Facts and questions about the maximum deviation just-in-time scheduling problem

N. BRAUNER[1], Y. CRAMA[2]

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[1]Laboratoire Leibniz-IMAG, 46 avenue Félix Viallet, 38031 Grenoble cedex, France, Nadia.Brauner@imag.fr

[2]Ecole d'Administration des Affaires, University of Liège, Bld. du Rectorat 7 (B31), 4000 Liège, Belgium, Y.Crama@ulg.ac.be

Abstract

This note revisits the maximum deviation just-in-time (MDJIT) scheduling problem previously investigated by Steiner and Yeomans (1993). Its main result is a set of algebraic necessary and sufficient conditions for the existence of a MDJIT schedule with a given objective function value. These conditions are used to provide a finer analysis of the complexity of the MDJIT problem. The note also investigates various special cases of the MDJIT problem and suggests several questions for further investigation.

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1 Introduction

This paper revisits an optimization model originally motivated by scheduling issues arising in just-in-time (JIT) manufacturing environments. This model, to be called the *maximum deviation just-in-time problem*, or *MDJIT problem*, has been analyzed by Steiner and Yeomans (1993). These authors gave a very interesting, quite complete analysis of the problem, including structural and algorithmic results. We believe, however, that several intriguing complexity issues are still open with respect to this problem (see Kovalyov, Kubiak, and Yeomans (2000) for additional open questions). The main purpose of the present note is to clarify some of these issues. On our way, we propose alternative proofs for some of the results stated by Steiner and Yeomans (1993), we establish several new results and we propose a number of conjectures.

Section 2 contains a more precise description of JIT scheduling problems and of related computational complexity issues. In Section 3, we focus on the maximum deviation JIT problem. We recast it as a matching problem in a bipartite graph and we derive necessary and sufficient conditions for the existence of a schedule with a given objective function value. In Section 4, we use the previous result to establish that the MDJIT problem is in Co-NP and to prove that the problem can be solved in polynomial time when the number of part types is fixed. Section 5 establishes lower and upper bounds on the optimal value of the MDJIT problem. Section 6 proposes some results and conjectures concerning the structure of instances with small deviation. Finally, Section 7 provides a complete solution of the MDJIT problem for the special case where there are only two distinct part types.

2 Just-in-time scheduling problems

2.1 Position of the problem

An instance of a generic just-in-time scheduling problem consists of a number n of different part types and of the demand $d_i \in \mathbb{N}$ for part type i ($i = 1, 2 \dots n$). All part types are produced on the same equipment (typically, a mixed-model assembly line) and the production of each part requires one unit of time. We denote by $D = \sum_{i=1}^{n} d_i$ the total demand and by $r_i = d_i/D$ the *ideal production rate* for parts of type i. The term "ideal" refers here to the fact that, at each instant, we would like the line to have assembled part type i in proportion r_i. Such a schedule would be uniformly "leveled". Obviously, perfectly leveled schedules are never attainable, but the aim of JIT control systems is to keep the actual production of each part as close as possible to its "ideal rate". Monden (1983) states that this is a main objective of Toyota's JIT systems.

We can formulate this generic question as an optimization problem of the form (see Miltenburg (1989), Kubiak and Sethi (1991), Kubiak (1993)):

$$\text{minimize} \left\langle \begin{array}{l} \max_{k,i} \\ \\ \sum_k \sum_i \end{array} \right. \quad F^i(x_{ik} - kr_i) \quad \begin{array}{l} \longleftarrow \text{Maximum deviation} \\ \\ \longleftarrow \text{Total deviation} \end{array}$$

subject to

$$\begin{array}{llll} \sum_{i=1}^{n} x_{i,k} = k & k = 1, \ldots, D & & (1) \\ x_{i,D} = d_i & i = 1, \ldots, n & & (2) \\ 0 \quad x_{i,k} - x_{i,k-1} & i = 1, \ldots, n; & k = 2, \ldots, D & (3) \\ x_{i,k} \in \mathbb{N} & i = 1, \ldots, n; & k = 1, \ldots, D. & (4) \end{array}$$

In this formulation, $x_{i,k} = q$ if q parts of type i are produced in the first k periods. So, equation (1) means that k parts have to be produced during the first k periods, and equation (2) translates the demand constraints. Inequality (3) indicates that the number of parts of type i increases with time. Furthermore, for each i, the function F^i is a nonnegative convex function such that $F^i(x) = 0$ if and only if $x = 0$. Its interpretation is that $F^i(x_{ik} - kr_i)$ penalizes the deviation between the actual production x_{ik} and the target kr_i.

The above formulation emphasizes the distinctive number-theoretic flavor of JIT scheduling problems: given n rational numbers $r_1, \ldots, r_n$ with common denominator D, the problem is to find nD integers x_{ik} which 'optimally' approximate the sequence (kr_i) under the 'cardinality' and 'monotonicity' restrictions (1)-(4). Diophantine approximation problems of a similar nature are investigated for instance in Grötschel et al. (1993).

2.2 Literature review

We now briefly review some of the main results concerning the above JIT models. For more comprehensive surveys, we refer to Kubiak (1993) and Kovalyov, Kubiak, and Yeomans (2000).

The total deviation model has been proposed in Miltenburg (1989), together with some heuristics for its resolution. Kubiak and Sethi (1991, 1994) reformulated this model as an assignment problem. Their approach leads to an algorithm whose complexity is polynomial in n and D. Inman and Bulfin (1991) also considered the total deviation objective with $F^i(x) = x^2$ or $|x|$. They described a heuristic which runs in time $O(nD)$. The heuristic is based on a reduction to the one-machine scheduling problem with penalties for advance and tardiness.

The maximum deviation problem has been investigated in (Steiner and Yeomans 1993), with $F^i(x) = |x|$. These authors reduced the JIT problem to a one-machine scheduling problem with release dates and due dates, which they solved by an exact pseudo-polynomial algorithm with complexity $O(D \log D)$. This algorithm will be reviewed in Section 3.

Some of the connections between the above models are examined in Kovalyov, Kubiak, and Yeomans (2000). For the sake of completeness, let us also mention that multilevel extensions of the basic model have been investigated in the literature. Kubiak (1993) and Kubiak et al. (1997) establish that several of these extensions (including total and maximum deviation objectives) are NP-hard.

Several authors have also noted the connections between JIT scheduling problems and apportionment problems, i.e. problems dealing with the allocation of seats of a legislature among the states or provinces of a nation, in close proportion to their respective populations; see e.g. Bautista, Companys, and Corominas (1996) or Balinski and Shahidi (1998). From this connection, and from known results concerning apportionment problems, it is not too difficult to deduce that simple-minded procedures of a greedy nature do not provide an optimal solution of JIT scheduling problems (see Appendix).

2.3 The maximum deviation JIT problem: Complexity issues

Our main goal in this paper is to initiate further investigations into the complexity of JIT scheduling problems. We shall concentrate on (what seems to be) one of the simplest variants of the problem, namely the maximum deviation (MDJIT) problem with $F^i(x) = |x|$ previously investigated by Steiner and Yeomans (1993).

The input of the generic JIT scheduling problem is essentially the list of integers $d_1, d_2, \ldots, d_n$, so that its input size is $O(\sum_{1 \leq i \leq n} \log d_i) = O(n \log D)$. Hence, an algorithm which is polynomial in n and D is only *pseudo-polynomial*, but not polynomial in the input size. This limitation has already been noted by Kubiak (1993), who also mentioned in his survey that the question of the exact complexity of JIT scheduling problems remains open. Similar issues actually arise for a larger class of so-called *high multiplicity* optimization problems – see for instance Psaraftis (1980), Hochbaum and Shamir (1991) or Brauner et al. (2001).

The recognition version of this problem can be stated as follows:

MDJIT:

Input:

- $n \in \mathbb{N}$: number of part types;
- $d_i \in \mathbb{N}$: demand for part type i, $i = 1, 2 \ldots n$;

- $B \in \mathbb{Q}$: a bound.

Question: Does there exist an $n \times D$ matrix $X = (x_{i,k})$ such that:

$$\max_{1 \leq i \leq n, 1 \leq k \leq D} |x_{ik} - k\frac{d_i}{D}| \leq B \tag{5}$$

$$\sum_{i=1}^{n} x_{i,k} = k \qquad k = 1, \ldots, D \tag{6}$$

$$x_{i,D} = d_i \qquad i = 1, \ldots, n \tag{7}$$

$$0 \leq x_{i,k} - x_{i,k-1} \qquad i = 1, \ldots, n; \quad k = 2, \ldots, D \tag{8}$$

$$x_{i,k} \in \mathbb{N} \qquad i = 1, \ldots, n; \quad k = 1, \ldots, D? \tag{9}$$

It is actually interesting to observe that even this simple formulation of the problem is pseudo-polynomial in size, since it involves nD variables and $O(nD)$ constraints. So, obtaining truly polynomial algorithms for JIT scheduling problems is far from trivial (if possible at all) and requires deep insight into the structural properties of the problems. In particular, it is not even obvious whether MDJIT is in NP (or in Co-NP), i.e. whether there exists a polynomial-size certificate for every "Yes" (or "No") instance of MDJIT. We shall come back to this issue in Section 4.

2.4 Notations

We shall use the following notations:

- $[x]$ is the rounding of x to the closest integer; when the fractional part of x is equal to $1/2$, we round downward unless otherwise specified; so, $x - \frac{1}{2} \leq [x] < x + \frac{1}{2}$;
- $\lfloor x \rfloor$ is the largest integer smaller than x: $x - 1 < \lfloor x \rfloor \leq x$;
- $\lceil x \rceil$ is the smallest integer larger than x: $x \leq \lceil x \rceil < x + 1$;
- $[x_1..x_2]$ is the set of all integers between x_1 and x_2.

3 Maximum deviation just-in-time problem

In this section, we establish several structural properties of the MDJIT problem. We consider the recognition version (5)-(9) of the problem for a fixed value of B. These results constitute an alternative approach to the work of Steiner and Yeomans (1993). They lead to different insights and, in some cases, complete the arguments provided by these authors.

For $i = 1, \ldots, n$ and $j = 1, \ldots, d_i$, we use the shorthand (i, j) to denote the $j - th$ part of type i. A *schedule* for the MDJIT problem is a bijection $\{(i, j) \mid i = 1, \ldots, n,\ j = 1, \ldots, d_i\} \to [1..D]$. We say that a schedule is *feasible* if the inequality (5) holds for the implied variables $X = (x_{i,k})$.

The following statement is essentially due to Steiner and Yeomans (1993):

Proposition 1 *Consider an instance $(n, d_1, d_2, \ldots, d_n, B)$ of the MDJIT problem. A schedule for MDJIT is feasible if and only if, for all $i = 1, \ldots, n$ and $j = 1, \ldots, d_i$, this schedule assigns part (i, j) to the interval $[E(i, j)..L(i, j)]$ where*

$$E(i,j) = \left\lceil \frac{j - B}{r_i} \right\rceil$$
$$L(i,j) = \left\lfloor \frac{j - 1 + B}{r_i} + 1 \right\rfloor.$$

Proof (a) Consider any feasible schedule and suppose that part (i, j) is produced at time $k < \frac{j-B}{r_i}$ in this schedule. Then, $x_{i,k} = j$ and

$$|x_{i,k} - kr_i| \geq x_{i,k} - kr_i > j - j + B = B,$$

contradicting feasibility. Therefore, part (i, j) cannot start before time $\frac{j-B}{r_i}$, and hence before time $E(i, j) = \left\lceil \frac{j-B}{r_i} \right\rceil$.

Similarly, assume that part (i, j) starts at time $k > \frac{j-1+B}{r_i} + 1$. Then, at time $k - 1$, there holds $x_{i,k-1} = j - 1$ and

$$|x_{i,k-1} - (k-1)r_i| \geq (k-1)r_i - x_{i,k-1} > j - 1 + B - j + 1 = B,$$

a contradiction. This shows that part (i, j) cannot start after time $L(i, j) = \left\lfloor \frac{j-1+B}{r_i} + 1 \right\rfloor$. Hence, the necessary condition holds.

(b) Let us show that, if each part (i, j) is assigned to some time period in $[E(i, j)..L(i, j)] \cap [1..D]$, and all parts are assigned to different time periods, then the resulting schedule is feasible.

Consider a fixed part type i and a time period k. Let $j \in \{1, 2, \ldots, d_i\}$ be the number of parts of type i which have been produced up to (and including) time k, i.e. $x_{i,k} = j$. We must show that $|j - kr_i| \quad B$.

On one hand, $k \geq E(i, j)$. Therefore,

$$j - kr_i \quad j - E(i,j)r_i \quad j - (\frac{j - B}{r_i})r_i = B. \tag{10}$$

If $j = d_i$, then $j - kr_i = d_i - kr_i \geq 0$, so (10) implies that $|j - kr_i| \quad B$ as required.

On the other hand, if $j < d_i$, then $k < L(i, j+1)$ since the $(j+1)$-st part of type i is produced after time k. Thus,

$$kr_i - j \quad (L(i,j+1)-1)r_i - j \quad \left\lfloor \frac{j+B}{r_i} \right\rfloor r_i - j \quad j + B - j = B. \qquad (11)$$

So, $|j - kr_i| \quad B$ follows from (10) and (11). □

Steiner and Yeomans (1993) rely on Proposition 1 to solve the MDJIT problem. First, they set up a bipartite graph $G = (\mathcal{V}_1 \cup \mathcal{V}_2, E)$, where $\mathcal{V}_1 = [1..D]$, $\mathcal{V}_2$ is the set of all parts (i, j), for $i = 1, \ldots, n$, $j = 1, \ldots, d_i$, and $(k, (i, j))$ is an edge in E if and only if $k \in [E(i, j)..L(i, j)]$. For any subset X of vertices, denote by $N(X)$ the neighborhood of X, i.e. the set of all vertices adjacent to at least one vertex in X. Observe that the neighborhood of every vertex $(i, j) \in \mathcal{V}_2$ is an interval (namely, the interval $[E(i, j)..L(i, j)]$). Therefore, in agreement with Glover's terminology (Glover 1967), we say that the graph G is $\mathcal{V}_1$-*convex*.

Let us illustrate this concept on an example. Consider the instance $n = 3$, $d_1 = d_2 = 3$, $d_3 = 1$ and $B = \frac{5}{7}$. The convex bipartite graph G associated with this instance is represented in Figure 1. Observe that this instance is feasible, as it admits the feasible production sequence $(1, 2, 1, 3, 2, 1, 2)$. Furthermore, this feasible sequence corresponds in a natural way to the perfect matching of G indicated by the thick edges in Figure 1. This is no mere coincidence. Indeed, the following proposition is a simple corollary of Proposition 1 (it is implicit in Steiner and Yeomans (1993)).

Proposition 2 *The MDJIT problem has a feasible solution if and only if the graph G has a perfect matching.*

Proof Any solution of the MDJIT problem defines a perfect matching in G. More precisely, any such solution corresponds to an *order preserving* matching, i.e. a perfect matching such that, when $j_1 < j_2$, part (i, j_1) is matched to an earlier instant than part (i, j_2).

Conversely, if G has a perfect matching M, then it necessarily has an order preserving one, which corresponds therefore to a feasible solution of MDJIT. Indeed, if (i, j_1) is matched to k_1 and (i, j_2) is matched to k_2 in M, where $k_1 > k_2$, then matching (i, j_1) to k_2 and (i, j_2) to k_1 is also feasible (this is due to the convexity of G and to the fact that both $E(i, j)$ and $L(i, j)$ are nondecreasing in j). □

In view of the fact that G is convex, Steiner and Yeomans (1993) suggest to use Glover's (1967) *Earliest Due Date* algorithm to check the existence of a perfect matching in G. The algorithm runs through the time instants $k = 1, 2, \ldots, D$, in order, and assigns to k the part (i, j) with smallest value of $L(i, j)$ among all the available parts such that $(k, (i, j)) \in E$ (see also Lipski Jr. and Preparata

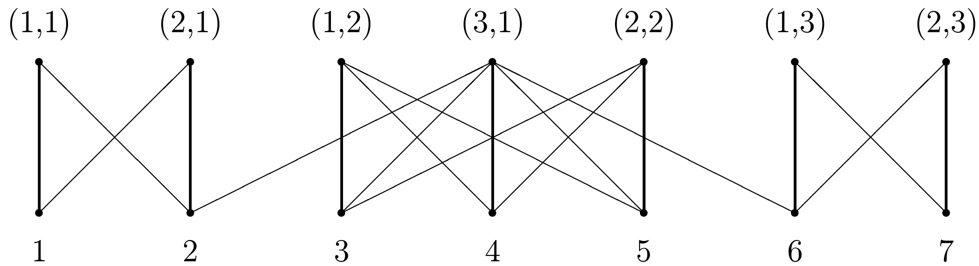

Figure 1: Convex bipartite graph for $d_1 = d_2 = 3$, $d_3 = 1$ and $B = \frac{5}{7}$

(1981), Gallo (1984), Gabow and Tarjan (1985) for successive improvements in the efficiency of Glover's algorithm).

We now proceed to develop necessary and sufficient conditions for the existence of a perfect matching in G. These conditions build on Hall's theorem, which implies that a bipartite graph $H = (V_1 \cup V_2, E)$ with $|V_1| = |V_2|$ has a perfect matching if and only if

$$|N(X)| \geq |X| \ \text{ for all } X \subseteq V_1 \tag{12}$$

(see e.g. Bondy and Murty (1976)).

We first show that, for convex graphs, Hall's conditions can be restricted to bear on special subsets of vertices. Let us define the following sets. First,

$$\mathcal{I}_1 \ = \ \{I : I \text{ is an interval of } \mathcal{V}_1\}.$$

Then, for each $I \in \mathcal{I}_1$, let $U(I)$ be the largest subset of $\mathcal{V}_2$ whose neighborhood is completely contained in I, i.e.

$$U(I) \ = \ \{v \in \mathcal{V}_2 : N(v) \subseteq I\}$$

(in terms of the original MDJIT problem, $U(I)$ is the set of all parts which must be processed during the time interval I). Finally, let

$$\mathcal{I}_2 \ = \ \{U(I) : I \in \mathcal{I}_1\}.$$

Proposition 3 *A $\mathcal{V}_1$-convex bipartite graph $G = (\mathcal{V}_1 \cup \mathcal{V}_2, E)$ has a perfect matching if and only if*

$$\forall X \in \mathcal{I}_1 \cup \mathcal{I}_2, \quad |N(X)| \geq |X|. \tag{13}$$

Proof The necessity of condition (13) is a consequence of Hall's theorem.

To establish sufficiency, suppose now that condition (13) is verified and that there exists $X_1 \subseteq \mathcal{V}_1$ such that $|N(X_1)| < |X_1|$. The set X_1 is a union of disjoint intervals:

$$X_1 = I_1 \cup I_2 \cup \ldots \cup I_p$$

where $I_i = [x_i..x'_i]$ and $x'_i < x_{i+1} - 1$ $(i = 1, \dots, p)$. We assume that X_1 is chosen so that p is minimal.

Case 1: Assume that $N(I_i) \cap N(I_j) = \emptyset$ for all $i, j \in \{1, \dots, p-1\}$ with $i \neq j$.

This implies that $|N(X_1)| = \sum_{i=1}^{p} |N(I_i)|$. But since I_i is an interval, condition (13) implies that $|N(I_i)| \geq |I_i|$ for $i = 1, \dots, p$. Therefore,

$$|N(X_1)| = \sum |N(I_i)| \geq \sum |I_i| = |X_1|$$

which is in contradiction with the hypothesis on X_1.

Case 2: There exist i and j $(i < j)$ such that $N(I_i) \cap N(I_j) \neq \emptyset$.

Let $u \in N(I_i) \cap N(I_j)$. Since G is $\mathcal{V}_1$-convex, there also holds $u \in N(I_i) \cap N(I_{i+1})$. Therefore, we can suppose that $j = i+1$. Let I be the interval nested between I_i and I_{i+1}, i.e. $I = [x'_i + 1..x_{i+1} - 1]$. Note that $X_1 \cap I = \emptyset$.

Let $U = U(I)$ be the set of vertices of $\mathcal{V}_2$ whose neighborhood is included in I. Since $U \in \mathcal{I}_2$, condition (13) implies that $|U| \quad |N(U)|$. Moreover, $|N(U)| \quad |I|$ (by definition of U), so that $|U| \quad |I|$. By definition of U, one has $U \cap N(X_1) = \emptyset$ and by convexity of the graph one has $N(I) \subseteq U \cup N(X_1)$. Therefore,

$$|N(I) \cup N(X_1)| \quad |U \cup N(X_1)| = |U| + |N(X_1)| < |I| + |X_1| = |I \cup X_1|.$$

This implies that $|N(X_1 \cup I)| < |X_1 \cup I|$. Let $Y = X_1 \cup I$. The set Y is a union of $p - 1$ intervals of $\mathcal{V}_1$ and satisfies $|N(Y)| < |Y|$. This is in contradiction with the minimality of p.

Therefore, condition (13) implies Hall's condition (12), and thus (13) implies the existence of a perfect matching in G. □

One may be tempted to conjecture that the conditions on $\mathcal{I}_2$ are superfluous in Proposition 3 and that Hall's conditions on intervals of $\mathcal{V}_1$ would be sufficient, by themselves, to ensure the existence of a perfect matching in a $\mathcal{V}_1$-convex bipartite graph. But this conjecture actually fails even in the special case where the convex graph is associated with an instance of the MDJIT problem, as illustrated by the graph in Figure 2. Indeed, this convex graph (which corresponds to the instance $d_1 = d_2 = 3$, $d_3 = 1$, $B = \frac{4}{7}$) satisfies the conditions

$$\forall X \in \mathcal{I}_1, \quad |N(X)| \geq |X|,$$

but does not have a perfect matching.

On the other hand, the conditions (13) can be strengthened by restricting them to those sets of $\mathcal{I}_2$ whose neighborhood is an interval. Define

$$\mathcal{I}'_2 = \{U \in \mathcal{I}_2 : N(U) \in \mathcal{I}_1\}.$$

Then, we can prove:

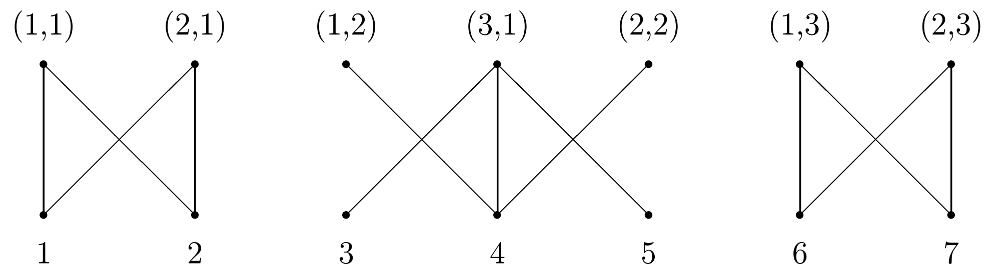

Figure 2: Convex bipartite graph for $d_1 = d_2 = 3$, $d_3 = 1$ and $B = \frac{4}{7}$

Proposition 4 *Conditions (14) and (15) are equivalent:*

$$\forall X \in \mathcal{I}_2, \qquad |N(X)| \geq |X|; \tag{14}$$

$$\forall X \in \mathcal{I}_2', \qquad |N(X)| \geq |X|. \tag{15}$$

Proof Since, $\mathcal{I}_2' \subseteq \mathcal{I}_2$, (14) trivially implies (15).

Conversely, suppose that (15) holds and let $U \in \mathcal{I}_2$. If $N(U)$ is an interval, then $U \in \mathcal{I}_2'$ and $|N(U)| \geq |U|$. So, suppose that $N(U)$ is a union of disjoint intervals:

$$N(U) = [x_1..x_1'] \cup \ldots \cup [x_p..x_p']$$

with $x_i' < x_{i+1} - 1$ $(i = 1, \ldots, p-1)$. Let $Y_i = \{v \in \mathcal{V}_2 : N(v) \subseteq [x_i..x_i']\}$. By definition, $\cup_{i=1}^p Y_i \subseteq U$. Suppose that there exists $k \in U$ such that $k \notin \cup_{i=1}^p Y_i$. This would mean that k has neighbors in at least two distinct intervals $[x_i..x_i']$ and $[x_j..x_j']$, with $i < j$. Since the graph is $\mathcal{V}_1$-convex, this would imply that all vertices between x_i' and x_j are in $N(U)$, a contradiction. Therefore $\cup_{i=1}^p Y_i = U$.

Since every element of $[x_i..x_i']$ is the neighbor of some element of U, there follows immediately that $N(Y_i) = [x_i..x_i']$ and hence $Y_i \in \mathcal{I}_2'$, for $i = 1, \ldots, p$. Therefore,

$$|N(U)| = \sum_{i=1}^{p} |N(Y_i)| \geq \sum_{i=1}^{p} |Y_i| = |U|$$

and we conclude that (15) implies (14). $\square$

Remark 1. From the above results, one easily concludes that, in the case of convex graphs, Hall's conditions need only to be applied to those sets X such that X is either an interval in $\mathcal{V}_1$, or the neighborhood of an interval in $\mathcal{V}_1$. Although we shall not use it in this form, this compact statement may be of independent interest.

We are now in a position to apply Hall's conditions to the convex bipartite graph associated with an instance of the MDJIT problem.

Theorem 1 *The MDJIT problem has a feasible solution if and only if, for all* x_1*,* x_2 *in* $[1..D]$ *with* $x_1 \quad x_2$*, the following inequalities are both valid:*

$$\sum_{i=1}^{n} \max(0, \lfloor x_2 r_i + B \rfloor - \lceil (x_1 - 1) r_i - B \rceil) \;\geq\; x_2 - x_1 + 1 \tag{16}$$

$$\sum_{i=1}^{n} \max(0, \lceil x_2 r_i - B \rceil - \lfloor (x_1 - 1) r_i + B \rfloor) \qquad x_2 - x_1 + 1. \tag{17}$$

Proof We know that the MDJIT problem has a feasible solution if and only if G has a perfect matching. This is equivalent to

$$\forall X \in \mathcal{I}_1 \cup \mathcal{I}_2'; \quad |N(X)| \geq |X|. \tag{18}$$

We want to express these conditions algebraically.

(a) Let $X = [x_1..x_2]$ be an interval of $\mathcal{V}_1$. Part (i, j) is in the neighborhood of X if and only if

$$\begin{aligned} & E(i,j) \quad x_2 \text{ and } L(i,j) \geq x_1 \\ \Leftrightarrow\; & \frac{j - B}{r_i} \quad x_2 \text{ and } \frac{j - 1 + B}{r_i} + 1 \geq x_1 \\ \Leftrightarrow\; & (x_1 - 1) r_i + 1 - B \quad j \quad x_2 r_i + B. \end{aligned}$$

For a given part type $i \in \{1, \ldots, n\}$, the number of copies j which verify the previous inequality is

$$\max(0, \lfloor x_2 r_i + B \rfloor - \lceil (x_1 - 1) r_i + 1 - B \rceil + 1).$$

Therefore, for $X = [x_1..x_2] \in \mathcal{I}_1$, $|N(X)| \geq |X|$ if and only if

$$\sum_{i=1}^{n} \max(0, \lfloor x_2 r_i + B \rfloor - \lceil (x_1 - 1) r_i - B \rceil) \geq x_2 - x_1 + 1.$$

(b) Given x_1, x_2 in $\mathcal{V}_1$, with $x_1 \quad x_2$, and given $(i, j) \in \mathcal{V}_2$, the following equivalences hold:

$$\begin{aligned} & [E(i,j)..L(i,j)] \subseteq [x_1..x_2] \\ \Leftrightarrow\; & E(i,j) \geq x_1 \text{ and } L(i,j) \quad x_2 \\ \Leftrightarrow\; & E(i,j) > x_1 - 1 \text{ and } L(i,j) < x_2 + 1 \\ \Leftrightarrow\; & \frac{j - B}{r_i} > x_1 - 1 \text{ and } \frac{j - 1 + B}{r_i} + 1 < x_2 + 1 \\ \Leftrightarrow\; & (x_1 - 1) r_i + B < j < x_2 r_i + 1 - B \\ \Leftrightarrow\; & \lfloor (x_1 - 1) r_i + B + 1 \rfloor \quad j \quad \lceil x_2 r_i - B \rceil. \end{aligned}$$

For a given part type i, the number of j which verify the previous equivalences is

$$\max(0, \lceil x_2 r_i - B \rceil - \lfloor (x_1 - 1) r_i + B \rfloor).$$

Thus, the cardinality of $U([x_1..x_2]) = \{(i,j) \in \mathcal{V}_2 : [E(i,j)..L(i,j)] \subseteq [x_1..x_2]\}$ can be computed as

$$|U([x_1..x_2])| = \sum_{i=1}^{n} \max(0, \lceil x_2 r_i - B \rceil - \lfloor (x_1 - 1) r_i + B \rfloor). \tag{19}$$

(c) Assume now that the inequalities (17) hold for all values of $x_1 \quad x_2$ and consider $X \in \mathcal{I}_2'$. By definition, $N(X)$ is an interval of $\mathcal{V}_1$, say $N(X) = [x_1..x_2]$. So, $X \subseteq U([x_1..x_2])$. In view of (19) and (17), we conclude that

$$|X| \quad |U([x_1..x_2])| \quad x_2 - x_1 + 1 = |N(X)|$$

as required by (18).

(d) Conversely, assume next that $|X| \quad |N(X)|$ for all $X \in \mathcal{I}_2'$. Then, we know from Proposition 4 that $|X| \quad |N(X)|$ for all $X \in \mathcal{I}_2$. Consider now x_1, x_2 in $\mathcal{V}_1$ with $x_1 \quad x_2$. Let $I = [x_1..x_2]$ and let $U = U(I)$. By definition of U, $N(U) \subseteq I$, and by definition of $\mathcal{I}_2$, $U \in \mathcal{I}_2$. So,

$$\begin{aligned}
|U| &= \sum_{i=1}^{n} \max(0, \lceil x_2 r_i - B \rceil - \lfloor (x_1 - 1) r_i + B \rfloor) \quad \text{(by (19))} \\
& \quad |N(U)| \qquad \text{(because } U \in \mathcal{I}_2\text{)} \\
& \quad |I| \qquad \text{(because } N(U) \subseteq I\text{)} \\
&= x_2 - x_1 + 1,
\end{aligned}$$

and (17) follows. □

In Section 5, we will see that the MDJIT problem always has a solution with $B < 1$ (see Theorem 5). If we slightly anticipate on this result, we can reformulate Theorem 1 in a simpler form by relying on the next observation.

Proposition 5 *When $B < 1$, the following statements (a)-(d) are equivalent:*

(a) for all x_1, x_2 in $[1..D]$ with $x_1 \quad x_2$,

$$\sum_{i=1}^{n} (\lfloor x_2 r_i + B \rfloor - \lceil (x_1 - 1) r_i - B \rceil) \geq x_2 - x_1 + 1; \tag{20}$$

(b) for all x_1, x_2 in $[1..D]$ with $x_1 \quad x_2$,

$$\sum_{i=1}^{n} \max(0, \lfloor x_2 r_i + B \rfloor - \lceil (x_1 - 1) r_i - B \rceil) \geq x_2 - x_1 + 1; \tag{21}$$

(c) for all x in $[1..D]$,

$$\sum_{i=1}^{n} \lfloor x r_i + B \rfloor \geq x; \tag{22}$$

(d) for all x in $[1..D]$,

$$\sum_{i=1}^{n} \lfloor x r_i + B \rfloor \;\geq\; x \tag{23}$$

$$\sum_{i=1}^{n} \lceil x r_i - B \rceil \qquad x. \tag{24}$$

Proof We are going to show that the following chain of implications is valid:

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a).$$

The first implication $(a) \Rightarrow (b)$ is trivial. To derive the next implication, just set $x_1 = 1$ in (21) and observe that $\lfloor x_2 r_i + B \rfloor \geq 0$ and $\lceil -B \rceil = 0$ (since $B < 1$).

The third implication is obtained by verifying that (23) holds for all $x \in [1..D]$ if and only if (24) holds for all $x \in [1..D]$ (just write out each inequality at the point $x' = D - x$, the case $x = D$ being trivial).

Finally, to derive the implication $(d) \Rightarrow (a)$, just write (23) at $x = x_2$, write (24) at $x = x_1 - 1$ and subtract the resulting inequalities. □

As a corollary, we obtain the simpler form of Theorem 1:

Theorem 2 *When $B < 1$, the MDJIT problem has a feasible solution if and only if, for all x_1, x_2 in $[1..D]$ with $x_1 \quad x_2$, the following inequalities are both valid:*

$$\sum_{i=1}^{n} \lfloor x_1 r_i + B \rfloor \;\geq\; x_1 \tag{25}$$

$$\sum_{i=1}^{n} \max(0, \lceil x_2 r_i - B \rceil - \lfloor (x_1 - 1) r_i + B \rfloor) \qquad x_2 - x_1 + 1. \tag{26}$$

Proof This is an immediate consequence of Theorem 1 and Proposition 5. □

In view of the equivalence of statements (c) and (d) in Proposition 5 and of the easy observation that (24) arises by setting $x_1 = 1$ and by dropping the max-operator in (26), one may be led to conjecture that the inequalities (26) actually are redundant in Theorem 2 and that the following compact statement holds true:

Conjecture 1 *The MDJIT problem has a feasible solution if and only if the following inequality is valid for all $x \in [1..D]$:*

$$\sum_{i=1}^{n} \lfloor x r_i + B \rfloor \;\geq\; x. \tag{27}$$

However, this conjecture is wrong, as evidenced by the instance $I = (d_1 = 3, d_2 = 3, d_3 = 1, B = \frac{4}{7})$. Indeed, the conditions (27) hold for this instance (for all $x \in [1..7]$), but the instance is infeasible since the graph in Figure 2 has no perfect matching or, alternatively, since inequality (26) fails when $x_1 = x_2 = 4$.

It may be interesting to note that Conjecture 1 is closely related to the so-called "Deficiency cases" described by Steiner and Yeomans (1993). These authors assert that the EDD "algorithm can stop at time $k < D - 1$ for one of two reasons": if less than k parts are available to schedule in $[1..k]$, or more than k parts must be scheduled in $[1..k]$. We observe that, for the instance $I = (d_1 = 3, d_2 = 3, d_3 = 1, B = \frac{4}{7})$, neither of the deficiency cases holds, even though I is infeasible.

4 Complexity results

As discussed in Section 2.3, the complexity of the MDJIT problem is not exactly known. Namely, it would be possible for its recognition version to be solvable in polynomial time despite the fact that the formulation (5)-(9) is of pseudo-polynomial size (see Brauner et al. (2001) for a finer discussion of this issue). Such a result, if true, would necessarily imply that the MDJIT problem is both in NP and in Co-NP. Here, we prove that at least one of these conditions holds:

Theorem 3 *The MDJIT problem is in Co-NP.*

Proof If an instance of the MDJIT problem is not feasible, then, by Theorem 2, there exist values of x_1 and x_2 such that one of the inequalities (25) or (26) is not valid. Given x_1 and x_2, this can be checked in time $O(n \log D)$. $\square$

We can also use Theorem 2 to reduce the MDJIT problem to an integer linear program (ILP) whose size is polynomial in n and $\log D$. Indeed, consider an instance $(d_1, \ldots d_n, B)$ of MDJIT. By Theorem 2, the instance has no feasible solution if and only if either there exists $x \in [1..D]$ such that

$$\sum_{i=1}^{n} \lfloor x r_i + B \rfloor < x \tag{28}$$

or there exist $x_1, x_2 \in [1..D]$ with x_1 x_2 such that

$$\sum_{i=1}^{n} \max(0, \lceil x_2 r_i - B \rceil - \lfloor (x_1 - 1) r_i + B \rfloor) > x_2 - x_1 + 1. \tag{29}$$

For $i = 1, \ldots, n$, introduce the decision variable X_i to represent the quantity $\lfloor x r_i + B \rfloor$. Then, inequality (28) holds exactly when the following system of inequalities is feasible:

$$\begin{aligned}
\sum_{i=1}^{n} X_i &< x \\
X_i &> x r_i + B - 1 \qquad (i = 1, \ldots, n) \\
x &\in [1..D] \\
X_i &\in \mathbb{N} \qquad (i = 1, \ldots, n).
\end{aligned}$$

Observe that, from the expression (5) (or from the definition of the "deviation" concept), it is clear that we can restrict the values of B to integer multiples of $1/D$. Thus, let $B = \beta/D$, where β is an integer. After some manipulations, the above system can be rewritten as

$$\begin{aligned}
\sum_{i=1}^{n} X_i &\quad x - 1 && (30) \\
D X_i &\geq x d_i + \beta - D + 1 \qquad (i = 1, \ldots, n) && (31) \\
x &\quad D && (32) \\
x, X_i &\in \mathbb{N} \qquad (i = 1, \ldots, n). && (33)
\end{aligned}$$

In particular, the smallest value of B such that inequality (25) holds for all values of x can be computed by solving the ILP

$$\max \left(\frac{\beta}{D} + \frac{1}{D} \right) \qquad \text{subject to (30)-(33) and } \beta \in \mathbb{N}.$$

Inequality (29) can be handled in a similar way, except that the presence of the max-operators creates some additional difficulties. So, for $i = 1, \ldots, n$, introduce the decision variables x_1, x_2, X_i^1, X_i^2 and X_i, where X_i^1 stands for $\lfloor (x_1 - 1) r_i + B \rfloor$, X_i^2 stands for $\lceil x_2 r_i - B \rceil$ and X_i stands for $\max(0, X_i^2 - X_i^1)$. For each i, we also introduce a binary variable δ_i which takes value 1 when $X_i = X_i^2 - X_i^1$ and value 0 when $X_i = 0$.

Inequality (29) holds exactly when the following system of inequalities is feasible:

$$\begin{aligned}
\sum_{i=1}^{n} X_i &> x_2 - x_1 + 1 \\
X_i^1 &> (x_1 - 1) r_i + B - 1 \qquad (i = 1, \ldots, n) \\
X_i^2 &< x_2 r_i - B + 1 \qquad (i = 1, \ldots, n) \\
X_i &\geq X_i^2 - X_i^1 \qquad (i = 1, \ldots, n) \\
X_i &\geq 0 \qquad (i = 1, \ldots, n) \\
X_i &\quad X_i^2 - X_i^1 + D(1 - \delta_i) \qquad (i = 1, \ldots, n) \\
X_i &\quad \delta_i D \qquad (i = 1, \ldots, n)
\end{aligned}$$

$$\begin{aligned} x_1 & \quad x_2 \\ x_1, x_2 & \in [1..D] \\ X_i^1, X_i^2, X_i & \in \mathbb{N}, \quad \delta_i \in \{0,1\} \qquad (i = 1, \ldots, n). \end{aligned}$$

After some manipulations, we obtain again that the smallest value of B such that inequality (26) holds for all values of (x_1, x_2) can be computed by solving an appropriate ILP problem. So, we conclude that the optimization version of the MDJIT problem (i.e., the problem of finding the smallest value of B such that (5)-(9) is feasible) can be reduced to a pair of integer linear programs involving $O(n)$ variables. In particular:

Theorem 4 *For fixed n, the minimum value of B such that the MDJIT problem is feasible can be computed in time polynomial in $O(\log D)$.*

Proof This is a straightforward consequence of the above discussion and of Lenstra's results (Lenstra Jr. 1983) on integer programming problems with a fixed number of variables. □

In Section 7, we will show that the MDJIT problem is actually very easy to solve when $n = 2$. But for fixed $n > 2$, we are not aware of a more direct proof of the fact that MDJIT is polynomially solvable.

These observations also leave open the more challenging question: is it possible to solve the (recognition version of the) MDJIT problem in time polynomial in n and $\log D$ or is the problem co-NP-complete?

5 Bounds on the maximum deviation

In this section, we provide an upper bound and several lower bounds on the optimal value of the MDJIT problem.

5.1 Upper bound

Steiner and Yeomans (1993) proved the interesting fact that the optimal value of the MDJIT problem is always less than or equal to 1. The same bound can also be deduced from the description of the quota apportionment method (see Balinski and Young (1975); in fact, the quota method can be interpreted as a version of the EDD algorithm, applied with the bound $B = 1$). Here, we state a slightly stronger version of this result, whose proof is a rather easy consequence of Theorem 2.

Theorem 5 *The optimal value B^* of the MDJIT problem satisfies*

$$B^* \quad 1 - \frac{1}{D}.$$

Proof Let $B = 1 - \frac{1}{D}$. Since $B < 1$, we are in a position to apply Theorem 2, i.e., we only need to prove that inequalities (25) and (26) hold for this value of B.

We first prove that, for any integer $x \geq 0$, one has

$$\lfloor xr_i + B \rfloor \geq xr_i. \tag{34}$$

Indeed, if xr_i is an integer, then $\lfloor xr_i + B \rfloor = xr_i$. If xr_i is not an integer, then $\{xr_i\} \geq \frac{1}{D}$ (where $\{\alpha\}$ denotes the fractional part of α). This implies

$$\{xr_i\} + B \geq \frac{1}{D} + 1 - \frac{1}{D} = 1$$

and hence

$$\lfloor xr_i + B \rfloor = \lfloor xr_i \rfloor + 1 > xr_i,$$

which establishes (34). Inequality (25) follows now immediately by summing (34) over $i = 1, \ldots, n$.

For inequality (26), fix $x_1, x_2 \in [1..D]$, with $x_1 \quad x_2$, and consider the set $J \subseteq \{1, 2 \ldots n\}$ defined by

$$i \in J \Leftrightarrow \lceil x_2 r_i - B \rceil - \lfloor (x_1 - 1) r_i + B \rfloor \geq 0.$$

Note that substituting $D - x_2$ for x in inequality (34) leads to $\lceil x_2 r_i - B \rceil \quad x_2 r_i$ for $i = 1, \ldots, n$. Then, we derive successively

$$\begin{aligned} \sum_{i=1}^{n} \max(0, \lceil x_2 r_i - B \rceil - \lfloor (x_1 - 1) r_i + B \rfloor) &= \sum_{i \in J} (\lceil x_2 r_i - B \rceil - \lfloor (x_1 - 1) r_i + B \rfloor) \\ & \sum_{i \in J} (x_2 r_i - (x_1 - 1) r_i) \\ & x_2 - x_1 + 1 \end{aligned}$$

and hence inequality (26) holds. □

Note that the bound $1 - \frac{1}{D}$ is attained when $d_i = 1$ for all $i = 1, \ldots, n$.

Theorem 5 has interesting side-implications regarding the periodicity of optimal sequences (this issue has also been considered, for instance, by Miltenburg (1989), and Kubiak (2000), for total deviation objective functions). Assume that kr_i is an integer for some $k \in [1..D]$ and $i \in \{1, \ldots, n\}$. Then, in every optimal solution of MDJIT, the number of parts of type i produced up to time k must be equal to $x_{i,k} = kr_i$, since otherwise the deviation at time k would be at least $1 > B^*$. Now, if $\gcd(d_1, \ldots, d_n, D) = \alpha$ and $D = \alpha\Delta$, then $\Delta\, r_i = \frac{d_i}{\alpha}$ is integral for *all* values of i. Thus, necessarily, $\frac{d_i}{\alpha}$ parts of type i must have been produced up to time Δ, for *all* $i \in \{1, \ldots, n\}$. A similar argument also applies at times 2Δ, 3Δ... and this implies that the optimal schedule must consist of α subsequences, where each subsequence involves exactly $\frac{d_i}{\alpha}$ parts of type i for $i = 1, \ldots, n$. Pushing the reasoning a little bit further actually leads to the conclusion that, when $\alpha > 1$, there exists an optimal production sequence of the form $(S, S, \ldots, S)$, where S is an optimal production sequence for the reduced instance $(\frac{d_1}{\alpha}, \ldots, \frac{d_n}{\alpha})$ (we omit the details).

5.2 Lower bounds

We now turn to lower bounds on the optimal value B^*. We start with an easy result.

Proposition 6 *Assume that* $d_1 \quad d_2 \quad \dots \quad d_n$. *Then, the optimal value* B^* *of the MDJIT problem satisfies*

$$
\begin{aligned}
r_i & \quad B^* \quad \text{for } i = 1, 2, \dots, n-1 && (35)\\
1 - r_i & \quad 2B^* \quad \text{for } i = 1, 2, \dots, n-1 && (36)\\
r_n & \quad 2B^* && (37)\\
1 - r_n & \quad B^* && (38)\\
\frac{n-1}{2n-1} & \quad B^*. && (39)
\end{aligned}
$$

Proof (a) In any feasible schedule, some part type i must be produced in the first time period $k = 1$. From inequality (5), we deduce that $|1 - r_i| = 1 - r_i \quad B^*$ and that $|0 - r_j| = r_j \quad B^*$ for all $j \neq i$. Conditions (35) and (38) follow.

(b) If MDJIT has a feasible schedule, then necessarily $E(i, j) \quad L(i, j)$ for all parts (i, j). Hence:

$$
\begin{aligned}
0 & \quad L(i,j) - E(i,j)\\
& \quad \frac{j - 1 + B^*}{r_i} + 1 - \frac{j - B^*}{r_i}\\
& = \frac{2B^* + r_i - 1}{r_i}
\end{aligned}
$$

and condition (36) follows. Moreover, condition (37) also holds, since

$$2B^* \geq 1 - r_1 = \sum_{i=2}^{n} r_i \geq r_n.$$

(c) For the last bound, just add the inequalities (36) ($i = 1, 2 \dots, n-1$) to inequality (38). □

Interestingly, Kovalyov, Kubiak, and Yeomans (2000) mention that, for small examples, the optimal value of MDJIT very often coincides with the lower bound (38). In their computational experiments, however, they provide examples showing that the bound is not always attained. As a matter of fact, combining bounds (37) and (38) immediately implies that (38) cannot be tight as soon as $r_n > \frac{2}{3}$. In particular, the instance $d_1 = 1$, $d_2 = 3$, whose optimal value is $B^* = \frac{1}{2}$, suffices to show that none of the lower bounds in Proposition 6 needs to be attained (see also Theorem 9 hereunder).

Theorem 6 *Let $\Delta_i = D/\gcd(d_i, D)$ $(i = 1, \ldots, n)$. The optimal value B^* of the MDJIT problem satisfies*

$$B^* \geq \frac{1}{\Delta_i} \left\lfloor \frac{\Delta_i}{2} \right\rfloor \qquad \textit{for } i = 1, \ldots, n.$$

Proof Consider some part type i. Its ideal production rate is $r_i = \frac{d_i}{D} = \frac{\delta_i}{\Delta_i}$ where $\gcd(\Delta_i, \delta_i) = 1$. Note that any feasible solution (x_{ik}) of the MDJIT problem satisfies $|x_{ik} - kr_i| \geq |[kr_i] - kr_i|$ for all $k \in [1..D]$. We shall prove the theorem by showing that there exists a value of k such that $|[kr_i] - kr_i| = \frac{1}{\Delta_i} \left\lfloor \frac{\Delta_i}{2} \right\rfloor$.

If Δ_i is even, then, for $k = \frac{\Delta_i}{2}$,

$$|kr_i - [kr_i]| = \left| \frac{\Delta_i}{2} \frac{\delta_i}{\Delta_i} - \left[\frac{\Delta_i}{2} \frac{\delta_i}{\Delta_i} \right] \right| = \left| \frac{\delta_i}{2} - \left[\frac{\delta_i}{2} \right] \right|.$$

Since $\gcd(\delta_i, \Delta_i) = 1$ and Δ_i is even, δ_i is odd and hence $|\frac{\delta_i}{2} - [\frac{\delta_i}{2}]| = 0.5$. Therefore, for Δ_i even, the optimal value of the objective function is at least $\frac{1}{\Delta_i} \left\lfloor \frac{\Delta_i}{2} \right\rfloor = 0.5$.

Suppose now that Δ_i is odd. We prove that there exists a value of k such that $|[kr_i] - kr_i| = \frac{\Delta_i - 1}{2\Delta_i}$. Since δ_i and Δ_i are relatively prime, Bezout's identity implies that there exist two integers u and v such that $u\Delta_i + v\delta_i = 1$. Multiply this inequality by $\frac{\Delta_i - 1}{2\Delta_i}$ and set $k = |v|\frac{\Delta_i - 1}{2}$ thus obtaining

$$k\frac{\delta_i}{\Delta_i} = |u|\frac{\Delta_i - 1}{2} \pm \frac{\Delta_i - 1}{2\Delta_i}.$$

Since $\frac{\Delta_i - 1}{2\Delta_i} < 0.5$, one has $[k\frac{\delta_i}{\Delta_i}] = |u|\frac{\Delta_i - 1}{2}$ and $|[k\frac{\delta_i}{\Delta_i}] - k\frac{\delta_i}{\Delta_i}| = \frac{\Delta_i - 1}{2\Delta_i}$. □

We will see in Section 7 that the bound presented in Theorem 6 is attained for $n = 2$.

6 Small deviations

Observe that the bound in Theorem 6 is usually close to $1/2$, and that $B^* \geq 1/2$ as soon as there exists some i such that Δ_i is even. Note also that the bound (39) goes to $1/2$ when n goes to $+\infty$. These observations suggest to look more closely at those instances for which the maximum deviation does not exceed the value $1/2$.

In the remainder of this section, we state a few conjectures regarding the structure of the instances with B^* $1/2$, and we identify all instances with optimal value $B^* < 1/2$ for n 6. In order to formulate these statements in the simplest possible form, we restrict our attention to *standard* instances of the MDJIT problem, i.e. instances $(d_1, \ldots, d_n)$ such that d_1 d_2 $\ldots$ d_n and $\gcd(d_1, \ldots, d_n, D) = 1$ (remember the comments following Theorem 5).

6.1 Instances with $B^* < 1/2$

When $B^* < 1/2$, the condition $|x_{ik} - kr_i| \le B^*$ forces x_{ik} to be equal to $[kr_i]$, so that the problem becomes highly constrained. As a matter of fact, we conjecture that, for $n > 2$, only a handful of very special instances have this property, namely the instances of the form $(1, 2, 4 \ldots 2^{n-1})$ (the case $n = 2$ will be dealt with in the next section). We will prove this conjecture for $n \le 6$ using connections with so-called balanced words (Tijdeman 2000a).

Conjecture 2 *For $n > 2$, a standard instance $(d_1, \ldots, d_n)$ of the MDJIT problem has optimal value $B^* < 1/2$ if and only if $d_i = 2^{i-1}$ for $i = 1, 2 \ldots, n$, and $B^* = \frac{2^{n-1}-1}{2^n-1}$.*

We first establish the sufficiency of the conditions.

Proposition 7 *The instance $(d_1, \ldots, d_n)$, with $d_i = 2^{i-1}$ for $i = 1, 2 \ldots, n$, has optimal value $B^* = \frac{2^{n-1}-1}{2^n-1}$.*

Proof Consider the instance $d_i = 2^{i-1}$ for $i = 1, \ldots, n$, and let $B^* = \frac{D-1}{2D} = \frac{2^{n-1}-1}{2^n-1} < \frac{1}{2}$. We want to show that this value B^* is feasible. In view of Theorem 6, this will imply that B^* is optimal.

The ideal rates are defined by $r_i = \frac{2^{i-1}}{2^n-1}$. Let $s_{ij} = 2^{n-i}(2j-1)$ for $i \in [1..n]$ and $j \in [1..2^{i-1}]$. We first prove that

$$s_{ij} \geq \frac{j - B^*}{r_i} \quad \text{and} \quad s_{ij} \le \frac{j - 1 + B^*}{r_i} + 1.$$

Indeed,

$$\begin{aligned} s_{ij} - \frac{j - B^*}{r_i} &= \frac{2^{n-1}(2j-1) - j(2^n - 1) + 2^{n-1} - 1}{2^{i-1}} \\ &= \frac{j-1}{2^{i-1}} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \frac{j - 1 + B^*}{r_i} + 1 - s_{ij} &= \frac{(j-1)(2^n - 1) + 2^{n-1} - 1 + 2^{i-1} - 2^{n-1}(2j-1)}{2^{i-1}} \\ &= \frac{2^{i-1} - j}{2^{i-1}} \geq 0. \end{aligned}$$

Since s_{ij} is integer, one has $E(i,j) = \left\lceil \frac{j-B^*}{r_i} \right\rceil \le s_{ij} \le \left\lfloor \frac{j-1+B^*}{r_i} + 1 \right\rfloor = L(i,j)$. Moreover, $s_{ij} \neq s_{i',j'}$ for $i \neq i'$ or $j \neq j'$ and $1 \le s_{ij} \le D = 2^n - 1$. Therefore, by Proposition 1, there exists a feasible schedule: just produce part (i,j) at time s_{ij} for $i \in [1..n]$ and $j \in [1..2^{i-1}]$.

□

In order to prove the unicity of the instance in Conjecture 2 for n 6, let us introduce some terminology from Tijdeman (2000a). A *balanced word* on $\{1, 2, \ldots, n\}$ is an infinite sequence $\sigma = (s_1, s_2, \ldots)$ such that
(1) $s_j \in \{1, 2, \ldots, n\}$ for all $j \in \mathbb{N}_0$, and
(2) if σ_1 and σ_2 are two subsequences consisting of t consecutive elements of σ ($t \in \mathbb{N}$), then the number of occurrences of i in σ_1 and σ_2 differs by at most 1, for all $i = 1, 2, \ldots, n$.
If σ is a balanced word on $\{1, 2, \ldots, n\}$, then the *density* of i in σ is

$$\theta_i = \lim_{t \to \infty} \frac{|\{j \in [1..t] : s_j = i\}|}{t},$$

for $i = 1, 2, \ldots, n$ (it can be shown that the limit always exists).

Proposition 8 *Let S be an optimal sequence with $B^* < 1/2$ for an instance $(d_1, \ldots, d_n)$ of MDJIT. Then, the infinite sequence obtained by repeating S is a balanced word with distinct densities $(\frac{d_1}{D}, \ldots, \frac{d_n}{D})$.*

Proof Consider an optimal sequence S for the instance $(d_1, d_2 \ldots d_n)$, with $B^* < 1/2$, and consider the infinite sequence σ obtained by repeating S, i.e. $\sigma = (S, S, S, \ldots)$. We want to show that σ, viewed as a sequence on $\{1, 2, \ldots n\}$, is a balanced word. Let $t \in \mathbb{N}$, let σ_1, σ_2 be two subsequences consisting of t consecutive elements of σ, and let us establish condition (2) in the definition of balanced words.
Assume that σ_j ranges from time $t_j + 1$ to time $t_j + t$, for $j = 1, 2$. Fix $i \in \{1, 2, \ldots n\}$ and denote by $|I|$ the number of occurrences of i in any time interval I. Then, for $j = 1, 2$,

$$\begin{aligned} |[t_j + 1, t_j + t]| &= |[1, t_j + t]| - |[1, t_j]| \\ &< \left((t_j + t)r_i + \frac{1}{2}\right) - \left(t_j r_i - \frac{1}{2}\right) \\ &= tr_i + 1, \end{aligned}$$

and similarly

$$\begin{aligned} |[t_j + 1, t_j + t]| &> \left((t_j + t)r_i - \frac{1}{2}\right) - \left(t_j r_i + \frac{1}{2}\right) \\ &= tr_i - 1. \end{aligned}$$

Thus, $|[t_1 + 1, t_1 + t]|$ and $|[t_2 + 1, t_2 + t]|$ are two integers lying strictly between $tr_i - 1$ and $tr_i + 1$. There follows that $|[t_1 + 1, t_1 + t]|$ and $|[t_2 + 1, t_2 + t]|$ differ at most by 1, and hence σ is balanced.

For $i = 1, 2 \ldots n$, the density of i in σ is equal to $r_i = \frac{d_i}{D}$. If $r_i = r_j$ for $i \neq j$, then $x_{ik} = [kr_i] = [kr_j] = x_{jk}$ for all k, which is clearly impossible. Hence, the densities in σ are pairwise distinct. □

Some attention has been devoted in number theory to the properties of balanced words and related concepts, such as Beatty covers and Sturmian words. Without going into the details, let us just mention here that Beatty covers and Sturmian words are special types of balanced words. Fraenkel conjectured that, when $n \geq 3$, every Beatty cover on n letters with distinct densities has densities $\frac{2^{i-1}}{2^n-1}$, for $i = 1, \ldots, n$. Tijdeman (2000a) presents a thorough discussion of the state-of-the-art concerning this conjecture. Although the conjecture is still open, it is known to hold when $n \quad 6$, even in the more general case of balanced words (see Tijdeman (1996) for $n = 3$, Altman, Gaujal, and Hordijk (1998) for $n = 4$, Tijdeman (2000a) for $n = 5$ and Tijdeman (2000a), Tijdeman (2000b) for $n = 6$). As a corollary, we obtain:

Theorem 7 *Conjecture 2 is valid for* $n \quad 6$.

Proof This follows from Proposition 8 and from the above discussion. □

More generally, Conjecture 2 would follow from the validity of Fraenkel's conjecture for balanced words.

6.2 Instances with $B^* \quad 1/2$

If we only impose $B \quad 1/2$, then the structure of feasible instances becomes more complex, as x_{ik} may now be equal either to $[kr_i] = kr_i - 1/2$ or to $[kr_i] + 1 = kr_i + 1/2$ when kr_i is half-integral.

Conjecture 3 *A standard instance* $(d_1, \ldots, d_n)$ *of the MDJIT problem has optimal value* $B^* \quad 1/2$ *if and only if it satisfies one of the following conditions:*

- d_1 *and* d_2 *are arbitrary and for* $i \geq 3$, d_i *is the sum of all demands with smaller index:*

$$d_i = \sum_{j<i} d_j = 2^{i-3}(d_1 + d_2) \quad \text{for all } i \in [3..n],$$

- $d_1 = 1$, $d_2 = 2$, $d_3 = 9$ *and for* $i \geq 4$, d_i *is the sum of all demands with smaller index:*

$$d_1 = 1; \quad d_2 = 2; \quad d_3 = 9; \quad d_i = \sum_{j<i} d_j = 2^{i-4} \times 12 \quad \text{for all } i \in [4..n],$$

- $d_1 = 2$, $d_2 = 3$, $d_3 = 7$ *and for* $i \geq 4$, d_i *is the sum of all demands with smaller index:*

$$d_1 = 2; \quad d_2 = 3; \quad d_3 = 7; \quad d_i = \sum_{j<i} d_j = 2^{i-4} \times 12 \quad \text{for all } i \in [4..n],$$

- *the demands are successive powers of two:*

$$d_i = 2^{i-1} \quad \textit{for all } i \in [1..n],$$

- *there exists an index* $l \in [3..n]$ *such that*

$$\begin{aligned} d_i &= 2^{i-1} \quad \textit{for all } i \in [1..l]; \\ d_i &= \sum_{j<i} d_j = 2^{i-l-1}(2^l - 1) \quad \textit{for all } i \in [l+1..n], \end{aligned}$$

- *there exists an index* $l \in [3..n]$ *such that*

$$\begin{aligned} d_i &= 2^{i-1} \quad \textit{for all } i \in [1..l-1]; \\ d_l &= 2^{l-1} + 1; \\ d_i &= \sum_{j<i} d_j = 2^{i-1} \quad \textit{for all } i \in [l+1..n]. \end{aligned}$$

The sufficiency of the conditions in Conjecture 3 can be verified using some case-by-case analysis together with the general result below (we skip the details).

Proposition 9 *Let* $I = (n, d_1, \ldots, d_n, B)$ *be a feasible instance of the MDJIT problem with* $B \geq 1/2$. *Then, the following instance, involving an additional part type, is also feasible:*

$$I' = (n+1,\ d_1, \ldots, d_n,\ d_{n+1} = \sum_{i=1}^{n} d_i,\ B).$$

Proof Let (x_{ik}) be a solution of the MDJIT for the instance I. For each $k = 0, \ldots, D-1$, one can determine the index of the part type which is produced at time $k+1$, namely the unique index i_k such that

$$x_{i_k,k+1} = x_{i_k,k} + 1 \quad \text{and} \quad x_{i,k+1} = x_{i,k} \quad \text{for all} \quad i \neq i_k$$

(for simplicity of notations, we assume here that $x_{i0} = 0$).

For the extended instance, define a solution x'_{ij} $(i = 1, 2 \ldots n{+}1, j = 1, 2 \ldots 2D)$ as follows:

$$\begin{array}{lrcll} \text{for } k = 1..D \text{ and } i = 1..n: & x'_{i,2k} & = & x_{ik} & \\ & x'_{n+1,2k} & = & k & \\ \text{for } k = 0..D-1 \text{ and } i \neq i_k: & x'_{i,2k+1} & = & x_{ik} & \\ \text{for } k = 0..D-1 \text{ and } i = i_k: & x'_{i,2k+1} & = & x_{ik} & \text{if } |x_{ik} - (k+1/2)r_i| \quad B \\ & & = & x_{ik} + 1 & \text{otherwise} \\ \text{for } k = 0..D-1 \text{ and } i = i_k: & x'_{n+1,2k+1} & = & k+1 & \text{if } |x_{ik} - (k+1/2)r_i| \quad B \\ & & = & k & \text{otherwise.} \end{array}$$

It remains to prove that x'_{ik} is a solution of the MDJIT problem for the instance I'. Since $x_{i,k+1} \geq x_{ik}$ for any i and k, one may verify that $x'_{i,k+1} \geq x'_{ik}$ for any i and k. Moreover, $x'_{i,2D} = x_{i,D} = d_i$ and $x'_{n+1,2D} = D = d_{n+1}$ imply equality (2). We will now prove that equality (1) applies for $x'_{i,k}$:

$$\begin{aligned}
\sum_{i=1}^{n+1} x'_{i,2k} &= \sum_{i=1}^{n} x_{i,k} + k \\
&= 2k \qquad \text{for } k = 1, 2 \ldots D \\
\sum_{i=1}^{n+1} x'_{i,2k+1} &= \sum_{i=1, i \neq i_k}^{n} x_{i,k} + x'_{i_k,2k+1} + x'_{n+1,2k+1} \\
&= \sum_{i=1, i \neq i_k}^{n} x_{i,k} + x_{i_k,k} + k + 1 \\
&= 2k + 1 \qquad \text{for } k = 0, 1 \ldots D - 1.
\end{aligned}$$

Note that for the instance I', the ideal production rate r'_i of part i is equal to $r_i/2$, where r_i is the rate for part i in the instance I ($i = 1, \ldots, n$), and $r'_{n+1} = 1/2$. Using these remarks, one can verify that $|x'_{ik} - kr'_i| \quad B$ for $i = 1, 2 \ldots n + 1$ and $k = 1, 2 \ldots 2D$. This concludes the proof. □

7 Two-part type problem

In this last section, we consider the MDJIT problem for two part types ($n = 2$) with ideal production rates r_1 and $r_2 = 1 - r_1$. We assume without loss of generality that $0 < r_1 \quad 0.5$. The optimization version of the MDJIT problem can be written as

Input: d_1, $d_2 \in \mathbb{N}$.

As usual, denote the total demand by $D = d_1 + d_2$ and the ideal production rate for part type 1 by $r_1 = \frac{d_1}{D}$. Then the ideal production rate for part type 2 satisfies $r_2 = 1 - r_1$.

Question: find a $2 \times D$ matrix $X = (x_{i,k})$ which minimizes the maximum deviation $\max_{1 \ k \ D}(\,|x_{1k} - kr_1|, |x_{2k} - kr_2|\,)$ subject to

$$x_{1,k} + x_{2,k} = k \qquad k = 1, \ldots, D \tag{40}$$

$$x_{i,D} = d_i \qquad i = 1, 2; \tag{41}$$

$$0 \quad x_{i,k} - x_{i,k-1} \qquad i = 1, 2; \quad k = 2, \ldots, D \tag{42}$$

$$x_{i,k} \in \mathbb{N} \qquad i = 1, 2; \quad k = 1, \ldots, D. \tag{43}$$

Theorem 8 *The matrix X defined by $x_{1,k} = [kr_1]$ and $x_{2,k} = k - [kr_1]$ ($k = 1, \ldots, D$) is an optimal solution of the 2-part type MDJIT problem.*

Proof Note that $x_{2,k}$ might be different from $[kr_2]$ whenever the fractional part of $x_{1,k}$ is 0.5. One can observe that for $k = 1, \ldots, D$,

$$x_{1,k} + x_{2,k} = [kr_1] + k - [kr_1] = k,$$

$$x_{i,D} = \quad D\frac{d_i}{D}\Big] = d_i,$$

$$x_{1,k} - x_{1,k-1} = [kr_1] - [(k-1)r_1] \geq 0,$$

and

$$\begin{aligned} x_{2,k} - x_{2,k-1} &= k - [kr_1] - k + 1 + [(k-1)r_1] \\ &= 1 + [(k-1)r_1] - [kr_1] \\ &\geq 0. \end{aligned}$$

Therefore, X satisfies the constraints (40)-(43). We now prove that X is an optimal solution for the 2-part type problem. Clearly,

$$|x_{1,k} - kr_1| = |[kr_1] - kr_1| \quad 1/2$$

and

$$|x_{2,k} - kr_2| = |k - [kr_1] - k + kr_1| = |-[kr_1] + kr_1| \quad 1/2.$$

Let $X' = (x'_{1,k}; x'_{2,k})$ be another feasible solution of the 2-part MDJIT problem. We want to show that X' has maximum deviation larger than or equal to 1/2, which implies that X' is not better than X. Assume that X' differs from $X = (x_{1,k}; x_{2,k})$ at period l. Because of constraint (40), it must be the case that $x'_{1,l}$ is not equal to $x_{1l} = [lr_1]$. Thus, by definition of the [.] operator, $x'_{1,l}$ is at distance at least 1/2 from lr_1, i.e. $|x'_{1,l} - lr_1| \geq 1/2$, as needed. □

Remark 2. Note that the matrix defined in Theorem 8 actually minimizes the deviation $x_{ik} - kr_i$ for all k and i. Therefore, it is optimal for the maximum deviation problem and for the total deviation problem with any penalty function F^i such as introduced in Section 2.1.

Theorem 8 solves the two-part type MDJIT problem in polynomial time, in the following sense: at every instant k, the theorem allows to determine efficiently which part type should be produced at time k. Furthermore, the optimal value of the two-part type problem can be computed very easily, as shown by our next result.

Theorem 9 *Let* $\Delta = D/\gcd(d_1, D) = D/\gcd(d_2, D)$. *The optimal value* B^* *of the objective function of the 2-part type MDJIT problem is*

$$B^* = \frac{1}{\Delta}\left\lfloor \frac{\Delta}{2} \right\rfloor.$$

Proof By Theorem 6, we know that $B^* \geq \frac{1}{\Delta}\left\lfloor\frac{\Delta}{2}\right\rfloor$. There remains to prove that $B^* \quad \frac{1}{\Delta}\left\lfloor\frac{\Delta}{2}\right\rfloor$, i.e., that $|[kr_i] - kr_i| \quad \frac{1}{\Delta}\left\lfloor\frac{\Delta}{2}\right\rfloor$ for $i = 1, 2$ and for every k.

If Δ is even, then $|kr_i - [kr_i]| \quad 0.5 = \frac{1}{\Delta}\left\lfloor\frac{\Delta}{2}\right\rfloor$. Suppose now that Δ is odd and let us prove that $|[kr_i] - kr_i| \quad \frac{\Delta-1}{2\Delta}$.

After simplification, the ideal rate $r_i = \frac{d_i}{D}$ can be rewritten as $\frac{\delta_i}{\Delta}$. Therefore, the deviation $|[kr_i] - kr_i|$ is an integral multiple of $\frac{1}{\Delta}$, say $|[kr_i] - kr_i| = \frac{c}{\Delta}$ with $c \in \mathbb{N}$. Suppose that $\frac{c}{\Delta} = 0.5$. Then, $\Delta = 2c$, in contradiction with the hypothesis that Δ is odd.

Thus, $\frac{c}{\Delta} < 0.5$, which implies that $2c < \Delta$ and hence $2c \quad \Delta - 1$. Therefore $\frac{c}{\Delta} \quad \frac{\Delta-1}{2\Delta}$ as required. □

8 Conclusions

In this paper, we have revisited one of the most basic scheduling models of JIT production systems. In spite of its apparent simplicity, this model is not completely understood, yet. In particular, its computational complexity is not exactly known. We have shown that the model is in co-NP and that it is polynomially solvable when the number of part-types is fixed, but its general version may still turn out to be either co-NP-complete or polynomially solvable. We believe that the algebraic characterization of feasible instances presented in Theorem 2 may provide a useful tool for the analysis of such issues. Finally, obtaining a full description of instances with small max-deviation also presents an interesting challenge for future research.

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Appendix: Greedy algorithm

One may want to suggest the following greedy algorithm for the MDJIT problem: at each instant k, produce the part for which $x_{i,k-1}$ is the farthest from kr_i. But this algorithm is not optimal, as shown by the following simple example. Assume that 6 parts of type A, 6 parts of type B and 1 part of type C are to be produced. Then, the greedy algorithm would schedule the part of type C at time 5 since $\frac{5}{13} - 0 > 5 * \frac{6}{13} - 2$, thus leading to a maximum deviation of $\frac{10}{13}$ (at time 6). But the optimal solution $ABABABCABABAB$, where part C is produced at time 7, has a maximum deviation of $\frac{9}{13}$. (The famous Alabama paradox similarly similarly establishes the shortcomings of greedy approaches in the apportionment context.) Note also that the greedy algorithm is pseudo-polynomial, but not polynomial.