New analytic unitarization schemes

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We consider two well-known classes of unitarization of Born amplitudes of hadron elastic scattering. The standard class, which saturates at the black disk limit includes the standard eikonal representation, while the other class, which goes beyond the black disk limit to reach the full unitarity circle, includes the U matrix. It is shown that the basic properties of these schemes are independent of the functional form used for the unitarisation, and that U-matrix and eikonal schemes can be extended to have similar properties. A common form of unitarization is proposed interpolating between both classes. The correspondence with different nonlinear equations are also briefly examined.

I. INTRODUCTION

At low energies, hadron scattering can be described by one-Reggeon exchange terms. But as the Pomeron term(s) grow(s) with energy, these exchanges will eventually violate unitarity. To see this, one can switch to partial waves [1], or to the representation in impact parameter $\mathbf{b}$. As s grows, one needs to sum many partial waves with $l \sim |\mathbf{b}|\sqrt{s} \to \infty$. The summation over l then becomes an integration over $\mathbf{b}$.

The partial wave $G(s, \mathbf{b})$ has two regimes. First of all, it can reach maximum inelasticity. In this case, $G(s, \mathbf{b}) = 1$, and half of the interactions are inelastic. The center of the protons then becomes black, and multiple exchanges, *i.e.* cuts in the complex J plane, become important. Second, the partial wave can later reach the full unitarity limit $G(s, \mathbf{b}) = 2$.

The maximum inelasticity limit may be reached in pp or $p\bar{p}$ scattering a little above the Tevatron energy [2], so that one expects cuts to be important in the description of soft interactions at the CERN LHC. The inclusion of these goes under the name of unitarization. It is a formidable task to calculate the contribution of cuts, as not only multiple Pomeron exchanges must be calculated, but also multiple Pomeron vertices.

Different schemes have been proposed, and we want in this paper to show that the general properties of the amplitude do not heavily depend on the scheme, but rather on what assumes for the inelastic contribution at high energy. We shall limit ourselves to two popular schemes: the eikonal and the U matrix, and show that simple extensions of each lead to similar properties.

In Sec. I, we remind the reader of the simple requirements coming from unitarity, and examine in Sec. II the two schemes. In Sec. III, we show that it is possible to obtain the properties of the eikonal by extending the U-matrix scheme, whereas in Sec. IV, we show the reverse,

i.e. that one can extend the eikonal to mimic the behavior of the U matrix, at least for physical amplitudes.

II. UNITARITY

At high energy, we can start with the elastic scattering amplitude $a(s, t)$, related to the elastic cross section though

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}|a(s, t)|^2. \quad (1)$$

One can then Fourier transform a to $\mathbf{b}$ space

$$G(s, \mathbf{b}) = \int \frac{d^2\mathbf{\Delta}}{(2\pi)^2} \frac{a(s, t)}{2s} e^{i\mathbf{\Delta}\cdot\mathbf{b}}, \quad (2)$$

which leads to the expressions

$$\sigma_{tot} = 2 \int d^2\mathbf{b}\ \mathrm{Im}\ G(s, \mathbf{b}), \quad (3)$$

$$\sigma_{el} = \int d^2\mathbf{b}\ |G(s, \mathbf{b})|^2, \quad (4)$$

where we have assumed that the spin-flip contribution to the elastic cross section is negligible. One can then write the square of the S-matrix density $S(s, \mathbf{b}) = 1 + iG(s, \mathbf{b})$ as

$$|S(s, \mathbf{b})|^2 = 1 - 2\mathrm{Im}\ G(s, \mathbf{b}) + |G(s, \mathbf{b})|^2. \quad (5)$$

Unitarity demands that $|S(\mathbf{b})|^2| \leq 1$, the difference coming from inelastic channels:

$$\eta_{in}(s, \mathbf{b}) = 1 - |S(s, \mathbf{b})|^2 \geq 0. \quad (6)$$

There are several ways to represent the unit circle. First of all, one can map the upper complex plane into a circle via a complex exponential

$$S(s, \mathbf{b}) = \exp(iz(s, \mathbf{b})) \text{ with } \mathrm{Im}z(s, \mathbf{b}) \geq 0, . \quad (7)$$

This maps in fact an infinite number of strips with $2n\pi < \mathrm{Re}z(s, \mathbf{b}) < 2(n+1)\pi$ each onto the unit circle.

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It is also possible to use a one-to-one map through a Möbius transform and write

$$S(s, \mathbf{b}) = \frac{1 + iz'(s, \mathbf{b})}{1 - iz'(s, \mathbf{b})}, \text{ with } \mathrm{Im} z'(s, \mathbf{b}) \geq 0. \quad (8)$$

Other representations are possible, but we shall concentrate on these two in the following section.

III. UNITARIZATION

The physical amplitude lies within the unitarity circle, so that the associated S matrix can always be represented by Eqs. (7) and (8). The unitarization scheme comes in once one identifies z or z' with the one-Reggeon exchange amplitude. One then considers (7) and (8) as series expansions in n-Reggeon exchanges, so that their first term must give $1 + i\chi(s, b)$.

Indeed, if one writes the one-Reggeon exchange amplitude as $\chi(s, \mathbf{b})$, then assuming $z = \chi$ in (7) leads to the well-known eikonal representation:

$$G(s, \mathbf{b}) = i(1 - \exp(i\chi(s, \mathbf{b}))). \quad (9)$$

This scheme can be derived in QED and other field theories [3, 4, 5, 6] or in potential theory [7]. It can be extended to include diffractive channels [8]. It leads at asymptotic energies ($s \to \infty$) to the limit $\sigma_{\mathrm{el}}/\sigma_{\mathrm{inel}} = 1$, i.e. to maximum inelasticity.

The other unitarization scheme considered here is the U-matrix representation [9, 10, 11] where one identifies z' in (8) with $\chi(s, \mathbf{b})/2$, to match the one-Reggeon exchange

$$G(s, \mathbf{b}) = \frac{\chi(s, \mathbf{b})}{1 - i\chi(s, \mathbf{b})/2}. \quad (10)$$

In this scheme, $S(s, \mathbf{b})$ tends to -1 when $s \to \infty$ and $\mathbf{b}$ is finite, so that the inelastic partial wave $\eta_{\mathrm{in}}(s, \mathbf{b})$ tends to 0: the ratio $\sigma_{\mathrm{el}}/\sigma_{\mathrm{inel}}$ vanishes asymptotically.

Both schemes have the same development at second order in χ, and differ only in the rest of the series.

It must be noted however that the resummation must lead to an amplitude within the unitarity circle, but there is no reason to assume that it maps the entire complex plane to the circle. Hence, one can easily extend both schemes through a change in the strength of successive scattering. This gives the extended eikonal schemes [12, 13, 14]

$$G(s, \mathbf{b}) = \frac{i}{\omega}(1 - \exp(i\omega\chi(s, \mathbf{b}))) \quad (11)$$

and the extended U-matrix schemes

$$G(s, \mathbf{b}) = \frac{\chi(s, \mathbf{b})}{1 - i\omega'\chi(s, \mathbf{b})}. \quad (12)$$

It is straightforward to check that using $\omega \geq 1$ or $\omega' \geq 1/2$ maps any amplitude χ into the unitarity circle.

We shall now show that the various possibilities can be grouped into two wide classes of unitarization schemes, and that the exact form matters little.

IV. SHADOWING

As we have seen, the eikonal predicts that at high energy the inelastic component of the cross section will be maximal, $\eta_{\mathrm{in}} = 1$. This in turn leads to $|S(\infty, \mathbf{b})| = 0$ and $G(\infty, \mathbf{b}) = i$. To reach this regime via an extended U matrix, one needs to choose $\omega' = 1$ in (12).

The inelasticity will then be

$$\eta_{\mathrm{in}}(s, \mathbf{b}) = \frac{2\mathrm{Im}\chi(s, \mathbf{b}) + |\chi(s, \mathbf{b})|^2}{1 + 2\mathrm{Im}\chi(s, \mathbf{b}) + |\chi(s, \mathbf{b})|^2}. \quad (13)$$

It can easily be seen that $s \to \infty$ leads to $\sigma_{\mathrm{inel}}/\sigma_{\mathrm{el}} \to 1$ and $\sigma_{\mathrm{el}}/\sigma_{\mathrm{tot}} \to 1/2$.

Differently stated, this extended U matrix representation has the standard black disk limit.
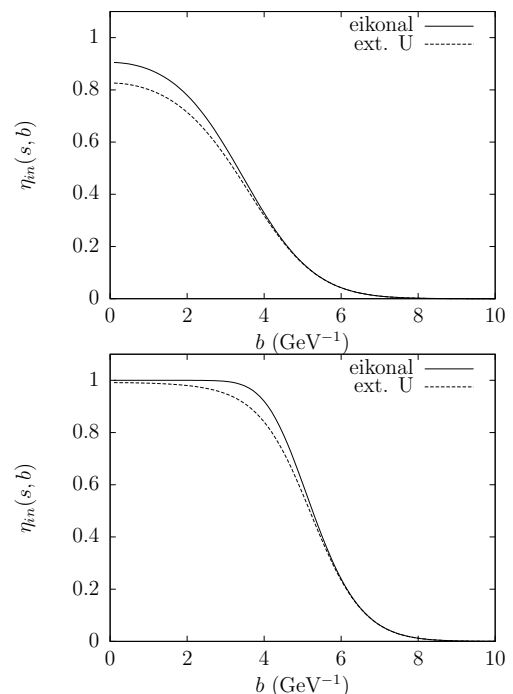

FIG. 1: Inelasticity for the eikonal and extended U matrix at $\sqrt{s} = 1.8$ TeV (upper figure) and $\sqrt{s} = 14$ TeV (lower figure).

Throughout this paper, we shall use as an example of one-Reggeon exchange amplitude a hard Pomeron term, with a parametrization

$$\chi(s, b) = \left(\frac{\sqrt{s}}{1500 \text{ GeV}}\right)^{0.9} exp(-\mathbf{b}^2/(9 \text{ } GeV^{-2}))$$

$$\times \left((i + \tan\left(\frac{0.45\pi}{2}\right)\right). \quad (14)$$

This amplitude reaches the black disk limit at 1500 GeV, similarly to the model of ref. [15], and has a dependence in t similar to that of pp scattering. We also neglect the effect of shrinkage, which is small for a hard Pomeron.

In Fig. 1, one can compare the inelasticity in the case of the eikonal and in that of the extended U matrix. One clearly sees that the generic features of both schemes are very close.

The extended U matrix and the eikonal scheme are different representations of a wider class of unitarization procedures with a standard black disk limit. Indeed, we can extend (12) to

$$G(s, \mathbf{b}) = i[1 - \frac{1}{(1 - i\chi(s, \mathbf{b})/\gamma)^\gamma}]. \qquad (15)$$

If $\gamma = 1$ this form leads to the extended U matrix while, for $\gamma \to \infty$, we obtain the standard eikonal. When γ varies from 1 to ∞, we obtain different forms of unitarization which all lead to a black disk limit, and the amplitude G does not change anymore once it has reached its maximum value. In Fig. 2, we show the inelasticity $\eta(s, \mathbf{b})$ for different values of γ, again in the case of the hard Pomeron input of Eq. (14).
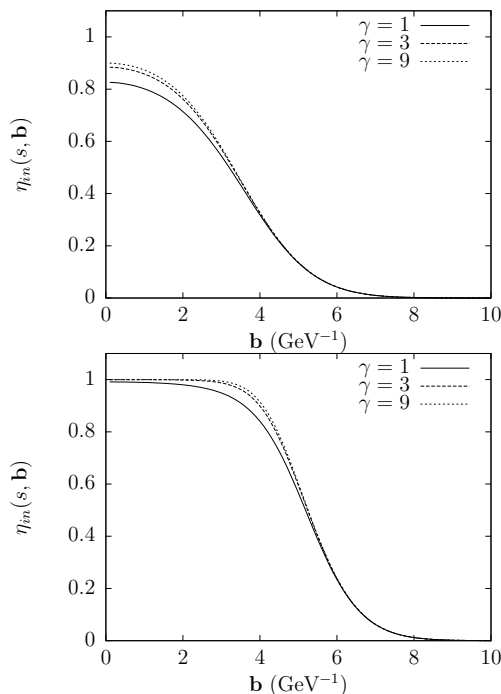

FIG. 2: Inelasticity $\eta_{\rm in}(s, \mathbf{b})$ for various values of γ, for $\sqrt{s} = 1.8$ TeV (upper line) and $\sqrt{s} = 14$ TeV (lower line).

It may be worth pointing out that Eq. (15) can also lead to a noninteracting theory in the limit $n \to 0$, as in this case $G(s, \mathbf{b}) \to 0$.

V. FULL UNITARITY LIMIT

The standard U-matrix scheme (10) was intensively explored in [11] in the partial-wave language. In the impact parameter representation, the properties of the U-matrix are explored in [16].

For a purely imaginary one-Reggeon exchange, when $\mathrm{Im}\chi(s, \mathbf{b})$ goes from 0 to ∞, the S-matrix varies in the interval $[-1, +1]$, and the amplitude G goes from 0 to $2i$. Hence in this case the full unitarity limit can be reached.

This form of unitarization leads to unusual properties at super-high energies as was shown in [16]. In this representation

$$\sigma_{\rm el}(s) = 4 \int_0^\infty \frac{\left|\frac{\chi(s, \mathbf{b})}{2}\right|^2}{\left|1 - i\left(\frac{\chi(s, \mathbf{b})}{2}\right)\right|^2} \, d\mathbf{b} \qquad (16)$$

and

$$\sigma_{\rm inel}(s) = 4 \int_0^\infty \mathrm{Im}\left(\frac{\left(\frac{\chi(s, \mathbf{b})}{2}\right)}{\left[1 - i\left(\frac{\chi(s, \mathbf{b})}{2}\right)\right]^2}\right) \, d\mathbf{b}. \qquad (17)$$

so that, when $\mathrm{Im}\chi(s, \mathbf{b}) \to \infty$, one gets $\sigma_{\rm inel}/\sigma_{\rm tot} \to 0$ and $\sigma_{\rm el}/\sigma_{\rm tot} \to 1$.

It is often considered [17] that these properties are intrinsic to the Möbius projection of χ onto the unitarity circle. We want to show now that, in fact, extended eikonals can lead to the same properties for the unitarized amplitude.

We have seen that choosing $\omega > 1$ in (11) guarantees that any amplitude would be unitarized. However, one must be concerned with the physical amplitude, and it is not needed to map the whole complex plane into the unitarity circle. This means that for some specific choices of one-reggeon exchange χ, one can extend the range of values of ω, and restrict oneself to part of the complex χ plane. Unitarity, in this case, leads to the condition

$$\cos(\omega \mathrm{Re}\chi) \geq \frac{e^{-\omega \mathrm{Im}\chi} - (2\omega - 1)e^{\omega \mathrm{Im}\chi}}{2(1 - \omega)}. \qquad (18)$$

So we see that for $\omega < 1/2$, the second term of the numerator will guarantee the inequality for sufficiently large $\mathrm{Im}\chi$.

We show in Fig. 3 the region allowed in the case $\omega = 0.525$ together with a curve showing the amplitude corresponding to the exchange of one hard pomeron with intercept 1.45. This is of course an extreme curve, corresponding to a ratio $\mathrm{Re}\chi/\mathrm{Im}\chi$ of 0.73. Any physical amplitude will include softer intercepts, and will lie above the hard pomeron line. So we see that, in practice, eikonals can be extended to values of ω between 1/2 and 1.

But at high energy, such eikonals have all the basic properties of the U-matrix unitarization. For instance, the inelasticity reaches the asymptotic value

$$\eta_{in} \to \frac{2\omega - 1}{\omega^2} \text{ as } s \to \infty \qquad (19)$$

which is close to 0 for ω close to 1/2.

Our calculations for $\eta_{\rm in}(s, b)$ in the cases of the U-matrix and of the extended eikonal are shown in Fig.
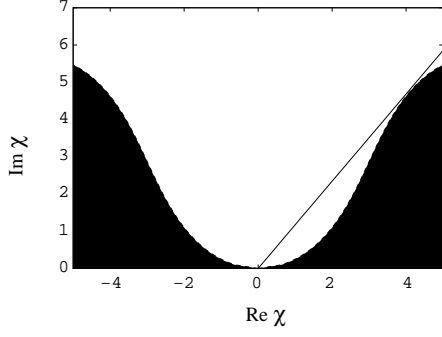
FIG. 3: The allowed region (in white) for amplitudes χ to be unitarised by an extended eikonal with $\omega = 0.525$, together with the line corresponding to a hard pomeron amplitude with intercept 1.45.
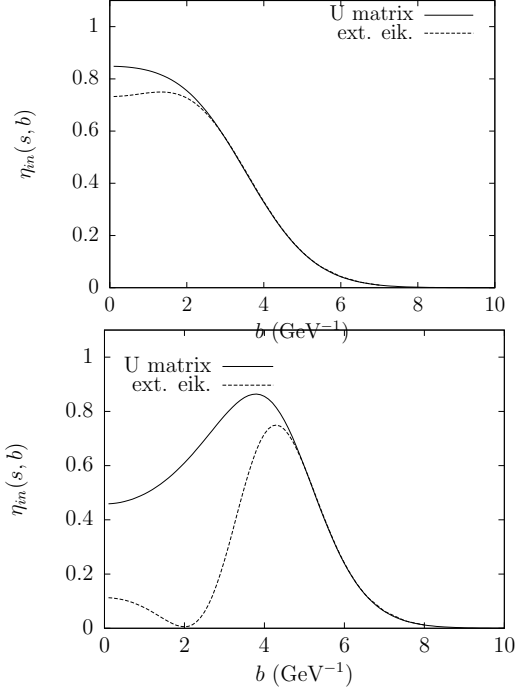

FIG. 4: Antishadowing effects in extended eikonal and U matrix schemes, for $\sqrt{s} = 1.8$ TeV (upper figure) and $\sqrt{s} = 14$ TeV (lower figure).

4. We see that both solutions have the same behavior in s and b, but also that the extended eikonal has sharper anti-shadowing properties.

Hence, the peculiar asymptotic properties of (17) are not unique to the U-matrix (10). The extended eikonal (11) has a similar asymptotic behavior for ω between $1/2$ and 1.

Again, we can find a scheme that interpolates between these two forms [18]:

$$G(s,b) = \frac{i}{\omega}[1 - \frac{1}{(1 - i\omega\chi(s,b)/\gamma)^\gamma}]. \qquad (20)$$

If $\gamma = 1$ this form coincides with the standard U-matrix

representation for $\omega = 1/2$. If, on the contrary, we let $\gamma \to \infty$, we recover the extended form of the eikonal representation. Hence, when γ varies from 1 to ∞, and $\omega \approx 1/2$, we obtain different forms of unitarization belonging to a wide class with the similar asymptotic properties, which we shall examine more closely in the next section.

VI. COMPARISON OF THE BORN TERMS

Another way to make the unitarization schemes lead to similar results is to use different inputs for z in (8) or (7). We can solve the equation

$$\frac{i}{\omega}(1 - \exp(i\omega\chi_e) = \frac{\chi_u}{1 - i\omega'\chi_u}. \qquad (21)$$

to find which Born term in the U-matrix representation would give results similar to those of the eikonal. Writing $\chi_u = \chi_u^R + i\chi_u^I$, and $\chi_e = \chi_e^R + i\chi_e^I$, we obtain

$$2\omega\chi_e^I = \log\left(\frac{|\chi_u|^2\omega'^2 + 2\chi_u^I\omega' + 1}{|\chi_u|^2(\omega - \omega')^2 - 2\chi_u^I(\omega - \omega') + 1}\right) \qquad (22)$$

$$\tan(\omega\chi_e^R) = \frac{\chi_u^R\omega}{|\chi_u|^2\omega'(\omega' - \omega) + \chi_u^I(2\omega' - \omega) + 1} \qquad (23)$$

This simplifies to a particularly simple expression in the case of a purely imaginary χ_u:

$$\omega\chi_e^I = \log\left|\frac{\chi_u^I\omega' + 1}{1 - \chi_u^I(\omega - \omega')}\right| \qquad (24)$$

and the real part goes from 0 to $\omega\chi_e^R = \pi$ if $(\omega - \omega')\chi_u^I$ crosses 1. This relation is clearly discontinuous if $\omega \neq \omega'$. We illustrate this in Fig. 5 in the case $\omega = 1$ (eikonal) and $\omega' = 1/2$ (U matrix). At low energy, the phases are approximately the same. But at high energy when $\chi_u^I(s,b) \to 2$, χ_e has a discontinuity: its imaginary part goes to infinity, and its real part jumps by $i\pi$. On the other hand, the extended U matrix does not lead to such a singularity if $\omega' = 1$.

VII. NON-LINEAR EQUATIONS

All previous schemes can be recast as non-linear equations, which may be reminiscent of those obtained in QCD from gluon saturation.

The simplest way [19] to get these is first to take the derivative of G with respect to χ in (11) and (12):

$$\frac{dG}{d\chi} = 1 + i\omega G \qquad (25)$$

in the eikonal case, and
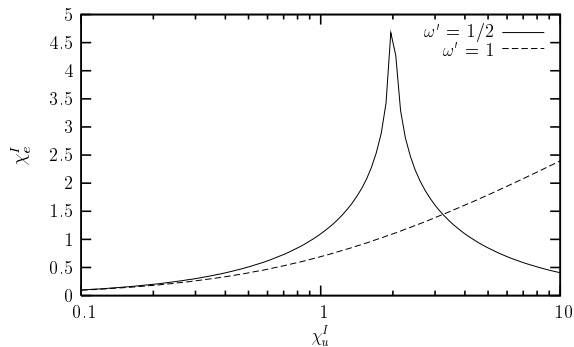
$$\frac{dG}{d\chi} = (1 + i\omega' G)^2 \qquad (26)$$

FIG. 5: The relation between the Born terms of the U-matrix scheme and of the eikonal scheme, in the case of a purely imaginary χ_u. The real part of $\omega\chi_e$ is discontinuous and goes from 0 to $i\pi$ at $\chi_u = 2$.

for the U matrix. To make a tentative connection with saturation, we shall consider a purely imaginary Born term, and we shall write $G = ig$. Assuming $\chi = ig_0 x^{-\Delta}$, we can write the above equations as an evolution in $y = \log(1/x)$ at fixed $\mathbf{b}$:

$$\frac{dg}{dy} = \frac{\Delta}{\omega}\log(1 - \omega g)(1 - \omega g).$$ (27)

in the eikonal case and

$$\frac{dg}{dy} = \Delta g(1 - \omega' g)$$ (28)

for the U matrix.

We see that the U matrix schemes lead to equations which look more natural than the corresponding ones in the eikonal case, as it is hard to imagine how saturation would lead to a log containing the amplitude.

One can further generalize these equation to reproduce Eqs. (15) and (20). The corresponding nonlinear equation will be

$$\frac{dg}{dy} = \frac{\gamma\Delta}{\omega}(1 - (1 - g)^{1/\gamma})(1 - \omega g).$$ (29)

It is easily seen that when $\gamma = 1$ we obtain the non-linear equation for the U matrix. In the case $\omega = 1$ Eq.(28) amounts to the standard logistic equation, and leads to the extended U-matrix unitarization scheme.

VIII. CONCLUSION

In this paper we presented two new unitarization schemes which generalize the usual eikonal and U matrix unitarization schemes. We showed that they belong to two wide classes which cannot be mapped analytically one onto the other. We showed however that it is possible to build a more general scheme which interpolates between the two.

The basic behavior of $G(s, \mathbf{b})$ as a function of s is mostly constrained by the value of ω (or ω'), but not

by the details of the unitarizing map. To illustrate this point, we show in Fig. 6 the behavior of $G(s, \mathbf{b} = 0)$ in our interpolating scheme (20), for $\gamma = 9$ (close to an eikonal) and for $\gamma = 1$ (U matrix), for $\omega = 1$ or $1/2$. Clearly, the large-s behavior of the amplitude is controlled by ω, and not by γ.
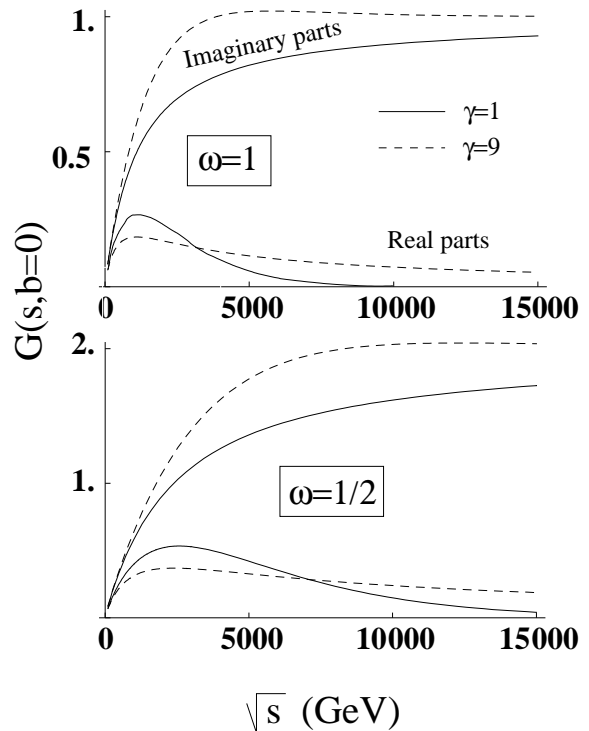

FIG. 6: The behavior of $G(s, \mathbf{b} = 0)$ from (20) for various choices of γ and ω.

So the question of the asymptotic behavior of the elastic amplitude remains open. It is possible to build an infinite number of schemes in which the amplitude will saturate at the black disk limit, but there also exists an infinite number of schemes in which it will exceed it and eventually converge to the full unitarity limit.

Up to now we do not have a decisive argument to choose one class of unitarization over the other. One possibility would be to fit the existing data to determine γ and ω. It is however known that this is possible in eikonal schemes, so that it is unlikely that the constraints will be very stringent. However, the prediction for the total cross sections in these two classes of unitarization have large differences (see, for example [15, 20]) for the LHC energy region, and hence we may know soon which is realized.

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