Abelian bordered factors and periodicity

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Bordered words

- A finite word u is *bordered*, if there exists a non-empty word z which is a proper prefix and a suffix of u.
  Example: $aababbaaab$
- A *periodic* infinite word: $w = u^\omega = uuu\cdots$.

Theorem [A. Ehrenfeucht, D. Silberger, 1979]

An infinite word w is periodic if and only if there exists a constant C such that every factor v of w with $|v| \geq C$ is bordered.

We study

- abelian version
- weak abelian version

Abelian periodicity and borders

Σ^* – finite words over the alphabet Σ
$a \in \Sigma$, $|u|_a$ – the number of occurrences of the letter a in u

- $u, v \in \Sigma^*$ are abelian equivalent if $|u|_a = |v|_a$ for all $a \in \Sigma$.
- An abelian (ultimately) periodic infinite word: $w = v_0 v_1 \cdots$, where $v_k \in \Sigma^*$ for $k \geq 0$, and $v_i \sim_{ab} v_j$ for all integers $i, j \geq 1$.

  Example: Thue-Morse word is abelian periodic with period 2
- $u \in \Sigma^*$ is abelian bordered, if it has a non-empty prefix which is abelian equivalent to its suffix.

  Example: *abababbaabb* abelian bordered
  Example: *abababbaabbb* abelian unbordered

  - the number of abelian bordered words [N. Rampersad, M. Rigo, P. Salimov, 2013]

Abelian bordered words and periodicity

finitely many abelian unbordered factors

relations
?
$\Leftrightarrow$

(abelian) periodicity

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Proposition

There exists an infinite aperiodic word w and a constant C such that every factor v of w with $|v| \geq C$ is abelian bordered.

Example

Any aperiodic $w \in \{010100110011, 0101001100110011\}^\omega$ satisfies the condition with $C = 15$.

Abelian bordered words

Abelian periodicity $\not\Rightarrow$ finitely many abelian unbordered factors:

Example

Thue-Morse word $t = 0110100110010110\cdots$

- abelian periodic with period 2
- has infinitely many abelian unbordered factors: $0p1$, where p is a palindrome

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Is the converse true?

Open question 1

Let w be an infinite word with finitely many abelian unbordered factors; does it follow that w is abelian periodic?

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We do not know. But we are able to answer the question in weak abelian setting.

Weak abelian periodicity

frequency $\rho_a(w)$ of a letter $a \in \Sigma$ in $w \in \Sigma^*$ is $\rho_a(w) = \frac{|w|_a}{|w|}$.

$w \in \Sigma^\omega$ is called weakly abelian (ultimately) periodic (WAP), if $w = v_0 v_1 \ldots$, where $v_i \in \Sigma^*$, $\rho_a(v_i) = \rho_a(v_j)$ for all $a \in \Sigma$ and all integers $i, j \geq 1$.

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$w \in \Sigma^\omega$ is called bounded weakly abelian periodic, if it is WAP with bounded lengths of blocks, i. e., there exists C such that for every i we have $|v_i| \leq C$.

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- Avoidability of weak abelian powers
  - J. L. Gerver, L. T. Ramsey, 1979: an infinite ternary word having no weak abelian $(5^{11}+1)$-powers.
  - V. Krajnev, 1980: an upper bound for length of binary word which does not contain weak abelian k-powers, $k \in \mathbb{N}$
- weak abelian periodicity
  - S. Avgustinovich, S. P., 2013

Geometric interpretation of WAP

$w = w_1 w_2 \cdots \in \Sigma^\omega \to$ graph g_w in $\mathbb{Z}^{|\Sigma|}$

In the binary case: $0 \to$ move by $\mathbf{v}_0$, e.g., $(1, -1)$
$1 \to$ move by $\mathbf{v}_1$, e.g., $(1, 1)$

Start at the origin $(x_0, y_0) = (0, 0)$

Step n, $n > 0$:

we are at a point (x_{n-1}, y_{n-1})
move by $\mathbf{v}_{w_n}$: $(x_n, y_n) = (x_{n-1}, y_{n-1}) + \mathbf{v}_{w_n}$,
(x_{n-1}, y_{n-1}) and (x_n, y_n) are connected with a line segment

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WAP: infinitely many integer points on a line with rational slope

bounded WAP: infinitely many integer points on a line with bounded gaps

For a k-letter alphabet one can consider a similar graph in $\mathbb{Z}^k$.

Example: Paperfolding word

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$$w = 00100110001101100010011100110110\ldots$$

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It can be defined, e.g., as a Toeplitz word with pattern 0?1?.

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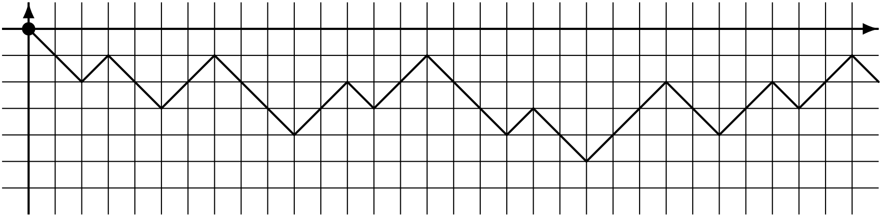
The graph g_w with $\mathbf{v}_0 = (1, -1)$, $\mathbf{v}_1 = (1, 1)$



The paperfolding word is not balanced and is WAP along the line $y = -1$ (and actually along any line $y = C$, $C = -1, -2, \dots$).

Example: bounded WAP word

A word obtained as an image of the morphism

$$0 \mapsto 01,$$

$$1 \mapsto 000111$$

of any nonperiodic binary word is bounded WAP.

Abelian periodicity and borders

$u \in \Sigma^*$ is weakly abelian bordered, if it has a non-empty proper prefix and a suffix with the same frequencies of letters

Example: $abaabbaabb$ weakly abelian bordered

Example: $abaabbaabbb$ weakly abelian unbordered

finitely many weakly abelian unbordered factors

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weak abelian periodicity

Main theorem

finitely many weakly abelian unbordered factors $\Longrightarrow$ WAP

Theorem

Let w be an infinite binary word. If there exists a constant C such that every factor v of w with $|v| \geq C$ is weakly abelian bordered, then w is bounded weakly abelian periodic. Moreover, its graph lies between two rational lines and has points on each of these two lines with bounded gaps.

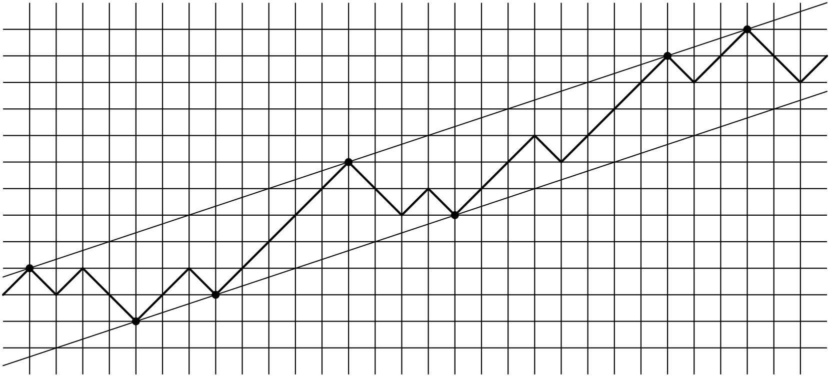
In terms of Avgustinovich, P., 2013, the word is of bounded width.

Open question 2

Is the converse true?

True for $\rho_0 = \rho_1 = 1/2$.

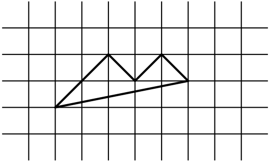
Example



the graph of w lies between two rational lines and has points on each of these two lines with bounded gaps

Lemma

Let w be an infinite binary word, i, j be integers, $i < j$. If $g_w(k) > \frac{g_w(j)-g_w(i)}{j-i}k + \frac{g_w(i)j-g_w(j)i}{j-i}$ for each $i < k < j$, then the factor $w[i..j]$ is weakly abelian unbordered.



Lemmas

A word w is called K-balanced if for each two its factors u and v of equal length $||u|_a - |v|_a| \leq K$ for any $a \in \Sigma$.

Lemma

Let w have finitely many weakly abelian unbordered factors. Then w is K-balanced for some K.

Lemma

Let w have finitely many weakly abelian unbordered factors. Then the frequencies of letters are rational.

Additional result on abelian critical factorization theorem

A critical factorization theorem implies for infinite words:

Theorem

A biinfinite word w is periodic if and only if there exists an integer l such that w has at every position a centered square with period at most l.

S. Avgustinovich, J. Karhumäki, P., 2012: what about abelian squares (powers) and periodicity?

Additional result

A non-abelian periodic word with bounded abelian square at each position

Question [S. Avgustinovich, J. Karhumäki, S. P., 2012]

Let w be an infinite word and C be an integer such that each position in w is a centre of an abelian square of length at most C. Is w abelian periodic?

The answer is NO:

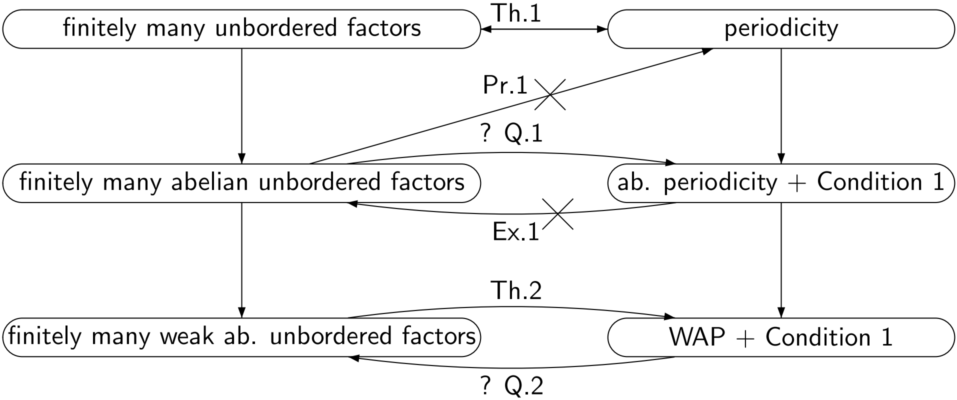
Example

Consider a family of infinite words of the following form:

$$(000101010111000111000(111000)^{*}111010101)^{\omega}$$

- such words have abelian square of length at most 12 at each position
- this family contains abelian aperiodic words

Results and open questions



Condition 1: the graph of the word lies between two rational lines and has points on each of these two lines with bounded gaps.

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