Regularity of functions: Genericity and multifractal analysis

Dissertation presented by

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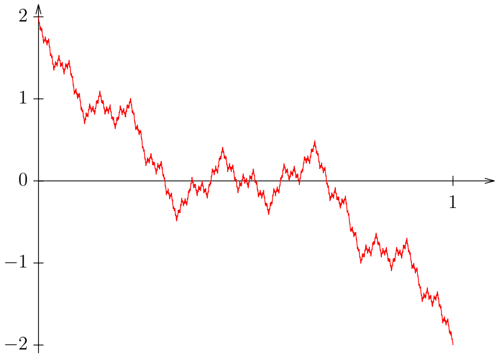

Figure: Weierstraß function for $a = 0.5$ and $b = 3$

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Content of the presentation.

1. Notions of genericity
   a) Residuality, prevalence and lineability
   b) Denjoy-Carleman classes
2. Multifractal analysis
   a) Hölder regularity and multifractal spectrum
   b) Multifractal formalism
   c) Leaders profile method
   d) $\mathcal{L}^\nu$ spaces

Notions of genericity

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- **Prevalence (Christensen, 1974 / Hunt, Sauer, Yorke, 1992).** Let X be a complete metrizable vector space. A Borel subset M of X is shy if there exists a Borel measure μ on X with compact support such that

$$\mu(M + x) = 0, \quad x \in X.$$

  More generally, a subset V is called shy if it is contained in a shy Borel set. The complement of a shy set is called a prevalent set.

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- **Lineability (Aron, Gurariy, Seoane-Sepúlveda, 2005).** Let X be a topological vector space and μ a cardinal number. A subset M of X is (dense-)lineable if $M \cup \{0\}$ contains an infinite dimensional vector subspace (dense) in X. If the dimension of this subspace is μ, M is said to be μ-(dense-)lineable.

Existence of nowhere analytic functions. An example was given by Cellérier (1890) by the function

$$f(x) := \sum_{n=1}^{+\infty} \frac{\sin(a^n x)}{n!}, \quad x \in \mathbb{R}$$

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- Genericity of the set of nowhere analytic functions in $C^\infty([0,1])$.
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- Genericity of the set of nowhere analytic functions in $C^\infty([0,1])$.
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Question. Similar results in the context of classes of ultradifferentiable functions?

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An arbitrary sequence of positive real numbers $M = (M_k)_{k \in \mathbb{N}_0}$ is called a weight sequence.

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Let Ω be an open subset of $\mathbb{R}$ and M be a weight sequence. The space $\mathcal{E}_{\{M\}}(\Omega)$ is defined by

$$\mathcal{E}_{\{M\}}(\Omega) := \left\{ f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega \text{ compact } \exists h > 0 \text{ such that } \|f\|^M_{K,h} < +\infty \right\},$$

where

$$\|f\|^M_{K,h} := \sup_{k\in\mathbb{N}_0} \sup_{x\in K} \frac{|D^k f(x)|}{h^k M_k}.$$

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Particular case. The weight sequences $(k!)_{k\in\mathbb{N}_0}$ and $((k!)^\alpha)_{k\in\mathbb{N}_0}$ with $\alpha > 1$.

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If $f \in \mathcal{E}_{(M)}(\Omega)$, we say that f is M-ultradifferentiable of Beurling type on Ω and we use the representation

$$\mathcal{E}_{(M)}(\Omega) = \underset{\overleftarrow{K \subseteq \Omega}}{\text{proj}} \ \underset{\overleftarrow{h>0}}{\text{proj}} \ \mathcal{E}_{M,h}(K)$$

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Questions.

- When do we have $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$?
- In that case, "how small" is $\mathcal{E}_{\{M\}}(\Omega)$ in $\mathcal{E}_{(N)}(\Omega)$?

General assumptions.

- We assume that any weight sequence M is logarithmically convex, i.e.

$$M_k^2 \leq M_{k-1} M_{k+1}, \quad \forall k \in \mathbb{N} .$$

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  It implies that the space $\mathcal{E}_{\{M\}}(\Omega)$ is an algebra.
- We assume that any weight sequence M is such that $M_0 = 1$.
- We usually assume that any weight sequence M is non-quasianalytic, i.e.

$$\sum_{k=1}^{+\infty}(M_k)^{-1/k} < +\infty.$$

  By Denjoy-Carleman theorem, it implies that there exists non-zero functions with compact support in $\mathcal{E}_{\{M\}}(\mathbb{R})$.

Inclusions between Denjoy-Carleman classes

Notation. Given two weight sequences M and N, we write

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Let M, N be two weight sequences and let Ω be an open subset of $\mathbb{R}$. Then

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Keys.

- If $M \lhd N$, then there exists a weight sequence L such that $M \lhd L \lhd N$.
- There exists $\theta \in \mathcal{E}_{\{M\}}(\mathbb{R})$ such that $|D^k\theta(0)| \geq M_k$ for all $k \in \mathbb{N}_0$. In particular, this function does not belong to $\mathcal{E}_{(M)}(\mathbb{R})$.

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Idea. Construct a sequence $(L^{(p)})_{p\in\mathbb{N}}$ of weight sequences such that

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For every $p \in \mathbb{N}$, consider a function $f_p \in \mathcal{E}_{\{L^{(p)}\}}(\mathbb{R})$ such that $|D^k f_p(0)| \geq L_k^{(p)}$, $\forall k \in \mathbb{N}_0$.

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$$f(x) = \sum_{p=1}^{+\infty} f_p(x - x_p)\Phi_p(x), \quad x \in \mathbb{R}$$

where Φ_p is a compactly supported function well chosen.

Generic results

Proposition

Assume that N and M are two weight sequences such that $M \lhd N$. If M is non quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is

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Idea. The set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is the complement of

$$\underbrace{\bigcup_{I\subseteq\mathbb{R}} \bigcup_{m\in\mathbb{N}} \bigcup_{s\in\mathbb{N}} \underbrace{\left\{ f \in \mathcal{E}_{(N)}(\mathbb{R}) : \sup_{x\in I} |D^k f(x)| \leq s m^k M_k, \ \forall k \in \mathbb{N}_0 \right\}}_{\text{closed set with empty interior}}}_{\text{proper linear subspace of } \mathcal{E}_{(N)}(\mathbb{R})}.$$

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Idea. Construct for every $t \in (0,1)$ a weight sequence $L^{(t)}$ such that

$$M \lhd L^{(t)} \lhd N \quad \text{and} \quad L^{(t)} \lhd L^{(s)} \text{ if } t < s.$$

Then, we have for every $t \in (0,1)$

$$M \lhd L^{(\frac{t}{2})} \lhd L^{(\frac{2t}{3})} \lhd L^{(\frac{3t}{4})} \lhd \cdots \lhd L^{(t)} \lhd N$$

and we construct as before a function of $\mathcal{E}_{(N)}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{M\}}$.

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More with countable unions

Let N be a weight sequence and let $(M^{(n)})_{n\in\mathbb{N}}$ be a sequence of weight sequences such that $M^{(n)} \lhd N$ for every $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that the weight sequence $M^{(n_0)}$ is non quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}$ is prevalent, residual and $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{(N)}(\mathbb{R})$.

Idea. Construct a weight sequence P such that

$$\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}(\Omega) \subseteq \mathcal{E}_{\{P\}}(\Omega) \subsetneq \mathcal{E}_{(N)}(\Omega).$$

An important example of ultradifferentiable functions of Roumieu type is given by the classes of Gevrey differentiable functions of order $\alpha > 1$. They correspond to the weight sequences

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Particular case of Gevrey classes

Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1, \alpha)$, is prevalent, residual and $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.

It suffices to take the weight sequences $M^{(n)}$ ($n \in \mathbb{N}$) given by

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Proposition (Schmets, Valdivia, 1991)

Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1,\alpha)$ is residual in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.

Other results.

- Similar results have been obtained with classes of ultradifferentiable functions defined using weight functions and weight matrices.

Perspectives.

- What about the algebrability?
- Other notions of genericity (such as porosity)?
- More with Pringsheim singularities?

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2. Multifractal analysis
   a) Hölder regularity and multifractal spectrum
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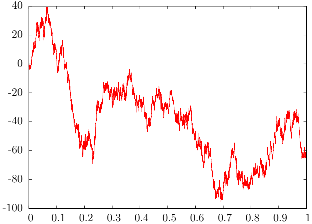
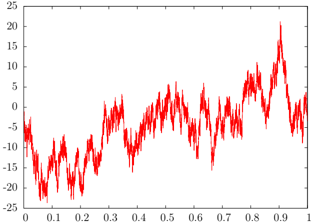
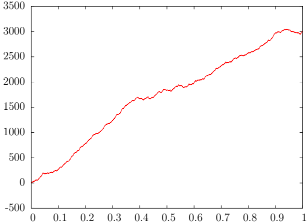
Hölder regularity and multifractal spectrum

Recall. Is it possible to characterize the local regularity of an irregular function?







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Recall. Is it possible to characterize the local regularity of an irregular function?

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, $\alpha \geq 0$ and $x \in \mathbb{R}$. The function f belongs to the Hölder space $C^\alpha(x)$ if there exist a constant $C > 0$ and a polynomial P of degree strictly smaller than α such that

$$|f(y) - P(y)| \leq C|y - x|^\alpha$$

for all y in a neighborhood of x. Then, the Hölder exponent $h_f(x)$ of f at x is defined by

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Weierstraß function. $h_f(x) = -\frac{\log a}{\log b}, \ \forall x \in \mathbb{R}$.

- Since $h_f(x)$ can change widely from a point to another, we will characterize the size of the sets of points which have the same local regularity.
- The iso-Hölder sets of f are $E_h := \{x \in \mathbb{R} : h_f(x) = h\}$.

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Definition

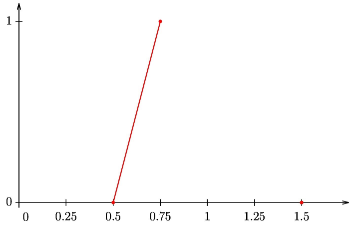
The multifractal spectrum d_f of f is defined by

$$d_f(h) := \dim_{\mathcal{H}} E_h, \quad \forall h \in [0, +\infty],$$

with the convention that $\dim_{\mathcal{H}} \emptyset = -\infty$.

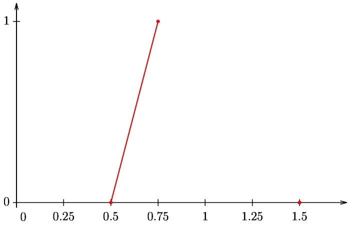
$\longrightarrow$ d_f gives a geometrical idea about the distribution of the singularities of f

Examples



Riemann function

Examples



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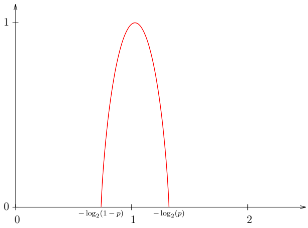
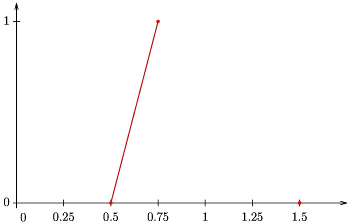


Cascade

Examples



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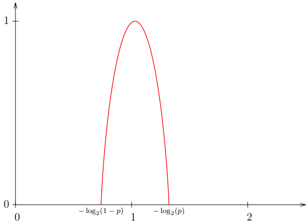

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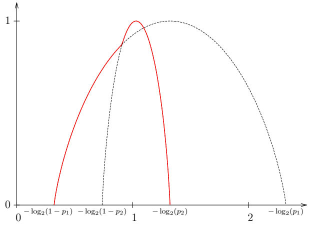
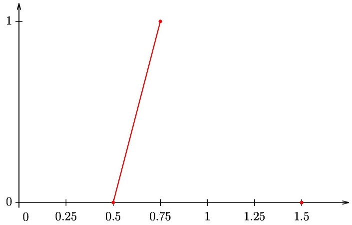


Sum of two cascades

Examples



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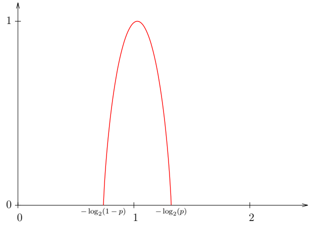
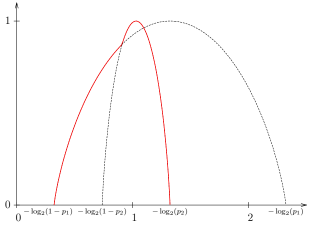
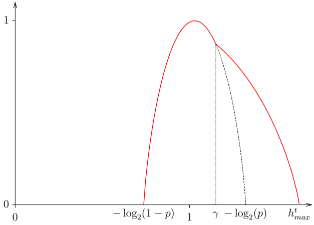


Cascade



Sum of two cascades



Threshold of a cascade

Multifractal formalism

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$$f = \sum_{j \in \mathbb{N}_0} \sum_{k=0}^{2^j-1} c_{j,k} \psi_{j,k} \; + \; C$$

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Characterization of the Hölder exponent using wavelet coefficients

If f is uniformly Hölder, the Hölder exponent of f at x is

$$h_f(x) = \liminf_{j \to +\infty} \inf_{k \in \{0,\dots,2^j-1\}} \frac{\log(|c_{j,k}|)}{\log(2^{-j} + |k2^{-j} - x|)}.$$

Advantage. Easy to compute and relatively stable from a numerical point of view.

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- Introduction of spaces of type $\mathcal{S}^\nu$ (J.M. Aubry, S. Jaffard, 2005), based on histograms of wavelet coefficients.
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- Combination of the two previous methods to obtain the leaders profile method and the spaces of type $\mathcal{L}^\nu$.
  $\longrightarrow$ Detection of increasing and decreasing parts of concave and non-concave spectra.

Wavelet leaders

Standard notation. For $j \in \mathbb{N}_0,\ k \in \{0, \dots, 2^j - 1\}$,

$$\lambda(j,k) := \{x \in \mathbb{R} : 2^j x - k \in [0,1[\} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right),$$

and for all $j \in \mathbb{N}_0$, Λ_j denotes the set of all dyadic intervals (of $[0,1)$) of length 2^{-j}.
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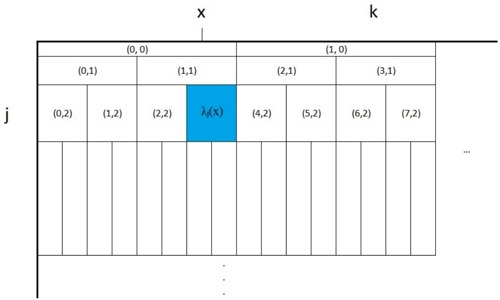
Definition

The wavelet leaders of a function $f \in L^2([0,1])$ are defined by

$$d_\lambda := \sup_{\lambda' \subseteq 3\lambda} |c_{\lambda'}|, \quad \lambda \in \Lambda_j,\ j \in \mathbb{N}_0 .$$

$\longrightarrow$ their decay properties are directly related with the Hölder exponent.

If $x \in [0,1)$, let $\lambda_j(x)$ denote the dyadic interval of length 2^{-j} which contains x.



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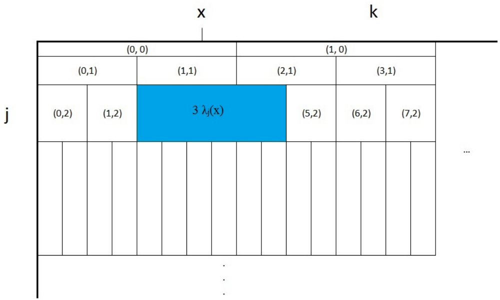

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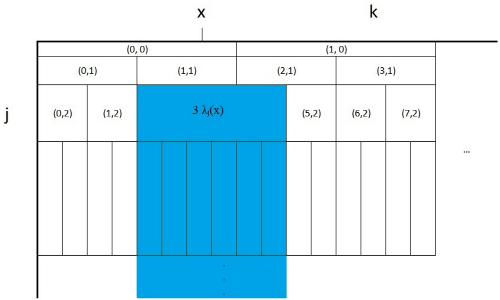

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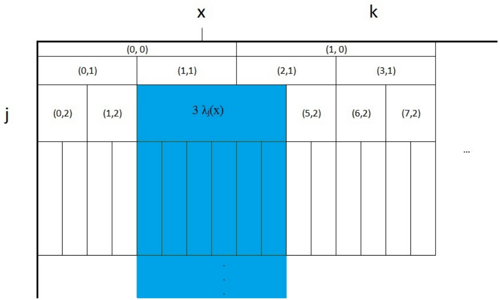

Hölder regularity and wavelet leaders

If f is uniformly Hölder, the Hölder exponent of f at x is given by

$$h_f(x) = \liminf_{j \to +\infty} \frac{\log d_{\lambda_j(x)}}{\log 2^{-j}}.$$

Interpretation.

$$d_{\lambda_j(x)} \sim 2^{-h_f(x)j}$$

Method based on $\mathcal{S}^\nu$ spaces

The wavelet profile ν_f of a locally bounded function f is defined for every $h \geq 0$ by

$$\nu_f(h) := \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \#\{\lambda \in \Lambda_j \ : \ |c_\lambda| \geq 2^{-(h+\varepsilon)j}\}}{\log 2^j}.$$

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Interpretation.

- There are approximatively $2^{\nu_f(h)j}$ coefficients greater in modulus than 2^{-hj}.

Properties.

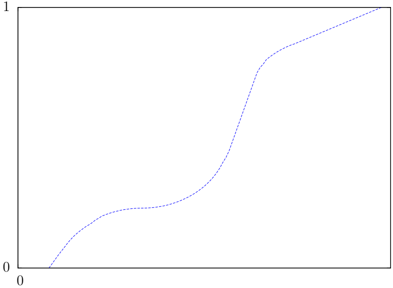
- ν_f is a right-continuous increasing function.
- ν_f is independent of the chosen wavelet basis.
- If f is uniformly Hölder,

$$d_f(h) \leq d^{\nu_f}(h) := \min\left\{ h \sup_{h' \in (0,h]} \frac{\nu_f(h')}{h'}, 1 \right\}, \quad \forall h \geq 0.$$

Definition

Take $0 \leq a < b \leq +\infty$. A function $g : [a,b] \mapsto [0,+\infty)$ is with increasing-visibility if g is continuous at a and $\sup_{y\in(a,x]} \frac{g(y)}{y} \leq \frac{g(x)}{x}$ for all $x \in (a,b]$.

In other words, a function g is with increasing-visibility if for all $x \in (a,b]$, the segment $[(0,0),(x,g(x))]$ lies above the graph of g on $(a,x]$.



Example of ν_f (---) and d^{ν_f} (—)

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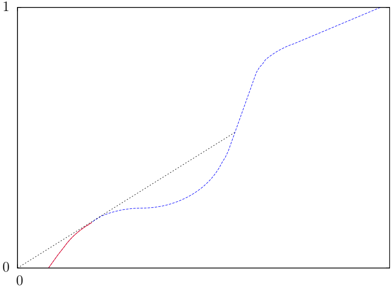

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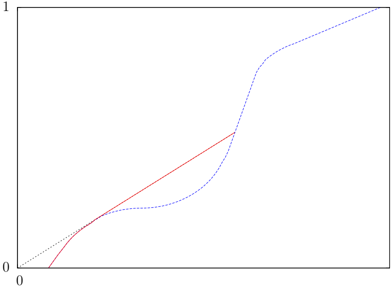

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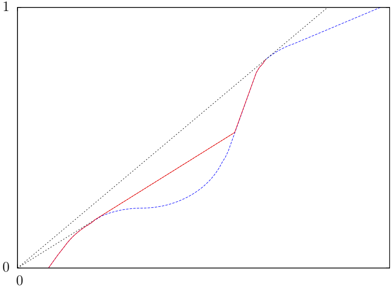

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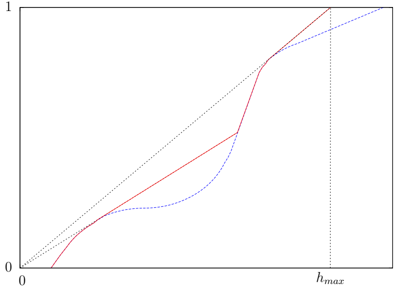

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Particular case

Assumption. Assume that the wavelet coefficients of f are given by $c_\lambda = \mu(\lambda)$ where μ is a finite Borel measure on $[0, 1]$.

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In this case, one has

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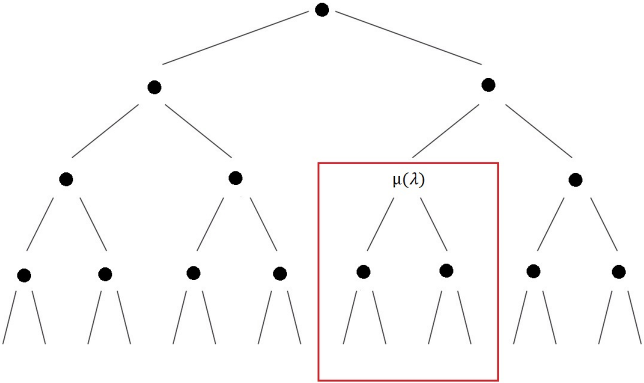
Moreover, if

$$\inf\left\{\frac{\nu_f(x)-\nu_f(y)}{x-y} : x,y \in [h_{\min}, h'_{\max}],\ x<y\right\} > 0,$$

where $h_{\min} = \inf\{\alpha : \nu_f(\alpha) \geq 0\}$, $h'_{\max} = \inf\{\alpha : \nu_f(\alpha) = 1\}$, then there exists $\beta > 0$ such that the function ν_{f_β} is with increasing-visibility on $[h_{\min}, h'_{\max}]$. In this case, $d^{\nu_{f_\beta}} = \nu_{f_\beta}$ approximates d_{f_β}. Therefore the increasing part of d_f can be approximated by ν_f.



There is a tree-structure in the repartition of the wavelet coefficients

Wavelet leaders density

The wavelet leaders density of f is defined for every $h \geq 0$ by

$$\widetilde{\rho}_f(h) := \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \# \left\{ \lambda \in \Lambda_j : 2^{-(h+\varepsilon)j} \leq d_\lambda < 2^{-(h-\varepsilon)j} \right\}}{\log 2^j}.$$

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Heuristic argument. We consider the points x such that $h_f(x) = h$.

- $d_{\lambda_j(x)} \sim 2^{-hj}$ and there are about $2^{\widetilde{\rho}_f(h)j}$ such dyadic intervals.
- If we cover each singularity x by dyadic intervals of size 2^{-j}, from the definition of the Hausdorff dimension, there are about $2^{d_f(h)j}$ such intervals.

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Problems.

- The wavelet leaders density may depend on the chosen wavelet basis.
- The definition of the wavelet leaders density is numerically extremely unstable.

Wavelet leaders profile

Let h_s be the smallest positive real number such that $\widetilde{\rho}_f(h_s) = 1$. The wavelet leaders profile of f is defined by

$$\widetilde{\nu}_f(h) := \begin{cases} \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \dfrac{\log \#\{\lambda \in \Lambda_j \ : \ d_\lambda \geq 2^{-(h+\varepsilon)j}\}}{\log 2^j} & \text{if } h \leq h_s, \\ \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \dfrac{\log \#\{\lambda \in \Lambda_j \ : \ d_\lambda \leq 2^{-(h-\varepsilon)j}\}}{\log 2^j} & \text{if } h \geq h_s. \end{cases}$$

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Properties.

- $\widetilde{\nu}_f$ is independent of the chosen wavelet basis.
- $\widetilde{\nu}_f$ takes values in $\{-\infty\} \cup [0, 1]$, it is increasing and right-continuous on $[0, h_s]$, decreasing and left-continuous on $[h_s, +\infty)$, $\widetilde{\nu}_f(h_s) = 1$ and the function

$$h \in [h_s, +\infty) \mapsto \frac{\widetilde{\nu}_f(h) - 1}{h}$$

  is decreasing.
- Moreover, any function ν which satisfies these properties is the wavelet leaders profile of a function.

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- An implementation of this method has been proposed and tested on several examples.

$\mathcal{L}^\nu$ spaces

Let ν be a function which has the same properties as any wavelet leaders profile.

Definition

The space $\mathcal{L}^\nu$ is the set of functions $f \in L^2([0,1])$ such that $\widetilde{\nu}_f \leq \nu$.

This space has been endowed with a complete metrizable topology.

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Results. If there is $\alpha_{\min} > 0$ such that $\nu(\alpha) = -\infty$ if $\alpha < \alpha_{\min}$, then

- $\mathcal{L}^\nu$ is also separable;
- The set of functions f such that $\widetilde{\nu}_f = \nu$ is residual and dense-lineable in $\mathcal{L}^\nu$.

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Perspectives.

- Generic validity of the leaders profile method;
- More with oscillating singularities.

References (Part I)

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