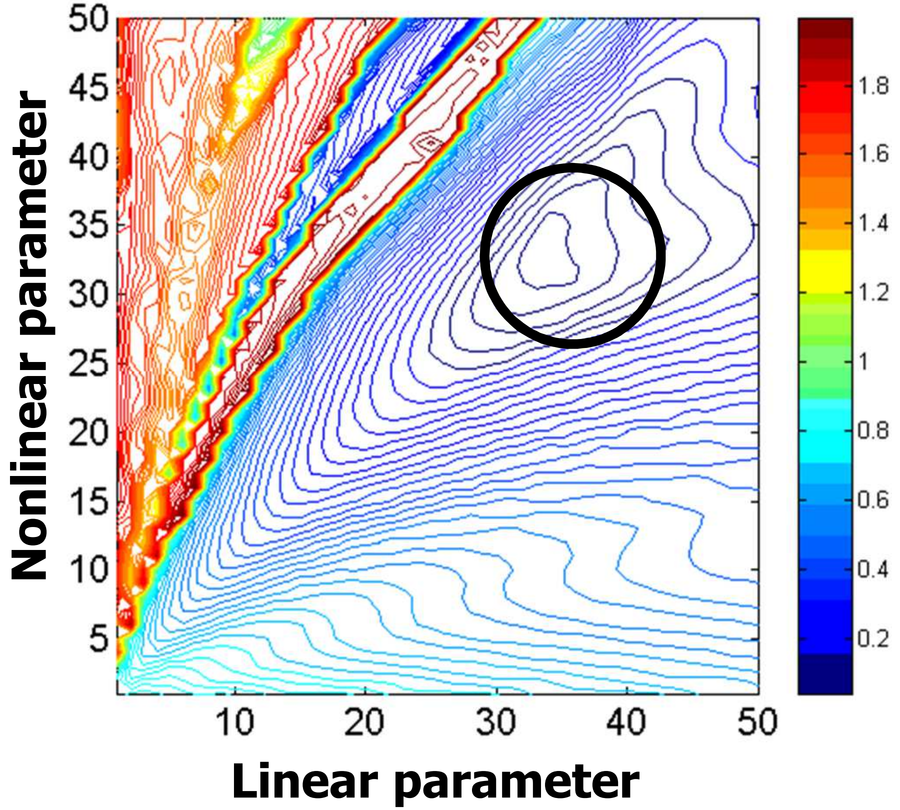
Before updating

After updating

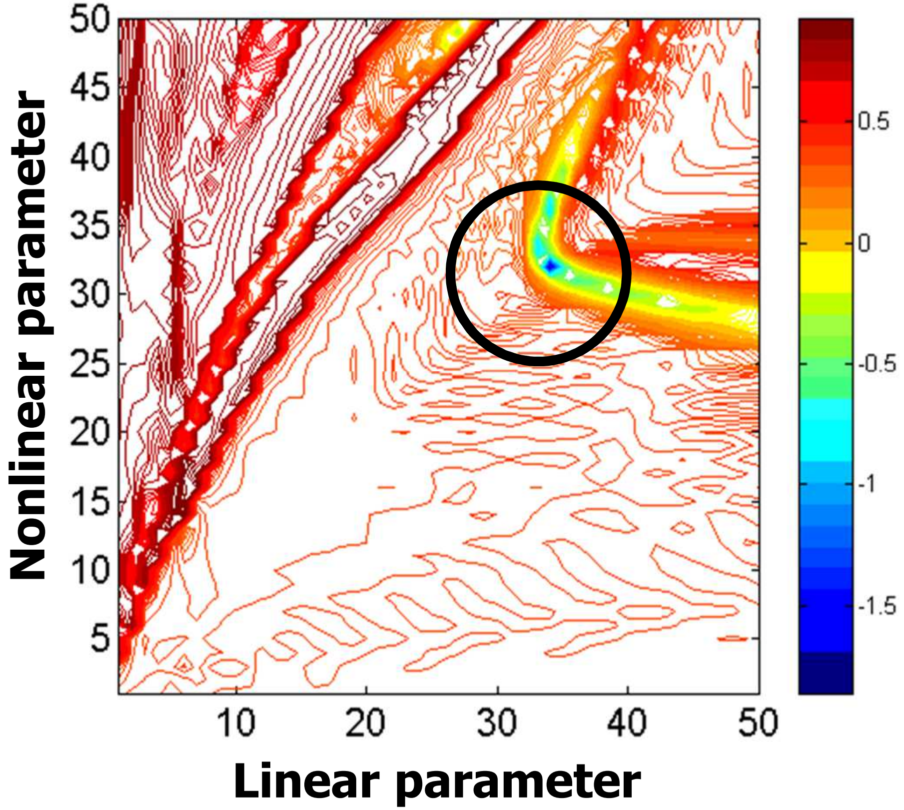
Example of a nonlinear beam

Simulated results

Contour Plot


Penalty Function (use of WT)

Well-conditioning


Penalty Function (no WT)

Ill-conditioning

Example of a nonlinear beam

Experimental results (Vertical set-up)

Model of the nonlinear stiffness

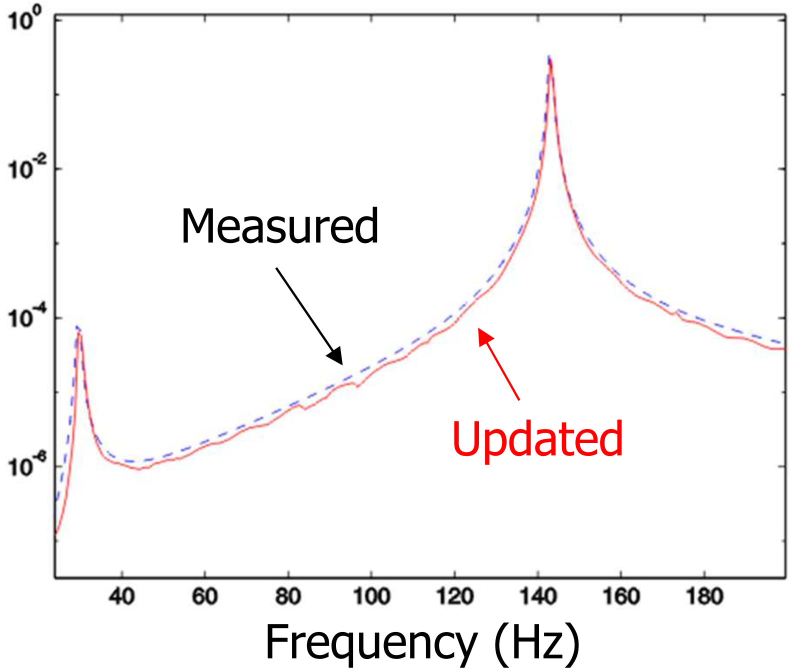
$$\boxed{f_{nl}(x) = A|x|^{\alpha} sign(x)}$$

Results of the identification of the nonlinear parameters based on the model updating method:

$$\alpha = 2.8$$

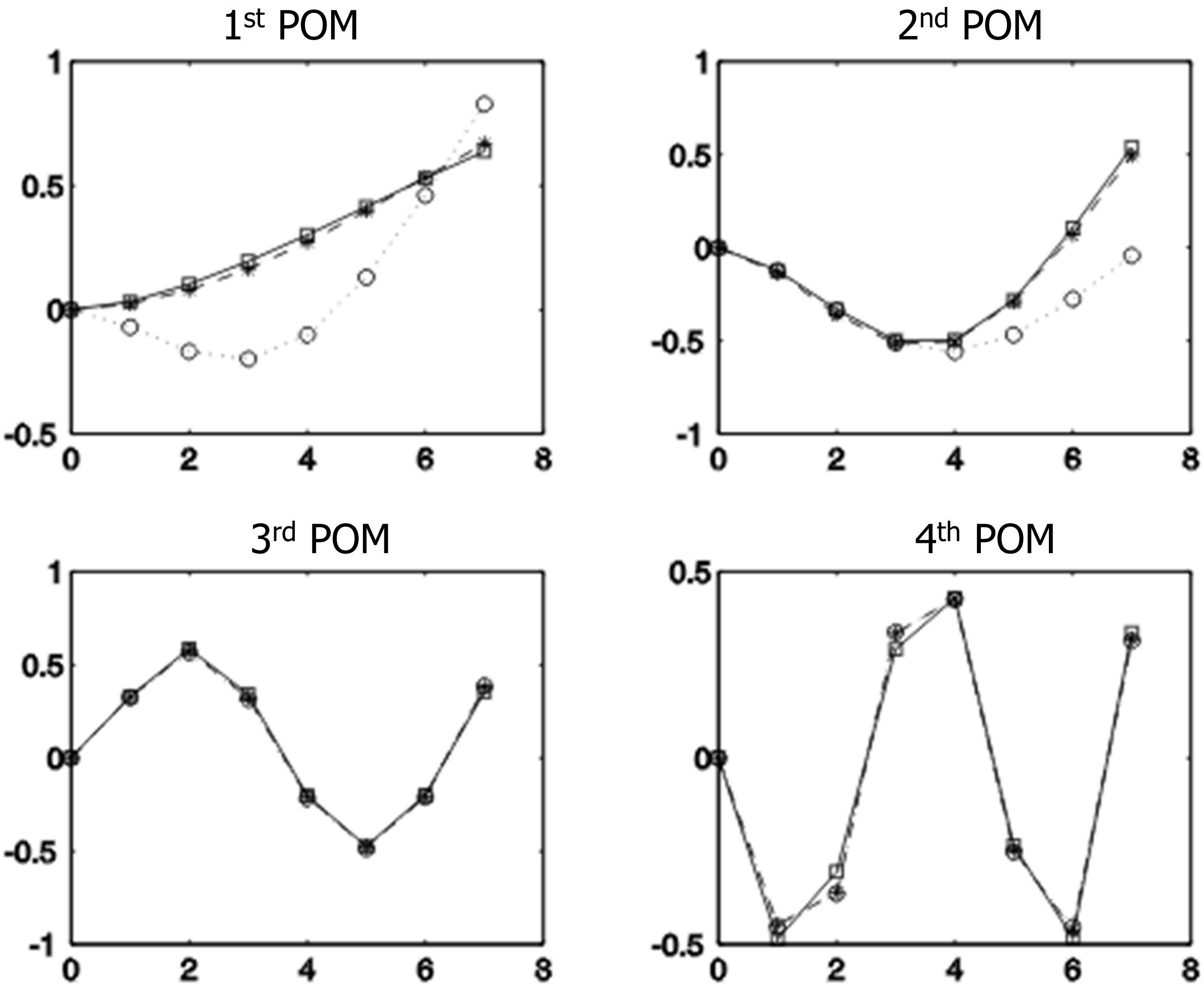
$$A = 1.65\ 10^9\ \mathrm{N/m}^{2.8}$$

PSD of the time evolution of the 1st POM



Example of a nonlinear beam

Experimental results (Vertical set-up)



Comparison of the POM

□ experimental

* nonlinear model
(after updating)

o linear model
(before updating)

Conclusion

- Use of PCA for 3 goals:
  - damage detection problem based on the concept of subspace angle;
  - elimination of environmental effects.
  - Identification of nonlinear parameters
- Good results obtained on an intentionally damaged bridge.
- Testing of the method on many other bridges is currently in progress.

Thank you for your attention.