A general recurrence relation between the moments of a scaling function

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Abstract. Under natural and weak hypotheses, we prove a reproducing formula for polynomials. Then we obtain a new recurrence relation between the moments of a scaling function and a new exact formula for the computation of moments of even order.

1. Introduction

In [7], W. Sweldens and R. Piessens present the following result:

$$M_2 = (M_1)^2$$

where

$$M_j = \int_{\mathbb{R}} x^j \varphi(x)\ dx$$

are the moments of a compactly supported scaling function associated to a multiresolution analysis and in case the associated wavelet has at least three vanishing moments. Then, considering the shifted moments, they cancel the first and the second error terms in approximations and obtain an interesting quadrature formula. Their result about moments comes from a reproducing formula for polynomials.

A general result leading to a formula of that kind has been obtained by Y. Meyer in [6] but under rather strong regularity hypothesis.

In [1], under natural hypothesis and Strang-Fix conditions on a function φ (not necessarily a scaling function), we prove the reproducing formula for polynomials with absolute uniform convergence on compact sets and obtain the unicity of the coefficients. The proof we give does not follow the lines of the one of Y. Meyer and only use trigonometric Fourier series. Moreover, our result leads to relations showing that moments M_j of even order can be expressed in terms of a linear combination of products of moments of smaller order, with coefficients directly computable. In particular, we obtain $M_2 = (M_1)^2$.

Recurrence relations to compute the moments or approximations of them can be found in [2],[7]. These relations involve approximations or computation of auxiliary numbers related to the specific property of scaling functions. Here, we present relations leading to the exact computation of moments of even order using only combinatory coefficients.

In what follows, the set of natural numbers greater or equal to 0 (resp. strictly greater than 0) is denoted $\mathbb{N}$ (resp. $\mathbb{N}_0$) and the set of all integers (resp. all integers not equal to 0) is denoted $\mathbb{Z}$ (resp. $\mathbb{Z}_0$).

We also use the following notation $C_m^n = \frac{m!}{n!\ (m-n)!}$ where $m, n \in \mathbb{N}$, $m \geq n$.

2. Reproducing formula

Here is the result concerning the reproducing formula (see [1]).

Proposition 2.1 *Let φ be a function defined on $\mathbb{R}$ satisfying*

$$|\varphi(x)| \leq \frac{C}{(1+|x|)^{m+1+\varepsilon}}$$

for some $m \in \mathbb{N}_0, C, \varepsilon > 0$ and such that the functions $\varphi(.-k),\ k \in \mathbb{Z}$ satisfy

$$\int_{\mathbb{R}} \varphi(x-k)\ \varphi(x-j)\ dx \ = \ \delta_{kj},\ j,k \in \mathbb{Z}.$$

If in addition φ is such that $M_0 = \widehat{\varphi}(0) = 1$ and satisfy the Strang-Fix conditions

$$D^j \widehat{\varphi}(2k\pi) = 0 \quad \text{for } k \in \mathbb{Z}_0, 1 \leq j \leq m,$$

then for every $j = 0, \ldots, m$, there is a unique sequence $(a_k^{(j)})_{k\in\mathbb{Z}}$ such that

$$x^j = \sum_{k\in\mathbb{Z}} a_k^{(j)} \varphi(x-k) \quad ae$$

where the serie is absolutely and uniformly convergent on every compact subset of $\mathbb{R}$ and where $a_k^{(j)}$ is a polynomial of degree j in the variable k. These coefficients are

$$a_k^{(j)} = \int_{\mathbb{R}} x^j \varphi(x-k)\ dx, \quad j = 0, \ldots, m,\ k \in \mathbb{Z}.$$

In particular we have

$$a_0^{(j)} = M_j.$$

Another expression of the polynomials $a_k^{(j)}$ is obtained below. The proof can be found in [1]; it uses the previous result and recurrence technique. This expression leads to new relations between moments.

We use some definitions and notations: for $j, l \in \mathbb{N}_0$, we define

$$K_l(j) = \{(i_1, \ldots, i_l) \in \mathbb{N}_0^l : \sum_{k=1}^{l} i_k = j\}$$

and

$$K(j) = \bigcup_{l=1}^{j} K_l(j).$$

For $(i_1, \ldots, i_l) \in K(j)$, we write $i \in K(j)$. For $j \geq i_1 + \ldots + i_l$ we define

$$F_j(i_1, \ldots, i_l) = F_j(i) = (-1)^{i_1+1} \cdots (-1)^{i_l+1} C_j^{i_1} C_{j-i_1}^{i_2} \ldots C_{j-\sum_{k=1}^{l-1} i_k}^{i_l} M_{i_1} \ldots M_{i_l}$$

where

$$M_j = \int_{\mathbb{R}} x^j \varphi(x)\ dx = a_0^{(j)}.$$

For $j \in \mathbb{N}$, we also set

$$\sum_{i\in K(0)} F_j(i) = 1. \qquad (*)$$

Proposition 2.2 *Under the same hypothesis as in Proposition 2.1 and using the notations introduced above, we have the following relations*

$$a_k^{(j)} = \sum_{l=0}^{j} k^l \sum_{i\in K(j-l)} F_j(i), \quad k \in \mathbb{Z}, j = 1, \ldots, m. \tag{1}$$

3. Relation between moments

We can deduce from the previous relations that the moments of even order can be expressed in terms of a linear combination of products of moments of smaller order in which the coefficients are of type C_m^l.

Corollary 3.1 *Under the same hypothesis as in Proposition 2.1 and using the same notations, we have*

$$M_j = \sum_{i \in K(j)} F_j(i) = \sum_{i \in \cup_{l=1}^{j} K_l(j)} F_j(i), \quad j = 1, \ldots, m.$$

In particular, if j is even, we have

$$2M_j = \sum_{i \in \cup_{l=2}^{j} K_l(j)} F_j(i)$$

Proof. It suffices to take $k = 0$ in the relations (1) giving $a_k^{(j)}$ in the previous proposition.

For j even, we have

$$F_j(j) = (-1)^{1+j} M_j = -M_j$$

hence the conclusion.

As example, we obtain

$$K_2(2) = \{(1,1)\}, \ F_2((1,1)) = 2$$

hence

$$2M_2 = 2(M_1)^2;$$

in the same way

$$M_4 = -3(M_1)^4 + 4M_1M_3,$$

$$M_6 = 45(M_1)^6 - 60(M_1)^3 M_3 + 6M_1M_5 + 10(M_3)^2.$$

4. A numerical use

The previous relations can be used for numerical applications. In the following example, a recurrence formula can be replaced by simple exact formulas which directly lead to the same precision.

Following the ideas of [2], for a filter of the form

$$m_0(\omega) = 2^{-1/2} \sum_{k=1}^{2M} h_k \exp(i(k-1)\omega),$$

we can approximate the m-th moment with the relation

$$\mathcal{M}_m^{(r)} = i^{-m} [D^m \prod_{j=1}^{r} m_0(2^{-j}\omega)]_{\omega=0}. \tag{2}$$

Here, r represents the number of factors in the approximation. From (2), we can obtain the recurrence formula

$$\mathcal{M}_m^{(r+1)} = \sum_{k=0}^{m} C_m^k 2^{-rk} \mathcal{M}_{m-k}^{(r)} \mathcal{M}_k^{(1)}.$$

When m is even, we can replace this approximation with our formula where the preceding moments, which have been estimated, are involved. So, in this case,we replace r evaluations by 1.

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